Valuing Multiple Employee Stock Options Issued by the Same Company

PATRICK J. DENNIS AND RICHARD J. RENDLEMAN, Jr.

PATRICK J. DENNIS
is an associate professor in the McIntire School of Commerce at the University of Virginia in Charlottesville, VA.
pjd9v@virginia.edu

RICHARD J. RENDLEMAN, Jr.
is a visiting professor in the Tuck School of Business at Dartmouth in Hanover, NH, and a professor emeritus at the University of North Carolina, Kenan-Flagler Business School in Chapel Hill, NC.
richard_rendleman@unc.edu

Current techniques employed in valuing multiple employee stock options issued by the same company treat each option as being independent of the others and do not reflect that the option values should be determined simultaneously. In this article, we develop a model to simultaneously value all employee options issued by the same company and study the valuation impact associated with ignoring this simultaneity when more conventional approaches to employee option valuation are used. Using a set of valuation examples, we demonstrate that if each employee stock option is issued on a small to moderate fraction of a firm's outstanding shares, the standard Black–Scholes/binomial and multiple employee option pricing models give values that are not materially different from each other. However, if employee options are issued on a larger portion of outstanding shares, using the Black–Scholes/binomial approach can result in valuation errors as great as 20% to 50%. Therefore, in cases for which there is the potential for significant equity dilution resulting from the exercise of employee options, there can be a material difference between option values computed via the multiple employee option pricing model and those computed using more standard option valuation methods.

Modeling the effects of dilution in the valuation of employee stock options is not as simple as it may seem. In a recent article, Hull and White [2004a] assumed that the per share stock price after (employee) options are issued is lognormal while recognizing that

> [a]n alternative, more theoretically correct, assumption is that the value of the stock plus all outstanding employee stock options (as well as warrants and convertibles, if any) follows geometric Brownian motion. This assumption is consistent with Galai and Schneller's [1978] approach. Unfortunately, it is extremely difficult to extend their work to the situation in which more than one option issue is outstanding (p. 118).

Extending the work of Galai and Schneller [1978] and others, we take the "more theoretically correct" approach, as suggested by Hull and White, by assuming that the distribution of the value of a company's common stock plus outstanding employee options follows geometric Brownian motion and develop a general model for simultaneously valuing any number of outstanding employee options.

Consider a company that has issued multiple employee options with maturities that range over several years. When the nearby option matures and the holders of the option must decide whether or not to exercise, the values of all the remaining options must be factored into the exercise decision. The outcome of this exercise decision, in turn, will affect the exercise decisions, related payoffs, and values of the remaining options. Thus, the optimal exercise decisions, payoffs, and values of all the options are interdependent and must be computed jointly. While some authors have addressed the problem of simultaneous option valuation for a firm that has only two employee options outstanding (Darsinos and Satchell [2002], Kapadia and Willette [2002], Bodurtha [2002], and Lim and Terry [2003]), the more general and realistic case of simultaneously valuing the options of firms that have issued more than two employee options has not been examined.[1]

The problem of joint valuation also arises with other types of financial claims. For example, valuing multiple warrants issued by the same company is directly analogous to valuing multiple employee options, since warrants and employee options are essentially the same except for vesting features and marketability. Also, most firms that borrow are likely to have numerous debt instruments outstanding with different maturities and different terms. Some instruments may be senior, others may be subordinated, some may be callable or noncallable, and others may be convertible to common stock or into other securities. As with multiple employee options, all debt instruments of the same firm should be valued simultaneously, since the potential disposition of each instrument should affect the values of the others.[2] The simultaneous valuation of multiple debt instruments represents a more complex pricing problem, since the various debt instruments issued by the same company are not likely to be as homogeneous in their structure and contractual features as employee options and warrants.

In this article, we develop a model for simultaneously valuing multiple employee options issued by the same company and study the impact of ignoring this simultaneity in the pricing of each individual option. The model as formulated can accommodate any number of employee options, although the practical limit to the number of distinctly different options that the model can handle, based on present desktop computing standards, is around 30. While we focus on the valuation of employee stock options, our method is general and can be applied in other situations when there are multiple claims on assets, such as convertible debt or warrants where the disposition of any one claim has a potentially dilutive effect on common equity and the other outstanding securities.

Our work advances the literature of employee option valuation in two distinct ways. First, it provides a methodology for valuing up to 30 employee options issued by the same company within a reasonable amount of time. This represents a significant advancement in relation to previous models that have been limited to two outstanding employee options.

The second contribution of our work is to show how much difference it makes if one uses our model of employee option pricing rather than a simpler Black–Scholes or standard binomial model that ignores the interdependence among optimal exercise decisions and resulting option values. In some situations, especially those that involve options on a relatively small number of shares, there is very little difference between the option prices of our model and those using more conventional models. However, in other situations the value differences can be large.

The remainder of this article is organized as follows. In the next section, we summarize prior work on the simultaneous valuation of multiple employee options or warrants. The third section develops a two-period example of the multiple employee option pricing model as applied to the valuation of two employee options that have been issued by the same firm and shows how the model can be generalized to value a larger number of options over a larger number of time periods. In the fourth section, the model is applied to some realistic valuation scenarios to determine the extent to which option values computed via the model compare with those computed using more standard methods. The last section provides concluding comments.

PRIOR WORK ON MULTIPLE EMPLOYEE (OR WARRANT) VALUATION

Darsinos and Satchell [2002], Bodurtha [2002], Kapadia and Willette [2002], and Lim and Terry [2003] addressed the issue of the simultaneous valuation of multiple employee stock options or warrants. Only Darsinos and Satchell formulated a model for valuing more than two options simultaneously. However, both their model and that of Bodurtha ignored that the conditional value of an option depends on the values of options that mature later. The models of Kapadia and Willette and Lim and Terry assume lognormal returns to total equity and require a search for the critical equity value at each option maturity date that causes the maturing option to have nonnegative exercise value.[3] In contrast to these two models, our methodology is cast in a path-dependent binomial framework and, therefore, can easily accommodate valuation structures that might otherwise create path dependencies that the other two models cannot easily handle (e.g., nonproportional dividends, state- and time-dependent volatility structures, and so forth).[4]

When pricing options with two different maturities, the lognormal multiple employee option pricing models of Kapadia and Willette and Lim and Terry require one to evaluate cumulative bivariate normal probabilities. The same models, if extended to the valuation of options with three distinctly different maturities, would require that cumulative trivariate normal probabilities be evaluated, and so on. In addition, if extended versions of these models were used to value m options with different maturities, one would need to solve iteratively and simultaneously for $2^{m-1} + 2^{m-2} + \cdots + 2^{m-m} = 2^m - 1$ unique values of total equity, denoted as A^*, that will cause the exercise values of expiring options to be zero. For an $m = 10$ option problem, one would have to solve for 1,023 unique values of A^*.

Presently, numerical routines for directly evaluating cumulative multivariate normal probabilities are available for bivariate and trivariate normals only (see Genz [2004, p. 151]). For multivariate normals of higher dimension, m, methods involving numerical integration via Monte Carlo simulation (Genz [1992]) or Simpson's Rule (Schervish [1984, 1985]) are generally employed. For example, the multivariate normal routine in R, the programming language of choice among academic statisticians, is based on the Genz method.[5] In testing the two methods, Genz [1993] concluded that the practical limit to m using the Schervish method is six. Extrapolating computational timings given by Genz [1993] to 2006 computing standards, we estimate the present day limit to be $m = 8$ for the Schervish method. The Genz [1992] Monte Carlo method is much faster, but it produces different values each time a given cumulative multivariate normal probability is evaluated. Therefore, the Genz method would present significant computational problems when used in conjunction with iterative numerical search for $2^m - 1$ unique A^* values, where $m > 8$.

Inspection of Lim and Terry's formulation as applied to two options with different maturities indicates that four bivariate normal probabilities must be evaluated along with two univariate normal probabilities. Extending their methodology to three options, one would have to evaluate two univariate normal probabilities, four bivariate probabilities, and eight trivariate probabilities or, in general, 2^1 univariates + 2^2 bivariates + $\cdots$ + 2^m m-dimensional multivariate normal probabilities.

Based on estimates of computation times that we will provide upon request, we believe that it is infeasible to implement the Kapadia and Willette [2002] and Lim and Terry [2003] models for anything close to the problem size that we analyze in this article. For example, we have estimated a computation time of over 37 days for an 8-option problem using the models of Kapadia and Willette and Lim and Terry. In estimating this computation time, we have made assumptions most favorable to the two models. By contrast, an 8-option version of the pricing model summarized in Exhibit 3 takes only 0.0075 seconds of computation time.

MULTIPLE EMPLOYEE OPTION PRICING MODEL

Employee options are complex financial instruments whose proper valuation is complicated by the following issues/features:

1. Employee options are not transferable and, therefore, cannot be traded on an exchange or other organized market.[6]
2. Typically, employee options are issued with original maturities of 10 years. Estimating equity volatility over such a long period can be problematic.
3. The proper valuation of employee options should reflect employee vesting and the termination of employment prior to vesting.[7]
4. The presence of reload features (new option issuances as others expire or are exercised) and repricing features (potential changes to the terms of an option over its life) can affect the estimation of an option's fair value.

We do not wish to minimize the importance of these special features related to employee options and their potential impact on valuation. However, we believe it is important to begin the process of valuing employee options by treating the options as if they were fully marketable securities, such as warrants, not subject to the special valuation problems just listed, and then adjust these values along the lines of Rubinstein [1995] to take account of the special features. Therefore, our model might be more properly thought of as a model for valuing multiple warrants issued by the same company, rather than multiple employee options, recognizing that this should also be the starting point, but not the ending point, for valuing employee options.

We begin with the assumption that the value of a company's total equity, including common stock and any employee options that it has issued, follows a standard binomial approximation to a constant-volatility lognormal distribution. The valuation of employee options within this framework results in an implied distribution of common per share stock values that is not lognormal and whose volatility is a function of the level of the stock price and of time.

The problem of valuing multiple options over multiple maturities is inherently path dependent, meaning that the prices of options associated with a given value of equity at any future date depends upon the prior pattern of movements in equity value. For example, suppose a company with \$100 million in total equity today potentially can reach a value of \$120 million five years from now. If the firm's equity value eventually reaches \$120 million by first falling and then rising toward the end, options with maturities less than five years are less likely to have been exercised than if the firm's value reaches \$120 million by rising first and then falling. Therefore, the number of shares outstanding when the firm's equity value eventually reaches \$120 million will depend upon the pattern of prior movements in equity value. This path-dependency causes the pricing of employee options to be computationally intensive. Using a conventional binomial valuation structure, we have estimated that the computation time required for a 30-period (and 30-option) valuation problem is on the order of 130,000 years using a 2-Ghz Pentium 4 computer, and the storage requirements for such a problem would be well beyond the capacity of modern-day computers.[8] However, by recasting the calculations in terms of the depth-first technique (Dennis [2001]) and employing pruning methods for eliminating large portions of unnecessary or redundant computations within the binomial tree, we are able to eliminate the storage problem and reduce the computation time for a 30-period problem to approximately four hours.

Ownership Structure, Partial Exercise, and Premature Exercise

Two extremes characterize the potential ownership structure of a company's outstanding employee options and warrants. At the one extreme, a single individual owns all the company's outstanding options. At the other extreme, the options are held in small amounts by a large number of competing individuals. As shown in Emanuel [1983], Constantinides [1984], Constantinides and Rosenthal [1984], and Cox and Rubinstein [1985], if a single individual owns all of a company's outstanding options, a strategy of partial exercise may create more value than a strategy of block exercise, that is, exercising all or none of a given option issue. However, as shown by Constantinides [1984] and Constantinides and Rosenthal [1984], in a competitive equilibrium in which options are held in small amounts by a large number of competing individuals, option prices will be the same as those established by a single owner who is constrained to a policy of block exercise. In this article, we assume that the ownership structure of employee options is sufficiently diverse that the pricing of options is consistent with a block exercise strategy of a single individual. As such, this rules out having to consider pricing resulting from strategic gaming among competing option holders.

Given the block exercise assumption, it can be shown that in the presence of dividends, premature exercise of a single option issue among the many that a company may have issued is irrational if $X_t(n_0/(n_0 + n_t))(([1+r]^t-1)/[1+r]^t)>div$, where X_t is the strike price of the option that matures t periods from today, n_0 denotes the total number of shares outstanding at the time of possible premature exercise, n_t denotes the number of shares associated with the option that matures at time t, r is the risk-free interest rate per period, and div is the dollar value of the dividend to be paid immediately after the option is exercised. (A proof will be provided upon request.) Except for the dilution term, $(n_0/(n_0 + n_t))$, this is consistent with the standard condition of premature exercise for exchange-traded call options.

The methodology for valuing multiple employee options that we present in this article applies specifically to situations for which dividends are expected to be sufficiently low that it would not be rational for a given option issue to be exercised prematurely. As such, American style options can be valued as if they are European. In all the examples we present, dividends are assumed to be zero.

General Framework of Valuation

Given the path-dependency in the pricing of employee options, we employ the binomial model as the basis for valuation. As is well documented, the binomial model can be parameterized to provide approximate values when no closed-form solutions exist for Black Scholes—type option valuation problems. It is in that sense that the binomial model is used here.

In the multiple employee option pricing model, the value of total equity, defined as the value of common equity plus the value of all employee options prior to the infusion of new equity capital from option exercise, is assumed to follow a standard multiplicative binomial process with constant up and down factors of u and d, respectively. The company issues m employee options. Option 1 expires at binomial Time $t = 1$, Option 2 at binomial Time $t = 2$, and so on. The strike prices of the m options are $X_i = X_1, X_2 \ldots X_m$, and the number of shares associated with each is $n_i = n_1, n_2 \ldots n_m$, with n_0 denoting the number of common shares originally outstanding. If a company has issued no employee options for a given maturity, the number of shares for that maturity can be set to zero. Thus, having option expiration time correspond precisely with binomial time is not as restrictive as it may seem.[9] Also, if a company has issued options with the same terms to more than one employee, for valuation purposes we treat that collection of options as a single option issued to a single individual.

The key to valuing options via the multiple employee option pricing model is simultaneously keeping track of binomial-based increases and decreases in total equity value and the many possible combinations of exercise decisions for the m options issued by the firm. To facilitate this process, let j_t denote a binary index value applicable to binomial Time t that indicates whether total equity increases or decreases in value from Time $t - 1$ to Time t. A value of $j_t = 1$ indicates an increase in the value of total equity at Time t, and a value of $j_t = 0$ indicates a decrease in value. Also, let k_t denote a binary index value that indicates if the option that matures at Time t is exercised at Time t. If $k_t = 1$, option t is exercised, and if $k_t = 0$, option t is not exercised. Although we could present a detailed mathematical formulation of an m-period model, we have chosen, instead, to present a two-period example that is easily generalized for m periods.

Two-Period Example

A two-period example is presented in Exhibit 1. Before the specific details of the example are presented, however, it is useful to consider the general structure of Exhibit 1.

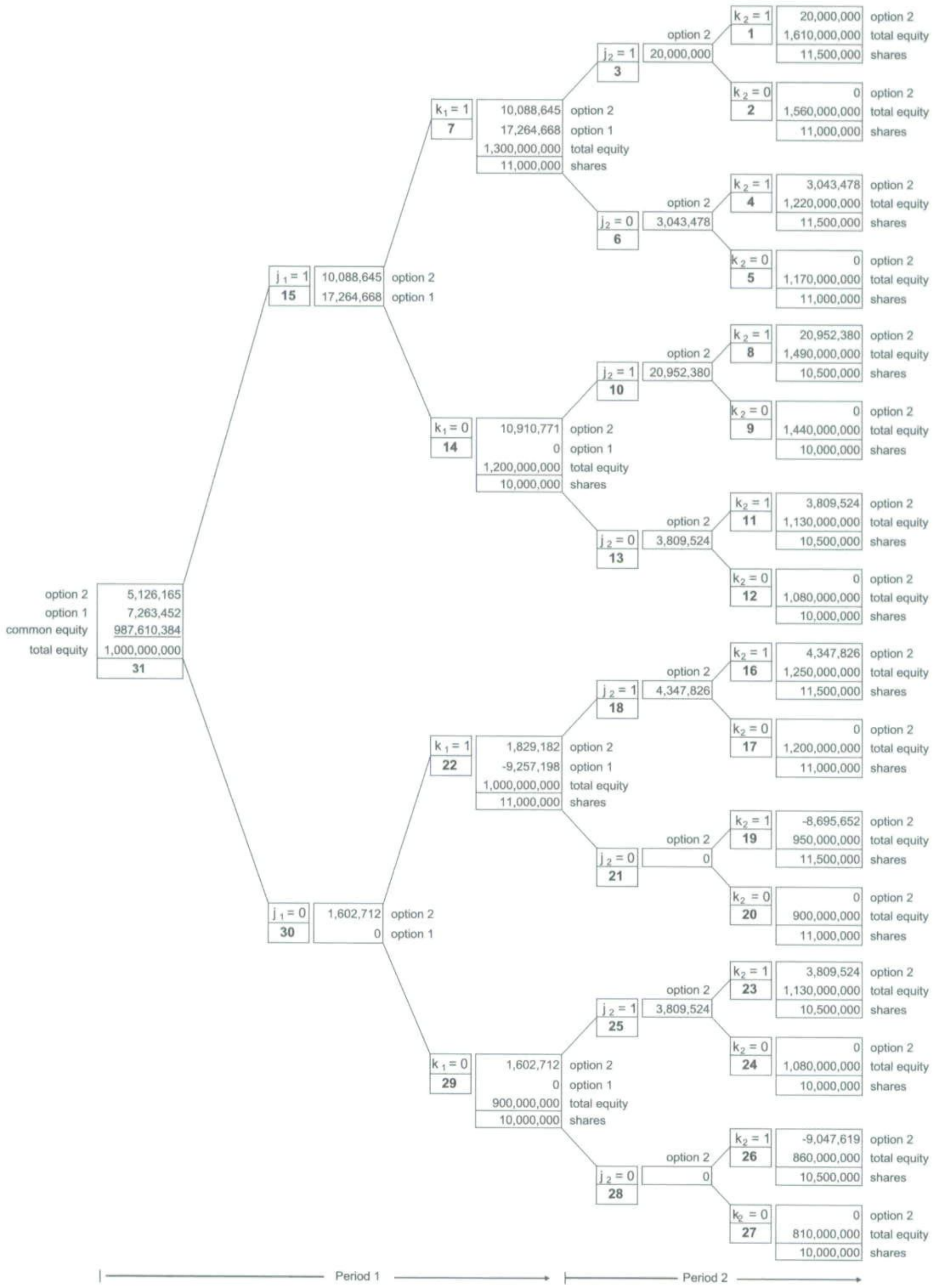
The structure of Exhibit 1 is that of a non-recombining binomial tree. The first branch of the tree indicates a potential change in the value of total equity as of binomial Time 1. The upper portion of the initial branch is associated with an increase in total equity value ($j_1 = 1$), and the lower portion is associated with a decrease in the value of total equity ($j_0 = 1$). Note that each value of j_1 is associated with two possible values of k_1, with $k_1 = 1$ indicating that the option that matures at Time 1 is exercised, and $k_1 = 0$ indicating that the same option is not exercised. It is also important to note that passage of time from Time $t = 0$ to Time $t = 1$ is implied by the j_1 index values, but no further passage of time is implied by the values of k_1.

At Time 1 there are four possible states, or sequences of events, $\{j_1 = 0, k_1 = 0\}$, $\{j_1 = 0, k_1 = 1\}$, $\{j_1 = 1, k_1 = 0\}$, and $\{j_1 = 1, k_1 = 1\}$. From each of these sequences, the value of total equity can, again, go up ($j_2 = 1$) or down ($j_2 = 0$), as of Time 2, and from each resulting up or down state, Option 2 can be exercised ($k_2 = 1$) or not exercised ($k_2 = 0$). Note that there are 16 possible sequences of events as of Time 2, $\{j_1 = 0, k_1 = 0, j_2 = 0, k_2 = 0\}$, $\{j_1 = 0, k_1 = 0, j_2 = 0, k_2 = 1\}$, $\cdots$ $\{j_1 = 1, k_1 = 1, j_2 = 1, k_2 = 1\}$. This general structure, when extended to m binomial time periods, results in 4^m possible sequences through Time m and a total of $2 \times 4^m - 1$ states within the binomial tree. In the two-period example, the total number of sequences through Time 2 is $4^m = 4^2 = 16$, and the total number of states within the binomial tree is $2 \times 4^m - 1 = 2 \times 4^2 - 1 = 31$.

In the example, a company has issued two employee stock options. Option 1 matures at binomial Time 1 and gives an employee the right to purchase 1,000,000 shares of stock at \$100 per share. Option 2 matures at binomial Time 2 and gives an employee the right to purchase 500,000 shares of stock at \$100 per share. Presently there are 10,000,000 shares of common stock outstanding, and the total value of equity, including common equity and employee options, is \$1 billion. The value of total equity follows a standard multiplicative binomial process with $u = 1.2$ and $d = 0.9$, and the risk-free rate of interest is 0.03 per binomial period. These parameters imply a risk-adjusted probability associated with an increase in total equity value of $\pi = (1 + r + d)/(u - d) = (1.03 - 0.9)/(1.2 - 0.9) \approx 0.433$, where r denotes the risk-free rate of interest per binomial period.

In the model, the value of each option is determined recursively using the depth-first technique of Dennis [2001]. To illustrate the valuation process, consider the upper-most portion of Exhibit 1 as of Time 2, labeled as state 1 (shown as 1 in the exhibit). This portion of the

EXHIBIT 1
Two-Period Example of Multiple Employee Stock Option Pricing Model



Note: Option 1 matures at binomial time 1 and option matures at time 2. Both options have a strike price of \$100 per share. Option 1 is for 1,000,000 shares of common stock and option 2 is for 500,000 shares. There are 10,000,000 shares of stock originally outstanding. Total equity follows a standard multiplicative binomial process with $u = 1.2$ and $d = 0.9$. The risk-free interest rate is 0.03 per binomial period.

tree establishes the maturity value of Option 2 for sequence $\{j_1 = 1, k_1 = 1, j_2 = 1, k_2 = 1\}$, the sequence for which the value of total equity increases at Time 1 and Time 2, and Options 1 and 2 are both exercised. Note that if both options are exercised, there will be $n_0 + n_1 + n_2 =$ 10,000,000 + 1,000,000 + 500,000 = 11,500,000 shares outstanding in State 1. Also, as shown in State 1, the value of total equity is $1,610,000,000, reflecting two increases in equity value *and* the exercise of both options. The $1,610,000,000 total equity value is computed as follows.

Let A_t denote the value of total equity as of Time t, with A_0 denoting the initial value of total equity, or $1 billion in this example. Then, the value of total equity as of Time t is given by $A_t = A_{t-1}u^{j_t}d^{1-j_t} + n_tX_tk_t$ for $t = 1, 2, \ldots m$. This value reflects natural changes in equity value as implied by the value sequence $j_1, j_2, \ldots j_m$, plus the infusion of new capital associated with exercise sequence $k_1, k_2, \ldots k_m$. Any new capital obtained though the exercise of options is assumed to be invested back into the firm and to follow the same binomial process as previously existing equity capital.[10] Noting that $j_1 = 1$, $k_1 = 1$, $j_2 = 1$, and $k_2 = 1$ in State 1, the value of total equity as of Time 1 is $A_1 = A_0u^{j_1}d^{1-j_1} + n_1X_1k_1 = \$1{,}000{,}000{,}000 \times 1.2^1 \times 0.9^{1-1} + 1{,}000{,}000 \times \$100 \times 1 = \$1{,}200{,}000{,}000 + \$100{,}000{,}000 = \$1{,}300{,}000{,}000$, reflecting a natural increase in total equity value from $1 billion to $1.2 billion plus the infusion of $100,000,000 in new equity capital from the exercise of Option 1. With $1.3 billion in total equity value as of Time 1, the total value of equity, as of Time 2, becomes $A_2 = A_1u^{j_2}d^{1-j_2} + n_2X_2k_2 = \$1{,}300{,}000{,}000 \times 1.2^1 \times 0.9^{1-1} + 500{,}000 \times \$100 \times 1 = \$1{,}560{,}000{,}000 + \$50{,}000{,}000 = \$1{,}610{,}000{,}000$. This value, when distributed among 11,500,000 shares, results in a stock price of $1,610,000,000/11,500,000 = $140 per share. Therefore, when Option 2 is exercised, the employee nets $140 – $100 = $40 per share in State 1. Since the option is for 500,000 shares, the total value of Option 2 in State 1 is 500,000 × $40 = $20,000,000.

In State 2, the value of Option 2 is computed for the sequence $\{j_1 = 1, k_1 = 1, j_2 = 1, \text{and } k_2 = 0\}$. Since $k_1 = 1$ and $k_2 = 0$, the entries for State 2 reflect that Option 1 is exercised, but Option 2 is not exercised. Therefore, the value of Option 2 in State 2 is $0.

The value of Option 2 as shown in State 3 reflects the optimal exercise decision with respect to Option 2, given that the value of total equity increases twice and Option 1 is exercised at Time 1. Inasmuch as the $20,000,000 obtained from exercising Option 2 exceeds $0, the value associated with not exercising, the relevant value of Option 2 in State 3 is its exercise value of $20,000,000. Similarly, the $3,043,478 value of Option 2 in State 6 reflects the optimal exercise decision for Option 2, given that the value of total equity increases at Time 1 ($j_1 = 1$), Option 1 is exercised at Time 1 ($k_1 = 1$), and the value of total equity decreases at Time 2 ($j_2 = 0$).

Next, consider States 8 and 9. Since both states are part of the branch that evolves from State 14, a state for which $k_1 = 0$, both are associated with Option 1 *not* being exercised. The value of Option 2 in State 8 reflects that Option 1 is not exercised at Time 1, but that Option 2 *is* exercised at Time 2. Under these circumstances, the total value of equity is $1,490,000,000, and there are 10,500,000 shares of stock outstanding, resulting in a stock price of $1,490,000,000/10,500,000 = $141.904 per share. Therefore, the net value of exercising Option 2 is $141.904 – $100 = $41.904 per share or $41.904 × 500,000 = $20,952,380 total. The value of Option 2 in State 9 reflects that Option 2 is not exercised. Therefore, the value of Option 2 in State 9 is $0. The $20,952,380 value for Option 2 in State 10 reflects that if Option 1 is not exercised ($k_1 = 0$), it will be optimal to exercise Option 2, since $20,952,380 > $0. Similarly, the 3,809,524 value of Option 2 in State 13 reflects that it is optimal to exercise Option 2 in this state.

Now consider the entries for State 7. Since $j_1 = 1$ and $k_1 = 1$, this state is associated with an increase in equity value for the first binomial period and Option 1 being exercised. In this state, the relevant up-state value of Option 2 is $20,000,000 and the relevant down-state value is $3,043,478. Therefore, in State 7, the value of Option 2 is [$20,000,000(0.433) + $3,043,478(1 – 0.433)]/1.03 = $10,088,645. Similarly, the entries for State 14 show that if Option 1 is not exercised ($k_1 = 0$), the value of Option 2 is [$20,952,380(0.433) + $3,809,524 (1 – 0.433)]/1.03 = $10,910,711.

We now turn to the valuation of Option 1. In State 7, a state for which Option 1 is exercised, the total value of equity is $1.3 billion, reflecting a natural increase in total equity value from $1 billion to $1.2 billion, plus $100,000,000 received from the exercise of Option 1. Of this $1.3 billion, $10,088,645 is the value of Option 2, which leaves $1,300,000,000 – $10,088,645 = $1,289,911,355 of value for common stock and Option 1. Upon the exercise of Option 1, there will be 10,000,000 + 1,000,000 = 11,000,000 shares outstanding, resulting in a stock price of $1,289,911,355/11,000,000 = $117.264 per share.

Therefore, the net value of exercising Option 1 is \$117.264 – \$100 = \$17.264 per share or \$17.264 × 1,000,000 = \$17,264,668 for all 1,000,000 shares.

In State 14, Option 1 has a value of \$0, since State 14 is a state for which Option 1 is not exercised. Inasmuch as exercising creates greater value than not exercising, the relevant value of Option 1 in the initial up state ($j_1 = 1$) is \$17,264,668, and the corresponding value of Option 2 is \$10,088,645. These values are shown in State 15.

Option values in the lower portion of Exhibit 1, branching from State 30, are determined in similar fashion. This same valuation process leads to values of \$0 and \$1,602,712 in State 30 for Options 1 and 2, respectively. The initial \$7,263,452 value of Option 1, shown in State 31, reflects an up-state value of \$17,264,668 and a down-state value of \$0, while the initial \$5,126,165 value of Option 2 reflects up- and down-state values of \$10,088,645 and \$1,602,712, respectively.

Generalization and Computational Considerations

The valuation model described above can be generalized for m options and m periods. Although the model is inherently path dependent, and the number of calculations required is on the order of 4^m, we have developed two "pruning" techniques that substantially reduce the model's computational requirements. With pruning, the computation time required for a 30-option (year) model using a 2-Ghz Pentium 4 computer is three to four hours. Therefore, we consider 30 options (and 30 periods) to be the practical limit of our model under present computing standards. A detailed description of the m-period valuation procedure and associated pruning methods is provided in Appendix A. In addition, Appendix B describes how model computations can be carried out when the time interval associated with each binomial period is relatively short and options do not expire at the end of each period. For example, each binomial period might represent one month, but each successive option maturity date might be one year, or 12 binomial periods, apart. Applying pruning to the methodology described in Appendix B, and also recognizing that the binomial tree recombines at all times except for the period immediately following each option maturity, we show that the model can easily accommodate the valuation of 10 options within a 50-period binomial tree (24 minutes of computation time) or r options within a 100-period tree (4.6 minutes of computation time).

APPLICATION OF MULTIPLE EMPLOYEE OPTION PRICING MODEL TO REALISTIC VALUATION SCENARIOS

Example Setup

According to Henry [2002], on average, the 200 largest U.S. companies grant options on approximately 3% of outstanding shares each year. Henry also reports an average annual grant of 1.9% for companies in the S&P Index and grants of over 10% for some high-tech companies, including a 16.5% average annual grant by Siebel Systems. Based on these figures, the multiple employee option pricing model is evaluated for annual option grants of 15% and 3% of the *current* number of shares outstanding.

The multiple employee option pricing model also requires an estimate of the volatility of a firm's returns to total equity which, in turn, is reflected in the values of u and d. Although there are various ways u and d can be expressed in terms of volatility, we assume $u = e^{\ln(1+r)-\frac{1}{2}\sigma^2+\sigma}$ and $d = e^{\ln(1+r)-\frac{1}{2}\sigma^2-\sigma}$, where r is the risk-free rate of interest per year with annual compounding and σ is an estimate of the standard deviation of the annual logarithmic return to total equity.[11] A standard deviation of 0.30 is often used as an estimate of volatility for a typical NYSE stock and 0.60 appears to be a reasonable estimate of volatility for the stock of a high-tech firm.[12] For each valuation scenario, the company is assumed to have issued options with maturities of 1 to 10 years inclusively. Inasmuch as most employee options are issued with an initial maturity of 10 years, the 10-year option is assumed to be newly issued, and the remaining options with maturities of 1 to 9 years are assumed to have been issued in past years.

In valuing the various employee options, we assume three distinctly different strike price patterns, using the parameter α to characterize a given pattern. In general, the newly issued option with a maturity of m years is assumed to be issued at the money (approximately) at a strike price of X_m. A previously issued option with a current maturity of t years has an exercise price of X_m/α^{10-t}. We let α take on values of 1.1, 1.0, and 1/1.1. When α = 1.1, all previously issued options are in the money, with each strike price being 10% lower than that of the option issued a year later. When α = 1.0, the newly issued option and all options issued in the past are assumed to have the same strike price. When α = 1/1.1, all previously issued options are out of the money, with each strike price being

10% higher than that of the option issued a year later. For each scenario, an initial strike price is computed iteratively such that the strike price equals the initial (Time 0) theoretical stock price rounded to the nearest dollar.[13]

Obviously, there are an infinite number of possible strike price patterns for employee options with maturities ranging from 1 to 10 years, and we cannot possibly evaluate them all. However, we feel that the option values computed for these three scenarios capture many of the important pricing relationships that one might observe in company-specific patterns actually observed in practice.

For each scenario, the initial value of total equity is \$1 billion, and there are 10,000,000 shares of common stock outstanding. The risk-free rate of interest, r, is 0.05 per year compounded annually and each binomial period represents one full year.

Exhibits 2 to 4 summarize valuation results for the multiple employee option pricing model, assuming a volatility for total equity of 0.30. The three exhibits are identical in structure and underlying assumptions, but they differ by the strike price patterns of options issued in past years. In Exhibit 2, $\alpha = 1.1$, implying that all outstanding options are in the money, with the longer-term options being more in the money than the shorter-term options. In Exhibit 3, $\alpha = 1$, implying that all outstanding options were issued with the same strike price. In Exhibit 4, $\alpha = 1/1.1$, which implies that all outstanding options are out of the money. In the remainder of this section, we focus on the entries in Exhibit 3, but the conclusions reached apply to the strike price scenarios summarized in Exhibits 2 and 4 as well. If anything, the pricing errors from ignoring that all option exercise decisions must be determined jointly for the in-the-money and out-of-the money exercise patterns of Exhibits 2 and 4 are smaller than those documented in detail for Exhibit 3.

As in Exhibit 1, the entries in Exhibits 2 to 4 are based on a simple binomial model in which there is only 1 binomial period per year. Later, we show that transforming the option values in Exhibits 2 to 4 to control-variate equivalents results in approximately the same values as those obtained using the control-variate technique with 5 binomial periods per year. Although we would like to apply the control-variate technique in a binomial setting with a larger number of time periods, we estimate that it would take over three months of computation time to repeat the control-variate analysis using 10 binomial periods per year.

As in the example illustrated in Exhibit 1, the initial value of total equity is \$1 billion in all versions of Exhibit 2, and there are 10,000,000 shares of stock outstanding. Values in Panel A of Exhibit 3 are computed as if each option has a strike price of \$68 and is for 1,500,000 shares, or 15% of outstanding shares. Values in Panel B are based on each option being issued on 300,000 shares with a strike price of \$91. In both panels, the same strike price applies to all 10 options and is selected to equal the theoretical stock price rounded to the nearest dollar.

Consider the third column of Panel A of Exhibit 3 which shows option values computed using the multiple option pricing model. The sum of the option values in this column is \$321,301,625. Since the total value of equity is \$1 billion, the theoretical value of common stock computed via the multiple employee option pricing model is $\$1,000,000,000 - \$321,301,625 = \$678,698,375$. With 10,000,000 shares of stock initially outstanding, the stock price per share is $\$678,698,375/10,000,000 \approx \$67.870$. The \$67.870 stock price is approximately equal to the \$68 common strike price assumed in the computation of option values. Thus, the new 10-year option would be issued approximately at the money.

If the multiple employee option pricing model is formulated properly, it should produce option values identical to the values of exchange-traded options whose values are determined directly from the implied distribution of underlying stock prices (see Cox and Rubinstein [1985, p. 395], Galai and Schneller [1978, p. 1336], and Hull and White [2004a, p. 117]). Exhibit 5 illustrates the equivalence of the two pricing methods for the option that matures at Time 3. (With a non-recombining binomial tree, this is the longest-term option among the 10 whose time- and state-dependent values can be represented within a single-page exhibit.) Throughout the tree the stock price per share reflects the simultaneous pricing of all 10 options and their associated optimal exercise strategies. Option prices at Time 3 reflect the optimal exercise of a three-period exchange-traded option with a strike price of \$68. Option prices at Times 0 through 2 are computed using the standard binomial model, assuming a risk-free interest rate of 5% per binomial period and a risk-adjusted probability associated with an increase in the stock price of 0.50113.[14] The initial option price is \$15.086, equivalent to the \$22,628,678 value for the 3-year employee option shown in the third column of Exhibit 3 (Panel A) divided by the 1,500,000 shares on which the employee option is issued.

Exhibit 2

Comparison of Option Values Computed via Multiple Employee Option Pricing Model and Standard Binomial Model with $\sigma = 0.3$

Panel A: Shares per option = 1,500,000; strike price of newly issued option = \$56; strike price factor = 1.1

Option maturity (t)	Strike	Cumulative Logarithmic Returns to Common Equity: Mean (1)	Std. Dev. (2)	Option Value Computed Using Multiple Option Pricing Model (3)	Standard Binomial Option Value Computed with Common Stock Volatility of: 1-year (4)	t-years (5)	10-years (6)
1	23.749	0.02409	0.22092	49,764,855	49,764,855	49,764,855	49,764,855
2	26.124	0.02344	0.22386	48,149,245	48,149,245	48,149,245	48,149,245
3	28.737	0.02270	0.22721	46,456,701	46,456,701	46,456,701	46,740,453
4	31.611	0.02181	0.23111	45,202,949	45,158,975	45,254,194	45,483,910
5	34.772	0.02083	0.23537	44,007,627	43,756,699	44,161,518	44,757,143
6	38.249	0.01978	0.23981	42,769,576	42,451,139	42,761,831	43,465,424
7	42.074	0.01869	0.24441	42,303,132	41,629,920	42,443,759	42,873,833
8	46.281	0.01757	0.24901	41,666,880	40,359,430	41,935,054	42,402,180
9	50.909	0.01646	0.25348	40,934,768	39,420,724	40,825,840	41,147,069
10	56.000	0.01537	0.25779	40,793,171	38,835,325	41,012,235	41,012,235
		Total Option Value		442,048,904	435,983,011	442,765,233	445,796,345
		Common Equity Value		557,951,096	557,951,096	557,951,096	557,951,096
		Total Equity Value		1,000,000,000	993,934,107	1,000,716,329	1,003,747,442
		Stock Price		55.795	55.795	55.795	55.795

Panel B: Shares per option = 300,000; strike price of newly issued option = \$86; strike price factor = 1.1

Option maturity (t)	Strike	Cumulative Logarithmic Returns to Common Equity: Mean (1)	Std. Dev. (2)	Option Value Computed Using Multiple Option Pricing Model (3)	Standard Binomial Option Value Computed with Common Stock Volatility of: 1-year (4)	t-years (5)	10-years (6)
1	36.472	0.01167	0.27190	15,382,810	15,382,810	15,382,810	15,382,810
2	40.120	0.01139	0.27296	14,886,587	14,886,587	14,886,587	14,886,587
3	44.132	0.01103	0.27428	14,522,170	14,518,928	14,530,091	14,581,383
4	48.545	0.01063	0.27577	14,109,624	14,099,900	14,109,277	14,130,443
5	53.399	0.01020	0.27737	13,979,061	13,946,039	13,986,692	14,042,742
6	58.739	0.00975	0.27899	13,715,298	13,640,061	13,738,888	13,823,937
7	64.613	0.00931	0.28059	13,464,334	13,389,864	13,468,514	13,507,862
8	71.074	0.00888	0.28217	13,465,636	13,330,469	13,485,861	13,529,741
9	78.182	0.00846	0.28369	13,250,308	13,033,332	13,286,688	13,317,540
10	86.000	0.00805	0.28515	13,107,697	12,912,920	13,121,317	13,121,317
		Total Option Value		139,883,524	139,140,910	139,996,726	140,324,363
		Common Equity Value		860,116,476	860,116,476	860,116,476	860,116,476
		Total Equity Value		1,000,000,000	999,257,386	1,000,113,202	1,000,440,838
		Stock Price		86.012	86.012	86.012	86.012

Note: The risk-free rate is 5% a year, and each binomial period represents one full year. Values of u and d for the multiple option pricing model, as applied to total equity are computed as $u = \exp(\ln[1.05] - \frac{1}{2}[0.3^2] + 0.3)$ *and* $d = \exp(\ln[1.05] - \frac{1}{2}[0.3^2] - 0.3)$*, respectively. There are 10,000,000 shares of stock initially outstanding. For the standard binomial values computed using* $t > 1$*-year volatility, the values of u and d are computed as* $u = \exp(\ln[1.05] - \frac{1}{2}\sigma^2_{com} + \sigma_{com})$ *and* $d = \exp(\ln[1.05] - \frac{1}{2}\sigma^2_{com} - \sigma_{com})$*, where* σ_{com} *is the standard deviation of the cumulative annual logarithmic return to common equity. For the standard binomial values computed using 1-year volatility, u and d are the ratios of the value of common equity in the immediate up and down states, respectively, to the initial value of common equity.*

EXHIBIT 3

Comparison of Option Values Computed via Multiple Employee Option Pricing Model and Standard Binomial Model with $\sigma = 0.3$

Panel A: Shares per option = 1,500,000; strike price of newly issued option = \$68; strike price factor = 1.0

Option maturity (t)	Strike	Cumulative Logarithmic Returns to Common Equity: Mean (1)	Std. Dev. (2)	Option Value Computed Using Multiple Option Pricing Model (3)	Standard Binomial Option Value Computed with Common Stock Volatility of: 1-year (4)	t-years (5)	10-years (6)
1	68.000	0.02777	0.20352	12,550,415	12,550,415	12,550,415	14,908,556
2	68.000	0.02657	0.20933	16,387,936	16,142,578	16,379,128	17,976,473
3	68.000	0.02528	0.21546	22,628,678	21,978,515	22,807,521	25,332,204
4	68.000	0.02407	0.22105	26,739,017	25,898,185	26,873,677	28,506,089
5	68.000	0.02282	0.22668	31,146,453	29,615,468	31,527,288	33,590,375
6	68.000	0.02161	0.23202	35,252,486	33,655,471	35,511,241	36,746,217
7	68.000	0.02038	0.23733	38,770,062	36,166,052	39,246,693	40,563,490
8	68.000	0.01916	0.24245	42,679,984	40,225,761	42,994,484	43,644,822
9	68.000	0.01795	0.24748	45,833,374	42,653,389	46,164,569	46,610,729
10	68.000	0.01675	0.25234	49,313,220	45,939,974	49,587,432	49,587,432
		Total Option Value		321,301,625	304,825,808	323,642,447	337,466,388
		Common Equity Value		678,698,375	678,698,375	678,698,375	678,698,375
		Total Equity Value		1,000,000,000	983,524,183	1,002,340,822	1,016,164,763
		Stock Price		67.870	67.870	67.870	67.870

Panel B: Shares per option = 300,000; strike price of newly issued option = \$91; strike price factor = 1.0

Option maturity (t)	Strike	Cumulative Logarithmic Returns to Common Equity: Mean (1)	Std. Dev. (2)	Option Value Computed Using Multiple Option Pricing Model (3)	Standard Binomial Option Value Computed with Common Stock Volatility of: 1-year (4)	t-years (5)	10-years (6)
1	91.000	0.01226	0.26969	4,166,105	4,166,105	4,166,105	4,362,230
2	91.000	0.01168	0.27187	4,948,255	4,911,651	4,933,620	5,055,253
3	91.000	0.01115	0.27386	7,094,511	7,040,534	7,113,829	7,312,754
4	91.000	0.01065	0.27572	7,881,627	7,795,679	7,875,109	7,994,975
5	91.000	0.01016	0.27749	9,425,560	9,300,153	9,458,619	9,615,024
6	91.000	0.00970	0.27918	10,180,987	10,039,146	10,181,726	10,269,642
7	91.000	0.00926	0.28080	11,401,730	11,197,157	11,440,528	11,537,085
8	91.000	0.00883	0.28235	12,108,726	11,909,163	12,112,833	12,157,974
9	91.000	0.00841	0.28384	13,120,766	12,834,400	13,156,391	13,188,278
10	91.000	0.00802	0.28527	13,769,170	13,514,204	13,773,197	13,773,197
		Total Option Value		94,097,438	92,708,192	94,211,957	95,266,412
		Common Equity Value		905,902,562	905,902,562	905,902,562	905,902,562
		Total Equity Value		1,000,000,000	998,610,754	1,000,114,519	1,001,168,974
		Stock Price		90.590	90.590	90.590	90.590

Note: The risk-free rate is 5% a year, and each binomial period represents one full year. Values of u and d for the multiple option pricing model, as applied to total equity are computed as $u = \exp(\ln[1.05] - \frac{1}{2}[0.3^2] + 0.3)$ and $d = \exp(\ln[1.05] - \frac{1}{2}[0.3^2] - 0.3)$, respectively. There are 10,000,000 shares of stock initially outstanding. For the standard binomial values computed using $t > 1$-year volatility, the values of u and d are computed as $u = \exp(\ln[1.05] - \frac{1}{2}\sigma^2_{com} + \sigma_{com})$ and $d = \exp(\ln[1.05] - \frac{1}{2}\sigma^2_{com} - \sigma_{com})$, where σ_{com} is the standard deviation of the cumulative annual logarithmic return to common equity. For the standard binomial values computed using 1-year volatility, u and d are the ratios of the value of common equity in the immediate up and down states, respectively, to the initial value of common equity.

Potential Pricing Error Using Standard Option Pricing Models to Value Multiple Employee Stock Options

In contrast to the multiple employee option pricing model, both the Black–Scholes and standard binomial models are designed to value single options issued on an underlying asset whose returns are assumed to follow a log-normal distribution. It is interesting to consider how much error is introduced into option valuation when the multiple employee option pricing model ought to be used to simultaneously value all options issued by the same company, but a simpler standard binomial (or Black–Scholes) approach to valuation is used instead. Inasmuch as the multiple employee option pricing model is based on binomial pricing, potential error should be measured in relation to the standard binomial model rather than to the Black–Scholes model. Otherwise, the usual small differences between standard binomial prices and Black–Scholes prices would get reflected in the errors.[15]

The standard binomial option pricing model requires an estimate of the volatility of the underlying stock's returns over the option's life. The model can accommodate situations for which volatility is a deterministic function of time. If the options to be valued are of the American type, the branching structure of the binomial tree should reflect the anticipated period-by-period volatility structure. However, if options are of the European type, or if they are American options that can be valued as if they are European, one can employ a single "average" volatility when constructing the binomial tree that takes account of the expected time-dependent volatility structure.[16] Since we assume that dividends are zero, and, therefore, all employee options are valued as if they are European, standard binomial option prices computed with time-dependent volatilities are computed using single average volatilities.

Estimating volatility is the key to computing an option's fair value. However, for the purposes of estimating the error from using a standard binomial approach to employee option valuation when it is otherwise correct to use the multiple employee option pricing model, we compute standard binomial option prices assuming that one can estimate the "average" of the time- and state-dependent volatility of the underlying stock's returns with perfect precision. Obviously, this represents a best-case scenario. Therefore, the pricing error resulting from this scenario would represent the minimum pricing error, since it is unlikely that one would be able to estimate the time-dependent volatility structure of stock returns with such precision. Consider Exhibit 6. As in Exhibit 5, this exhibit shows how the price of the underlying stock will evolve over the first three years if the stock price is expected to reflect the employee option values shown in the third column of Exhibit 3, Panel A. In addition, the exhibit shows the logarithmic return associated with each stock price starting at Time 1.

The means, variances, and standard deviations shown at the bottom of the exhibit are incremental means, variances, and standard deviations computed for each specific time period, assuming up and down stock returns are equally likely to occur. For example, the Time 1 mean return of 0.02777 is computed as $\frac{1}{2}(0.23129)+\frac{1}{2}(-0.17575) = 0.02777$, the Time 2 mean return is computed as $\frac{1}{4}(0.23944) + \frac{1}{4}(-0.18818) + \frac{1}{4}(0.24129) + \frac{1}{4}(-0.19104) = 0.02538$ and so on. Using the average of the incremental means for Periods 1 and 2, the mean return per period over the first two periods is $(0.02777 + 0.02538)/2 = 0.02657$. For the first three periods, the mean return per period is $(0.02777 + 0.02538 + 0.02269)/3 = 0.02528$, etc. Note that these two- and three-period mean returns are the same as the cumulative means shown in the first column of Exhibit 3, Panel A.

Incremental return variances and standard deviations shown at the bottom of Exhibit 6 and cumulative standard deviations shown in the second column of Exhibit 3 are computed in similar fashion. For example, the three-period cumulative standard deviation shown in Exhibit 3, Panel A, is computed as $\sqrt{(0.04142+0.04622+0.05162)/3} = 0.21546$. Note, in particular, that the standard deviations of the cumulative logarithmic return to common equity, shown in both panels of Exhibit 3, are increasing functions of time, although there is not a great deal of difference among the various time-dependent standard deviation values.[17]

With this as background, suppose one were interested in pricing the 10-year option but did not have access to the multiple employee option pricing model and used the standard binomial model instead. For the purposes of pricing the 10-year option, assume that one knows that the 10-year cumulative standard deviation of equity returns will be 0.25234 as shown in the second column of Exhibit 3, Panel A. According to the binomial model,

EXHIBIT 4

Comparison of Option Values Computed via Multiple Employee Option Pricing Model and Standard Binomial Model with $\sigma = 0.3$

Panel A: Shares per option = 1,500,000; strike price of newly issued option = \$78; strike price factor = 1/1.1

		Cumulative Logarithmic Returns to Common Equity		Option Value Computed Using Multiple Option Pricing Model	Standard Binomial Option Value Computed with Common Stock Volatility of:		
Option Maturity (t)	Strike	Mean (1)	Std. Dev. (2)	(3)	1-year (4)	t-years (5)	10-years (6)
1	183.920	0.02092	0.23493	0	0	0	0
2	167.200	0.02065	0.23611	0	0	0	0
3	152.000	0.02036	0.23735	2,102,768	2,533,766	2,698,584	3,928,882
4	138.182	0.01998	0.23898	5,726,054	6,086,509	6,286,046	7,307,395
5	125.620	0.01955	0.24078	14,342,211	14,788,774	15,361,988	16,956,827
6	114.200	0.01902	0.24302	23,215,616	23,247,269	24,174,302	25,722,526
7	103.818	0.01835	0.24580	31,469,746	30,858,154	31,977,342	33,057,778
8	94.380	0.01754	0.24911	39,490,238	38,661,044	40,412,916	41,355,931
9	85.800	0.01663	0.25278	49,036,186	47,727,744	49,624,226	50,034,443
10	78.000	0.01564	0.25674	56,998,582	55,348,903	57,105,150	57,105,150
		Total Option Value		222,381,400	219,252,163	227,640,553	235,468,931
		Common Equity Value		777,618,600	777,618,600	777,618,600	777,618,600
		Total Equity Value		1,000,000,000	996,870,763	1,005,259,153	1,013,087,531
		Stock Price		77.762	77.762	77.762	77.762

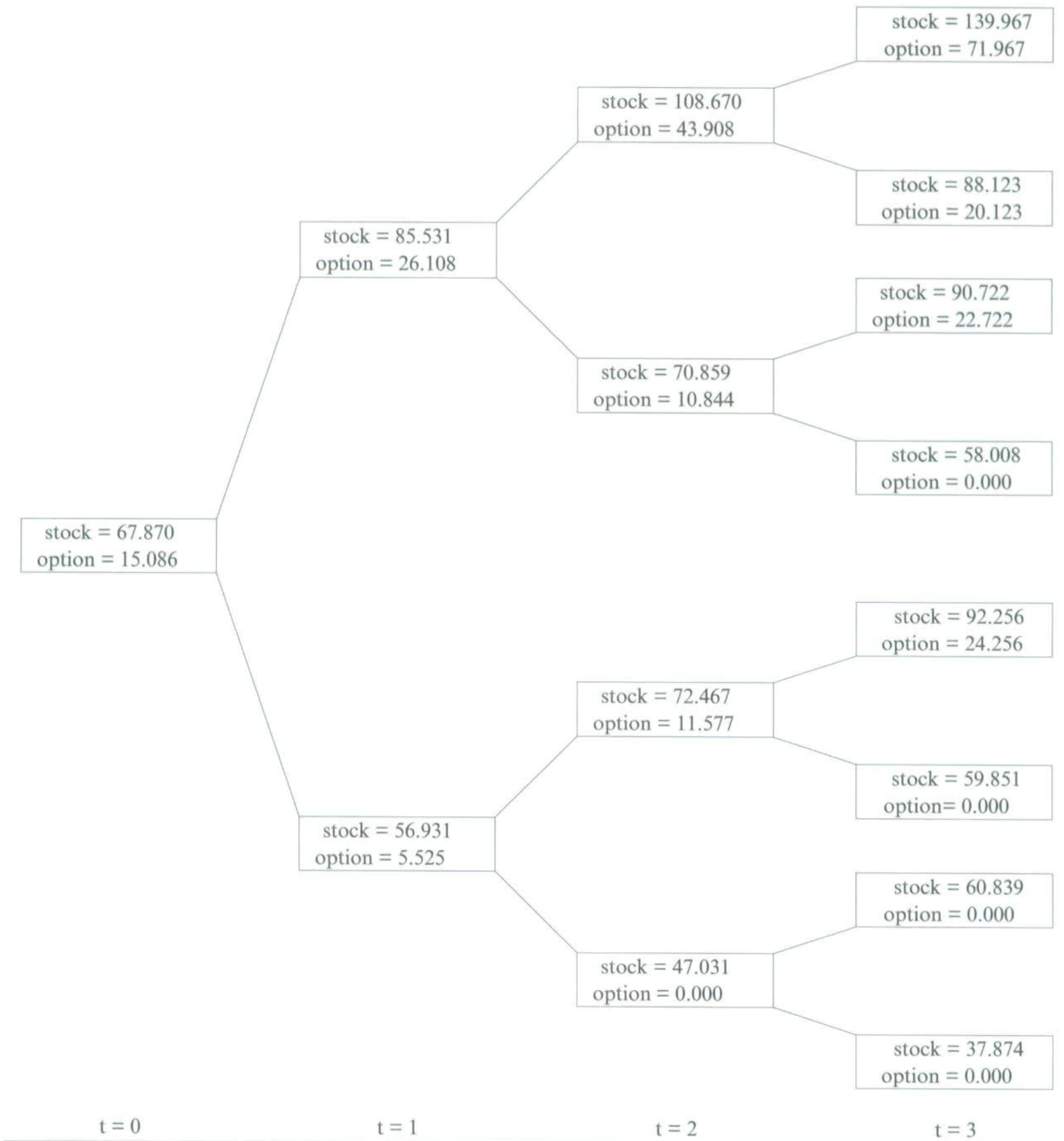
Panel B: Shares per option = 300,000; strike price of newly issued option = \$94; strike price factor = 1/1.1

		Cumulative Logarithmic Returns to Common Equity		Option Value Computed Using Multiple Option Pricing Model	Standard Binomial Option Value Computed with Common Stock Volatility of:		
Option Maturity (t)	Strike	Mean (1)	Std. Dev. (2)	(3)	1-year (4)	t-years (5)	10-years (6)
1	221.647	0.00921	0.28098	0	0	0	0
2	201.497	0.00906	0.28151	0	0	0	0
3	183.179	0.00891	0.28205	1,290,872	1,334,060	1,351,833	1,450,834
4	166.527	0.00873	0.28271	2,324,297	2,332,498	2,375,228	2,502,159
5	151.388	0.00852	0.28346	4,660,273	4,691,755	4,751,652	4,869,895
6	137.625	0.00829	0.28429	6,891,440	6,869,590	6,958,247	7,063,435
7	125.114	0.00803	0.28521	8,624,413	8,547,590	8,644,274	8,711,849
8	113.740	0.00776	0.28619	10,777,439	10,700,010	10,847,736	10,907,047
9	103.400	0.00747	0.28722	12,791,689	12,671,051	12,815,726	12,839,993
10	94.000	0.00717	0.28828	14,351,280	14,221,376	14,344,062	14,344,062
		Total Option Value		61,711,704	61,367,930	62,088,757	62,689,273
		Common Equity Value		938,288,296	938,288,296	938,288,296	938,288,296
		Total Equity Value		1,000,000,000	999,656,226	1,000,377,053	1,000,977,569
		Stock Price		93.829	93.829	93.829	93.829

Note: The risk-free rate is 5% a year, and each binomial period represents one full year. Values of u and d for the multiple option pricing model, as applied to total equity are computed as $u = exp(ln[1.05] - \frac{1}{2}[0.3^2] + 0.3)$ and $d = exp(ln[1.05] - \frac{1}{2}[0.3^2] - 0.3)$, respectively. There are 10,000,000 shares of stock initially outstanding. For the standard binomial values computed using $t > 1$-year volatility, the values of u and d are computed as $u = exp(ln[1.05] - \frac{1}{2}\sigma^2_{com} + \sigma_{com})$ and $d = exp(ln[1.05] - \frac{1}{2}\sigma^2_{com} - \sigma_{com})$, where σ_{com} is the standard deviation of the cumulative annual logarithmic return to common equity. For the standard binomial values computed using 1-year volatility, u and d are the ratios of the value of common equity in the immediate up and down states, respectively, to the initial value of common equity.

EXHIBIT 5

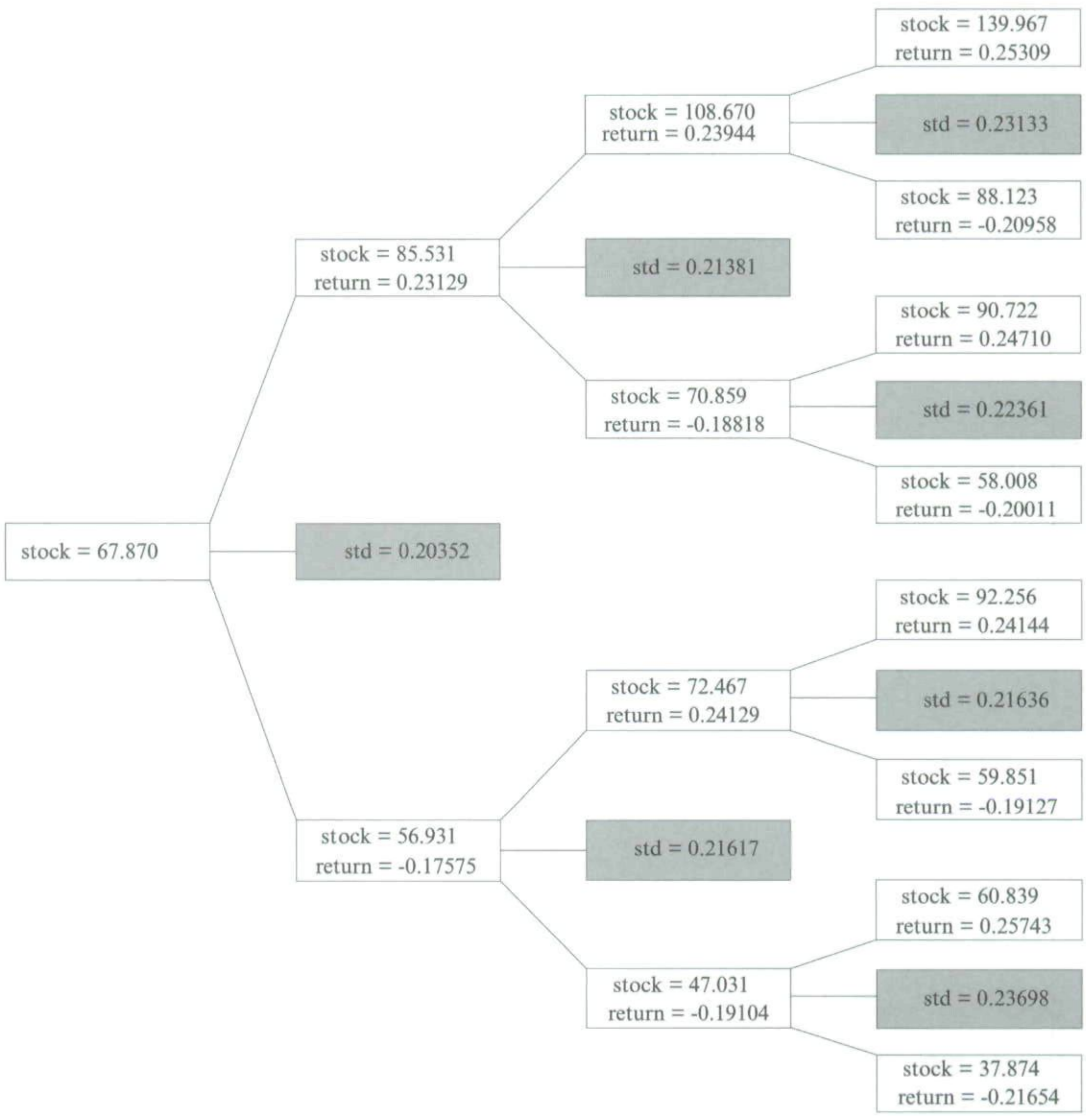
Pricing the Three-Period Option Using Distribution of Stock Prices Implied by Multiple Employee Option Pricing Model



Note: The option's strike price is $68, the risk-free rate is 5% per binomial period, and the risk-adjusted probability associated with an increase in the stock price from one binomial period to the next is 0.50113.

EXHIBIT 6

Evolution of Stock Price, Logarithmic Stock Return, and Volatility over Three Binomial Periods



t = 0	t = 1	t =2	t = 3
Mean	0.02777	0.02538	0.02269
Variance	0.04142	0.04622	0.05162
Std	0.20352	0.21499	0.22721

Note: Stock prices and logarithmic returns are shown for the first 3 of the 10 binomial periods associated with the valuation of 10 options using the multiple option pricing model, as summarized in panel A of Exhibit 3. The means, variances, and standard deviations shown at the bottom of the exhibit are computed incrementally. For example, the standard deviation (std) of 0.22731, shown in the summary section at the bottom of the exhibit for Time 3 is computed using Time 3 returns only under the assumption that up and down binomial outcomes are equally likely. This implies a probability of 0.125 associated with each outcome at Time 3. The shaded standard deviations within the exhibit are one-period standard deviations associated with adjacent up and down states only. For example, the shaded standard deviation of 0.21617 at time 2 is the standard deviation associated with a 0.5 probability of a 0.24219 return and a 0.5 probability of a −0.19104 return.

the price of a single 10-year call option on a stock worth \$67.870 per share with an expected volatility of 0.25234 per year is \$33.058 per share.[18] Since the 10-year option is for 1,500,000 shares, the total standard binomial-based value of the 10-year option is \$33.058 × 1,500,000 = \$49,587,432. This is the value shown for the 10-year option in the fifth and sixth columns of Exhibit 3, Panel A. Note that there is almost no difference between this value and the theoretically correct value of \$49,313,220 computed using the multiple option pricing model. Note also that there is very little difference between the theoretically correct values of all the options, as shown in the third column, and the values, as shown in the fifth column, computed via the standard binomial model using a *maturity-specific* time-dependent estimate of volatility.[19]

Potential Pricing Error Using the Incorrect Volatility Estimate in Standard Option Pricing Models

For the options with maturities of 1 to 9 years, there is very little difference between their theoretically correct values computed via the multiple employee option pricing model and the standard binomial option values shown in the fifth column of Exhibit 3, Panel A. Note, however, that the standard binomial values in the fifth column are computed using maturity-specific "average" volatilities which differ slightly for each maturity and are assumed to be known in advance. If, instead, one were to estimate a 10-year volatility and employ this volatility when valuing all 10 options, as shown in the sixth column, more error would be introduced into the values of the 1- to 9-year options. For example, the values for the 5-year option in the third and sixth columns of Panel A differ by almost 8%. Although it is unlikely that anyone who is charged with valuing employee options would be able to precisely estimate the 10-year (or comparable) volatility implied by the multiple employee option pricing model, it is even less likely that one would be able to precisely estimate time- or maturity-specific volatilities for each of the next 10 years without actually using our model, or something like it. Thus, from a practical standpoint, the standard binomial values shown in the sixth column, all of which are computed using the 10-year volatility, may be as relevant as those computed using maturity-specific volatility.

It is also useful to consider how much error would be produced using standard binomial-based valuation if one were able to precisely estimate the stock's *immediate* instantaneous volatility, and for the purposes of valuing employee options, assumed that this volatility would remain constant over the life of each option. In the binomial model, "instantaneous" is one binomial period, and in the context of the option values summarized in Exhibit 3, one binomial period is one year. If one were using a recent history of realized returns to estimate a stock's volatility, and the statistical procedure employed was able to capture the potentially changing volatility structure of stock returns over time (such as a model within the GARCH family), the volatility estimate evolving from such a procedure would be an estimate of short-term (immediate) volatility rather than long-term volatility. Therefore, binomial values computed using a one-year (constant) stock volatility would be most relevant when the volatility estimate is based on a recent history of stock returns. Standard binomial option values computed using an estimate of 1-year stock return volatility are shown in the fourth column of Exhibit 3.

As shown in the fourth column of the first row of Exhibit 3, Panel A, if the option that expires at Time 1 is valued as an option on common equity using the standard binomial approach, the option value is \$12,550,415, the same as the \$12,550,415 value for the same option computed via the multiple employee option pricing model. Therefore, this option can be valued via the standard binomial model as an option on common equity, ignoring the presence of the other options, without introducing any error into the valuation process.

The remaining entries in the fourth column represent standard binomial option values computed under the assumption that the equity volatility calculated for the first year remains constant for the next 10 years. Note that the remaining standard binomial option values are all somewhat lower than those computed via the multiple employee option pricing model, and some might view the differences as material. For example, when the 1-year volatility of common equity is used in connection with the standard binomial model, the value of the 10-year option of \$45,939,974 compared with a theoretically correct value is \$49,313,220, a difference of almost 7%. Corresponding differences are somewhat less for the shorter-term options. Nevertheless, these differences highlight the importance of using a maturity-specific volatility in connection with standard binomial pricing (or a valuation model such as ours) rather than an estimate that reflects the anticipated near-term volatility characteristics of the underlying stock.

Pricing Error when Options Are Less Dilutive

Thus far we have focused our attention on Panel A of Exhibit 3, which is based on each option being issued for 1,500,000 new shares when there are 10,000,000 shares of stock outstanding, a scenario that is close to the outer limit of potential dilution as reported by Henry [2002]. In Panel B, each option is issued for only 300,000 shares, or 3% of outstanding shares, which represents the average of option grants as reported by Henry among the 200 largest U.S. companies. Thus, the analysis as summarized in Panel B is likely to be more applicable to a typical large company.

One can see that the differences between theoretically correct option values shown in the third column of Panel B, and those in the fourth through sixth columns computed using the standard binomial model with various volatility estimates implied by the multiple employee option pricing model, are not large. Regardless of the option's maturity, little error would be made by using either 1- or 10-year volatility in connection with the standard binomial model or, by implication, a volatility estimate applicable to any period of time between 1 and 10 years.

Pricing Error as a Function of Underlying Firm Volatility and Strike Price Patterns

Consider the graph in the middle panel of Exhibit 7 titled "Strike Price Pattern: $\alpha = 1$ (At-the-Money)." This graph shows how the pricing error associated with standard binomial pricing for the same 10-year option in Panel A of Exhibit 3 relates to the volatility of the firm's total equity. As in Panel A of Exhibit 3, the firm, with 10,000,000 outstanding shares, has issued 10 options, with successive maturities of 1 to 10 years, and each option is for 1,500,000 shares. Also, all 10 options have identical strike prices equal to the initial stock price rounded to the nearest dollar. The dotted line shows percentage pricing error as a function of total equity volatility when the 10-year option is evaluated with the standard binomial model using a volatility estimate equal to the 10-year "average" volatility of common equity computed from the multiple option pricing model. The solid line shows percentage pricing error when the 10-year option is evaluated with the standard binomial model using a volatility estimate equal to the 1-year volatility of common equity.

It is interesting to note that the error associated with using 1-year volatility in connection with a standard binomial model is at its maximum for a total equity volatility of approximately 0.45. At extremely high and low levels of volatility, the errors appear to converge to zero. Errors associated with using 10-year volatility in connection with the standard binomial model appear to oscillate between 0% and 2%. The oscillation most likely reflects the coarseness of the binomial time interval (a full year) assumed in valuing each of the 10 options.

The graph in the upper panel of Exhibit 7 reflects an in-the-money scenario for options issued in the past ($\alpha = 1.1$), and the graph in the lower panel reflects an out-of-the-money scenario ($\alpha = 1.1/1$).

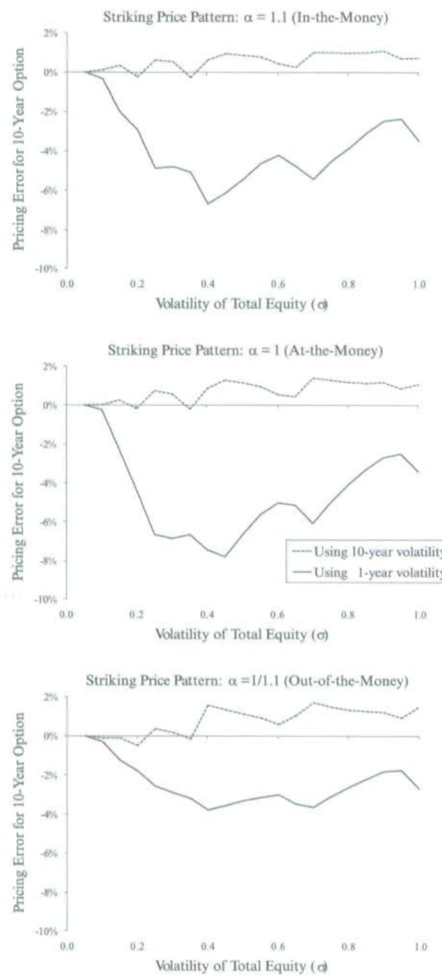
The graph in the upper panel shows an error pattern associated with using the 10-year average volatility in connection with the standard binomial model that is somewhat lower than that for at-the-money options. By contrast, the graph in the lower panel, reflecting previously issued options being out of the money, shows an error pattern associated with using the 10-year average volatility in connection with the standard binomial model that is similar to that in the middle panel for at-the-money options, but generally higher than that for in-the-money options. But in all three graphs, the error never exceeds 2%.

The errors associated with using 1-year volatility in connection with the standard binomial model are similar for both at-the-money options in the middle panel and in-the-money options in the upper panel, but somewhat lower for out-of-the money options in the lower panel. Regardless of the strike price patterns and the volatility of total equity, these errors never exceed 8%.

Exhibit 8, similar in structure to Exhibit 7, summarizes percentage pricing errors for the 5-year option as a function of total equity volatility and the same strike price patterns. In this exhibit, three plots of pricing errors are presented in each of the three panels. One plot shows the error assuming that the 5-year option is priced using the standard binomial model with a volatility estimate equal to the 10-year "average" volatility for common equity implied by the multiple employee option pricing model. A second plot shows the pricing error assuming that the 5-year "average" volatility is used in connection with the standard binomial model, and a third plot shows the pricing error assuming that 1-year common equity volatility is used.

EXHIBIT 7

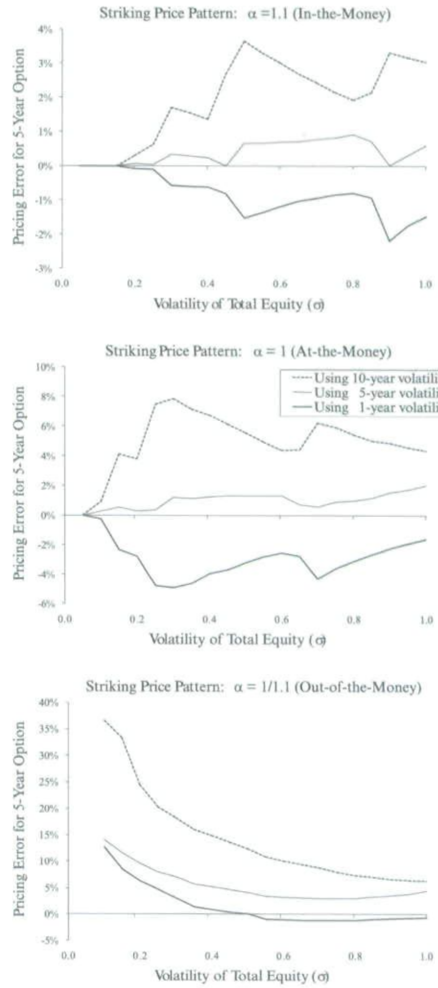
Percentage Error Relative to Multiple Employee Option Pricing Model in Pricing Newly Issued 10-Year Employee Stock Option with Standard Binomial Model



Note: In each graph, the risk-free rate is 5% a year, and each binomial period represents one full year. Values of u and d for the multiple option pricing model, as applied to total equity, are computed as $u = \exp(\ln[1.05] - \frac{1}{2}[0.3^2] + 0.3)$ *and* $d = \exp(\ln[1.05] - \frac{1}{2}[0.3^2] - 0.3)$*, respectively. There are 10,000,000 shares of stock initially outstanding, the company has issued 10 employee options with successive maturities of 1 to 10 years, and each option is for 1,500,000 shares. The strike price for the newly issued 10-year option is determined iteratively to equal the current stock price per share rounded to the nearest dollar. The strike price for the option maturing in t years equals the strike price of the 10-year option divided by* α^{10-t}*. For the standard binomial values computed using 10-year volatility, the values of u and d are computed as* $u = \exp(\ln[1.05] - \frac{1}{2}\sigma^2_{com} + \sigma_{com})$ *and* $d = \exp(\ln[1.05] - \frac{1}{2}\sigma^2_{com} - \sigma_{com})$*, where* σ_{com} *is the standard deviation of the cumulative annual logarithmic return to common equity. For the standard binomial values computed using 1-year volatility, u and d are the ratios of the value of common equity in the immediate up and down states, respectively, to the initial value of common equity.*

EXHIBIT 8

Percentage Error Relative to Multiple Employee Option Pricing Model in Pricing 5-Year Employee Stock Option with Standard Binomial Model



Note: In each graph, the risk-free rate is 5% a year, and each binomial period represents one full year. Values of u and d for the multiple option pricing model, as applied to total equity, are computed as $u = \exp(\ln[1.05] - \frac{1}{2}[0.3^2] + 0.3)$ *and* $d = \exp(\ln[1.05] - \frac{1}{2}[0.3^2] - 0.3)$*, respectively. There are 10,000,000 shares of stock initially outstanding, the company has issued 10 employee options with successive maturities of 1 to 10 years, and each option is for 1,500,000 shares. The strike price for the newly issued 10-year option is determined iteratively to equal the current stock price per share rounded to the nearest dollar. The strike price for the option maturing in t years equals the strike price of the 10-year option divided by* α^{10-t}*. For the standard binomial values computed using 5- and 10-year volatility, the values of u and d are computed as* $u = \exp(\ln[1.05] - \frac{1}{2}\sigma^2_{com} + \sigma_{com})$ *and* $d = \exp(\ln[1.05] - \frac{1}{2}\sigma^2_{com} - \sigma_{com})$*, where* σ_{com} *is the standard deviation of the cumulative annual logarithmic return to common equity. For the standard binomial values computed using 1-year volatility, u and d are the ratios of the value of common equity in the immediate up and down states, respectively, to the initial value of common equity.*

Generally, for each strike price pattern, the pricing error for the 5-year option associated with the use of 10-year volatility is substantially higher than the corresponding pricing error for the 10-year option as illustrated in Exhibit 7. The greatest pricing error is shown for the out-of-the-money strike price pattern in the lower panel of Exhibit 8 and can be on the order of 20% to 50%. Even if one employed the 5-year volatility when pricing the 5-year option, the plot in the lower panel shows a pricing error on the order of 10% to 20%. The plot in the lower panel also shows a similar, but slightly lower, error if 1-year volatility is used.

Hull and White [2004b] took issue with the IFRS 2 and FASB proposals, which have now become SFAS 123R, that require companies to compute an option's fair value at the time of issuance and then recognize an expense year-by-year over the option's vesting period without making any further adjustments to the option's fair value over time. They propose that the option's fair value be expensed in its entirety at the time of issuance and that changes in the option's fair value also be recognized and expensed over the option's lifetime.

In the context of the present analysis, the 10-year option represents a newly issued option, and the remaining options with maturities ranging from 1 to 9 years would have been issued in past years. The analysis summarized in Exhibit 7 indicates that there is not likely to be a great deal of error from using standard binomial pricing methods to price a newly issued option. However, the analysis summarized in Exhibit 8 indicates that the use of standard binomial pricing methods could result in substantial error as an option is repriced through time, especially if the future value of the stock causes the option to become out of the money prior to maturity.

Valuation of Multiple Options Using the Control-Variate Technique

The control-variate technique is often used to value options in a lognormal, Black–Scholes-type setting when the features of the option or options do not lend themselves to closed-form solutions. The basic idea is to use the Black–Scholes model to obtain the "bulk" of the option value and then use the binomial or similar numerical method to approximate the incremental value associated with the feature that prevents a closed-form solution from being obtained. This incremental value is then added to the Black–Scholes value to obtain a final numerical option value. Often, the incremental value can be computed using substantially fewer time steps than would be required if the same numerical method were used alone to value the option.

We compare control-variate prices for the same employee options evaluated in Exhibits 2 to 4 using both 1 binomial period per year, the same pricing structure assumed in the various versions of Exhibit 2, and 5 binomial periods per year (using the binomial pricing methodology described in Appendix B). Thus, at one extreme, the 10-year options are evaluated using a total of 10 and 50 binomial periods, respectively, while at the other extreme, the 1-year options are evaluated with only 1 and 5 binomial periods, respectively. In each case, binomial values are computed using up and down factors for common equity, u and d, equal to the ratio of the value of common equity in the immediate up and down states, respectively, to the initial value of common equity. All Black–Scholes values that comprise the control-variate values are computed with the annualized volatility associated with the respective values of u and d for common equity. Exhibit 9 summarizes the results.

Exhibit 9 is divided into six sections. In each section, the computation time required to evaluate the complete set of options using 5 binomial periods per year ($\gamma = 5$) is approximately 24 minutes. Computation times appear to increase by a factor of 4 (approximately) as γ is increased by 1 unit. Thus, we estimate that it would take approximately 24,756 minutes, or 17 days per section, if the number of periods per year were increased to 10.

With just a few exceptions, there is very little difference between the control-variate values associated with any of the exercise patterns using 1 and r binomial periods per year. Stated differently, the difference between option values computed via the multiple option pricing model and corresponding standard binomial values does not depend a great deal on whether 1 or 5 binomial periods per year are employed in the computations. There are four exceptions where there appear to be large percentage differences (1.579% to 27.646%) in the pricing of individual options, but these are options for which the dollar value is quite small and, therefore, the dollar difference is also relatively small. Thus, we conclude that the basic story, based on the multiple employee option pricing model being evaluated using 1 binomial period per year, is sound and would not change to any significant extent if a larger number of binomial periods were used.

EXHIBIT 9

Control-Variate Employee Option Values Using Five and One Binomial Periods a Year

Panel A: shares per option = 1,500,000

	α = 1.1				α = 1				α = 1/1.1			
		Control-Variate Value				Control-Variate Value				Control-Variate Value		
Maturity (years)	Strike	y = 5	y = 1	% Diff	Strike	y = 5	y = 1	% Diff	Strike	y = 5	y = 1	% Diff
1	23.75	49,766,373	49,764,911	-0.003	68.00	10,724,829	10,605,640	-1.111	183.92	2,680	2,835	5.780
2	26.12	48,166,841	48,164,858	-0.004	68.00	16,782,283	16,610,923	-1.021	167.20	346,248	441,972	27.646
3	28.74	46,591,794	46,586,776	-0.011	68.00	22,001,627	21,868,309	-0.606	152.00	2,470,315	2,412,183	-2.353
4	31.61	45,156,782	45,151,119	-0.013	68.00	26,642,715	26,388,643	-0.954	138.18	7,176,986	7,363,167	2.594
5	34.77	43,928,249	44,002,122	0.168	68.00	30,984,082	31,036,349	0.169	125.62	14,020,089	14,065,586	0.325
6	38.25	42,922,823	42,834,777	-0.205	68.00	35,160,753	34,766,875	-1.120	114.20	22,205,290	22,395,868	0.858
7	42.07	42,134,541	42,063,737	-0.168	68.00	38,896,051	39,193,775	0.765	103.82	31,047,861	31,429,117	1.228
8	46.28	41,550,097	41,667,379	0.282	68.00	42,587,650	42,252,172	-0.788	94.38	40,059,167	40,045,689	-0.034
9	50.91	41,116,550	40,925,718	-0.464	68.00	46,054,328	45,998,349	-0.122	85.80	48,678,420	48,627,917	-0.104
10	56.00	40,724,608	40,492,418	-0.570	68.00	49,259,663	49,042,665	-0.441	78.00	56,689,673	56,592,070	-0.172
	Total	442,058,657	441,653,817	-0.092		319,093,980	317,763,700	-0.417		222,696,729	223,376,403	0.305

Panel B: shares per option = 300,000

	α = 1.1				α = 1				α = 1/1.1			
		Control-Variate Value				Control-Variate Value				Control-Variate Value		
Maturity (years)	Strike	y = 5	y = 1	% Diff	Strike	y = 5	y = 1	% Diff	Strike	y = 5	y = 1	% Diff
1	36.47	15,384,742	15,383,302	-0.009	91.00	3,488,227	3,478,704	-0.273	221.65	6,585	6,689	1.579
2	40.12	14,916,489	14,914,533	-0.013	91.00	5,293,134	5,286,667	-0.122	201.50	272,881	296,829	8.776
3	44.13	14,507,584	14,505,351	-0.015	91.00	6,756,020	6,748,305	-0.114	183.18	1,219,208	1,217,449	-0.144
4	48.54	14,172,197	14,165,395	-0.048	91.00	8,051,901	8,038,242	-0.170	166.53	2,742,807	2,784,976	1.537
5	53.40	13,902,156	13,897,935	-0.030	91.00	9,213,439	9,205,466	-0.087	151.39	4,626,626	4,628,487	0.040
6	58.74	13,686,039	13,693,178	0.052	91.00	10,274,264	10,249,217	-0.244	137.63	6,673,248	6,694,160	0.313
7	64.61	13,518,908	13,479,572	-0.291	91.00	11,266,493	11,257,191	-0.083	125.11	8,744,654	8,779,730	0.401
8	71.07	13,351,223	13,354,279	0.023	91.00	12,166,808	12,127,846	-0.320	113.74	10,752,634	10,749,335	-0.031
9	78.18	13,231,958	13,271,511	0.299	91.00	13,041,322	13,029,829	-0.088	103.40	12,644,667	12,652,824	0.065
10	86.00	13,142,776	13,102,071	-0.310	91.00	13,815,268	13,760,078	-0.399	94.00	14,392,396	14,388,359	-0.028
		139,814,073	139,767,127	-0.034		93,366,874	93,181,545	-0.198		62,075,706	62,198,839	0.198

Note: The risk-free rate is 5% a year. Values of u and d for the multiple option pricing model, as applied to total equity, are computed as $u = \exp([\ln(1.05)] - \frac{1}{2}(0.3^2)/y + 0.3/\sqrt{Y})$ *and* $d = \exp([\ln(1.05)] - \frac{1}{2}(0.3^2)/y - 0.3/\sqrt{y})$*, respectively, where y is the number of binomial periods per year. There are 10,000,000 shares of stock initially outstanding. All binomial values that comprise the control-variate values are computed using u and d values equal to the ratio of the value of common equity in the immediate up and down states, respectively, to the initial value of common equity. All Black–Scholes values that comprise the control-variate values are computed with the annualized volatility associated with the respective values of u and d for common equity.*

Before concluding, we acknowledge that the model does not incorporate the effect of potential future employee option issuances on the values of outstanding employee options. Although we have attempted to address this issue in our work, it is extremely complex, and we do not feel that we are in a position to formally address this issue here. However, it is clear that the distribution of a company's future stock price per share should be the same, whether the company gives future compensation in the form of cash or in the form of equity with equal value to what would otherwise be paid out in cash. Therefore, a policy of compensating employees in the future with new equity shares in lieu of cash should have no impact on the values of outstanding employee options. Since very long-term options are asymptotically the same as equity, it is reasonable to expect the effect of future issuances of long-term options to be relatively small. However, we are unable to actually quantify the valuation effects of issuing future finite-life employee options and leave this to future research.

SUMMARY AND CONCLUSION

Current methods recommended for valuing employee stock options issued by the same company treat each option as being independent of the others and do not reflect that the options should be valued simultaneously. In this article, we develop a model to simultaneously value all employee options issued by the same firm and study the valuation impact associated with ignoring this simultaneity when more conventional approaches to employee option valuation are used. The problem of simultaneously valuing multiple employee stock options is inherently path dependent and, therefore, computationally intensive. Even though the valuation model is path dependent, and the number of computations required after t binomial periods increases exponentially at a rate of 4^t, our computational method enables us to simultaneously value up to 30 employee options with different maturity dates, using a desktop PC.

We demonstrate that if each employee stock option is issued on a small-to-moderate fraction (3%) of a firm's outstanding shares, the standard binomial model and multiple option model give values that are not materially different from each other. However, if employee options are issued on a large portion of outstanding shares (15%), using the standard Black–Scholes/binomial approach can result in valuation errors as great as 20% to 50%. Therefore, in cases for which there is the potential for significant equity dilution resulting from the exercise of employee options, there can be a material difference between option values computed via the multiple employee option pricing model and those computed using more standard option valuation methods.

The methodology developed in this article can serve as a starting point for simultaneously valuing multiple debt issues of the same company. The simultaneous valuation of multiple debt instruments is more complex than the simultaneous valuation of multiple employee options, since the various debt instruments issued by the same company are not likely to be as homogeneous in their structure and contractual features. This is an interesting and potentially important valuation problem that others may want to address in the future.

APPENDIX A

VALUATION OF m OPTIONS OVER m BINOMIAL PERIODS

The two-period valuation procedure is easily generalized for m options expiring at the end of m successive binomial periods. Note that in the two-period illustration, computations are carried out in the same order as the states are numbered in Exhibit 1. This method of ordering the computations is referred to as the depth-first technique. Using this technique for a generalized m-period problem, only 4^m computer memory locations are needed to store intermediate option values. By contrast, if all the option and equity values in the final column of Exhibit 1 are computed first, followed by the calculation of values in the next-to-last column, and so on, $2 \times 4^m - 1$ computer memory locations are required. Although the depth-first technique solves the computer memory problem associated with path-dependent valuation, it does not reduce the number of required computations. With either method, computations are required.[20] This implies that an $m + 1$-option (period) problem will take approximately four times the computational time of an m-option problem. To put this into a practical perspective, if it takes one minute to compute option values for an m-option problem, it would take over a full day to compute values for an $m + 6$-option problem.

It is possible to reduce the number of computations required by pruning portions of the binomial tree once it becomes evident that those portions will not enter into the final option solution values. We employ two pruning methods that significantly reduce the computational requirements of the model.

Consider States 7 and 14 in Exhibit 1. Values of both options in State 7 are associated with Option 1 being exercised ($k_1 = 1$), while values in State 14 are associated with Option 1 not being exercised ($k_1 = 0$). In this particular example, the exercise value of Option 1, as shown in State 7, is positive. Therefore, once option values in State 7 have been computed, it is clear that Option 1 will be exercised, and it is not necessary to carry out the calculations branching from State 14. Thus, we "prune" the portion of the tree branching from State 14 and similar states, since these branches of the binomial tree are not needed in the calculations.

The second round of pruning can be illustrated by considering State 22. Before the calculations branching from State 22 (and others like it) are made, for pruning purposes the exercise value of Option 1 (or the option expiring in that state) is calculated under the assumption that the values of all non-expiring options (Option 2, in the example) are zero. If the exercise value of Option 1, calculated in this fashion, is negative, it will be potentially even more negative if the proper values of the non-expiring options are used when calculating exercise value of Option 1. Thus, even without carrying out the detailed calculations of option values in the portion of the binomial tree branching from State 22, if the value of Option 1 calculated in this fashion is negative, it is clear that Option 1 will not be exercised, and one can skip directly to the calculation of option values branching from State 29.

Without pruning, the calculations required for a 20-option (period) model would be projected to take 51 days using a 2-Ghz Pentium 4 computer. This projection is based on a 10-option model taking 4.23 seconds of computation time, with computation time increasing by a factor of 4 for each additional option. With pruning, computation time for a 20-option model takes only 11.9 seconds. In addition, computation times with pruning for 21-option and 22-option models are 23.9 seconds and 47.1 seconds, respectively.[21] Based on these results, it appears that computation time with pruning increases approximately by a factor of 2 per option. The computation time required for a 30-option model is three to four hours. Therefore, we consider 30 options to be the practical limit of our model under present computing standards.

Appendix B

VALUATION WHEN OPTIONS DO NOT MATURE AT THE END OF EACH BINOMIAL PERIOD

The model, as formulated, assumes the firm has issued m options, with Options 1 through m expiring at binomial Times 1 though m, inclusively. If one or more of the m binomial times are not associated with the maturity of an option, the model can easily accommodate this situation by specifying that the number of shares associated with options maturing at these times is zero. For example, suppose a firm has issued 1-, 3-, 4-, and 5-year options, each with 300,000 shares outstanding, but has issued no 2-year option. Then the model, as formulated, can accommodate this situation by assuming $n_1 = n_3 = n_4 = n_5 = 300{,}000$ and $n_2 = 0$. Although this method will produce correct option values, it may be computationally inefficient if the number of binomial periods is significantly greater than the number of actual options.

Consider the situation in which a firm has issued m options, each of which expires at the end of the next m years, but each year is partitioned into y binomial time periods. Therefore, at the end of the *first* $y - 1$ binomial periods of each year, there is no expiring option. Also, during the *last* $y - 1$ periods of each year, the binomial valuation process will not be path dependent.[22]

Computationally, it is inefficient to value the firm's m options in the last $y - 1$ binomial periods using a path-dependent valuation structure. Therefore, we modify our valuation procedure to employ a recombining binomial tree structure for the last $y - 1$ binomial periods between option maturity dates and to use a non-recombining structure only for the first of the y periods between maturities. With this modified structure, the number of paths after $m \times y$ binomial periods is $(2[y + 1])^m$, and the total number of states within the binomial tree is $(([2(y + 1)]^m - 1)/(2y + 1))([y + 1]\,[y + 2]/2) + (2[y + 1])^m$.[23] However, as in the previous specification, pruning can significantly reduce the number of states for which computations are required.

Consider a situation in which 4 options maturing in 1, 2, 3, and 4 years, respectively, are valued using 5 binomial time periods per year. Therefore, $m = 4$ and $y = 5$, and the options are valued using a 20-period binomial tree. If a non-recombining valuation structure is employed for each of the 20 binomial periods, the total number of states within the binomial tree would be $2 \times 4^{20} - 1 \approx 2.20 \times 10^{12}$ before pruning. However, using a recombining structure during the last 4 binomial periods between each maturity reduces the total number of states before pruning to $(([2(5 + 1)]^4 - 1)/(2 \times 5 + 1))([5 + 1][5 + 2]/2) + (2[5 + 1])^4 = 60{,}321$.[24]

Exhibit 10 shows computation times and illustrates model precision when additional binomial time intervals are included between option expirations. As in Exhibits 2 to 4, the volatility of total equity returns is assumed to be 0.30 per year, the risk-free interest rate is 5% a year, and the firm is assumed to have issued 10 employee options, the first of which expires at the end of Year 1, the second at the end of Year 2, and so on. Three sets of option values are computed. In the first set, each option is for 300,000 shares and the strike price of each option is \$91. In the second set, the number of shares and strike price are 1,500,000 and \$68, respectively. In the third set, the number of shares and strike price are 3,000,000 and \$53.

The first row of stock values shown in Exhibit 10 corresponds to those shown in the last row of Exhibit 3. As in Exhibit 3, option values in this row reflect that each binomial period represents a full year and that an option matures at the end of each year. As such, $m = 10$, $y = 1$, and there are a total of 10 binomial periods employed in the calculations.

In the second row, $m = 10$ and $y = 2$, resulting in a total of 20 binomial periods. Note that the option values in this row correspond very closely to those in the first row. Moreover, this same pattern is seen in the rows for which $y = 3$, $y = 4$, and $y = 5$. Although the option values are calculated with greater precision in these rows, the resulting stock prices per share are very close to those shown in the first row.

Computation times shown in Exhibit 10 are for the option prices in the first column only and are somewhat lower than those required for the options whose prices are shown in the other two columns. The computation time for the 50-period model shown in the last row is 1,451.5 seconds, or approximately 24 minutes, indicating that pricing 10 options within a 50-period binomial tree is quite feasible.

Exhibit 11 shows computation times assuming the final option being valued expires in 100 binomial periods, with the 100 periods being divided into $m = 1$, 2, 4, and 5 years. All computation times reflect that the strike price of each outstanding option is \$91, that each option is for 300,000 shares, and that all expiration dates are exactly 1 year apart.

EXHIBIT 10

Computation Times and Per Share Stock Prices with Model Modified to Reflect Recombining Binomial Trees between Option Expirations

				Stock price per share		
Options (m)*	Periods Per Year (y)	Total Periods ($m \times y$)	Time (seconds)	Strike = 91 Shares = 300,000	Strike = 68 Shares = 1,500,000	Strike = 53 Shares = 3,000,000
10	1	10	0.0	90.59	67.87	52.92
10	2	20	1.0	90.71	67.93	52.92
10	3	30	20.3	90.63	68.00	53.03
10	4	40	208.7	90.66	67.98	52.99
10	5		1,481.1	90.64	68.02	53.05

Note: The risk-free rate is 5% a year, or $1 + r = 1.05^{1/y}$, and there are 10,000,000 shares of stock initially outstanding. Values of u and d for the multiple option pricing model, as applied to total equity, are computed as $u = \exp(\ln[1.05]/y - \frac{1}{2}[0.3^2/y] + 0.3/\sqrt{y})$ and $d = \exp(\ln[1.05]/y - \frac{1}{2}[0.3^2/y] - 0.3/\sqrt{y})$, respectively, where y is the number of binomial periods per year. Computation times are for options on 300,000 shares with a strike price of \$91. The m options mature at the end of years 1 through m, inclusively. Therefore, m also represents the number of years until the expiration of the last option.

EXHIBIT 11

Computation Times for 100-Period Model with Modification to Reflect Recombining Binomial Trees between Option Expirations

Options (m)*	Periods Per Year (y)	Total Periods $(m \times y)$	Time (seconds)
1	100	100	0.0
2	50	100	0.0
4	25	100	36.9
5	20	100	272.1

Note: Each option has a strike price of \$91 and is issued on 300,000 shares of stock. The risk-free rate is 5% a year, or $1 + r = 1.05^{1/y}$*, and there are 10,000,000 shares of stock initially outstanding. Values of u and d for the multiple option pricing model, as applied to total equity, are computed as* $u = exp(ln[1.05]/y - \frac{1}{2}[0.3^2/y] + 0.3/\sqrt{y})$ *and* $d = exp(ln[1.05]/y - \frac{1}{2}[0.3^2/y] - 0.3/\sqrt{y})$*, respectively, where y is the number of binomial periods per year. Computation times are for options on 300,000 shares with a strike price of \$91. The m options mature at the end of years 1 through m, inclusively. Therefore, m also represents the number of years until the expiration of the last option.*

Note that the longest computation time is 277.9 seconds (or 4.6 minutes) for 5 options spread over 100 binomial periods, with 20 binomial periods included between each expiration. These results indicate that it is feasible to use 100 binomial periods in the calculation of up to five interdependent employee option prices.

ENDNOTES

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[1]Darsinos and Satchell [2002] actually formulated their model for more than two options, but they provided examples for two options only. As will be discussed in the next section, however, their model ignores that the conditional value of an option depends on the values of options that mature later.

[2]Barth, Landsman, and Rendleman [1998, 2000] discussed the problem of simultaneously valuing multiple debt instruments in greater detail and proposed a heuristic procedure for valuing each instrument individually, given the presence of the other instruments.

[3]Although unstated, we suspect that the computational complexity of this search procedure is what causes the authors to limit their analyses to only two options of different maturity.

[4]Certain path-dependent structures may prevent us from using the specific pruning techniques for reducing computation time, which are described later in the article.

[5]See http://www.r-project.org.

[6]From an accounting standpoint, Bodie, Kaplan, and Merton [2003, p. 65] argued that an option's lack of liquidity to its holder makes no difference in what it costs the issuer unless the issuer is able to benefit from the lack of liquidity.

[7]Drawing on the work of Rubinstein [1995], Hull and White [2004] concluded that any potential correlation between employee exit rates and the level of stock prices should have little impact on option values and can probably be ignored. In contrast, Bodie, Kaplan, and Merton [2003] argued that a model which properly reflects the correlation between employee forfeitures and the level of the stock price is likely to result in higher option values in relation to a model that ignores the correlation.

[8]The 130,000-year estimate assumes that computations would not be slowed by the use of virtual memory for storing intermediate option values.

[9]Later, we show how the computational requirements of the model can be reduced substantially when there are points in binomial time when options do not mature. But in its most general form, the model can accommodate gaps in option maturity times by simply assuming that the number of options issued for a particular maturity is zero.

[10]This is the same assumption made by Cox and Rubinstein [1985, p. 397] in their game-theoretic approach to optimal exercise strategy when a company has two warrants held by a single individual or by two competing individuals.

[11]Frequently, users of the binomial model assume a mean logarithmic return of zero when computing up and down factors. However, it can be shown that binomial option values converge most rapidly to corresponding Black–Scholes values when the mean return equals the continuously compounded risk-free rate per binomial period, or $\ln(1 + r)$, minus half the variance of the logarithmic return per binomial period (see Rendleman and Bartter [1979, p. 1101]) which results in $u = \exp(\ln[1 + r] - \frac{1}{2}\sigma^2 + \sigma)$ and $d = \exp(\ln[1 + r] - \frac{1}{2}\sigma^2 - \sigma)$, as specified above. With this specification, the expected instantaneous return to total equity equals the risk-free rate and all securities whose values depend upon the return to total equity are priced as if investors are risk neutral. Also, it should be noted that the assumption of a zero mean can result in the irrational set of binomial parameters $1 + r > u > d$ if $\ln(1 + r) > \sigma$.

[12]As of May 2002, 86 of 2,406 historical volatilities published by the CBOE exceeded 1.00 (100%) and 504 of 2,406

exceeded 0.50. The CBOE numbers reflect volatility computed with daily returns over a single month, and it is unlikely that volatility of this high a magnitude would continue over the long term. In addition, there are well-known statistical problems associated with using daily returns for estimating the standard deviation of stock returns (see, for example, Blume and Stambaugh [1983]). Nevertheless, as a back-of-the-envelope estimate, it would seem reasonable to expect some high-tech companies to have long-term equity volatilities on the order of 60% a year.

[13]The model produces an initial value for each of the 10 options and an initial value of common equity. Dividing the initial value of common equity by the number of shares outstanding produces an initial stock price per share. If this price, rounded to the nearest dollar, does not equal the initial strike price, a new initial strike price is assumed, and each option value and the initial stock price are recomputed. This process is continued until the initial stock price, rounded to the nearest dollar, equals the initial strike price.

[14]Binomial up and down factors for total equity are $u = \exp([\ln(1 + r)] - \frac{1}{2}(\sigma^2) + \sigma) = \exp([\ln(1.05)] - \frac{1}{2}(0.3^2) + 0.3) \approx 1.35498$ and $d = \exp(\ln[1 + r] - \frac{1}{2}\sigma^2 - \sigma) \approx 0.74363$. The risk-adjusted probability is $\pi = (1 + r - d)/(u - d) = (1.05 - 0.74363)/(1.35498 - 0.74363) = 0.50113$.

[15]As is well known and documented in Cox, Ross, and Rubinstein [1979] and Rendleman and Bartter [1979], as the number of binomial time intervals over the life of an option becomes infinitely large, standard binomial option values converge to Black–Scholes values. However, in practical applications of the binomial model in which the number of binomial periods used might be on the order of 100 to 500, depending upon the nature of the problem, there will be small differences between standard binomial and Black–Scholes option values.

[16]More specifically, the square root of the average of the time-dependent variance of stock returns would be used as the standard deviation estimate in the Black–Scholes (or binomial) model (see Merton [1973]).

[17]The calculation of the underlying stock's time- and state-dependent volatility structure requires that all stock values within the binomial tree be stored. Therefore, if employee option prices, and the initial value of common equity, are computed using the depth-first technique with pruning, it will be impossible to recover the volatility structure of common equity. This implies that it will be computationally infeasible to recover the volatility structure of stock returns for valuation problems of large scale.

[18]For the purposes of computing standard binomial option prices, the values of u and d are computed as $u = \exp(\ln[1.05] - \frac{1}{2}\sigma^2_{com} + \sigma_{com})$ and $d = \exp(\ln[1.05] - \frac{1}{2}\sigma^2_{com} - \sigma_{com})$, where σ_{com} is the standard deviation of the cumulative annual logarithmic return to common equity.

[19]For the standard binomial values computed $t > 1$-year volatility, the values of u and d are computed as $u = \exp([\ln(1 + r)] - \frac{1}{2}(\sigma^2_{com}) + \sigma_{com})$ and $d = \exp(\ln[1.05] - \frac{1}{2}(\sigma^2_{com}) - \sigma_{com})$, where σ_{com} is the standard deviation of the cumulative annual logarithmic return to common equity. For the standard binomial values computed using 1-year volatility, u and d are the ratios of the value of common equity in the immediate up and down states, respectively, to the initial value of common equity.

[20]$2 \times 4^m - 1$ is the total number of states in the binomial tree after m periods.

[21]Computation time with pruning for a given number of binomial periods depends upon the terms of the options being evaluated. The computation times reported here are for the types of options whose values are summarized later in Exhibit 3 and Exhibit 10.

[22]If an option expires at the end of a particular binomial period, it creates a path dependency at the end of the next binomial period only. Therefore, the remaining $y - 1$ periods will not be path dependent.

[23]The derivation of this quantity will be made available upon request.

[24]Note it is impossible to determine the number of relevant binomial states with pruning, as the number of states reduced by pruning will be determined by the specific structure of the valuation problem.

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