

Pricing American Interest Rate Options under the Jump-Extended Vasicek Model

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This article shows how to price American interest rate options under the exponential jumps-extended Vasicek model, or the Vasicek-EJ model. We modify the Gaussian jump-diffusion tree of Amin [1993] and apply it to the exponential jumps-based short rate process under the Vasicek-EJ model. The tree is truncated at both ends to allow fast computation of option prices. We also consider the time-inhomogeneous version of this model, denoted as the Vasicek-EJ++ model, that allows exact calibration to the initially observable bond prices. We provide an analytical solution to the deterministic shift term used for calibrating the short rate process to the initially observable bond prices, and show how to generate the jump-diffusion tree for the Vasicek-EJ++ model. Our simulations show fast convergence of European option prices obtained using the jump-diffusion tree to those obtained using the Fourier inversion method (for options on zero-coupon bonds, or caplets) and the cumulant expansion method (for options on coupon bonds, or swaptions).

Backus, Foresi, and Wu [1997], Das [1998, 2002], Das and Foresi [1996], Johannes [2004], and Piazzesi [1998, 2005] find that jumps caused by market crashes, interventions by the Federal Reserve, economic surprises, shocks in the foreign exchange markets, and other rare events, play a significant role in explaining the dynamics of interest rate changes. Das [2002] finds that a mix of ARCH processes with jumps explains the behavior of short rate changes well. The ARCH features are required

to capture the high persistence in short rate volatility, and jump features are required to explain the sudden large movements that lead to leptokurtosis. Das finds jumps to be more pronounced during the two-day meeting dates of the Federal Open Market Committee, and curiously also on many Wednesdays due to the option expiry effects on that day. In a recent article, Jarrow, Li, and Zhao [2007] find that a stochastic volatility extension of the LIBOR market model can be modeled to generate a symmetric-shaped smile in the Black-implied volatility of caps. However, the asymmetric hockey-shaped smile observed in the caps data can be generated only by using downward jumps under the risk-neutral measure.

Jump models assume that the size of the jump is a random variable with some probability distribution, which is independent of the length of the time interval over which the jump occurs. The probability of jump is assumed to be directly proportional to the length of the time interval. Though computational methods exist for pricing of European interest rate options under jump models, few numerical methods for pricing American interest rate options—such as options embedded in callable and convertible bonds, mortgage prepayment options, American swaptions, and so forth—exist in the fixed-income literature. In Nawalkha, Beliaeva, and Soto (NBS [2007]), we show how to price American interest rate options using the exponential jumps-extended Vasicek model of

Chacko and Das [2002], denoted as the Vasicek-EJ model. The advantage of the Vasicek-EJ model is that it allows separate distributions for the upward jumps and downward jumps. This can significantly reduce the probability of negative interest rates by allowing more flexibility in estimating parameters for different interest rate regimes. For example, when interest rates are at their lows at the bottom of a business cycle, the downward jumps may have a lower size and a lower frequency of occurrence than upward jumps.

To price American interest rate options under the Vasicek-EJ model, we construct a jump-diffusion tree by extending the work of Amin [1993]. The jump-diffusion tree allows an arbitrarily large number of nodes at each step to capture the jump component, while two local nodes are used to capture the diffusion component. We also demonstrate how to calibrate the jump-diffusion tree to fit an initial yield curve or the initially observable zero-coupon bond prices, by allowing a time-dependent drift for the short rate process. In this article, we build on the earlier results presented in NBS [2007], and show how to price options on coupon bonds (or swaptions) under the Vasicek-EJ model. We first extend the cumulant-expansion method of Collin-Dufresne and Goldstein [2002] to price European options on coupon bonds under the Vasicek-EJ model. Next, we demonstrate the application of the jump-diffusion tree to price American options on coupon bonds (or American swaptions) under this model.

Our simulations show fast convergence of European option prices obtained using the jump-diffusion tree to those obtained using the Fourier inversion method (for options on zero-coupon bonds, or caplets) and the cumulant expansion method (for options on coupon bonds, or swaptions).

We also consider the single-plus and double-plus extensions of these models given as the Vasicek-EJ+ and Vasicek-EJ++ models, based upon the new taxonomy of term structure models given by NBS [2007, pp. 106–112]. As shown by NBS, the Vasicek-EJ+ model allows independence from the market price of diffusion risk (representing the one plus), and Vasicek-EJ++ model allows independence from market price of diffusion risk, as well as calibration to the initial zero-coupon bond prices (representing the two plusses). Independence of these models from the market price of diffusion risk makes these models *partially* preference-free. These models still require the

market prices of jump risk, which cannot be eliminated using the single-plus and double-plus extensions.

THE VASICEK-EJ MODEL

Consider the short rate process given by Vasicek [1977] with an added Poisson-jump component, given as follows:

$$dr(t) = \alpha(m - r(t))dt + \sigma dZ(t) + JdN(\lambda) \quad (1)$$

where α , m , and σ define the speed of mean-reversion, long-term mean, and the diffusion volatility, respectively. The variable $Z(t)$ is the Wiener process distributed independently of $N(\lambda)$. The variable $N(\lambda)$ represents a Poisson process with an intensity λ . The variable λ gives the mean number of jumps per unit of time. The variable J denotes the size of the jump and is assumed to be distributed independently of both $Z(t)$ and $N(\lambda)$. The choice of jump distribution has a significant effect on the pricing of bonds. For example, the jump size can follow a Gaussian, a log-normal, or an exponential distribution. This article considers jumps that follow exponential distributions.

The probability of the occurrence of one Poisson event over the infinitesimal time-interval dt is given as λdt . Upon the occurrence of the Poisson event, the change in the Poisson variable, $dN(\lambda)$, equals 1, otherwise $dN(\lambda)$ equals 0. The probability that more than one jump occurs during the interval dt is less than $O(dt)$, hence as dt converges to an infinitesimally small number, only one jump with probability λdt needs to be modeled over each interval dt for numerical applications.

Chacko and Das [2002] extend the Vasicek model with two exponential jump processes, such that the jump size and the jump intensity of the upward jumps can be different from those of the downward jumps. The double exponential jump model, denoted as the Vasicek-EJ model, also allows an analytical closed-form solution to the bond price.

The short rate process under the Vasicek-EJ model is given as follows:

$$dr = \alpha(m - r(t))dt + \sigma dZ(t) + J_u dN_u(\lambda_u) - J_d dN_d(\lambda_d) \quad (2)$$

where α , m , and σ are the speed of mean-reversion, long-term mean, and the diffusion volatility, respectively.

The up-jump variable J_u and the down-jump variable J_d are exponentially distributed with positive means $1/\eta_u$ and $1/\eta_d$, respectively. The two Poisson variables $dN_u(\tilde{\lambda}_u)$ and $dN_d(\tilde{\lambda}_d)$ are distributed independently, with intensities $\tilde{\lambda}_u$ and $\tilde{\lambda}_d$, respectively. The probability density of the exponential variable J is defined as follows:

$$f(J) = \eta \times e^{-J\eta}, \quad \text{for } 0 \leq J \leq \infty \quad (3)$$

The advantage of this model is that it allows separate distributions for the upward jumps and downward jumps. This can significantly reduce the probability of negative interest rates caused by downward jumps by allowing more flexibility in choosing the parameters under different interest rate regimes. For example, when interest rates are low at the bottom of a business cycle, the downward jumps in interest rates can be chosen to be lower both in magnitude and the frequency of occurrence than the upward jumps.

Using the stochastic discount factor, the risk-neutral process for the short rate is given as follows:¹

$$dr = \tilde{\alpha}(\tilde{m} - r(t))dt + \sigma d\tilde{Z}(t) + J_u dN_u(\tilde{\lambda}_u) - J_d dN_d(\tilde{\lambda}_d) \quad (4)$$

where

$$\begin{aligned} \tilde{\alpha} &= \alpha + \gamma_1 \\ \tilde{m} &= \frac{\alpha m - \gamma_0}{\alpha + \gamma_1} \\ \tilde{\lambda}_u &= \lambda_u(1 - \gamma_{uj}) \\ \tilde{\lambda}_d &= \lambda_d(1 - \gamma_{dj}) \end{aligned} \quad (5)$$

where γ_0 and γ_1 give the more general forms of market prices of diffusion risk as in Duffee [2002], and γ_{uj} and γ_{dj} give the market prices of upward jumps and downward jumps, respectively.

Let the short rate be given as follows:

$$r(t) = \tilde{m} + Y(t) \quad (6)$$

where the state variable $Y(t)$ follows the following risk-neutral process,

$$dY(t) = -\tilde{\alpha}Y(t)dt + \sigma d\tilde{Z}(t) + J_u dN_u(\tilde{\lambda}_u) - J_d dN_d(\tilde{\lambda}_d) \quad (7)$$

Using Ito's lemma, it is easy to confirm that the short rate process given in Equation (4) is consistent with the state variable process given in the preceding equation.

Expressing the bond price as a function of $Y(t)$ and t , using Ito's lemma, and taking risk-neutral expectation of the bond price, the partial differential difference equation (PDDE) for the bond price can be given as follows:

$$\begin{aligned} \frac{\partial P}{\partial t} - \tilde{\alpha}Y(t) \frac{\partial P}{\partial Y} + \frac{1}{2} \sigma^2 \frac{\partial^2 P}{\partial Y^2} + \tilde{E}_t[P(Y(t) \\ + J_u, t, T) - P(Y(t), t, T)]\tilde{\lambda}_u + \tilde{E}_t[P(Y(t) \\ - J_d, t, T) - P(Y(t), t, T)]\tilde{\lambda}_d = r(t)P(t, T) \\ = (\tilde{m} + Y(t))P(t, T) \end{aligned} \quad (8)$$

subject to the boundary condition $P(T, T) = 1$.

Bond Price Solution

The bond price solution for the Vasicek-EJ model is given as follows:

$$P(t, T) = e^{A(\tau) - B(\tau)Y(t) - H(t, T)} \quad (9)$$

where $\tau = T - t$, and

$$H(t, T) = \int_t^T \tilde{m} du = \tilde{m}(T - t) = \tilde{m}\tau \quad (10)$$

$$\begin{aligned} A(\tau) = (\tau - B(\tau)) \left(\frac{\sigma^2}{2\tilde{\alpha}^2} \right) - \frac{\sigma^2 B^2(\tau)}{4\tilde{\alpha}} - (\tilde{\lambda}_u + \tilde{\lambda}_d)\tau \\ + \frac{\tilde{\lambda}_u \eta_u}{\tilde{\alpha} \eta_u + 1} \ln \left[\left(1 + \frac{1}{\tilde{\alpha} \eta_u} \right) e^{\tilde{\alpha}\tau} - \frac{1}{\tilde{\alpha} \eta_u} \right] \\ + \frac{\tilde{\lambda}_d \eta_d}{\tilde{\alpha} \eta_d - 1} \ln \left[\left(1 - \frac{1}{\tilde{\alpha} \eta_d} \right) e^{\tilde{\alpha}\tau} + \frac{1}{\tilde{\alpha} \eta_d} \right] \end{aligned} \quad (11)$$

and

$$B(t, T) = \left(\frac{1 - e^{-\tilde{\alpha}\tau}}{\tilde{\alpha}} \right) \quad (12)$$

This solution is identical to that given by Chacko and Das [2002], although expressed in a slightly different form. As shown later, expressing the solution in this form allows it to be extended to fit the initial bond prices by simply changing the definition of the term $H(t, T)$.

Jump Diffusion Tree

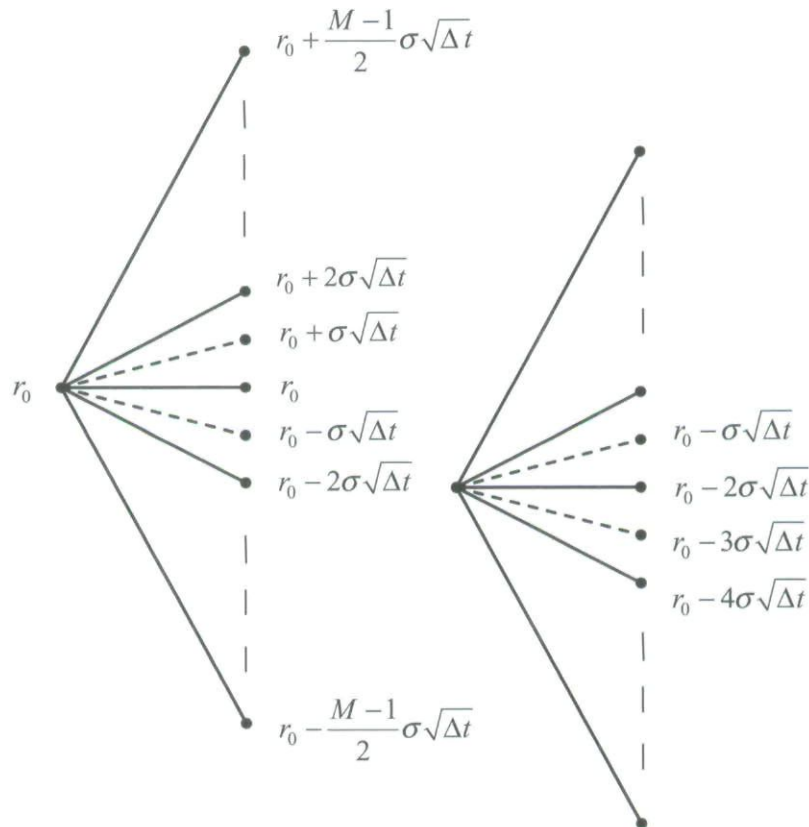
This section explains how to build a jump-diffusion tree for the Vasicek-EJ model for pricing American options. The following approximation based on Amin [1993] is used to model the short rate tree with exponential jumps:

$$dr(t) = \begin{cases} \tilde{\alpha}(\tilde{m} - r(t))dt + \sigma d\tilde{Z}(t), & \text{with probability } 1 - \tilde{\lambda}_u dt - \tilde{\lambda}_d dt, \\ J_u, & \text{with probability } \tilde{\lambda}_u dt, \\ -J_d, & \text{with probability } \tilde{\lambda}_d dt, \end{cases} \quad (13)$$

It can be shown that Equation (13) deviates from Equation (4) only in the order $o(dt)$ and, hence, the two equations converge in the limit as dt goes to zero. To approximate the jump-diffusion process in Equation (13), we use a single n -step *recombining* tree with M branches, where M is always an odd number. This tree is displayed in Exhibit 1. Given the initial value of the short rate is $r(0)$, the short rate can either undergo a local change due to the diffusion component shown through two local branches (represented by the segmented lines) or experience a jump shown through $M - 2$ remaining branches,

EXHIBIT 1

Jump-Diffusion Interest Rate Tree



including the central branch, where the short rate does not change (represented by the solid lines).

To keep the analysis general, we consider an arbitrary node $r(t)$, at time $t = i\Delta t$, for $i = 0, 1, 2, \dots, n-1$, where length of time T is divided into n equal intervals of Δt . At time $(i+1)\Delta t$, the M different values for the interest rate are given as follows:

$$r(t) \pm j\sigma\sqrt{\Delta t}, j = 0, 1, \dots, \frac{M-1}{2}$$

We specify M probabilities associated with the M nodes. The short rate can either undergo a local change with probability $1 - \tilde{\lambda}_u dt - \tilde{\lambda}_d dt$, or experience an up-jump with probability $\tilde{\lambda}_u dt$ or a down-jump with probability $\tilde{\lambda}_d dt$.

The two local probabilities are computed as follows:

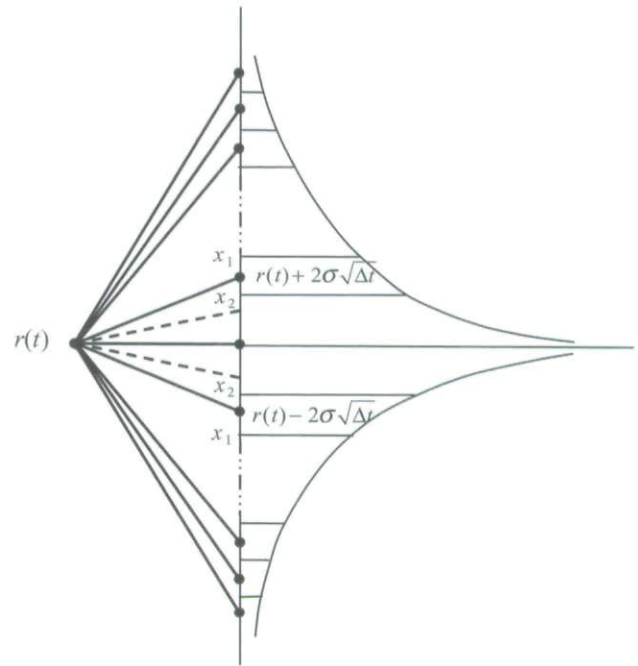
$$\begin{aligned} P[r(t) + \sigma\sqrt{\Delta t}] &= \left(\frac{1}{2} + \frac{1}{2} \frac{\tilde{\alpha}(\tilde{m} - r(t))}{\sigma} \sqrt{\Delta t} \right) \\ &\quad (1 - \tilde{\lambda}_u \Delta t - \tilde{\lambda}_d \Delta t) \\ P[r(t) - \sigma\sqrt{\Delta t}] &= \left(\frac{1}{2} - \frac{1}{2} \frac{\tilde{\alpha}(\tilde{m} - r(t))}{\sigma} \sqrt{\Delta t} \right) \\ &\quad (1 - \tilde{\lambda}_u \Delta t - \tilde{\lambda}_d \Delta t) \end{aligned} \quad (14)$$

The upward jump nodes and downward jump nodes are approximated separately with two exponential curves (see Exhibit 2).

The total risk-neutral probability of upward jumps is $\tilde{\lambda}_u \Delta t$ and the total risk-neutral probability of downward jumps is $\tilde{\lambda}_d \Delta t$. The upward jump probability and the downward jump probability are allocated to two separate exponential distributions, one drawn from zero to infinity above the center point in Exhibit 2, and the other drawn from zero to negative infinity below the center point. The entire line from negative infinity to positive infinity in Exhibit 2 consists of M points and M non-overlapping intervals that cover both the exponential distributions for upward and downward jumps. The probability mass over each of these M intervals is assigned

EXHIBIT 2

Approximation of the Jump Distribution Using Two Exponential Curves



to the point between each of these intervals using the following three steps.

Step 1:

The first step computes the probabilities of all jump nodes except the central node, the top node, and the bottom node, as follows. The probability mass for the intervals corresponding to the upward jump nodes is assigned to the points at the center of these intervals as follows:

$$\begin{aligned} P[r(t) + k\sigma\sqrt{\Delta t}] &= [F_u(x_1) - F_u(x_2)] \tilde{\lambda}_u \Delta t, \\ k &= 2, \dots, \frac{M-3}{2} \end{aligned} \quad (15)$$

where $F_u(x)$ is the cumulative exponential distribution for upward jumps defined as follows:

$$F_u(x) = \int_0^x \eta_u e^{-\eta_u J_u} dJ_u = 1 - e^{-\eta_u x} \quad (16)$$

and

$$\begin{aligned}x_1 &= (k + 0.5)\sigma\sqrt{\Delta t} \\x_2 &= (k - 0.5)\sigma\sqrt{\Delta t}\end{aligned}$$

The expression $F_u(x_1) - F_u(x_2)$ gives the difference between two cumulative exponential distribution functions and, hence, gives the probability measured by the area under the exponential curve over one of the regions corresponding to the upward jumps in Exhibit 2. Multiplication of this probability with the total risk-neutral jump probability of upward jumps $\tilde{\lambda}_u \Delta t$, gives the probability of the short rate node at the point between of x_1 and x_2 as shown in Equation (15).

Similarly, the probability mass for the intervals corresponding to the downward jump nodes is assigned to the points at the center of the following intervals as follows:

$$\begin{aligned}P[r(t) - k\sigma\sqrt{\Delta t}] &= [F_d(x_1) - F_d(x_2)]\tilde{\lambda}_d \Delta t, \\k &= 2, \dots, \frac{M-3}{2}\end{aligned}\quad (17)$$

where $F_d(x)$ is the cumulative exponential distribution for downward jumps defined as follows:

$$F_d(x) = \int_0^x \eta_d e^{-\eta_d J_d} dJ_d = 1 - e^{-\eta_d x} \quad (18)$$

where

$$\begin{aligned}x_1 &= (k + 0.5)\sigma\sqrt{\Delta t} \\x_2 &= (k - 0.5)\sigma\sqrt{\Delta t}\end{aligned}$$

Step 2:

Since we only consider the nodes in the range

$$\left[r(t) + \frac{M-1}{2}\sigma\sqrt{\Delta t}, r(t) - \frac{M-1}{2}\sigma\sqrt{\Delta t} \right]$$

in Exhibit 2 the probability mass outside this region should be assigned to the end nodes. The probability mass assigned to the top node corresponding to the upward jump is given as

$$P\left(r(t) + \frac{M-1}{2}\sigma\sqrt{\Delta t}\right) = [1 - F_u(x_2)]\tilde{\lambda}_u \Delta t \quad (19)$$

where

$$x_2 = \left(\frac{M-1}{2} - 0.5\right)\sigma\sqrt{\Delta t}$$

Similarly, the probability mass assigned to the bottom node corresponding to the downward jump is given as

$$P\left(r(t) - \frac{M-1}{2}\sigma\sqrt{\Delta t}\right) = [1 - F_d(x_2)]\tilde{\lambda}_d \Delta t \quad (20)$$

where

$$x_2 = \left(\frac{M-1}{2} - 0.5\right)\sigma\sqrt{\Delta t}$$

Step 3:

The remaining probabilities from both exponential distributions for upward and downward jumps are allocated to the central node as follows:

$$P[r(t)] = [F_u(x_1)]\tilde{\lambda}_u \Delta t + [F_d(x_1)]\tilde{\lambda}_d \Delta t \quad (21)$$

where

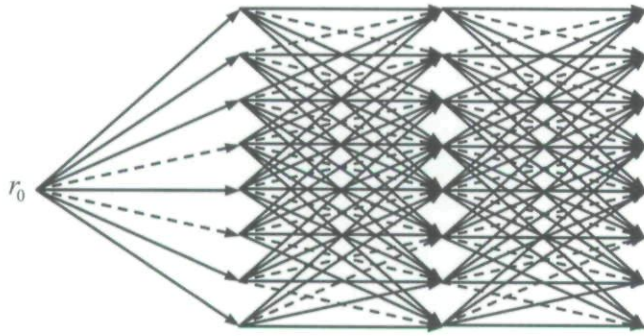
$$x_1 = 1.5\sigma\sqrt{\Delta t}$$

Equation (14) defines the probabilities at two nodes that give the local change due to the diffusion process, and Equations (15), (17), (19), (20), and (21) define the probabilities at the remaining $M-2$ nodes that give the change due to the two exponential jumps. Together, all of these probabilities sum up to one.

The number of nodes in a jump-diffusion tree is of the order $O(N^2M)$. To increase the efficiency we implement the following truncation methodology. The tree is truncated from above and below at $(M-1)/2$. This idea is illustrated in Exhibit 3. When M is large, the probability of reaching the top nodes is effectively zero. Therefore, the whole top part of the tree can be truncated without

EXHIBIT 3

Jump-Diffusion Tree Truncation



loss of accuracy. The probability mass above the truncation line is assigned to the node that lies on the truncation line. Because of truncation, the number of nodes in a tree is reduced to $O(N \times M)$.

THE VASICEK-EJ+ AND VASICEK-EJ++ MODELS

This section considers single-plus and double-plus extensions of the Vasicek-EJ model, given as the Vasicek-EJ+ model and the Vasicek-EJ++ model, based on the new taxonomy of term structure models given by NBS [2007]. As shown by NBS, the Vasicek-EJ+ model allows independence from the market price of diffusion risk (representing the one plus), and the Vasicek-EJ++ model allows not only independence from the market price of diffusion risk but also calibration to exogenously given initial zero-coupon bond prices (representing the two plusses).²

Though all of the valuation formulas and numerical solutions are identical under the Vasicek-EJ model and the Vasicek-EJ+ model, the empirical estimates of the risk-neutral parameters can be very different under these two models since the former model imposes restrictive linear functional forms on the diffusion-related market price of risk (MPR). In contrast, the latter model is independent of the diffusion-related MPR and can implicitly allow arbitrary nonlinear forms of diffusion-related MPR, which may even depend on multiple state variables (see NBS [2007, Chapter 4]).

The Vasicek-EJ+ model can be further extended to fit the initially observable bond prices at time zero. Under the extended model, denoted as the Vasicek-EJ++ model, the short rate process follows a time-inhomogeneous

process, given as a sum of a deterministic term and a state variable, as follows:

$$r(t) = \delta(t) + Y(t) \quad (22)$$

where $\delta(t)$ is the deterministic term used for calibration to the initial bond prices, and the risk-neutral stochastic process for the state variable $Y(t)$ is given as follows:

$$dY(t) = -\tilde{\alpha}Y(t)dt + \sigma d\tilde{Z}(t) + J_u dN_u(\tilde{\lambda}_u) - J_d dN_d(\tilde{\lambda}_d), \quad Y(0) = 0 \quad (23)$$

Expressing the bond price as a function of $Y(t)$ and t , using Ito's lemma, and taking a risk-neutral expectation of the bond price, the PDDE for the bond price can be given as follows:

$$\begin{aligned} \frac{\partial P}{\partial t} - \tilde{\alpha}Y(t)\frac{\partial P}{\partial Y} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial Y^2} + \tilde{E}_t[P(Y(t) \\ + J_u, t, T) - P(Y(t), t, T)]\tilde{\lambda}_u + \tilde{E}_t[P(Y(t) \\ - J_d, t, T) - P(Y(t), t, T)]\tilde{\lambda}_d = r(t)P(t, T) \\ = (Y(t) + \delta(t))P(t, T) \end{aligned} \quad (24)$$

The solution to the above PDDE is given as follows:

$$P(t, T) = e^{A(\tau) - B(\tau)Y(t) - H(t, T)} \quad (25)$$

where $\tau = T - t$, and

$$H(t, T) = \int_t^T \delta(u) du \quad (26)$$

subject to $A(0) = B(0) = 0$, and by definition $H(T, T) = 0$.

$$\begin{aligned}
A(\tau) = & (\tau - B(\tau)) \frac{\sigma^2}{2\tilde{\alpha}^2} - \frac{\sigma^2 B^2(\tau)}{4\tilde{\alpha}} - (\tilde{\lambda}_u + \tilde{\lambda}_d)\tau \\
& + \frac{\tilde{\lambda}_u \eta_u}{\tilde{\alpha} \eta_u + 1} \ln \left| \left(1 + \frac{1}{\tilde{\alpha} \eta_u} \right) e^{\tilde{\alpha} \tau} - \frac{1}{\tilde{\alpha} \eta_u} \right| \\
& + \frac{\tilde{\lambda}_d \eta_d}{\tilde{\alpha} \eta_d - 1} \ln \left| \left(1 - \frac{1}{\tilde{\alpha} \eta_d} \right) e^{\tilde{\alpha} \tau} + \frac{1}{\tilde{\alpha} \eta_d} \right| \quad (27)
\end{aligned}$$

and

$$B(t, T) = \left(\frac{1 - e^{-\tilde{\alpha} \tau}}{\tilde{\alpha}} \right) \quad (28)$$

Note that the solutions of the bond price under the Vasicek-EJ model (see Equation (9)) and the Vasicek-EJ++ model (see Equation (25)) are identical except for the $H(t, T)$ term, which becomes time-inhomogeneous under the latter model. To solve $\delta(t)$ and $H(t, T)$, under the Vasicek-EJ++ model, consider the initially observable zero-coupon bond price function given as $P(0, T)$. By fitting this price exactly to the Vasicek-EJ++ model, the log of the bond price evaluated at time 0 is given as follows:

$$\ln P(0, T) = A(T) - H(0, T) \quad (29)$$

where by definition, $Y(0) = 0$. Differentiating the preceding equation with respect to T , we get

$$\delta(T) = f(0, T) + \frac{\partial A(T)}{\partial T} \quad (30)$$

where $f(0, T) = -\partial \ln P(0, T) / \partial T$ is the initial forward rate curve at time 0. Substituting the partial derivative of $A(T)$ and replacing T with t , we get

$$\delta(t) = f(0, t) + \frac{1}{2} \sigma^2 B^2(t) + \tilde{\lambda}_u L_u(t) + \tilde{\lambda}_d L_d(t) \quad (31)$$

where

$$L_u(\tau) = \tilde{E}_t \left(e^{-B(\tau) J_u} \right) - 1 = \frac{-B(\tau)}{\eta_u + B(\tau)}$$

$$L_d(\tau) = \tilde{E}_t \left(e^{B(\tau) J_d} \right) - 1 = \frac{B(\tau)}{\eta_d - B(\tau)}$$

Using Equations (26) and (29), the function $H(t, T)$ can be expressed as follows:

$$\begin{aligned}
H(t, T) &= H(0, T) - H(0, t) \\
&= A(T) - A(t) - \ln P(0, T) + \ln P(0, t) \quad (32)
\end{aligned}$$

Since $P(0, t)$ and $P(0, T)$ are the initially observed bond prices, the preceding equation, together with Equations (27) and (28), gives the full closed-form solution of the bond price in Equation (25).

Jump Diffusion Tree

A slight modification of the jump diffusion tree derived for the Vasicek-EJ model immediately gives the jump diffusion tree for the Vasicek-EJ++ model. Consider the short rate process and the state variable process shown in Equations (4) and (23), respectively, given as follows:

$$dr = \tilde{\alpha}(\tilde{m} - r(t))dt + \sigma d\tilde{Z}(t) + J_u dN_u(\tilde{\lambda}_u) - J_d dN_d(\tilde{\lambda}_d) \quad (33)$$

$$\begin{aligned}
dY(t) &= \tilde{\alpha}(0 - Y(t))dt + \sigma d\tilde{Z}(t) + J_u dN_u(\tilde{\lambda}_u) \\
&\quad - J_d dN_d(\tilde{\lambda}_d), \quad Y(0) = 0 \quad (34)
\end{aligned}$$

The processes in the above two equations have identical forms except that the risk-neutral long-term mean $\tilde{m}$ equals 0 for the $Y(t)$ process, and the starting value for the $Y(t)$ process is given as $Y(0) = 0$. Hence, the jump diffusion tree for the $Y(t)$ process can be built exactly as it was built for the short rate under the Vasicek-EJ model, by equating $\tilde{m}$ to 0 and using the starting value $Y(0) = 0$. Once the jump diffusion tree for the $Y(t)$ process has been built, the jump diffusion tree for the short rate $r(t)$ can be obtained by adding the deterministic term $\delta(t)$ to $Y(t)$ at each node of the tree. The term $\delta(t)$ can be solved using two different methods. Using the first method, $\delta(t)$ is solved analytically using Equation (31). However, this solution is valid only in the continuous-time limit, and so it allows an exact match with the initial zero-coupon

bond prices only when using a large number of steps in the tree. If an exact match is desired for an arbitrary small number of steps, then $\delta(t)$ can be obtained numerically for $0 \leq t < T$ by defining a pseudo bond price $P^*(0, t)$, as follows:

$$P(0, t) = P^*(0, t)e^{-H(0, t)} = P^*(0, t)e^{-\int_0^t \delta(v)dv} \quad (35)$$

where $P(0, t)$ is the initially observed bond price r and $P^*(0, t)$ is the pseudo bond price obtained by taking discounted risk-neutral expectation using Y -process as the pseudo short rate. The jump diffusion tree for Y -process can be used to obtain $P^*(0, t)$ for different values of t (such that $0 \leq t < T$). Given the values of $P(0, t)$ and $P^*(0, t)$ for different values of t , $\delta(t)$ can be obtained numerically, as follows:

$$H(0, t) = \sum_{j=0}^{t/\Delta t - 1} \delta(j\Delta t)\Delta t = \ln P^*(0, t) - \ln P(0, t) \quad (36)$$

where $\Delta t = T/n$ and the integral in $H(0, t)$ is approximated as a discrete sum by dividing t into $t/\Delta t$ number of steps. The values of $\delta(j\Delta t)$, for $j = 0, 1, 2, \dots, n-1$, can be obtained iteratively by using successive values of t as $\Delta t, 2\Delta t, \dots$, and $n\Delta t$, such that an exact match is obtained with the initial bond price function $P(0, T)$. The values of $\delta(t)$ can then be added to the appropriate nodes on the n -step $Y(t)$ -tree to obtain the corresponding tree for the short rate.

VALUING OPTIONS

In this section, we value European options on zero-coupon bonds, or caplets, using the Fourier inversion method, and European options on coupon bonds, or European swaptions, using the cumulant expansion method.

Valuing European Options on Zero-Coupon Bonds, or Caplets: The Fourier Inversion Method

Heston [1993] introduced the Fourier inversion method to solve the price of an option when the underlying stock follows a stochastic volatility process. Recently, a number of researchers, including Duffie, Pan, and

Singleton [2000], Bakshi and Madan [2000], and Chacko and Das [2002], have shown this method to be versatile in obtaining quasi-analytical formulas for a variety of more difficult option pricing problems. For example, this method can be used to solve option prices when the underlying price processes include jumps and stochastic volatility, as well as to solve the prices of more complex derivatives like Asian options. We now demonstrate this method to solve for the price of an option on a zero-coupon bond under the Vasicek-EJ and Vasicek-EJ++ models. Because an interest rate caplet is equivalent to a European put option on a zero-coupon bond, and since an interest rate cap is a portfolio of caplets, this method can be also used to price caps.

Recall that we express the bond price solution in a slightly nontraditional form as follows:

$$P(t, T) = e^{A(\tau) - B(\tau)Y(t) - H(t, T)} \quad (37)$$

where $A(\tau)$ and $B(\tau)$ are time-homogenous functions of $\tau = T - t$. The motivation for using this form is that it allows a common framework for obtaining the formulas for the bond price under the Vasicek-EJ and Vasicek-EJ++ models, since the solutions of $A(\tau)$ and $B(\tau)$ are identical under both models and the only difference results from the $H(t, T)$ term. In this section, we derive formulas for pricing options under the Vasicek-EJ and Vasicek-EJ++ models and show that differences in the formulas between these models result only from the $H(t, T)$ term.

The price of a call option expiring on date S , written on a \$1 face-value zero-coupon bond maturing at time T , can be computed as follows (see Chacko and Das [2002]):

$$c(t) = P(t, T)\Pi_{1t} - KP(t, S)\Pi_{2t} \quad (38)$$

where

$$\Pi_{1t} = \int_{\ln K}^{\infty} f_{1t}(y)dy = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \text{Re} \left[\frac{e^{-i\omega \ln K} g_{1t}(\omega)}{i\omega} \right] d\omega \quad (39)$$

$$\Pi_{2t} = \int_{\ln K}^{\infty} f_{2t}(y)dy = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \text{Re} \left[\frac{e^{-i\omega \ln K} g_{2t}(\omega)}{i\omega} \right] d\omega \quad (40)$$

The terms Π_{1t} and Π_{2t} can be interpreted as *risk-neutral probabilities*. $\text{Re}(\cdot)$ function denotes the *real part* of the expression contained in the brackets. The expression inside the bracket of $\text{Re}(\cdot)$ contains complex numbers. All complex numbers can be written as $a + bi$, where i is the imaginary number, and a is the real part.

The characteristic functions $g_{1t}(w)$ and $g_{2t}(w)$ are given in closed-form as follows:³

$$g_{1t}(w) = \frac{e^{A_1^*(s) - B_1^*(s)Y(t) - H(S,T)(1+iw) - H(t,S)}}{P(t,T)}, \quad s = S - t \quad (41)$$

$$g_{2t}(w) = \frac{e^{A_2^*(s) - B_2^*(s)Y(t) - H(S,T)(iw) - H(t,S)}}{P(t,S)}, \quad s = S - t \quad (42)$$

where $H(t, S)$ and $H(S, T)$ are defined by Equation (10) under the Vasicek-EJ model and by Equation (32) under the Vasicek-EJ++ model, and $A_i^*(s)$ and $B_i^*(s)$ are given as

$$\begin{aligned} A_i^*(s) = & a_i + \frac{1}{2} \frac{\sigma^2}{\tilde{\alpha}^2} \left(\frac{s - 2(1 - \tilde{\alpha}b_i)B(s)}{+ \frac{1}{2}(1 - \tilde{\alpha}b_i)^2 B(2s)} \right) \\ & + \frac{\tilde{\lambda}_u \eta_u}{\tilde{\alpha} \eta_u + 1} \ln \left| \frac{(\tilde{\alpha} \eta_u + 1)e^{\tilde{\alpha}s} - (1 - \tilde{\alpha}b_i)}{\tilde{\alpha}(\eta_u + b_i)} \right| \\ & + \frac{\tilde{\lambda}_d \eta_d}{\tilde{\alpha} \eta_d - 1} \ln \left| \frac{(\tilde{\alpha} \eta_d - 1)e^{\tilde{\alpha}s} + (1 - \tilde{\alpha}b_i)}{\tilde{\alpha}(\eta_d - b_i)} \right| \\ & - (\tilde{\lambda}_u + \tilde{\lambda}_d)s \end{aligned} \quad (43)$$

and

$$B_i^*(s) = B(s) + b_i e^{-\tilde{\alpha}s} \quad (44)$$

where a_i and b_i for $i = 1$ and 2 are defined as

$$\begin{aligned} a_1 &= A(U)(1+iw) \\ b_1 &= B(U)(1+iw) \end{aligned} \quad (45)$$

and

$$\begin{aligned} a_2 &= A(U)(iw) \\ b_2 &= B(U)(iw) \end{aligned} \quad (46)$$

where $U = T - S$ and $A(\cdot)$ and $B(\cdot)$ are defined in Equations (11) and (12), respectively.

The solutions of the probabilities Π_{1t} and Π_{2t} can also be used to price a European put option by applying the put-call parity relation, as follows:

$$p(t) = KP(t, S)(1 - \Pi_{2t}) - P(t, T)(1 - \Pi_{1t}) \quad (47)$$

Note that the $H(\cdot)$ terms in Equations (41) and (42), under the Vasicek-EJ++ model, are given in *closed-form* in Equation (32). Hence, the preceding method has an analytical advantage over the calibration method suggested by Chacko and Das [2002], under which the time-dependent long-term mean is determined numerically using an iterative method for pricing options.

Valuing European Options on Coupon Bonds or European Swaptions: The Cumulant Expansion Method

Collin-Dufresne and Goldstein (CDG) [2002] proposed a fast and accurate cumulant expansion approximation for pricing European options on coupon bonds, or European swaptions, under multifactor affine models, giving explicit solutions for the case of the two-factor CIR model and the three-factor Vasicek model. Extending the CDG approach, NBS [2007] showed the *link* between the Fourier inversion-based approach for pricing caps and the cumulant expansion-based approach for pricing swaptions. Using this link, NBS derived explicit solutions for caps and swaptions under the *simple* multifactor affine models⁴ and also under the subfamily of quadratic term structure models with uncorrelated state variables and no interdependencies.

However, neither CDG nor NBS derived explicit analytical solutions for European options on coupon bonds (or European swaptions) for the case of the *jump-extended* Vasicek model of Chacko and Das [2002]. The previous section showed how to price options on zero-coupon bonds or caps under the jump-extended Vasicek model using the Fourier inversion-based approach. This section extends the results obtained in the previous section to

price European options on coupon bonds or European swaptions using the cumulant expansion-based approach.

As shown by CDG, the price of a European call option with expiration date S and exercise price K , written on a coupon bond with cash flows CF_i at time $T_i > S$ (for $i = 1, 2, \dots, n$), is given as follows:

$$c(t) = \sum_{i=1}^n CF_i P(t, T_i) \Pi_t^{T_i} (P_c(S) > K) - K P(t, S) \Pi_t^S (P_c(S) > K) \quad (48)$$

where $P_c(S) = \sum_{i=1}^n CF_i P(S, T_i)$ is the price of the coupon bond at the option expiration date, and $\Pi_t^W (P_c(S) > K) = E_t^W [1_{P_c(S) > K}]$ is defined as the probability of the option being in the money under the forward probability measure associated with maturity W .⁵

The method consists of approximating $n + 1$ forward probabilities $\Pi_t^W (P_c(S) > K)$ associated with maturity W , for $W = S, T_1, T_2, \dots, T_n$, using a cumulant expansion of the characteristic function associated with the random variable $P_c(S)$. Using the cumulant expansion, CDG showed that the forward probabilities can be approximated as follows:

$$\Pi_t^W (P_c(S) > K) \approx \sum_{j=0}^Q \kappa_j^{Q,W} \lambda_j^W \quad (49)$$

where Q can be chosen for the desired level of accuracy. CDG proposed a value of $Q = 7$ for most affine models. The appendix gives the solutions of the coefficients κ_j^Q and λ_j for $Q = 7$, as functions of the seven cumulants $c_1, c_2, c_3, \dots, c_7$ associated with the distribution of $P_c(S)$. Using the mathematical relationship between the cumulants and the moments of any distribution, the seven cumulants can be obtained using the first seven moments of the distribution of $P_c(S)$, as follows:

$$c_j = \mu_j - \sum_{i=1}^{j-1} \binom{j-1}{i} c_{j-i} \mu_i \quad (50)$$

The solutions of the coefficients κ_j^Q and λ_j (given in the appendix), and the moments and cumulants given in Equation (50) are independent of the forward measure being used. The forward-measure specific coefficients $\kappa_j^{Q,W}$ and

λ_j^W in Equation (49) can be obtained by simply replacing the seven moments $\mu_1, \mu_2, \mu_3, \dots, \mu_7$ in Equation (50) with the moments, $\mu_1^W, \mu_2^W, \mu_3^W, \dots, \mu_7^W$, defined as follows:

$$\mu_h^W = E_t^W [P_c(S)^h], \quad \text{for } h = 1, 2, \dots, Q \quad (51)$$

where the expectation is taken under the forward measure associated with the maturity W .

NBS [2007] provided a general solution for moments in Equation (51), which used solutions of the ordinary differential equations (ODEs) associated with the characteristic functions, *already obtained* for valuing options on zero-coupon bonds, or caps, using the Fourier-inversion method. Applying this method, the general solution for the moments under the jump-extended Vasicek models is given as follows:

$$\begin{aligned} \mu_h^W &= E_t^W [P_c(S)^h] \\ &= \sum_{l_1=1}^n \sum_{l_2=1}^n \dots \sum_{l_h=1}^n (CF_{l_1} CF_{l_2} \dots CF_{l_h}) \\ &\quad \times \exp \left(\frac{A_G^*(s) - B_G^*(s) Y(t) - H(t, W) - \sum_{k=1}^h H(S, T_{l_k})}{P(t, W)} \right) \end{aligned} \quad (52)$$

where $s = S - t$, and the solutions of $A_G^*(s)$ and $B_G^*(s)$ are identical to the solutions of $A_i^*(s)$ and $B_i^*(s)$, respectively, obtained in Equations (43) and (44), respectively, with a_i and b_i replaced by the following:

$$\begin{aligned} a_i &= A(W - S) + \sum_{k=1}^h A(U_{l_k}) \\ b_i &= B(W - S) + \sum_{k=1}^h B(U_{l_k}) \\ U_{l_k} &= T_{l_k} - S \end{aligned} \quad (53)$$

and $A(\cdot)$ and $B(\cdot)$ are the solutions of the ODEs associated with the PDDE of the bond price, given in Equations (27) and (28), respectively. The function $H(t, T)$ is given by Equation (10) for the Vasicek-EJ model and is given by Equation (32) for the Vasicek-EJ++ model.

Valuing American Options

The Fourier inversion method and the cumulant expansion method presented in the previous two subsections give prices for the European options on zero-coupon bonds (caplets) and European options on coupon bonds (European swaptions), respectively. For pricing American options, we use the truncated-tree approach introduced earlier for the Vasicek-EJ and Vasicek-EJ++ models. To compute American option prices, the value of the interest rate and the probability vector are computed at each node, according to Equations (14) through (21). The interest rate tree is then used to compute the terminal option values at each terminal node, and the option values at the remaining nodes are obtained by taking the maximum of the discounted option values and the corresponding intrinsic values.

The accuracy for pricing American options can be investigated by comparing the European put option prices obtained using the truncated tree approach and the Euro-

pean put prices obtained through the Fourier inversion approach, under the Vasicek-EJ model. Exhibit 4 makes this comparison using the following parameter values: current value of the interest rate, $r_0 = 10\%$; risk-neutral speed of mean reversion, $\tilde{\alpha} = 0.2$; risk-neutral long-run mean, $\tilde{m} = 10\%$; and interest rate volatility, $\sigma = 2\%$. The maturity of bonds was varied from $T = 1$ year to $T = 10$ years, the maturity of options was varied from 1 month to 5 years, and the bond face value, F , is assumed to be \$100. The intensities of the Poisson process, $\lambda_u = 3$, $\lambda_d = 3$, and the jump variables, J_u and J_d , are distributed exponentially with the means, $1/\eta_u = 0.005$ and $1/\eta_d = 0.005$. The option prices are solved using $N = 200$ steps and $N = 1,000$ steps, for which the truncation levels are at $M = 150$ and $M = 400$, respectively.

Exhibit 4 shows fast convergence between European put values computed using the truncated tree procedure and the Fourier inversion method. The accuracy falls as bond maturity increases, but improves with an increase in the number of steps, N , used in the tree.

EXHIBIT 4

European Put Option Prices under the Vasicek-EJ Model Using the Jump-Tree and Fourier Inversion Method

Bond Maturity (years)	Option Maturity (years)	Strike	Tree (N = 200)	Tree (N = 1000)	Fourier Inversion
1	0.0833	90	0.01	0.01	0.01
1	0.0833	100	8.71	8.68	8.68
1	0.0833	110	18.63	18.60	18.59
1	0.5	90	0.00	0.00	0.00
1	0.5	100	4.63	4.63	4.63
1	0.5	110	14.14	14.14	14.14
3	1	80	0.56	0.47	0.46
3	1	90	7.29	7.23	7.22
3	1	100	16.33	16.27	16.26
3	2	80	0.00	0.00	0.00
3	2	90	0.68	0.62	0.60
3	2	100	7.72	7.71	7.70
5	1	70	3.17	3.02	3.00
5	1	80	11.17	11.29	11.30
5	1	90	20.19	20.33	20.34
5	3	70	0.01	0.00	0.00
5	3	80	0.89	0.72	0.69
5	3	90	5.81	5.81	5.81

Exhibit 5 compares the prices for European swaptions obtained using the cumulant expansion method with $Q = 7$, and the truncated tree approach with $N = 200$ and $M = 150$, and with $N = 1000$ and $M = 400$. The parameter values are kept as in Exhibit 4. The swap length varies from one to five years, and the option expiration ranges from one month to three years. The swap strike rates are 2%, 6%, and 10%. The tenor of the fixed leg is 0.5 years, and the notional principal is assumed to be \$100.

Exhibit 5 shows the accuracy gains in the tree values when the number of steps increases and, also, in most of the cases, a convergence of the tree values to those obtained from the cumulant expansion approach. Also, the differences between the prices obtained using the truncated tree and the cumulant expansion approach are in line with the differences found between the tree approach and the Fourier expansion method in Exhibit 4.

SUMMARY AND CONCLUSION

In this article, we modify the Gaussian jump-diffusion tree of Amin [1993] and apply it to the Vasicek-EJ, Vasicek-EJ+, and Vasicek-EJ++ models for pricing American interest rate options. Our simulations show fast convergence of European option prices obtained using the *truncated* jump-diffusion tree, to those obtained using the Fourier inversion method (for options on zero-coupon bonds, or caplets), and the cumulant expansion method (for options on coupon bonds, or swaptions). As additional contributions, we also provide 1) an analytical solution for the deterministic drift of the short rate process under the Vasicek-EJ++ model, which allows this model to be exactly calibrated to the initially observable bond prices; and 2) new analytical solutions for pricing European interest rate swaptions under the Vasicek-EJ, Vasicek-EJ+, and Vasicek-EJ++ models, using the cumulant expansion method of Collin-Dufresne and Goldstein [2002].

EXHIBIT 5

European Swaption Prices under the Vasicek-EJ Model Using the Jump-Tree and Cumulant-Expansion Method

Swap Length (years)	Option Expiration (years)	Strike Rate (annual rate in %)	Tree (N = 200)	Tree (N = 1000)	Cumulant Expansion (Q = 7)
1	0.0833	2	7.61	7.60	7.59
1	0.0833	6	3.93	3.91	3.90
1	0.0833	10	0.39	0.38	0.42
1	0.5	2	7.31	7.27	7.26
1	0.5	6	3.78	3.74	3.73
1	0.5	10	0.72	0.70	0.70
3	1	2	18.60	18.59	18.59
3	1	6	9.46	9.46	9.45
3	1	10	1.89	1.88	1.88
3	2	2	16.68	16.68	16.68
3	2	6	8.49	8.48	8.48
3	2	10	2.11	2.09	2.08
5	1	2	28.12	28.05	28.02
5	1	6	14.20	14.13	14.09
5	1	10	2.46	2.39	2.36
5	3	2	22.56	22.59	22.60
5	3	6	11.27	11.26	11.27
5	3	10	2.73	2.61	2.58

APPENDIX

The solutions of the coefficients κ_j^Q and λ_j for $Q = 7$ as functions of the seven cumulants, $c_1, c_2, c_3, \dots, c_7$:

$$\begin{aligned}\kappa_0^7 &= 1 + \frac{3c_4}{4!c_2^2} - \frac{15}{c_2^3} \left(\frac{c_6}{6!} + \frac{c_3^2}{2(3!)^2} \right) \\ \kappa_1^7 &= -\frac{3c_3}{3!c_2^2} + \frac{15c_5}{5!c_2^3} - \frac{105}{c_2^4} \left(\frac{c_7}{7!} + \frac{c_3c_4}{3!4!} \right) \\ \kappa_2^7 &= -\frac{6c_4}{4!c_2^3} + \frac{45}{c_2^4} \left(\frac{c_6}{6!} + \frac{c_3^2}{2(3!)^2} \right) \\ \kappa_3^7 &= \frac{c_3}{3!c_2^3} - \frac{10c_5}{5!c_2^4} + \frac{105}{c_2^5} \left(\frac{c_7}{7!} + \frac{c_3c_4}{3!4!} \right) \\ \kappa_4^7 &= \frac{c_4}{4!c_2^4} - \frac{15}{c_2^5} \left(\frac{c_6}{6!} + \frac{c_3^2}{2(3!)^2} \right) \\ \kappa_5^7 &= \frac{c_5}{5!c_2^5} - \frac{21}{c_2^6} \left(\frac{c_7}{7!} + \frac{c_3c_4}{3!4!} \right) \\ \kappa_6^7 &= \frac{1}{c_2^6} \left(\frac{c_6}{6!} + \frac{c_3^2}{2(3!)^2} \right) \\ \kappa_7^7 &= \frac{1}{c_2^7} \left(\frac{c_7}{7!} + \frac{c_3c_4}{3!4!} \right)\end{aligned}\quad (54)$$

and

$$\begin{aligned}\lambda_0 &= D \\ \lambda_1 &= c_2 d \\ \lambda_2 &= c_2 D + c_2 (K - c_1) d \\ \lambda_3 &= (c_2 (K - c_1)^2 + 2c_2^2) d \\ \lambda_4 &= 3c_2^2 D + (c_2 (K - c_1)^3 + 3c_2^2 (K - c_1)) d \\ \lambda_5 &= (c_2 (K - c_1)^4 + 4c_2^2 (K - c_1)^2 + 8c_2^3) d \\ \lambda_6 &= 15c_2^3 D + (c_2 (K - c_1)^5 + 5c_2^2 (K - c_1)^3 + 15c_2^3 (K - c_1)) d \\ \lambda_7 &= (c_2 (K - c_1)^6 + 6c_2^2 (K - c_1)^4 + 24c_2^3 (K - c_1)^2 + 48c_2^4) d\end{aligned}\quad (55)$$

where

$$\begin{aligned}D &= \mathcal{N} \left[\frac{c_1 - K}{\sqrt{c_2}} \right] \\ d &= \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(K - c_1)^2}{2c_2}}\end{aligned}\quad (56)$$

$\mathcal{N}(\cdot)$ gives the cumulative normal distribution function, and c_j ($j = 1, 2, \dots, Q = 7$) are the cumulants of the distribution of $P_c(S)$.

ENDNOTES

¹The derivation is similar to Das and Foresi [1996] and can be obtained from the authors.

²The notation is borrowed from Brigo and Mercurio [2001] and NBS [2007, Chapter 5].

³The proof can be obtained from the authors upon request.

⁴NBS [2007, Chapter 9] defined simple multifactor affine models (i.e., simple $A_M(N)$ models) as models that have $N - M$ normally distributed processes, which may be correlated with each other but are uncorrelated with the M independently distributed square root processes, for $N \geq M$.

⁵ $1_{P_c(S) > K}$ is the indicator function, which equals 1 if $P_c(S) > K$, and 0 otherwise.

REFERENCES

- Amin, Kaushik I. "Jump Diffusion Option Valuation in Discrete Time." *Journal of Finance*, Vol. 48, No. 5 (1993), pp. 1833–1863.
- Backus, David K., Silverio Foresi, and Liuren Wu. "Macro-Economic Foundations of Higher-Order Moments in Bond Yields." Working paper, New York University, 1997.
- Bakshi, Gurdip, and Dilip Madan. "Spanning and Derivative Security Valuation." *Journal of Financial Economics*, Vol. 55, No. 2 (2000), pp. 205–238.
- Brigo, Damiano, and Fabio Mercurio. "Interest Rate Models: Theory and Practice." *Springer Finance*, 2001.
- Chacko, George, and Sanjiv R. Das. "Pricing Interest Rate Derivatives: A General Approach." *Review of Financial Studies*, Vol. 15, No. 1 (2002), pp. 195–241.
- Collin-Dufresne, Pierre, and Robert S. Goldstein. "Pricing Swaptions within Affine Framework." *The Journal of Derivatives*, Fall 2002.

- Das, Sanjiv R. "A Direct Discrete-Time Approach to Poisson-Gaussian Bond Option Pricing in the Heath-Jarrow-Morton Model." *Journal of Economic Dynamics and Control*, Vol. 23, No. 3 (1998), pp. 333-369.
- . "The Surprise Element: Jumps in Interest Rates." *Journal of Econometrics*, Vol. 106, No. 1 (2002), pp. 27-65.
- Das, Sanjiv R., and Silverio Foresi. "Exact Solutions for Bond and Option Prices with Systematic Jump Risk." *Review of Derivatives Research*, Vol. 1, No. 1 (1996), pp. 7-24.
- Duffee, Gregory R. "Term Premia and Interest Rate Forecasts in Affine Models." *Journal of Finance*, Vol. 57, No. 1 (2002), pp. 405-443.
- Duffie, Darrell, Jun Pan, and Kenneth J. Singleton. "Transform Analysis and Option Pricing for Affine Jump-Diffusions." *Econometrica*, Vol. 68, No. 6 (2000), pp. 1343-1376.
- Heston, Steven. "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options." *Review of Financial Studies*, 6 (1993), pp. 327-343.
- Jarrow, Robert, Haitao Li, and Feng Zhao. "Interest Rate Caps Smile Too! But Can the Libor Market Models Capture It." *Journal of Finance*, Vol. 62, No. 1 (2007).
- Johannes, Michael. "The Statistical and Economic Role of Jumps in Continuous-Time Interest Rate Models." *Journal of Finance*, 59 (2004), pp. 227-260.
- Nawalkha, Sanjay K., Natalia A. Beliaeva, and Gloria M. Soto. *Dynamic Term Structure Modeling: The Fixed-Income Valuation Course*. Hoboken, NJ: Wiley Finance, 2007.
- Piazzesi, Monika. "A Linear-Quadratic Jump-Diffusion Model with Scheduled and Unscheduled Announcements." Working paper, Stanford University, 1998.
- . "Bond Yields and the Federal Reserve." *Journal of Political Economy*, Vol. 113, No. 2 (2005), pp. 311-344.
- Vasicek, Oldrich. "An Equilibrium Characterization of the Term Structure." *Journal of Financial Economics*, Vol. 5, No. 2 (1977), pp. 177-188.

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