Analytical Valuation of Turbo Warrants under Double Exponential Jump Diffusion

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Turbo warrants have experienced huge growth since they first appeared 2005. In some European markets, buying and selling turbo warrants constitutes 50% of all derivative trading nowadays. In Asia, the Hong Kong Exchange and Clearing Limited (HKEx) introduced the callable bull/bear contracts, which are essentially turbo warrants, to the market in 2006. Turbo warrants are special types of barrier options in which the rebate is calculated as another exotic option. Under the Black-Scholes dynamics, Eriksson [2005] has derived the closed form solution for turbo warrants and shows that it is less sensitive to the change in volatility when compared to vanilla options. Wong and Chan [forthcoming] study the pricing under several stochastic volatility models and point out that turbo warrants can be very sensitive to the change in the shape of the volatility surface. As another possible factor contributing to the volatility surface is the jump in asset price, we obtain analytical solutions for turbo warrants under the double exponential jump diffusion model in terms of Laplace transforms in this article. Our simulation verifies that the analytical solution is accurate and very efficient. The computational time is less than one second. We then examine the impact of jumps on Turbo warrants.

A turbo warrant is similar to a barrier option, but the barrier is set to be below the current stock price and above the strike, so the option is in the money and a rebate is paid if the asset price passes the barrier. The rebate is usually calculated according to an exotic option payoff. At the end of February 2005, Societe Generale (SG) listed the first 40 turbo warrants on the Nordic Growth Market (NGM) and Nordic Derivatives Exchange. During February 2005, the turbo warrant trading revenue was 31 million kronor, 50% (31/55) of total NGM trading revenue of 55 million kronor. In June 2006, the Hong Kong Exchange and Clearing Limited (HKEx) introduced callable bull/bear contracts (CBBC), which are actually turbo warrants. There are two types of CBBC: N and R. The N-CBBC pays no rebate when the barrier is crossed whereas the R-CBBC pays an exotic rebate, see HKEx [2006]. Therefore, turbo warrants are no longer regarded as exotic options, and there is a practical need to establish an efficient pricing framework for this derivative security.

Eriksson [2005] derives explicit solutions to turbo call and put warrants using the Black-Scholes model. Wong and Lau [2008] derive an analytical solution for turbo warrants with mean reversion using Laplace transforms. The challenge of pricing a turbo warrant is that the option payoff involves two first passage times. When the barrier is crossed, the rebate paid to the turbo call (put) warrant holder is the difference between the lowest (highest) recorded stock price during a pre-specified period after the barrier is hit and the strike price. Therefore, the rebate itself can be viewed as a non-standard lookback option. Eriksson [2005] points out that a turbo warrant is not insensitive to the change in volatility but the sensitivity is a lot less than that of its vanilla counterpart.

When stochastic volatility is taken into account, Wong and Chan [forthcoming] find that turbo warrants can be very sensitive to the change in the shape of the volatility smile, in contrast to the result of Ericsson [2005] using the Black-Scholes model. In fact, Wong and Chan [forthcoming] find that the sensitivity is model-dependent. Under the CEV model, a turbo warrant is more sensitive to a parallel shift of the volatility smile than its vanilla counterpart but the sensitivity to a change in skewness is similar. For the fast mean-reverting stochastic volatility model, a turbo warrant is less sensitive to the change in volatility smile than its vanilla counterpart. For a two-scale volatility model, a turbo warrant can be more sensitive to the change in the volatility surface than a vanilla option.

Besides the stochastic volatility effect, it has been well documented that jump risk in asset price is an important factor for explaining the volatility smile, especially for short-term contracts. As there are no results on pricing turbo warrants under jump-diffusion, we investigate the pricing under the double exponential jump diffusion of Kou [2002, KJD model] in this article.

The KJD model is considered because of two reasons. Firstly, this model allows analytical tractability for path-dependent options. Kou and Wang [2004] and Kou et al. [2005] have derived analytical solutions for barrier and lookback options in terms of Laplace transform. We will see shortly that turbo warrants can be decomposed into a barrier option and a knock-in lookback option. The barrier option part can be valued using their solutions. Additional efforts should be paid to calculate the knock-in lookback component as it involves two first passage times: the first time of crossing the barrier and the first time of reaching an extreme value after the barrier is crossed. Secondly, the KJD model can produce a better statistical fit to the data but retains an efficient pricing formula, see comments of Leib [2000]. The KJD model is a popular alternative to the Black-Scholes model in the finance literature and in practice.

Despite deriving valuation formulas, we examine the pricing quality and pricing behavior of turbo warrants when jump appears in the asset price. As the solution involves several Laplace inversions, we show numerically that the accuracy and efficiency of our pricing formulas are superior to the Monte Carlo simulation. The computational speed is comparable to an explicit solution. Using our solution, we further examine the sensitivity of turbo warrants to the jump arrival rate and parameters of the jump size.

The remainder of the article is organized as follows. We next present some model-free characteristics of turbo warrants, including the replication of the rebate. The following section introduces the KJD model and derives the solution for turbo warrants. Then, we provide numerical examples and discusses the sensitivity issue. And, finally, we conclude the article.

CHARACTERISTICS OF TURBO WARRANTS

Let S_t be the underlying asset price at time t. A turbo call warrant pays the option holder $(S_T - K)^+$ at maturity T if a specified barrier $B \geq K$ has not been passed by S_t at any time prior to the maturity. Denote τ_b as the first time that the asset price crosses the barrier B, i.e., $\tau_b = \inf\{t \mid S_t \leq \mathrm{B} = S_0 e^{b}\}$. If $\tau_b \leq T$, then the contract is void and a new contract starts. The new contract is a call option on $m^h_{\tau_b} = \min_{\tau_b \leq t \leq \tau_b + h} S_t$, with the strike price K, and the time to maturity h. More precisely, the turbo call option can be expressed as

$$\begin{aligned}\mathrm{TC}(t,S) = e^{-rT}\mathbf{E}_t\left[(S_T - K)^+ 1_{\{\tau_b > T\}}\right] \\ + \mathbf{E}_t\left[e^{-r(\tau_b + h)}(m^h_{\tau_b} - K)^+ 1_{\{\tau_b \leq T\}}\right]\end{aligned} \quad (1)$$

where r is the constant interest rate and $\mathbf{E}_t$ represents the risk-neutral expectation given information up to time t.

Wong and Chan [forthcoming] show in their Proposition 2.1 that the turbo call warrant is less expensive than the European call if the underlying asset pays no dividend. As their proof is model-independent, this result is also true for jump-diffusion models.

It can be seen from Equation (1) that a turbo call warrant consists of two parts. The first part resembles a down-and-out call (DOC) option, with a zero rebate, and the second part is a down-and-in lookback (DIL) option. We now define:

$$\mathrm{DOC}(t,S) = e^{-rT}\mathbf{E}_t\left[(S_T - K)^+ 1_{\{\tau_H > T\}}\right] \quad (2)$$

$$\mathrm{DIL}(t,S,h) = \mathbf{E}_t\left[e^{-r(\tau_b + h)}(m^h_{\tau_b} - K)^+ 1_{\{\tau_b \leq T\}}\right] \quad (3)$$

such that TC = DOC + DIL.

As the DOC option pricing has been studied thoroughly in the literature, we focus on the DIL option. Using the tower property of conditional expectation, we know that

$$\mathrm{DIL} = \mathbf{E}_t\left\{e^{-r\tau_b}\mathbf{E}_{\tau_b}\left[e^{-rh}(m_{\tau_b}^h - K)^+\right]\right\}$$

The inner expectation represents a nonstandard lookback option,

$$\mathrm{LB}(\tau_b, S_{\tau_b}, h) = \mathbf{E}_{\tau_b}\left[e^{-rh}(m_{\tau_b}^h - K)^+\right]$$

We now link this lookback option to the standard floating strike lookback call option. It is useful to point out that

$$(m_{\tau_b}^h - K)^+ = [S_{\tau_b+h} - \min(m_{\tau_b}^h, K)] - (S_{\tau_b+h} - m_{\tau_b}^h)$$

By the no arbitrage pricing principle, we conclude that (see Wong and Kwok [2003]),

$$\begin{aligned}\mathrm{LB}(\tau_b, S_{\tau_b}, h) = {} & \mathrm{LC}_{fl}(S_{\tau_b}, \min(S_{\tau_b}, K), h) \\ & - \mathrm{LC}_{fl}(S_{\tau_b}, S_{\tau_b}, h) \qquad (4)\end{aligned}$$

where $\mathrm{LC}_{fl}(S, m, T)$ is the floating strike lookback call on S with realized minimum m, time to maturity T. The payoff of a floating strike lookback call option is $S_{\tau_b+h} - m_{\tau_b}^h$. In Equation (4), we have used $m_{\tau_b}^0 = S_{\tau_b}$.

When Equations (1)–(4) are put together, we have the following proposition.

Proposition 2.1. *At $t < \tau_b$, the model-free representation of the turbo call warrant is given by*

$$\mathrm{TC}(t,S) = \mathrm{DOC}(t,S) + \mathbf{E}_t\left[e^{-r\tau_b}1_{\{\tau_b \le T\}}LB(\tau_b, S_{\tau_b}, h)\right] \quad (5)$$

where the model-free representation of $LB(\tau_b, S_{\tau_b}, h)$ appears in Equation (4).

Remark: *For jump-diffusion models, we have $S_{\tau_b} \neq B$ in general because it is possible to have an overshoot at the first passage time (see comments in Kou and Wang [2003, 2004]).*

Wong and Chan [forthcoming] point out that turbo warrants can easily be valued in closed form when the asset price follows a time-independent local volatility model, in which the asset price dynamics is

$$\frac{dS}{S} = (r-q)dt + \sigma(S)dW$$

where $\sigma(S)$ is a deterministic function of S. This model nests the Black-Scholes and CEV models as special cases.

Corollary 2.1. (*Wong and Chan [forthcoming]*) *If the asset price follows the time-independent local volatility model, then, at $t < \tau_b$, the turbo call price reads*

$$\mathrm{TC}(t,S) = \mathrm{DOC}(t,S) + \mathrm{LB}(0,B,h)\mathbf{E}_t\left[e^{-r\tau_b}1_{\{\tau_b \le T\}}\right] \quad (6)$$

Specifically, the Black-Scholes (BS) price for turbo call warrant reads:

$$\begin{aligned}\mathrm{TC}_{BS} = {} & \mathrm{DOC}_{BS}(t,S) + \Big[\mathrm{LC}_{fl}^{BS}(B,K,h) \\ & - \mathrm{LC}_{fl}^{BS}(B,B,h)\Big]\mathrm{DR}_{BS}(t,S) \qquad (7)\end{aligned}$$

where DOC_{BS} is the BS price of the DOC option, LC_{fl}^{BS} is the BS price of the floating strike lookback call option and $\mathrm{DR}_{BS}(t,S) = \mathbf{E}_t(e^{-r\tau_b}1_{\{\tau_b \le T\}})$ is the expected present value of a unity rebate. Their formulas are provided in Appendix A. Notice that the payoff of LC_{fl}^{BS} is $S - m_{\tau_1}^h$. This pricing formula for turbo call warrant will be useful for comparison in a later section.

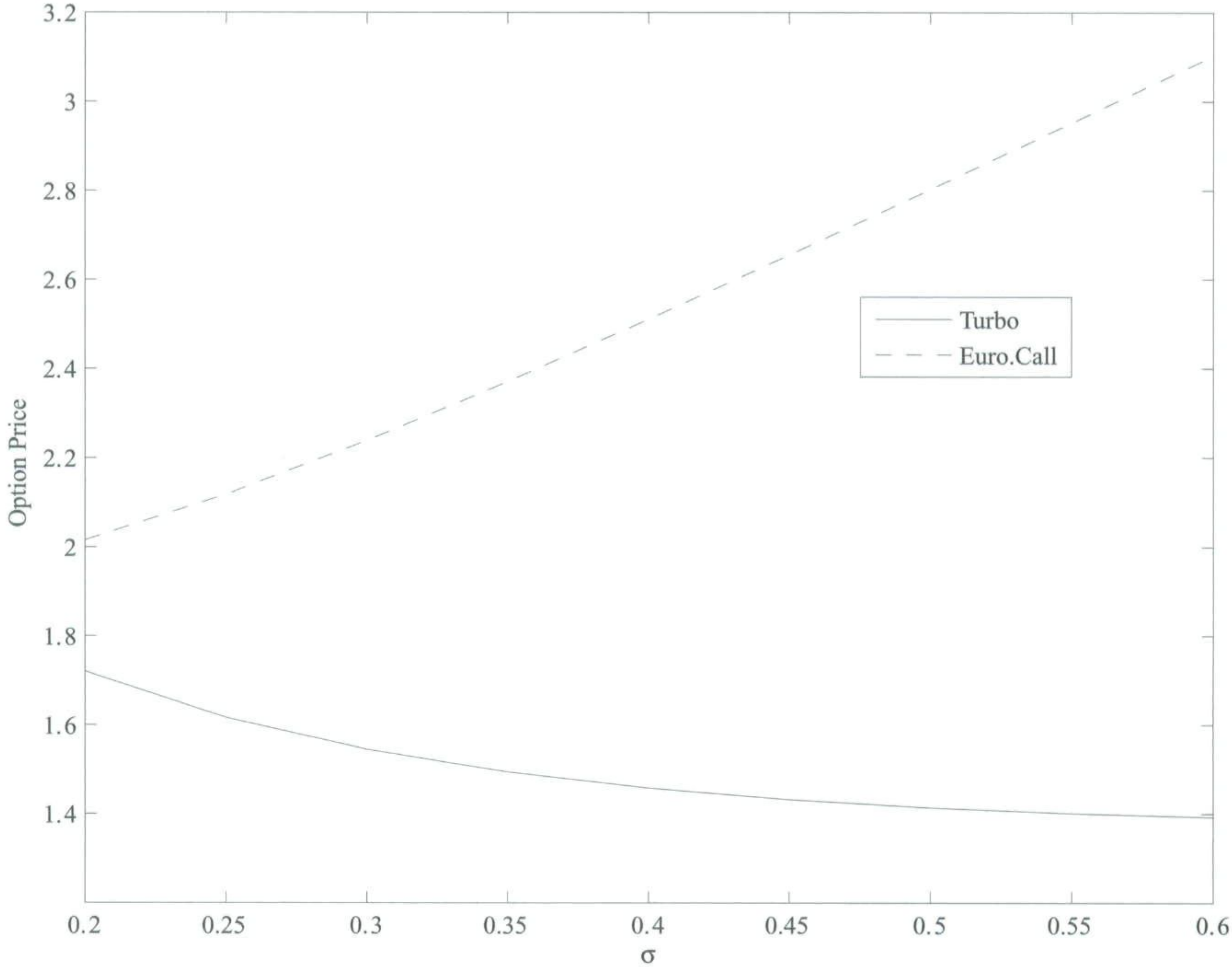
To examine the sensitivity to the volatility, we plot the vanilla and turbo call prices against the volatility in Exhibit 1. We have used $S_0 = 10$, $B = 9$, $K = 8$, $T = 1$, $T_0 = 0.2$, $r = 0.05$, and $q = 0.03$ in our computation. It can be observed that the turbo call is much less sensitive to the change in volatility when compared to the European call option. When the volatility increases, the vanilla call price rises whereas the turbo call price drops. Notice that the change in volatility here can be interpreted as the parallel movement of the implied volatility surface. In the Black-Scholes model, the surface is actually a flat plane.

TURBO PRICING UNDER KJD

We now introduce the KJD model. Under the risk-neutral probability measure, $\mathbb{Q}$, the underlying asset price process is postulated to be

Exhibit 1

Turbo Warrant under Black-Scholes



$$S_t = S_0 e^{X_t} \tag{8}$$

$$dX_t = (r - q - \frac{1}{2}\sigma^2 - \lambda k)dt + \sigma dW_t + Y_t dN_t$$

where r is the constant interest rate, q is the constant dividend yield, σ is the constant volatility for the diffusion component, N_t is the Poisson process having intensity λ, $k = E(e^Y - 1)$, W_t is the Wiener process, and Y_t is a series of independent random variables representing the jump size. The asset price dynamics in Equation (8) refer to the jump-diffusion model. In the finance literature, we always assume that W_t, Y_t, and N_t are independent when we talk about jump-diffusion models. A more general approach is called a Levy process.

Kou [2002] proposes a double exponential distribution to model the jump size Y. The probability density function for Y is then given by

$$f_Y(y) = p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + (1-p) \cdot \eta_2 e^{\eta_2 y} 1_{\{y<0\}},$$
$$p \in [0,1], \eta_1 > 1, \eta_2 > 0 \tag{9}$$

There are several advantages with the Kou jump-diffusion (KJD) model. Firstly, it allows analytical solutions for European options, American options, barrier options, and lookback options to be derived. Secondly, it captures the leptokurtic feature of market data and provides a psychological means for the empirical distribution. It is common in psychology to use double exponential distribution to describe underreactions and overreactions in human behavior, see Kou [2002]. Finally, the volatility smile implied by European options can be captured.

As a turbo warrant can be decomposed into a sum of a DOC option and a DIL option, see Proposition 2.1, we separately derive the solutions for each of them. It is important to notice that Kou and Wang [2003, 2004] and Kou et al. [2005] have developed a general pricing framework for barrier options. The pricing of DOC options can then be obtained through their approach but the DOC option pricing formula is not explicitly given in their papers. Deriving DOC options aims at producing a self-contained paper. More importantly, the pricing of the DIL option embedded in turbo warrants is completely new in the literature and the derivation is not an obvious extension of any other path-dependent options.

The DOC Component

Given the asset price process in Equations (8) and (9), we consider the moment-generating function of X_t:

$$\mathrm{E}\left[e^{\theta X_T}\right] = \exp[G(\theta)T] \tag{10}$$

where

$$G(x) = x\left(r - q - \frac{\sigma^2}{2} - \lambda(\mathrm{E}[e^Y]-1)\right) + \frac{x^2\sigma^2}{2} + \lambda\left(\frac{p\eta_1}{\eta_1 - x} + \frac{(1-p)\eta_2}{\eta_2 + x} - 1\right) \tag{11}$$

This moment-generating function has been obtained in Kou and Wang [2003]. They also show that for $z > 0$, the equation $G(x) = z$ has exactly four roots, namely $\beta_{1,z}$, $\beta_{2,z}$, $\beta_{3,z}$ and $\beta_{4,z}$, where

$$-\infty < \beta_{1,z} < -\eta_2 < \beta_{2,z} < 0 < \beta_{3,z} < \eta_1 < \beta_{4,z} < \infty \tag{12}$$

The following lemma will be useful for a later computation. Let $b = \log(B/S_0)$, which is negative as the barrier level B is smaller than the current asset value S_0.

Lemma 3.1.

$$A(z;b) := \mathrm{E}[e^{-z\tau_b} I_{\{X(\tau_b)<b\}}] = \frac{(\eta_2 + \beta_{1,z})(\beta_{2,z} + \eta_2)}{\eta_2(\beta_{2,z} - \beta_{1,z})}\left(e^{-b\beta_{1,z}} - e^{-b\beta_{2,z}}\right)$$

$$B(z;b) := \mathrm{E}[e^{-z\tau_b} I_{\{X(\tau_b)=b\}}] = -\frac{\eta_2 + \beta_{1,z}}{\beta_{2,z} - \beta_{1,z}} e^{-b\beta_{1,z}} + \frac{\eta_2 + \beta_{2,z}}{\beta_{2,z} - \beta_{1,z}} e^{-b\beta_{2,z}}$$

where η_2 is a parameter of the jump distribution in (9) and $\beta_{1,z}$ and $\beta_{2,z}$ are the roots defined in Equation (12).

Proof: The proof is presented in Appendix B

By a no arbitrage argument, it is easy to show that

$$\mathrm{C} = \mathrm{DOC} + \mathrm{DIC} \tag{13}$$

where C and DIC are pricing formulas of the European call option and down-and-in call option, respectively. Petrella and Kou [2005] has derived pricing formulas of the European call and up-and-in call (UIC) option, in terms of Laplace transforms. Thus, the European call price is readily available so that we only need to present the pricing formula for the DIC option.

According to Kou et al. [2005], the Laplace transform for a European call with respect to to k is given by

$$\mathcal{L}_{k,\zeta}\{\mathrm{C}\} = \int_{-\infty}^{\infty} e^{-\zeta k}\mathrm{C}\,dk = \frac{e^{-rT}S_0^{\zeta+1}}{\zeta(\zeta+1)}\exp(G(\zeta+1)T) \tag{14}$$

where $\zeta > 0$, $k = -\ln K$ and G is defined in Equation (11).

To derive an analytical solution of the DIC option, we follow the calculation of UIC option in Kou et al. [2005]. It is found that the double Laplace transform on the DIC option is given by

$$\begin{aligned}
&\mathcal{L}_{T,\alpha}\mathcal{L}_{k,\zeta}\{\mathrm{DIC}\} \\
&= \int_0^{\infty}\int_{-\infty}^{\infty} e^{-\zeta k - \alpha T}\mathrm{DIC}\,dk\,dT \\
&= \frac{\mathrm{E}[e^{-(r+\alpha)\tau_b}S(\tau_b)^{\zeta+1}]}{\zeta(\zeta+1)(\alpha + r - G(\zeta+1))} \\
&= \frac{\mathrm{E}[e^{-(r+\alpha)\tau_b}B^{\zeta+1}I_{\{X(\tau_b)<b\}}]\mathrm{E}[e^{(\zeta+1)(-y^-)}] + \mathrm{E}[e^{-(r+\alpha)\tau_b}B^{\zeta+1}I_{\{X(\tau_b)=b\}}]}{\zeta(\zeta+1)(\alpha + r - G(\zeta+1))} \\
&= \frac{B^{\zeta+1}\left[A(r+\alpha;b)\frac{\eta_2}{\eta_2+\zeta+1} + B(r+\alpha;b)\right]}{\zeta(\zeta+1)(\alpha + r - G(\zeta+1))}
\end{aligned} \tag{15}$$

where y^- is an exponential random variable with rate η_2 and the functions A and B are obtained in Lemma 3.1.

Using the in-out parity for barrier call options, Equation (13), we know that

$$\mathcal{L}_{T,\alpha}\mathcal{L}_{k,\zeta}\{\mathrm{DOC}\}=\mathcal{L}_{T,\alpha}\mathcal{L}_{k,\zeta}\{\mathrm{C}-\mathrm{DIC}\}$$

so that we are required to compute the double Laplace transform for the standard call. In fact, Equation (14) implies that

$$\mathcal{L}_{T,\alpha}\mathcal{L}_{k,\zeta}\{\mathrm{C}\}=\frac{S_0^{\zeta+1}}{\zeta(\zeta+1)(\alpha+r-G(\zeta+1))} \tag{16}$$

Combining Equations (15) and (16), it is clear that

$$\mathcal{L}_{T,\alpha}\mathcal{L}_{k,\zeta}\{\mathrm{DOC}\} =\frac{S_0^{\zeta+1}-B^{\zeta+1}\left[A(\alpha+r;\,b)\frac{\eta_2}{\eta_2+\zeta+1}+B(\alpha+r;\,b)\right]}{\zeta(\zeta+1)(\alpha+r-G(\zeta+1))} \tag{17}$$

The DIL Component

We recall that the present value of the DIL option has the following representation:

$$\mathrm{DIL}(T,S,K,h)=\mathrm{E}\left[e^{-r(\tau_b+h)}(m_{\tau_b}^h-e^{-k})^+1_{\{\tau_b\le T\}}\right]$$

Here, we assume that $S_0>B=S_0e^b$. Otherwise, the option has been knocked in and it becomes a linear combination of two floating strike lookback options as shown in Wong and Chan [forthcoming]. In the latter situation, the floating strike lookback options can be valued by the solutions in Kou and Wang [2005].

Theorem 3.1.

$$\mathcal{L}_{k,\zeta}\{e^{rh}\mathrm{DIL}(T,S,K,h)\}=\frac{1}{\zeta(\zeta+1)} \times\mathrm{E}\left[e^{-r\tau_b}I_{\{\tau_b\le T\}}S_{\tau_b}^{\zeta+1}\right]\mathrm{E}\left[\min_{0\le t\le h}e^{X_t(\zeta+1)}\right]$$

where the two expectations can be computed by using another two Laplace transforms:

$$\mathcal{L}_{T,\alpha}\mathrm{E}\left[e^{-r\tau_b}I_{\{\tau_b\le T\}}S_{\tau_b}^{\zeta+1}\right] =\frac{B^{\zeta+1}}{\alpha}\left[A(r+\alpha;\,b)\frac{\eta_2}{\eta_2+\zeta+1}+B(r+\alpha;\,b)\right],$$

$$\mathcal{L}_{h,\beta}\mathrm{E}\left[\min_{0\le t\le h}e^{X_t(\zeta+1)}\right] =\frac{1}{\beta}-\frac{(\zeta+1)\beta_{2,\beta}(\eta_2+\beta_{1,\beta})}{\beta\eta_2(\beta_{2,\beta}-\beta_{1,\beta})(\zeta+1-\beta_{1,\beta})} +\frac{(\zeta+1)\beta_{1,\beta}(\eta_2+\beta_{2,\beta})}{\beta\eta_2(\beta_{2,\beta}-\beta_{1,\beta})(\zeta+1-\beta_{2,\beta})}$$

Therefore, the triple Laplace transform for the DIL option respect to k, T, and h is the product of above two expressions and $\frac{1}{\zeta(\zeta+1)}$. Specifically,

$$\mathcal{L}_{h,\beta}\mathcal{L}_{T,\alpha}\mathcal{L}_{k,\zeta}\{e^{rh}\mathrm{DIL}(T,S,K,h)\} =\frac{B^{\zeta+1}}{\zeta(\zeta+1)\alpha\beta}\left[\frac{A(r+\alpha;b)\eta_2}{\eta_2+\zeta+1}+B(r+\alpha;b)\right] \times\left[1-\frac{\zeta+1}{\eta_2(\beta_{2,\beta}-\beta_{1,\beta})} \times\left(\frac{\beta_{2,\beta}(\eta_2+\beta_{1,\beta})}{\zeta+1-\beta_{1,\beta}}-\frac{\beta_{1,\beta}(\eta_2+\beta_{2,\beta})}{\zeta+1-\beta_{2,\beta}}\right)\right]$$

Proof: By the Fubini theorem, we have the following calculation:

$$\begin{aligned}\mathcal{L}_{k,\zeta}\{e^{rh}\mathrm{DIL}\}&=\int_{-\infty}^{\infty}e^{-\zeta k}\mathrm{E}\left[e^{-r\tau_b}I_{\{\tau_b\le T\}}\left(m_{\tau_b}^h-e^{-k}\right)^+\right]dk\\&=\mathrm{E}\left[e^{-r\tau_b}I_{\{\tau_b\le T\}}\int_{-\infty}^{\infty}\left(m_{\tau_b}^h-e^{-k}\right)^+e^{-\zeta k}\,dk\right]\\&=\mathrm{E}\left[e^{-r\tau_b}I_{\{\tau_b\le T\}}\left[\frac{m_{\tau_b}^h e^{-\zeta k}}{\zeta}-\frac{e^{-(\zeta+1)k}}{\zeta+1}\right]_{-m_{\tau_b}^h}^{\infty}\right]\\&=\mathrm{E}\left[e^{-r\tau_b}I_{\{\tau_b\le T\}}\frac{\left(m_{\tau_b}^h\right)^{\zeta+1}}{\zeta(\zeta+1)}\right]\\&=\frac{1}{\zeta(\zeta+1)}\mathrm{E}\left[e^{-r\tau_b}I_{\{\tau_b\le T\}}\mathrm{E}\left[\left(m_{\tau_b}^h\right)^{\zeta+1}\middle|\mathcal{F}_{\tau_b}\right]\right]\end{aligned} \tag{18}$$

where $\mathcal{F}_{\tau_b}$ is the information accumulated up to time τ_b. Substituting

$$m_{\tau_b}^h=S_{\tau_b}\min_{0\le t\le h}e^{(X_{\tau_b+t}-X_{\tau_b})}$$

into Equation (18), we have

$$
\begin{aligned}
\mathcal{L}_{k,\zeta}\{e^{rh}\mathrm{DIL}\} &= \frac{1}{\zeta(\zeta+1)}\mathrm{E}\left[e^{-r\tau_b} I_{\{\tau_b \le T\}} \mathrm{E}\left[S_{\tau_b}^{\zeta+1} \min_{0\le t\le h} e^{(X_{\tau_b+t}-X_{\tau_b})(\zeta+1)} \mid \mathcal{F}_{\tau_b}\right]\right] \\
&= \frac{1}{\zeta(\zeta+1)}\mathrm{E}\left[e^{-r\tau_b} I_{\{\tau_b \le T\}} S_{\tau_b}^{\zeta+1}\right] \mathrm{E}\left[\min_{0\le t\le h} e^{(X_{\tau_b+t}-X_{\tau_b})(\zeta+1)} \mid \mathcal{F}_{\tau_b}\right] \\
&= \frac{1}{\zeta(\zeta+1)}\mathrm{E}\left[e^{-r\tau_b} I_{\{\tau_b \le T\}} S_{\tau_b}^{\zeta+1}\right] \mathrm{E}\left[\min_{0\le t\le h} e^{X_t(\zeta+1)}\right]
\end{aligned}
$$

where we have used the locally independent property of the KJD model in the second line and the stationary property in the last line.

Next, we consider the Laplace transform of the first expectation.

$$
\begin{aligned}
&\mathcal{L}_{T,\alpha}\left\{\mathrm{E}\left[e^{-r\tau_b} I_{\{\tau_b \le T\}} S_{\tau_b}^{\zeta+1}\right]\right\} \\
&= \int_{-\infty}^{\infty} e^{-r\tau_b} \mathrm{E}\left[e^{-\alpha T} I_{\{\tau_b \le T\}} S_{\tau_b}^{\zeta+1}\right] dT \\
&= \mathrm{E}\left[\int_{-\infty}^{\infty} e^{-r\tau_b} e^{-\alpha T} I_{\{\tau_b \le T\}} S_{\tau_b}^{\zeta+1}\, dT\right] \\
&= \mathrm{E}\left[\int_{0}^{\infty} e^{-r\tau_b} e^{-\alpha(\tau_b+t)} S_{\tau_b}^{\zeta+1}\, dt\right] \\
&= \frac{1}{\alpha}\mathrm{E}\left[e^{-(r+\alpha)\tau_b} S_{\tau_b}^{\zeta+1}\right] \\
&= \frac{B^{\zeta+1}}{\alpha}\left[A(r+\alpha;\, b)\frac{\eta_2}{\eta_2+\zeta+1} + B(r+\alpha;\, b)\right]
\end{aligned}
$$

For the second expectation, we let $x_h = \min_{0\le t\le h} X_t \le 0$ and do the following calculation.

$$
\begin{aligned}
\mathrm{E}\left[\min_{0\le t\le h} e^{X_t(\zeta+1)}\right] &= \mathrm{E}\left[e^{x_h(\zeta+1)}\right] = \int_{-\infty}^{0} e^{y(\zeta+1)}\, d[P(x_h \le y)] \\
&= [e^{y(\zeta+1)} P(x_h \le y)]_{-\infty}^{0} - (\zeta+1)\int_{-\infty}^{0} e^{y(\zeta+1)} P(x_h \le y)\, dy \\
&= 1 - (\zeta+1)\int_{-\infty}^{0} e^{y(\zeta+1)} P(\tau_y \le h)\, dy
\end{aligned}
$$

We then consider the Laplace transform:

$$
\begin{aligned}
&\mathcal{L}_{h,\beta}\left\{\int_{-\infty}^{0} e^{y(\zeta+1)} P(\tau_y \le h)\, dy\right\} \\
&= \int_{0}^{\infty}\int_{-\infty}^{0} e^{-\beta h} e^{y(\zeta+1)} P(\tau_y \le h)\, dy\, dh \\
&= \int_{-\infty}^{0} e^{y(\zeta+1)}\left[\int_{0}^{\infty} e^{-\beta h} P(\tau_y \le h)\, dh\right] dy \\
&= \int_{-\infty}^{0} e^{y(\zeta+1)}\left[\int_{0}^{\infty} e^{-\beta h}\int_{0}^{h} P(\tau_y \in ds)\, dh\right] dy \\
&= \int_{-\infty}^{0} e^{y(\zeta+1)}\left[\int_{0}^{\infty} e^{-\beta(s+u)}\int_{0}^{\infty} P(\tau_y \in ds)\, du\right] dy, \quad h = s+u \\
&= \int_{-\infty}^{0} e^{y(\zeta+1)}\left[\int_{0}^{\infty} e^{-\beta s} P(\tau_y \in ds)\int_{0}^{\infty} e^{-\beta u}\, du\right] dy \\
&= \int_{-\infty}^{0} e^{y(\zeta+1)}\left[\mathrm{E}\left[e^{-\beta\tau_y} I_{\{\tau_y<\infty\}}\right]/\beta\right] dy
\end{aligned}
$$

As $\mathrm{E}[e^{-z\tau_b} I_{\{\tau_b<\infty\}}] = A(z;b) + B(z;b)$, we have

$$
\begin{aligned}
&\int_{-\infty}^{0} e^{y(\zeta+1)} \mathrm{E}\left[e^{-\beta\tau_y} I_{\{\tau_y<\infty\}}\right] dy \\
&= \int_{-\infty}^{0}\left[\frac{\beta_{2,\beta}(\eta_2+\beta_{1,\beta})}{\eta_2(\beta_{2,\beta}-\beta_{1,\beta})} e^{y(\zeta+1-\beta_{1,\beta})} - \frac{\beta_{1,\beta}(\eta_2+\beta_{2,\beta})}{\eta_2(\beta_{2,\beta}-\beta_{1,\beta})} e^{y(\zeta+1-\beta_{2,\beta})}\right] dy \\
&= \frac{\beta_{2,\beta}(\eta_2+\beta_{1,\beta})}{\eta_2(\beta_{2,\beta}-\beta_{1,\beta})(\zeta+1-\beta_{1,\beta})} - \frac{\beta_{1,\beta}(\eta_2+\beta_{2,\beta})}{\eta_2(\beta_{2,\beta}-\beta_{1,\beta})(\zeta+1-\beta_{2,\beta})}
\end{aligned}
$$

After recognizing $\mathcal{L}_{h,\beta}\{1\} = \frac{1}{\beta}$, we obtain an expression for $\mathcal{L}_{h,\beta}\mathrm{E}[e^{x_h(\zeta+1)}]$ as shown in the the Lemma.

From Theorem 3.1, the DIL option can be valued via a triple Laplace inversion. Appendix C presents a numerical triple Laplace inversion using the Euler summation, where we have reduced the computational burden using properties of the expectations in Theorem 3.1. More specifically, the computation is reduced to an order similar to that of the double Laplace inversion proposed by Petrella (2004).

NUMERICAL EXAMPLES AND SENSITIVITY ANALYSIS

Numerical examples are used to show that the analytical solution implemented with the triple Laplace inversion is efficient and accurate. We compare our numerical results with the Monte Carlo (MC) simulation. The simulation for jump diffusion models is classic, see Chapter 10.4 of Chan and Wong [2006]. It is found from the MC simulation that the approximated turbo price converges slowly with the time step, Δt. For instance, when we run the simulation with $r = 5\%$, $q = 0\%$, $\sigma = 0.3$, $\lambda = 2.0$, $\eta_1 = 25$, $\eta_2 = 20$, $p = 0.6$, $T = 1.0$, $S_0 = 100$, $B = 95$, and $h = 1/24$, the numerical result is converging to a stable value when the Δt is approaching to zero. However, the approximated value is not close enough to the true value with $\Delta t = 10^{-5}$ and 40,000 sample paths. Exhibit 2 plots the turbo warrant price against the time step. The points marked '$*$', at $\Delta t = 0$, are produced by the Richardson extrapolation using the four data points marked 'o'. When the extrapolated value is compared to the simulated price with $\Delta t = 10^{-5}$, the difference is around

0.03 for all strike prices. Thus, we will use the warrant price produced from the Richardson extrapolation as the benchmark to verify our analytical solution.

We check the performance of the analytical solution implemented with the triple Laplace inversion in Exhibit 3. The MC option price is obtained from the Richardson extrapolation based on the MC simulations with $\Delta t = 10^{-5}$, 5×10^{-5}, 25×10^{-5}, 0.001, and 0.005. It can be seen that using inverse Laplace transform (ILT) produces results consistent with the MC simulation. The difference between two approaches increases when the value of h increases. This is because the discretization error of the realized minimum asset price, after the barrier is crossed, accumulates with the value of h. The MC simulation is then biased upward once the h is getting large. We would like to stress that computing one option price only takes about 0.6 seconds using the triple Laplace transform. This computational time is remarkably efficient and useful for calibration.

As volatility smile can be partly explained by a jump diffusion model, we examine the sensitivity of turbo warrants to the volatility smile through examining the sensitivity to the jump parameters. Specifically in the KJD model, we are interested in the change in turbo price with respect to the marginal change in λ, η_1, η_2, and p. If implied volatility is insignificant to turbo pricing, then it should be the case that a turbo warrant is much less sensitive to the jump parameters than its vanilla counterpart.

EXHIBIT 2

MC Estimates and Extrapolation

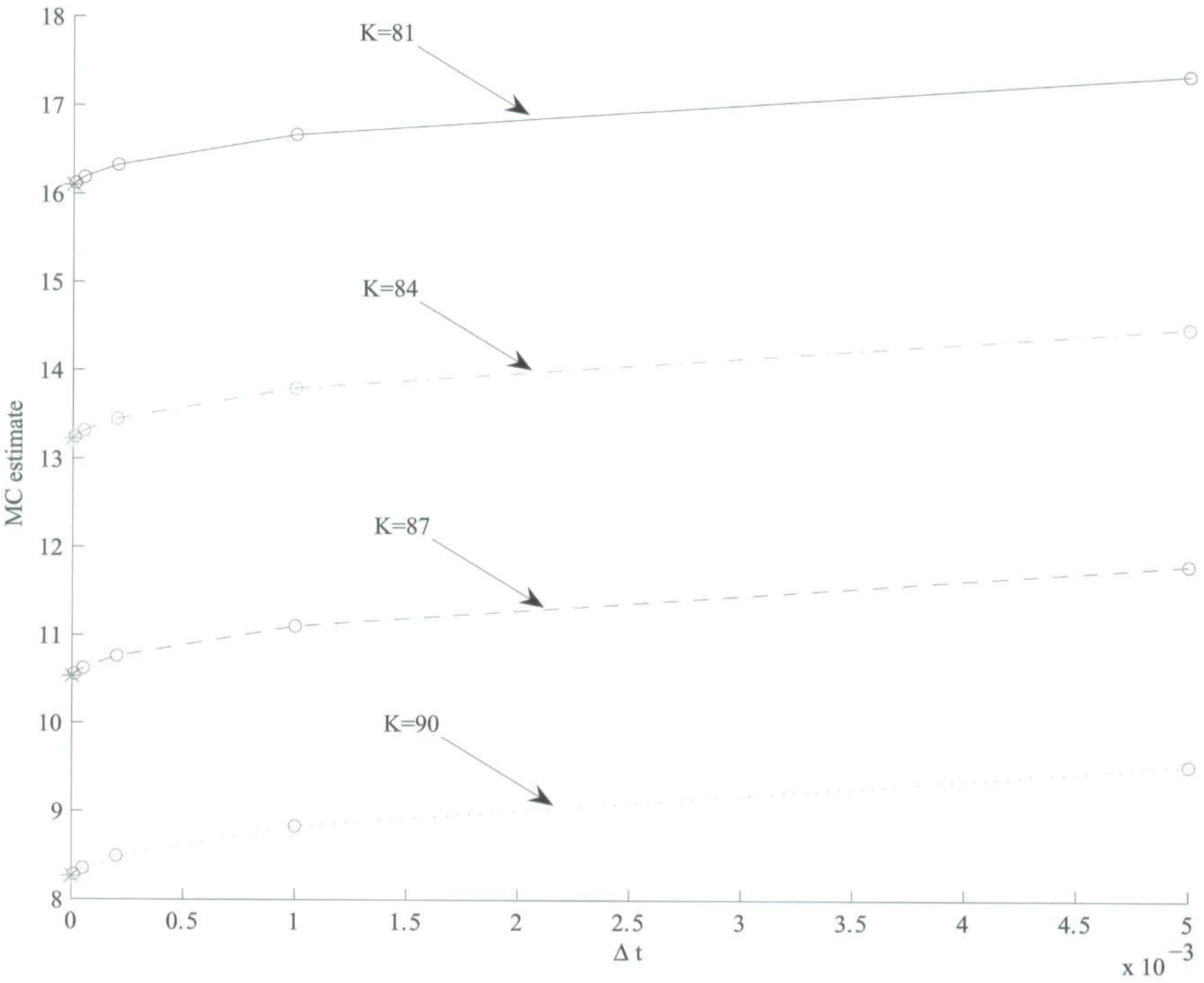


EXHIBIT 3
Simulation vs. Analytical Solution

Strike (K)	81		84		87		90	
Time (h)	ILT	MC	ILT	MC	ILT	MC	ILT	MC
1 week	17.179	17.202	14.235	14.260	11.347	11.371	8.708	8.731
2 weeks	16.073	16.100	13.199	13.227	10.508	10.535	8.241	8.265
1 month	14.685	14.714	12.041	12.070	9.713	9.739	7.865	7.887
2 months	13.209	13.242	10.952	10.983	9.050	9.076	7.580	7.603
3 months	12.399	12.439	10.397	10.432	8.733	8.761	7.450	7.474

Exhibit 4 plots the turbo and vanilla call prices against jump parameters. In all cases, we fix the strike price at 90 and vary the barrier level for the turbo warrant. It can be seen that the turbo warrant is much less sensitive to the jump arrival rate λ when compared with the vanilla call option. However, the sensitivities to the mean parameters η_1 and η_2 are similar to those of the vanilla option. The turbo warrant is more sensitive to the skewness parameter p. It is reasonable as the skewness of the asset distribution greatly affects the likelihood of falling into the knock-in region. In general, the vanilla call option is more sensitive to the jump parameters but is not very significant. Hence, the jump risk cannot be ignored in the valuation of turbo warrants.

Finally, we discuss the difference between the diffusion model and the jump diffusion model in pricing

EXHIBIT 4
Sensitivity Analysis

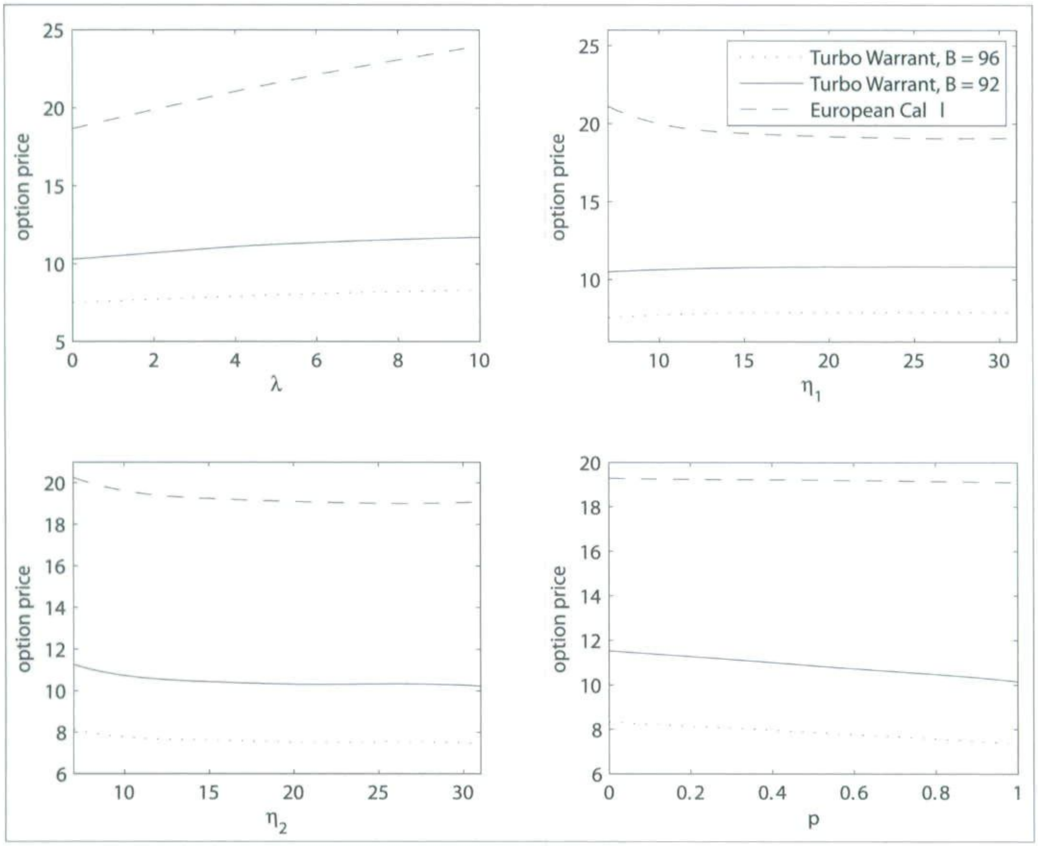
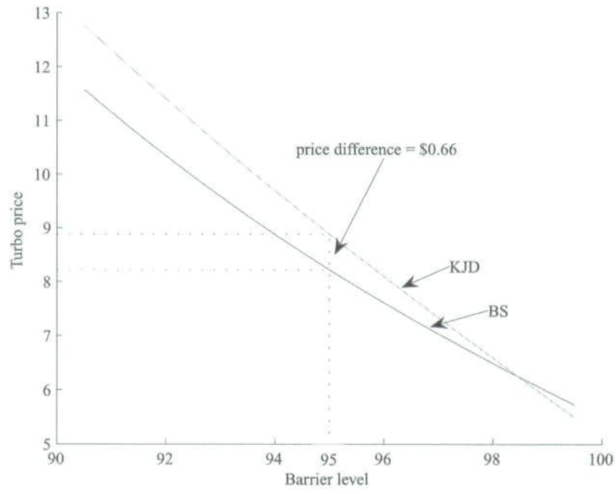


EXHIBIT 5
Comparing BS and KJD Turbo Prices



turbo warrants. Exhibit 5 plots the BS turbo price and KJD turbo price against the barrier level using the parameters: $r = 5\%$, $q = 0\%$, $\sigma = 0.3$, $\lambda = 5.0$, $\eta_1 = 10$, $\eta_2 = 10$, $p = 0.5$, $T = 1.0$, $S_0 = 100$, $K = 90$, and $h = 1/24$. Since a turbo call warrant requires the barrier level to be larger than the strike price, we plot the graph for the barrier level being larger than 90. It can be seen that the price difference can be quite large. For instance, when the barrier is set to 95, the price difference is 0.66 while the turbo price is in between 8 and 9. Thus, the pricing error is larger than 7.3%. When the barrier level gets closer to the strike price, the difference is even larger. Given that the turbo price can be computed within a second, it is worthwhile to include the KJD turbo price in practice.

CONCLUSION

In this article, we set up a framework to price turbo warrants under a jump diffusion model and, more precisely, derive the first analytical solution under the double exponential jump diffusion model of Kou [2002]. The pricing formula should be implemented with a triple Laplace inversion. To facilitate computation, we propose a triple Laplace transform that is specifically designed for turbo warrant in Appendix B. Our simulation verifies that the analytical solution implemented with the Laplace inversion is efficient and accurate. We then examine the sensitivity of the turbo warrant to jump parameters. It is found that the turbo warrant is less sensitive to the jump parameters than that of the vanilla option, but the sensitivity cannot be ignored. When we compare the BS and KJD turbo prices, it is found that the price difference can be significant. It implies that the jump risk in asset price materially affects the value of a turbo warrant.

APPENDIX A

Required BS Formulas

$$
\begin{aligned}
&\mathrm{DOC}_{BS}(t,S)\\
&= Se^{-q(T-t)}N\left[\frac{\log(\frac{S}{H})+\mu_1(T-t)}{\sigma\sqrt{T-t}}\right]\\
&\quad - H\left(\frac{H}{S}\right)^{\frac{2(r-q)}{\sigma^2}} e^{-q(T-t)}N\left[\frac{\log(\frac{H}{S})+\mu_1(T-t)}{\sigma\sqrt{T-t}}\right]\\
&\quad - Ke^{-r(T-t)}N\left[\frac{\log(\frac{S}{H})-\mu_2(T-t)}{\sigma\sqrt{T-t}}\right]\\
&\quad + K\left(\frac{H}{S}\right)^{\frac{2\mu_2}{\sigma^2}} e^{-r(T-t)}N\left[\frac{\log(\frac{H}{S})-\mu_2(T-t)}{\sigma\sqrt{T-t}}\right],
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{LC}_{fl}^{BS}(t,S)\\
&= Se^{-q(T-t)}N\left[\frac{\log(\frac{S}{m_0^t})+\mu_1(T-t)}{\sigma\sqrt{T-t}}\right]\\
&\quad - m_0^t e^{-r(T-t)}N\left[\frac{\log(\frac{S}{m_0^t})-\mu_2(T-t)}{\sigma\sqrt{T-t}}\right]\\
&\quad + \frac{\sigma^2}{2(r-q)}Se^{-r(T-t)}\left(\frac{S}{m_0^t}\right)^{-\frac{2(r-q)}{\sigma^2}} N\left[\frac{\log(\frac{m_0^t}{S})+\mu_2(T-t)}{\sigma\sqrt{T-t}}\right]
\end{aligned}
$$

$$-\frac{\sigma^2}{2(r-q)} S e^{-q(T-t)} N\left[\frac{\log(\frac{m_0^t}{S})-\mu_1(T-t)}{\sigma\sqrt{T-t}}\right]$$

$\mathrm{DR}_{BS}(t,S)$

$$=\left(\frac{S}{H}\right)^{\frac{\mu_3-\mu_1}{\sigma^2}} N\left[\frac{\log(\frac{H}{S})-\mu_3(T-t)}{\sigma\sqrt{T-t}}\right]$$

$$-\left(\frac{S}{H}\right)^{\frac{-\mu_3-\mu_1}{\sigma^2}} N\left[\frac{\log(\frac{H}{S})+\mu_3(T-t)}{\sigma\sqrt{T-t}}\right]$$

where $\mu_1=\left(r-q+\frac{\sigma^2}{2}\right), \mu_2=\left(r-q-\frac{\sigma^2}{2}\right), \mu_3=\sqrt{\mu_2^2+2r\sigma^2}$, and $N[\cdot]$ is the c.d.f. of a standard normal random variable.

Appendix B

Proof of Lemma 3.1

The proof is similar to the one in Kou and Wang [2003], but we consider $b<0$ in the present case.

Define

$$u(x)=\begin{cases}\gamma_1 e^{-\beta_1(b-x)}+\gamma_2 e^{-\beta_2(b-x)}, & x\ge b\\ 1, & x<b\end{cases}$$

with both β_1 and β_2 being negative, and constants γ_1 and γ_2 to be determined later. Denote infinitesimal generator of a jump-diffusion process X_t as

$$Au(x)=\mu u'(x)+\frac{1}{2}\sigma^2 u''(x)+\lambda\int_{-\infty}^{\infty}[u(x+y)-u(y)]f_Y(y)d(y)$$

where $f_Y(y)$ is the pdf of the jump size for a general jump distribution. Specifically, we substitute the double exponential distribution from Equation (9) in the expression. Simple algebra shows that

$$Au(x)=\sum_{k=1}^{2}\gamma_k G(\beta_k)e^{-\beta_k(b-x)} + \lambda(1-p)e^{\eta_2(b-x)}\left(1-\sum_{k=1}^{2}\gamma_k\frac{\eta_2}{\eta_2+\beta_k}\right)$$

Applying the Ito lemma to the process $\{e^{-zt}u(X_t)\}$, we find that the process

$$M(t)=e^{-z(t\wedge\tau_b)}u(X_{t\wedge\tau_b})-\int_0^{t\wedge\tau_b}e^{-zs}(Au(X_s)-zu(X_s))ds$$

is a local martingale with $M(0)=u(0)$. If $Au(X_s)-zu(X_s)=0$, then we have

$$u(0)=\mathrm{E}M(t)=\mathrm{E}[e^{-z(t\wedge\tau_b)}u(X_{t\wedge\tau_b})]\to \mathrm{E}[e^{-z(\tau_b)}I_{\{\tau_b<\infty\}}],\text{ as } t\to\infty$$

The limit is closely related to the function $A(z;b)$. Thus, it is our goal to determine the solution of the equation:

$$Au(X_s)-zu(X_s)=0$$

Solving the equation gives

$$G(\beta_k)-z=0,\ \beta_k<0$$
$$\gamma_1\frac{\eta_2}{\eta_2+\beta_1}+\gamma_2\frac{\eta_2}{\eta_2+\beta_2}=1,$$
$$\gamma_1+\gamma_2=1,\quad \text{by continuity of } u(x).$$

This allows us to solve for the values of γ_1 and γ_2 and to obtain

$$\mathrm{E}[e^{-z(\tau_b)}I_{\{\tau_b<\infty\}}]=\frac{\beta_{2,z}(\eta_2+\beta_{1,z})}{\eta_2(\beta_{2,z}-\beta_{1,z})}e^{-b\beta_{1,z}}-\frac{\beta_{1,z}(\eta_2+\beta_{2,z})}{\eta_2(\beta_{2,z}-\beta_{1,z})}e^{-b\beta_{2,z}} \tag{B1}$$

Using a similar idea, we define

$$v(x)=\begin{cases}\gamma_1^* e^{-\beta_1(b-x)}+\gamma_2^* e^{-\beta_2(b-x)}, & x\ge a\\ 0, & a-y\le x<a\\ 1, & x<a-y\end{cases}$$

where $y>0$. The infinitesimal generator applying to $v(x)$ gives

$$Av(x)=\sum_{k=1}^{2}\gamma_k^* G(\beta_k)e^{-\beta_k(b-x)} +\lambda(1-p)e^{\eta_2(b-x)}\left(e^{-\eta_2 y}-\sum_{k=1}^{2}\gamma_k^*\frac{\eta_2}{\eta_2+\beta_k}\right)$$

We construct a local martingale through the process $\{e^{-zt}v(X_t)\}$:

$$\widehat{M}(t)=e^{-z(t\wedge\tau_b)}v(X_{t\wedge\tau_b})-\int_0^{t\wedge\tau_b}e^{-zs}(Av(X_s)-zv(X_s))ds$$

where it can be easily shown that $\mathrm{E}(e^{-\alpha\tau_b}I_{\{a-X_{\tau_a}>y\}})=\lim_{t\to\infty}\times \mathrm{E}\widehat{M}(t)=\widehat{M}(0)=v(0)$ if $Av(X_s)-zv(X_s)=0$. The limit here is exactly the function $A(z;b)$ if we set $a=b$. Solving the equation, we arrive at

$$G(\beta_k)-z=0,\ \beta_k<0$$
$$\gamma_1^*\frac{\eta_2}{\eta_2+\beta_1}+\gamma_2^*\frac{\eta_2}{\eta_1+\beta_2}=e^{-\eta_2 y}$$
$$\gamma_1^*+\gamma_2^*=0,\quad \text{by continuity of } v(x) \text{ at } x=b.$$

Thus, the values of γ_1^* and γ_2^* are then obtained from the above equations.

By taking limit $y \to 0^{+}$ and setting $a = b$, we have

$$v(0) = \mathrm{E}(e^{-\alpha\tau_b} I_{\{b - X_{\tau_b} > 0\}}) = A(z; b)$$

The expression of $A(z; b)$ is given in the Lemma 3.1. For the function $B(z; b)$, it is clear that

$$\begin{aligned} B(z; b) &= \mathrm{E}[e^{-z\tau_b} I_{\{X(\tau_b) = b\}}] \\ &= \mathrm{E}[e^{-z(\tau_b)} I_{\{\tau_b < \infty\}}] - \mathrm{E}[e^{-z(\tau_b)} I_{\{X(\tau_b) < b\}}] \\ &= \mathrm{E}[e^{-z(\tau_b)} I_{\{\tau_b < \infty\}}] - A(z; b) \end{aligned}$$

Using the expression of $A(z; b)$ and Equation (B1), we obtain the solution of $B(z; b)$ as in Lemma 3.1.

APPENDIX C

Numerical Triple Laplace Inversion for Turbos

The price formula of down-and-out call option can be solved by using a double Laplace inversion proposed by Petrella [2004]. As we have a triple Laplace transform for the DIL option, the approach of Petrella [2004] should be modified.

First, we write the triple Laplace transform for DIL option as

$$\mathcal{L}_{h,\beta}\mathcal{L}_{T,\alpha}\mathcal{L}_{k,\zeta} e^{rh} \mathrm{DIL}(T, S, K, h) = F_\zeta(\zeta) F_\alpha(\alpha, \zeta) F_\beta(\beta, \zeta)$$

where

$$\begin{aligned} F_\zeta(\zeta) &= \frac{H^{\zeta+1}}{\zeta(\zeta+1)} \\ F_\alpha(\alpha, \zeta) &= \frac{1}{\alpha}\left[\frac{A(r+\alpha; b)\eta_2}{\eta_2 + \zeta + 1} + B(r+\alpha; b)\right] \\ F_\beta(\beta, \zeta) &= \frac{1}{\beta}\left[1 - \frac{\zeta+1}{\eta_2(\beta_{2,\beta} - \beta_{1,\beta})}\right. \\ &\quad \left.\times\left(\frac{\beta_{2,\beta}(\eta_2 + \beta_{1,\beta})}{\zeta + 1 - \beta_{1,\beta}} - \frac{\beta_{1,\beta}(\eta_2 + \beta_{2,\beta})}{\zeta + 1 - \beta_{2,\beta}}\right)\right] \end{aligned}$$

The analytical solution of expression of DIL is given by

$$\begin{aligned} \mathrm{DIL}(x, y, z) &= e^{-rh} \int_{c_1 - i\infty}^{c_1 + i\infty} \int_{c_2 - i\infty}^{c_2 + i\infty} \int_{c_3 - i\infty}^{c_3 + i\infty} \frac{e^{x\zeta + y\alpha + z\beta}}{(2\pi i)^3} \\ &\quad \times F_\zeta(\zeta) F_\alpha(\alpha, \zeta) F_\beta(\beta, \zeta)\, d\alpha\, d\beta\, d\zeta \end{aligned}$$

where c_1, c_2, c_3 are arbitrary positive real numbers, such that there is no singularity on the domain $\{\mathrm{Re}(\zeta) > c_1, \mathrm{Re}(\alpha) > c_2, \mathrm{Re}(\beta) > c_3\}$.

By discretization, the numerical inverse transform is

$$\frac{e^{-rh}}{8\pi^3} \sum_{j_1=-n_1}^{n_1} \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} e^{x\zeta_{j_1} + y\alpha_{j_2} + z\beta_{j_3}} F_\zeta(\zeta_{j_1}) F_\alpha(\alpha_{j_2}, \zeta_{j_1}) F_\beta(\beta_{j_3}, \zeta_{j_1}) h_1 h_2 h_3$$

for a large n_1, n_2 and n_3, where $\zeta_j = c_1 + jh_1 i$, $\alpha_j = c_2 + jh_2 i$ and $\beta_j = c_3 + jh_3 i$. By denoting $h_1 = \pi/x$, $h_2 = \pi/y$, $h_3 = \pi/z$, the expression becomes an alternating series,

$$A \sum_{j_1=-n_1}^{n_1} \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} (-1)^{j_1+j_2+j_3} F_\zeta(\zeta_{j_1}) F_\alpha(\alpha_{j_2}, \zeta_{j_1}) F_\beta(\beta_{j_3}, \zeta_{j_1})$$

where $A = \frac{e^{-rh} e^{xc_1 + yc_2 + zc_3}}{8xyz}$.

The order of computation is $O(n_1 n_2 n_3)$, which is less efficient than the case of double Laplace inversion of $O(n_1 n_2)$. To reduce the computational burden, we consider some factorization methods. Using the property that $F(\bar{z}) = \overline{F(z)}$, we simplify the expression as follows.

$$\begin{aligned} & A \sum_{j_1=-n_1}^{n_1} \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} (-1)^{j_1+j_2+j_3} F_\zeta(\zeta_{j_1}) F_\alpha(\alpha_{j_2}, \zeta_{j_1}) F_\beta(\beta_{j_3}, \zeta_{j_1}) \\ &= A \sum_{j_1=1}^{n_1} \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} (-1)^{j_1+j_2+j_3} F_\zeta(\zeta_{j_1}) F_\alpha(\alpha_{j_2}, \zeta_{j_1}) F_\beta(\beta_{j_3}, \zeta_{j_1}) \\ &\quad + A \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} (-1)^{j_2+j_3} F_\zeta(\zeta_0) F_\alpha(\alpha_{j_2}, \zeta_0) F_\beta(\beta_{j_3}, \zeta_0) \\ &\quad + A \sum_{j_1=-n_1}^{-1} \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} (-1)^{j_1+j_2+j_3} F_\zeta(\zeta_{j_1}) F_\alpha(\alpha_{j_2}, \zeta_{j_1}) F_\beta(\beta_{j_3}, \zeta_{j_1}) \\ &= A \sum_{j_1=1}^{n_1} \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} (-1)^{j_1+j_2+j_3} F_\zeta(\zeta_{j_1}) F_\alpha(\alpha_{j_2}, \zeta_{j_1}) F_\beta(\beta_{j_3}, \zeta_{j_1}) \\ &\quad + A F_\zeta(\zeta_0) \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} (-1)^{j_2+j_3} F_\alpha(\alpha_{j_2}, \zeta_0) F_\beta(\beta_{j_3}, \zeta_0) \\ &\quad + A \sum_{j_1=1}^{n_1} \sum_{j_2=n_2}^{-n_2} \sum_{j_3=n_3}^{-n_3} (-1)^{j_1+j_2+j_3} F_\zeta(\bar{\zeta}_{j_1}) F_\alpha(\bar{\alpha}_{j_2}, \bar{\zeta}_{j_1}) F_\beta(\bar{\beta}_{j_3}, \bar{\zeta}_{j_1}) \\ &= 2\mathrm{Re}\left\{ A \sum_{j_1=1}^{n_1} \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} (-1)^{j_1+j_2+j_3} F_\zeta(\zeta_{j_1}) F_\alpha(\alpha_{j_2}, \zeta_{j_1}) \right. \\ &\quad \left. \times F_\beta(\beta_{j_3}, \zeta_{j_1}) \right\} + A F_\zeta(\zeta_0) \sum_{j_2=-n_2}^{n_2} \sum_{j_3=-n_3}^{n_3} (-1)^{j_2+j_3} \\ &\quad \times F_\alpha(\alpha_{j_2}, \zeta_0) F_\beta(\beta_{j_3}, \zeta_0) \\ &= 2A \sum_{j_1=1}^{n_1} (-1)^{j_1} \mathrm{Re}\left\{ F_\zeta(\zeta_{j_1}) \left[\sum_{j_2=-n_2}^{n_2} (-1)^{j_2} F_\alpha(\alpha_{j_2}, \zeta_{j_1}) \right] \right. \\ &\quad \left. \times \left[\sum_{j_3=-n_3}^{n_3} (-1)^{j_3} F_\beta(\beta_{j_3}, \zeta_{j_1}) \right] \right\} + A F_\zeta(\zeta_0) \\ &\quad \times \sum_{j_2=-n_2}^{n_2} (-1)^{j_2} F_\alpha(\alpha_{j_2}, \zeta_0) \sum_{j_3=-n_3}^{n_3} (-1)^{j_3} F_\beta(\beta_{j_3}, \zeta_0) \end{aligned} \tag{C1}$$

Notice that $F_\alpha(\alpha_{j_2}, \zeta_{j_1})$ and $F_\beta(\alpha_{j_3}, \zeta_{j_1})$ are no longer conjugate functions of α and β, respectively, once ζ_{j_1} is not a real number. As the outer-most summation in the first line of Equation (C1) requires ζ_{j_1} to be a complex number, we cannot simplify the two inner summations in the first line of Equation (C1) using the conjugate property. For the last term, we can fortunately simplify it as follows,

$$\sum_{j_2=-n_2}^{n_2} (-1)^{j_2} F_\alpha(\alpha_{j_2}, \zeta_0) = F_\alpha(\alpha_0, \zeta_0) + 2\sum_{j_2=1}^{n_2} (-1)^{j_2} \mathrm{Re}[F_\alpha(\alpha_{j_2}, \zeta_0)]$$
$$\sum_{j_3=-n_3}^{n_3} (-1)^{j_3} F_\beta(\beta_{j_3}, \zeta_0) = F_\beta(\beta_0, \zeta_0) + 2\sum_{j_3=1}^{n_3} (-1)^{j_3} \mathrm{Re}[F_\beta(\beta_{j_3}, \zeta_0)]$$

To implement the Euler summation, we consider partial sums of the series, and aggregate the partial sums by binomial coefficient weights, see Petrella [2004]. The final result is

$$A \sum_{j_1=-n_1-m_1}^{n_1+m_1} \sum_{j_2=-n_2-m_2}^{n_2+m_2} \sum_{j_3=-n_3-m_3}^{n_3+m_3} W_{j_1} W_{j_2} W_{j_3} F_\zeta(\zeta_{j_1}) \times F_\alpha(\alpha_{j_2}, \zeta_{j_1}) F_\beta(\beta_{j_3}, \zeta_{j_1}) h_1 h_2 h_3$$

where

$$W_j = \begin{cases} (-1)^j, & 0 \le j \le n_i \\ (-1)^j \sum_{r=0}^{n_i+m_i-j} C_r^{m_i} \dfrac{1}{2^{m_i}}, & n_i + 1 \le j \le n_i + m_i \\ 0, & j > n_i + m_i \\ W_{-j}, & j < 0 \end{cases}$$

After incorporating the binomial weights, we have the following formula,

$$\begin{aligned} & 2A \sum_{j_1=1}^{n_1+m_1} W_{j_1} \mathrm{Re}\left\{ F_\zeta(\zeta_{j_1}) \left[\sum_{j_2=-n_2-m_2}^{n_2+m_2} W_{j_2} F_\alpha(\alpha_{j_2}, \zeta_{j_1}) \right] \right. \\ & \left. \times \left[\sum_{j_3=-n_3-m_3}^{n_3+m_3} W_{j_3} F_\beta(\beta_{j_3}, \zeta_{j_1}) \right] \right\} + A F_\zeta(\zeta_0) \\ & \times \left[F_\alpha(\alpha_0, \zeta_0) + 2 \sum_{j_2=1}^{n_2+m_2} W_{j_2} \mathrm{Re}[F_\alpha(\alpha_{j_2}, \zeta_0)] \right] \\ & \times \left[F_\beta(\beta_0, \zeta_0) + 2 \sum_{j_3=1}^{n_3+m_3} W_{j_3} \mathrm{Re}[F_\beta(\beta_{j_3}, \zeta_0)] \right] \end{aligned}$$

Yet, it is easy to count that the order of computation becomes $O(\hat{n}_1(\hat{n}_2 + \hat{n}_3))$, where $\hat{n}_j = n_j + m_j$ for $j = 1, 2, 3$. We find that the setting $n_1 = 500$, $m_1 = 250$, $n_2 = m_2 = 25$, and $n_3 = m_3 = 25$ produces an accurate estimate for the DIL option. In our numerical examples, we use this setting, and it takes less than one second to compute a turbo price.

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