General Equilibrium and Risk Neutral Framework for Option Pricing with a Mixture of Distributions

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This article develops a closed form risk-neutral valuation model for pricing European style options when the underlying has a mixture of transformed-normal distributions. Specifically, we introduce the mixture of g distributions which contains the mixture of normal and lognormal distributions as a special case. The risk neutral valuation relation is developed following Rubinstein [1976]; Brennan [1979] and Camara [2003] and is consistent with the framework of Heston [1993b] and Schroder [2004]. Our model encompasses several well known models and is particularly useful for pricing derivatives written on illiquid assets and derivatives that are themselves illiquid.

This article relaxes the lognormal distribution assumption in the Black-Scholes [1973] formula for pricing Europeans call and put options. It provides a risk neutral closed form solution for pricing options written on an underlying that has skewed and thick tailed distribution and where the resulting option pricing formula does not depend on investors' preferences. Our model, based on a mixture of g distributions, places very general and loose restrictions on preferences and our solution is invariant to the choice of CARA (constant absolute risk aversion) or CPRA (constant proportional risk aversion) utility function for the investor. The risk neutral valuation relation is developed following Rubinstein [1976], Brennan [1979], and particularly Camara [2003]. It is consistent with the sufficient conditions of Heston [1993b] and Schroder [2004]. Empirical examples based on S&P 500 future options and futures on Heating Degree Days show the flexibility of the model. However, its most important feature is the clear link between the real and the risk neutral distributions, constructed on the basis of the general equilibrium theory.

The well documented volatility smile effect is clear evidence that the lognormal distribution assumption in the Black-Scholes model does not correspond well to the empirical patterns of skewness and kurtosis of the real and the risk neutral distributions of the underlying (see for example, Jackwerth and Rubinstein [1996] and Ait-Sahalia and Lo [1998]). There has been a lot of effort made to tackle the volatility smile. The most notable achievements include the stochastic volatility and stochastic interest rate model by Heston [1993a], the jump diffusion model by Bates [1996] and more recently, the use of time changed Levy processes by Huang and Wu [2004] and Carr and Wu [2004]. These models make use of a continuous time hedging argument with some assumptions about the unhedgeable volatility and jump risks.

Under the general equilibrium framework, Camara [2003] provides an option pricing formula that encompasses all distributions that are transformed normal. Heston [1993b] and Schroder [2004] also provide very general discrete time extensions which can cope with a broader set of distributions.

In contrast to the continuous time no arbitrage models, a common feature of these discrete time general equilibrium models is the presence of a risk neutral valuation relationship (RNVR) between the option price and the forward price of the underlying. There is no preference parameter in the pricing formula.

In comparison with the continuous time framework, the discrete time preference-based approach is less common, possibly due to the notion that a preference-based approach requires strong assumptions about the representative investor's risk aversion. This is regardless of the fact that specific assumptions are required in the continuous time framework regarding the pricing dynamic of the underlying and the continuous trading mechanism. We argue that the assumption we make here about the representative investor's risk preference is very general and relatively less demanding compared with the strong assumptions required in the continuous time framework and especially those on the volatility and jump risk premia. The key advantage of the discrete time preference-based approach is the absence of commitment to a specific pricing dynamic, which is particularly interesting in cases where assets are illiquid.

The model we have developed here is based on a mixture of two g distributions. The g distribution, belonging to the g and h family of distributions introduced by Tukey [1977], is closely related to the lognormal distribution and possesses similar tractability. It also has the added advantage that it accommodates both positive and negative skewness and, with a mixture of g distributions, a wide range of kurtosis values also. The g and h family of distributions have been used in Badrinath and Chatterjee [1988] and Mills [1995] for modelling stock prices and in Dutta and Babbel [2002] for calibrating option prices to the risk neutral densities. Simulations produced in this article show that the g distribution is capable of generating a Black-Scholes volatility skew and volatility smirk.

The mixture of g distributions has the mixture of normal and the mixture of lognormal distributions as special cases. Given that the mixture of normal and the mixture of lognormal distributions have been the subjects of many empirical studies, a theoretically proven *risk neutral* option pricing relationship for a mixture of g distributions is particularly relevant. The double-lognormal distribution, for example, is one of the most popular models for extracting risk neutral densities from option prices by central banks (e.g., Bahra [1998]; Leahy and Thomas [1996]; Melick and Thomas [1998]; Söderlind [1997]; Söderlind and Svensson [1997], and Butler and Davies [1998]). Separately, a double-lognormal distribution was used to examine crude oil price behavior in Melick and Thomas [1996], to predict the ERM crisis in Mizrach [1996], and to gauge market reaction to the central bank's macroeconomic policy (Levin, McManus and Watt [1998]). Here we establish for the first time a formal link between the real and the risk neutral distributions when they are both mixtures of g distributions. Finally, as the mixture of g distributions covers a wide region on the skewness and kurtosis plane, our double-g option pricing model can cater for many more underlyings with distributions lying outside the double lognormal range.

The remainder of this article is organized as follows: We present the g and the double-g option pricing models, giving details of their characteristics and their relationships with models proposed by previous studies. Then, we present examples that demonstrate the flexibility of our model. Finally, we provide some discussion and conclude.

OPTION PRICING BASED ON g DISTRIBUTIONS

The starting point of the risk neutral option pricing in a general equilibrium framework is a risk-averse representative investor and the pricing kernel[1]

$$\phi(W_T) = \frac{U'(W_T)}{E^P\left[U'(W_T)\right]} \tag{1}$$

where W_T is the aggregate wealth at time T and $U'(\cdot)$ is the representative investor's marginal utility function. The superscript P of $E(\cdot)$ means that the expectation is taken with respect to the real probability.[2]

The forward price of the underlying, F, is given by

$$F = E^P\left[\phi(W_T)S_T\right] \tag{2}$$

where S_T is the value of the underlying at time T. The underlying could be an asset such as a stock or an indicator that attracts monetary value such as index futures or HDD/CDD (heating/cooling degree days) used in weather contracts. Note that here we have chosen to use the forward form of the pricing relationship. In the case of non-dividend paying stock, $F = Se^{rT}$, Equation (2) can be stated in spot price $S = e^{-rT} E^P[\phi(W_T) S_T]$. In other situations however, such as an electricity contract where it is not always possible (or practical) to store the underlying, the forward pricing relationship still holds. In this case the discounted forward price may not bear any relationship to the spot price used for spot delivery.

By taking conditional expectation, Equation (2) can be written as

$$F = E^P\left[\psi(S_T)S_T\right] \tag{3}$$

where

$$\psi(S_T) = E^P\left[\phi(W_T) \middle| S_T\right] \tag{4}$$

The expectation in Equation (2) is calculated using the joint probability distribution of wealth W_T and S_T, whereas the expectation in Equation (3) is calculated using the probability distribution of S_T and the conditional probability of W_T given S_T. The conditional pricing kernel $\psi(S_T)$ is termed the *asset specific pricing kernel* in Camara [2003] or the *conditional expected relative marginal utility function* in Brennan [1979]. It is the projection of the pricing kernel in Equation (1) onto the space of S_T.

Given the functional form of $U'(\cdot)$ and the distributions of S_T and W_T, $\psi(S_T)$ can be derived and used for pricing the uncertain cash flow S_T and any contracts or derivative securities written on S_T. Note also that if there is sufficient price information on S_T to infer the quantity $\psi(S_T)$, information on $U'(\cdot)$ and W_T is no longer needed for pricing derivative contracts written on S_T. This is the key idea behind spanning and, in the next section, the Risk Neutral Valuation Relationship. Note that this framework is perfectly consistent with a dynamically incomplete market. Additionally, it is interesting to emphasize that this framework holds without a specific assumption on the dynamics of F.

Distribution of the Underlying

Here we assume the underlying S_T has a g distribution.

Definition 1 *(The g distribution) The random variable $Y \sim G(\mu, \sigma, g)$ has a g distribution if*

$$Y = \delta + \lambda \frac{\exp(gz) - 1}{g} \tag{5}$$

where $z \sim N(\mu, \sigma)$ is a Gaussian random variable, δ, μ and g are real constants ($\delta, \mu, g \in \Re$), and λ and σ are positive constants ($\lambda, \sigma \in \Re^+$).

The parameter g is the key determinant of the skewness of the distribution. For $g > 0$ and $g < 0$ the distribution is right and left skewed respectively. For $g = 0$ there is no skewness and the distribution is a normal distribution.[3] It is possible to rewrite Equation (5) as $h_Y(Y) = z$, where

$$h_Y(Y) = \frac{1}{g}\ln\left[g\frac{(Y-\delta)}{\lambda} + 1\right] \tag{6}$$

is a monotonic differentiable function with $[g(Y-\delta)/\lambda + 1] > 0$. Since $z \sim N(\mu, \sigma)$, $h_Y(Y) \sim N(\mu, \sigma)$ for $\sigma > 0$. The log of a linear transform of Y is linear.

Given the definition above, the density function of Y is

$$f(Y) = \frac{1}{\sigma\sqrt{2\pi}\left(gY - g\delta + \lambda\right)} e^{-\frac{1}{2\sigma^2}\left(\frac{1}{g}\ln\left(g\frac{Y-\delta}{\lambda}+1\right)-\mu\right)^2} \tag{7}$$

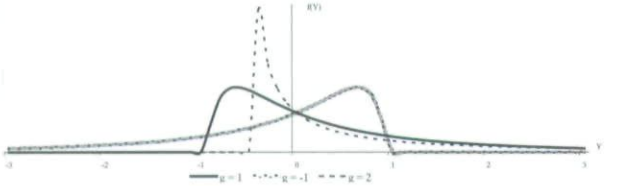
The shape of g distribution for different values of g is presented in Exhibit 1. Since the parameters δ, μ, and λ have no impact on skewness and kurtosis, these distributions are simulated by fixing $\delta = \mu = 0$ and $\lambda = \sigma = 1$.

Apart from the constraint $[g(Y-\delta)/\lambda + 1] > 0$ there is no other restriction on the range of values within which Y may lie. Some underlyings such as temperature may take both positive and negative values. In some cases we may wish to impose additional constraints on the parameter values to make sure that Y takes positive value only or lies within a specific range. For example, Y may represent stock price and hence has a lower bound at zero. In practice, the fitted parameter values of the distribution will be driven by the empirically observed values of Y or the empirically observed values of option prices, which are a function of Y.

The Pricing Kernel

To specify the pricing kernel we need some statements about wealth distribution and the preference

EXHIBIT 1
Shape of g Distribution with $\delta = \mu = 0$ and $\lambda = \sigma = 1$



function of the representative investor. Following Camara [2003], we assume the marginal utility function is given by

$$U'(W_T) = \exp\left[\gamma h_W(W_T)\right] \tag{8}$$

where γ is a constant preference parameter. Camara [2003] shows that the exact functional form of h_w is not crucial as long as $h_w(W_T)$ is normally distributed.[4] For instance, lognormally distributed wealth leads to a power marginal utility function $U'(W_T) = W_T^{\gamma}$ and Constant Proportional Risk Aversion (CPRA). This is the same assumption used in Rubinstein [1976] in his derivation of the general equilibrium version of the Black-Scholes option pricing formulas. If wealth is normally distributed, then the utility function is exponential $U'(W_T) = \exp[\gamma W_T]$, leading to Constant Absolute Risk Aversion (CARA). This is the assumption used by Brennan [1979].

With the distributions of S_T and W_T in place, we are now ready to derive the pricing kernel and the asset-specific pricing kernel following the approach of Brennan [1979]. Since we assume that the aggregate wealth has a transformed normal distribution, the derivations are the same as those in Camara [2003], so we simply state his results here.

Lemma 2 *(The pricing kernel) Assume a representative agent with a marginal utility function according to Equation (8). Assume also the end of period wealth has a transformed normal distribution $h_W(W_T) \sim N(\mu_W, \sigma_W)$. Then the pricing kernel is given by*

$$\phi(W_T) = \exp\left[\gamma h_W(W_T) - \gamma\,\mu_W - \frac{1}{2}\gamma^2\sigma_W^2\right] \tag{9}$$

Proof. See Camara [2003] Proposition 1.

Lemma 3 (The asset specific pricing kernel): Assume the pricing kernel has the form defined by Lemma 2. Assume also the underlying has a transformed normal distribution. Then the asset specific pricing kernel is given by

$$\psi(S_T) = \exp\left[-\gamma\,\rho\frac{\sigma_W}{\sigma}\mu + \gamma\,\rho\frac{\sigma_W}{\sigma}h_S(S_T) - \frac{1}{2}\gamma^2\rho^2\sigma_W^2\right] \tag{10}$$

where ρ is the correlation parameter between W_T and S_T.

Proof. See Camara [2003] Proposition 2.

The Risk Neutral Valuation Relationship

Given the density of S_T in Equation (7) and asset-specific pricing kernel $\psi(S_T)$ in Equation (10), we can now use Equation (3) to price the risky cash flow S_T and any derivatives based on S_T. Let $\hat{f}(S_T) = \psi(S_T) f(S_T)$ and substitute into Equation (3) to get

$$F = \int S_T \hat{f}(S_T) dS_T \tag{11}$$

The new density function $\hat{f}(S_T)$ is called the *risk-adjusted density*, and with the g-density function in Equation (7) and the asset-specific pricing kernel in Equation (10), $\hat{f}(S_T)$ has a functional form

$$\hat{f}(S_T) = \frac{1}{\sigma\sqrt{2\pi}\left(gS_T - g\delta + \lambda\right)} e^{-\frac{1}{2\sigma^2}\left[\frac{1}{g}\ln\left(g\frac{S_T-\delta}{\lambda}+1\right)-(\mu+\gamma\,\rho\sigma_W\sigma)\right]^2} \tag{12}$$

The pricing Equation (11) is not very useful in practice, as we can see from Equation (12) that it contains the growth parameter μ, the preference parameter γ, and the state variables ρ and σ_W. Since forward price always exists for a non-negative pricing kernel, it is information that one may use to infer the values of these variables. Note also that for pricing derivatives, only the composite amount $(\mu + \gamma\rho\sigma_W\sigma)$ is needed and not the individual value of each of these variables.

Derivations presented in the Appendix yield the following forward equilibrium relationship

$$\mu + \gamma\rho\sigma_W\sigma = \frac{1}{g}\left[\ln\left(g\frac{F-\delta}{\lambda}+1\right) - \frac{1}{2}g^2\sigma^2\right] = \mu^* \tag{13}$$

This relationship is an extension of the CAPM for g distributed assets. Substituting Equation (13) into Equation (12) leads to a preference-free version of $\hat{f}(S_T)$ denoted as $f^*(S_T)$, where

$$f^*(S_T) = \frac{1}{\sigma\sqrt{2\pi}\left(gS_T - g\delta + \lambda\right)} e^{-\frac{1}{2\sigma^2}\left\{\frac{1}{g}\ln\left(g\frac{S_T-\delta}{\lambda}+1\right)-\mu^*\right\}^2} \tag{14}$$

which is still a g density. From Equation (13), $\mu = \mu^*$ if the risk aversion parameter $\gamma = 0$. That is, $f^*(S_T)$ does not involve any preference parameters; the expected return under $f^*(S_T)$ is the riskless rate of return. Hence, $f^*(S_T)$ is called the *risk neutral probability*; it is identical to the real physical probability if the investor is risk neutral. This leads immediately to the risk neutral valuation relationship based on $f^*(S_T)$ for all derivatives written on S_T

$$F = e^{-rT}E^Q[S_T] = e^{-rT}\int S_T f^*(S_T)dS_T \tag{15}$$

These results prove the following Lemma:

Corollary 4 *(The RNVR for a g distribution) If the asset specific pricing kernel has the form specified by Equation (10) and if the end of period asset price has a g distribution according to Equation (5), then a risk neutral valuation relationship exists.*

An application of the RNVR result in Equation (15) is introduced in the next section, where the g-option pricing formula is presented.

The *g*-Option Pricing Formula

The risk neutral valuation relationship in Equation (15) is applied in this section to derive a closed form solution for pricing European options. In order to make the formula comparable with Black-Scholes and other continuous time models, the scale parameter σ is presented as an annualized figure. For pricing options we use $\sigma\sqrt{T}$, where T is the option's time to maturity.

Proposition 5 *(Option pricing formula with g distribution) Assume that Corollary 4 holds and that the underlying has a g distribution,* $S_T \sim G\,(\mu, \sigma\sqrt{T}, g)$. *Then, the price of a European call option C with payoff Max* $(S_T - K, 0)$ *and of a European put option P with payoff Max* $(K - S_T, 0)$ *are given respectively by*

$$C = (F + \lambda / g - \delta)e^{-rT}N(d_1) - (K + \lambda / g - \delta)e^{-rT}N(d_2) \tag{16}$$

$$P = (K + \lambda / g - \delta)e^{-rT}N(-d_2) - (F + \lambda / g - \delta)e^{-rT}N(-d_1) \tag{17}$$

where

$$d_1 = \ln\left[\frac{g(F-\delta)+\lambda}{g(K-\delta)+\lambda}\right]\frac{1}{g\sigma\sqrt{T}} + \frac{1}{2}g\sigma\sqrt{T},$$

$$d_2 = d_1 - g\sigma\sqrt{T}$$

F is the forward price, K is the strike price, T is the time to maturity, r is the risk free interest rate, $\mathrm{N}(\cdot)$ *is the cumulative distribution function of a standard normal random variable, and* δ, λ, g, *and* σ *are distribution parameters with* $\lambda > 0$, $g(F-\delta)/\lambda + 1 > 0$ *and* $g\,(K-\delta)/\lambda + 1 > 0$. *If* $g = 0$ *then the g distribution becomes a normal distribution, which is the case covered by Brennan [1979].*

Proof. See Appendix.

It is important to stress that Corollary 4 and Proposition 5 guarantee the existence of a risk neutral option pricing formula for a g distributed underlying. These enable us to distinguish the option pricing model provided in Proposition 5 from others that simply assume the existence of risk neutrality.

These option pricing formulae provide some interesting limiting cases. For instance, when $g = 1$ and $\lambda = \delta$ the resulting equation is the Black-Scholes formula. When $g = \lambda = 1$, $\theta = \delta - 1$ and $F = Se^{rT}$ then Equation (16) converges to the three-parameter lognormal option pricing formula of Camara [1999]. Since the three-parameter lognormal option pricing model nests Rubinstein's [1983] displaced diffusion model, the g-option pricing model developed here can also produce Rubinstein's displaced diffusion result. For $g = -1$, $\delta = 0$ and $\lambda = 1$, Equation (17) converges to the negatively skewed option pricing equation of Stapleton and Subrahmanyam [1993].

To show the flexibility of the g option pricing formula, the Black-Scholes volatility smile is obtained based on simulated g call option prices calculated according to Equation (16). The option prices are calculated for three different values of g, keeping the risk neutral mean and variance constant. Since δ and λ do not affect skewness and kurtosis, these two parameters are kept constant. Thus we set $\delta = \lambda = 100$, $Fe^{-rT} = 100$, $r = 5\%$, and $T = 1$. Then we apply Equation (16) to a range of strike prices. From the g call prices calculated, the Black-Scholes implied volatility is inverted and the result is presented in Exhibit 2.

In Exhibit 2(a), with $g = -1$, the implied volatility is a decreasing function of the strike prices. Compared with the Black-Scholes prices, the g option prices are higher for options with low strike prices and lower for options with high strike prices. The particular case in which $g = 1$, presented in Exhibit 2(b), results in a horizontal implied volatility line, which is consistent with the Black-Scholes assumption of lognormality. For $g = 2$, presented in Exhibit 2(c), the implied volatility is an increasing function of the strike price. The simulations show that our g option pricing model is capable of generating a volatility skew and a volatility smirk. However, without the mixture extension below, the level of convexity/concavity of the volatility skew/smirk generated by the single g distribution does not appear to be convexed/concaved enough compared to those observed empirically.

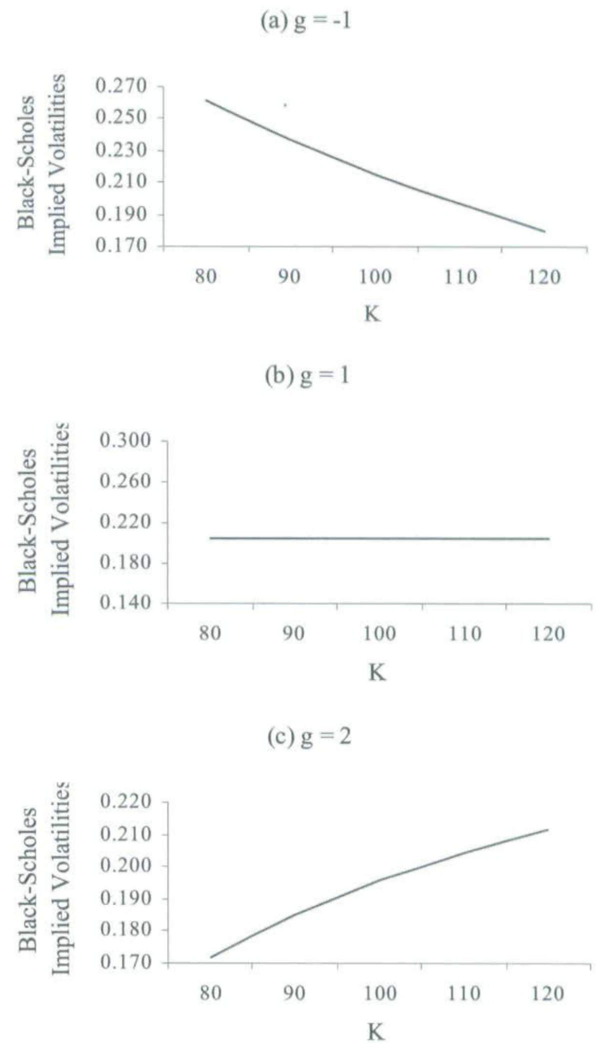
The Double-*g* Extension

In this section we introduce an extension based on a mixture of two g distributions. The double-g distribution offers a greater flexibility than the single-g distribution, and it encompasses the latter as a special case. It is also possible to extend the model further to include more than two g distributions, but that will introduce many more unknown parameters that are much harder to estimate in practice.

The flexibility of the double-g distribution can be observed from Exhibit 3, which shows the $(\beta_1 \times \beta_2)$ plane covered by the double-g distribution, where β_1 is the squared skewness and β_2 is the kurtosis. The dotted line is the boundary below which skewness-kurtosis combinations are theoretically impossible. The solid line represents the lognormal case and the g-distribution with $g > 0$. For $g < 0$ the skewness-kurtosis combinations of the the g-distribution are drawn on the quadrant, with negative skewness not shown in this diagram. The double-g distribution includes lognormal and single-g as special cases and covers all areas above the solid line. All three distributions have in common a kurtosis value that is equal to or greater than 3.

EXHIBIT 2

The Black-Scholes Volatility Smile Obtained from Simulated *g* Option Prices–add a note, as indicated on the fax



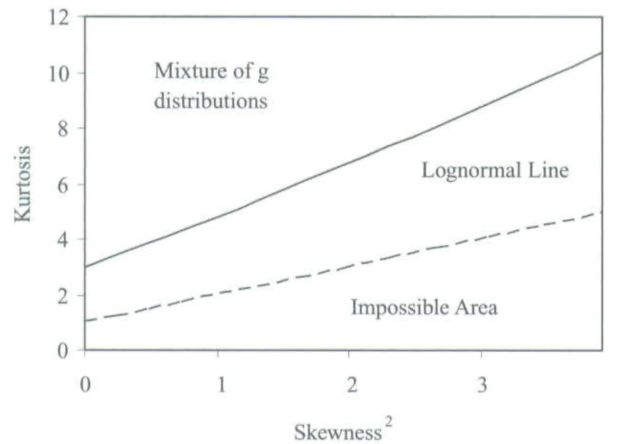
Note: Obtained from simulated g option prices using Equation (16), holding the firsts two moments constant and letting $T = 1$, $F\exp(-rT) = \delta = \lambda = 100$ and $r = 5\%$.

Assume that the underlying's distribution is made up of two g distributions:

$$f(S_T) = \sum_i w_i f_i(S_T) \tag{18}$$

where $f_i(S_T)$ for $i = 1, 2$ is defined according to Equation (7) and w_i is the weight of the i^{th} component with $\Sigma_i w_i = 1$ and $w_i \geq 0$. Then the equilibrium relationship in Equation (11) can be written as

EXHIBIT 3
The Skewness and Kurtosis Plane



$$F = \int S_T \sum_i w_i \psi_i(S_T) f_i(S_T) dS_T$$

and with the definition of risk adjusted density

$$F = \int S_T \sum_i w_i \hat{f}_i(S_T) dS_T \tag{19}$$

for $i = 1, 2$,

$$\hat{f}_i(S_T) = \frac{e^{-\frac{1}{2\sigma_i^2}\left[\frac{1}{g_i}\ln\left(g_i\frac{S_T-\delta_i}{\lambda_i}+1\right)-(\mu_i+\gamma\rho_i\sigma_W\sigma_i)\right]^2}}{\sigma_i\sqrt{2\pi}\left(g_i S_T - g_i\delta_i + \lambda_i\right)} \tag{20}$$

The details in the Appendix show that solving Equation (19) yields

$$F = w_1\left[\delta_1 + \frac{\lambda_1}{g_1}\left(e^{g_1(\mu_1+\gamma\rho_1\sigma_W\sigma_1)+\frac{1}{2}g_1^2\sigma_1^2} - 1\right)\right] + w_2\left[\delta_2 + \frac{\lambda_2}{g_2}\left(e^{g_2(\mu_2+\gamma\rho_2\sigma_W\sigma_2)+\frac{1}{2}g_2^2\sigma_2^2} - 1\right)\right] \tag{21}$$

In the previous section, the forward price of the underlying alone was sufficient to derive the quantity $\mu^* = (\mu + \gamma\rho\sigma_w\sigma)$, but now there are two sets of parameters in Equation (21), i.e., $(\mu_1 + \gamma\rho_1\sigma_W\sigma_1)$ and $(\mu_2 + \gamma\rho_2\sigma_W\sigma_2)$. To solve the above equation for both sets of preference parameters, we will need more than one price observation. If the forward price F is the only piece of information available, then to keep the option pricing model free of preference parameters we make the additional assumption that[5]

$$g_1(\mu_1 + \gamma\rho_1\sigma_W\sigma_1) = g_2(\mu_2 + \gamma\rho_2\sigma_W\sigma_2) = \kappa \tag{22}$$

in which case Equation (21) becomes

$$F = w_1\left[\delta_1 + \frac{\lambda_1}{g_1}\left(e^{\kappa+\frac{1}{2}g_1^2\sigma_1^2} - 1\right)\right] + w_2\left[\delta_2 + \frac{\lambda_2}{g_2}\left(e^{\kappa+\frac{1}{2}g_2^2\sigma_2^2} - 1\right)\right]$$

Solving for κ yields

$$\kappa = \ln\left[\frac{F - w_1\left(\delta_1 - \frac{\lambda_1}{g_1}\right) - w_2\left(\delta_2 - \frac{\lambda_2}{g_2}\right)}{w_1\frac{\lambda_1}{g_1}e^{\frac{1}{2}g_1^2\sigma_1^2} + w_2\frac{\lambda_2}{g_2}e^{\frac{1}{2}g_2^2\sigma_2^2}}\right] \tag{23}$$

Lemma 6 *(The RNVR for a mixture of g distributions) Assume that Equation (20) holds. If the end of period asset price density is given by a mixture of two g distributions according to Equation (18), then it is possible to obtain a risk neutral valuation relationship under the assumption specified in Equation (22).*

The formulas for pricing European calls and puts are presented in the next proposition. As before, to make the formulas comparable with Black-Scholes and other continuous time models the scale parameter σ is presented as an annualized figure and for pricing options we use $\sigma\sqrt{T}$, where T is the option's time to maturity.

Proposition 7 (Option pricing formula with a mixture of two g distributions) Assume that Lemma 6 holds and that the end of period stock price is given by a mixture of two g distributions with parameters μ_1, $\sigma_1\sqrt{T}$, g_1 and μ_2, $\sigma_2\sqrt{T}$, g_2. Then the price of a European call option C with payoff $Max[S_T - K, 0]$ and of a European put option P with payoff $Max[K - S_T, 0]$ are given respectively by

$$\begin{aligned} C = {} & w_1\frac{\lambda_1}{g_1}e^{\frac{1}{2}g_1^2\sigma_1^2T-rT}AN\left(d_1 + g_1\sigma_1\sqrt{T}\right) \\ & -w_1\left(K + \frac{\lambda_1}{g_1} - \delta_1\right)e^{-rT}N(d_1) \\ & +w_2\frac{\lambda_2}{g_2}e^{\frac{1}{2}g_2^2\sigma_2^2T-rT}AN\left(d_2 + g_2\sigma_2\sqrt{T}\right) \\ & -w_2\left(K + \frac{\lambda_2}{g_2} - \delta_2\right)e^{-rT}N(d_2) \end{aligned} \tag{24}$$

$$
\begin{aligned}
P = {} & w_1\left(K + \frac{\lambda_1}{g_1} - \delta_1\right)e^{-rT}N(-d_1) \\
& - w_1\frac{\lambda_1}{g_1}e^{\frac{1}{2}g_1^2\sigma_1^2T - rT}AN\left(-d_1 - g_1\sigma_1\sqrt{T}\right) \\
& + w_2\left(K + \frac{\lambda_2}{g_2} - \delta_2\right)e^{-rT}N(-d_2) \\
& - w_2\frac{\lambda_2}{g_2}e^{\frac{1}{2}g_2^2\sigma_2^2T - rT}AN\left(-d_2 - g_2\sigma_2\sqrt{T}\right) \qquad (25)
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \frac{F - w_1\left(\delta_1 - \frac{\lambda_1}{g_1}\right) - w_2\left(\delta_2 - \frac{\lambda_2}{g_2}\right)}{w_1\frac{\lambda_1}{g_1}e^{\frac{1}{2}g_1^2\sigma_1^2T} + w_2\frac{\lambda_2}{g_2}e^{\frac{1}{2}g_2^2\sigma_2^2T}}, \\
d_1 &= \frac{1}{g_1\sigma_1\sqrt{T}}\ln\left(\frac{\lambda_1 A}{g_1\left(K - \delta_1\right) + \lambda_1}\right), \\
d_2 &= \frac{1}{g_2\sigma_2\sqrt{T}}\ln\left(\frac{\lambda_2 A}{g_2\left(K - \delta_2\right) + \lambda_2}\right)
\end{aligned}
$$

F is the forward price, K is the strike price, T is the time to maturity, r is the risk free interest rate, $N(\cdot)$ is the cumulative distribution function of a standard normal random variable, and δ, λ, g and σ are distribution parameters with λ, $\sigma > 0$ and $g_i \neq 0$.

Proof. See Appendix.

APPLICATIONS

In this section we demonstrate the use and the performance of the double-g option pricing model with two examples. To price derivatives using the double-g model one could estimate the distributional parameters either from the real or the risk neutral distributions. The real and risk neutral location parameters are linked through Equation (22); all the other parameters are identical for the real and the risk neutral distributions.

The double-g model has nine unknown parameters, four for each g-distribution and a weighting parameter, ω. One way of reducing the number of parameters is to set $\delta = \lambda = F$. That is, for a g distributed S_T, if $(Y - \delta)/\lambda$ on the RHS of Equation (6) is written as $(S_T - F)/F$, then $(Y - \delta)/\lambda$ could be interpreted as the discrete T-period return. Note that this reformulation does not affect the skewness and kurtosis of the distribution.

In the first example below we use the double-g model to estimate the distributional parameters from the risk neutral density implied by traded prices of S&P 500 options. In the second example we use a Heating Degree Day (HDD) historical distribution to produce prices of options written on HDD. The S&P 500 futures and futures options are highly liquid. HDD itself is not traded. The associated futures and option markets for HDD are both very illiquid.

S&P 500

Our data sample consists of end-of-day prices of European out-of-the-money call and put options written on futures contracts of S&P 500 and traded on the Chicago Mercantile Exchange on the 28th of November 2007. The U.S. T-bill rate maturing closest to the option maturity date is used as the risk free rate. Data screening procedures were applied to the dataset, which resulted in the following options being excluded: first, options with moneyness K/F bigger than 1.2 or smaller than 0.8, as they usually have very low liquidity, which increases the probability of non-synchronous prices for the option and the underlying; second, options with prices that do not conform to option boundary conditions; and finally, options with prices lower than \$0.20. The final sample included 62 option prices. As futures contract holders are not entitled to dividends, a dividend adjustment is not required for pricing options written on futures.

The estimation involves finding an optimum set of distributional parameter values that minimizes the total percentage pricing errors between model price and market price of options with the same maturity T across a range of strike prices. The risk neutral density resulting from the maximization procedure is shown in Exhibit 4 with parameters $\omega_1 = 0.798$, $\omega_2 = 0.202$, $\sigma_1 = 0.1523$, $\sigma_2 = 0.2491$, $g_1 = -0.6291$, $g_2 = 0.7216$, $\lambda_i = \delta_i = 1426.40$ and $F = 1470.60$ for $i = 1,2$. It is interesting to note that the S&P 500 risk neutral density is left skewed as suggested by Jackwerth and Rubinstein [1996], and Brown and Jackwerth [2004], for instance. The respective Black-Scholes implied volatilities obtained from the market prices and from the double-g option prices are presented in Exhibit 5.

Heating Degree Day

The sample consists of monthly aggregate Heating Degree Day values at the Chicago O'Hare International Airport from December 1979 to December 2000 obtained from Wang [2002]. The final sample consists of 22 observations. Here we use the historical weather data to project the future temperature distribution for the month of December. As the editor has pointed out, to do this we assume that both the actual distribution of December temperature and the investor's utility function are constant over time, and hence we can use this projected objective distribution to map the risk neutral distribution. As the theory in the previous sections shows, the shape and all the distribution parameters, except for the location parameter, are the same for the objective and risk neutral mixture of g distributions.

The distributional parameters are estimated through a maximum likelihood estimation (MLE) using the risk-adjusted density given by Equations (7) and (18) with the restriction $g_1\mu_1 = g_2\mu_2$. The estimated double-g and the risk-neutral densities are shown in Exhibit 6. The actual density is obtained using the following parameters: $\omega_1 = 0.5$, $\omega_2 = 0.5$, $\mu_1 = 0.052$, $\mu_2 = -0.113$,

EXHIBIT 4

S&P 500 Risk-Neutral Density Estimated from Option Prices Using the Double-*g* Option Pricing Model

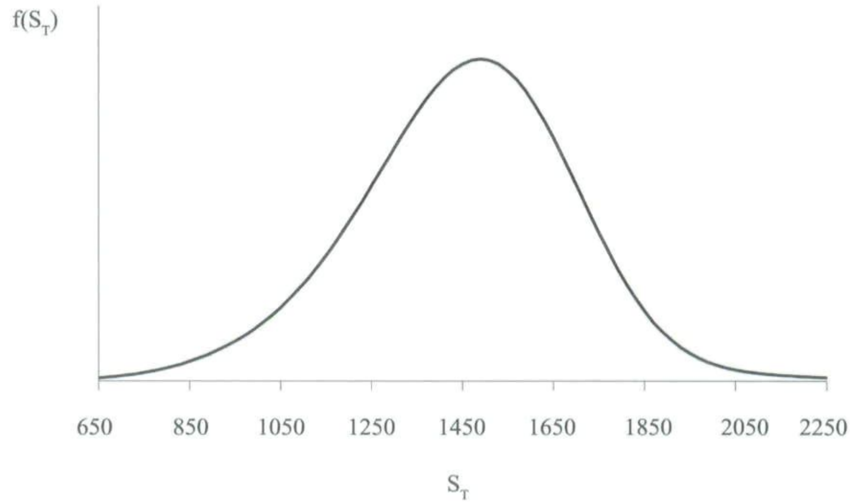
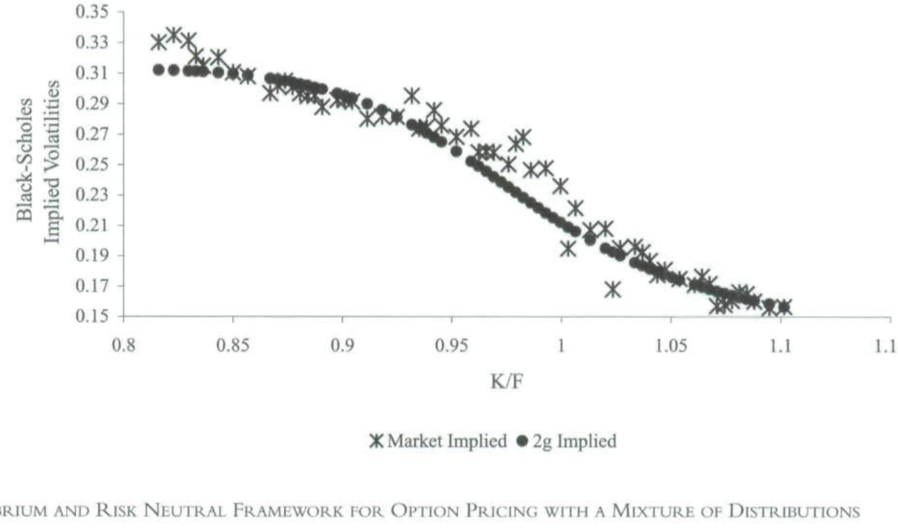


EXHIBIT 5

Black-Scholes Implied Volatility for S&P 500 Futures Options as Derived from Market Prices and from the Double-*g* Model Prices



$\sigma_1 = 0.179$, $\sigma_2 = 0.067$, $g_1 = 0.465$, $g_2 = -0.213$, $\lambda_i = \delta_i = F = 1200.20$, for $i = 1, 2$. The risk neutral density and the corresponding Black-Scholes implied volatility are presented in Exhibit 6 and Exhibit 7.

DISCUSSION ARTICLE AND CONCLUSION

In this article we develop a risk neutral discrete-time option pricing model based on a general equilibrium framework, the underlying having a *g* distribution or a mixture of two *g* distributions. As our model is developed without hedging argument or continuous trading assumption it can be used to price options written on underlying that cannot be hedged or replicated. It is also useful where the risk neutral density cannot be obtained with accuracy because the derivatives themselves are not actively traded. Finally, since the double-*g* distribution covers a wide region of the skewness-kurtosis plane it can be used for many new exotic underlyings and, at the same time, be consistent with the observed Black-Scholes volatility smile.

As the *g* distribution permits both positive and negative skewness our option pricing models can capture the negative tail documented in previous studies and the"crash-o-phobia" phenomenon observed by Rubinstein [1994]. The double-*g* model has the double-normal and double-lognormal models as special cases, the latter of which is very popular among central banks. However, the

EXHIBIT 6
Actual and Risk-Neutral Double-*g* Densities Estimated from HDD

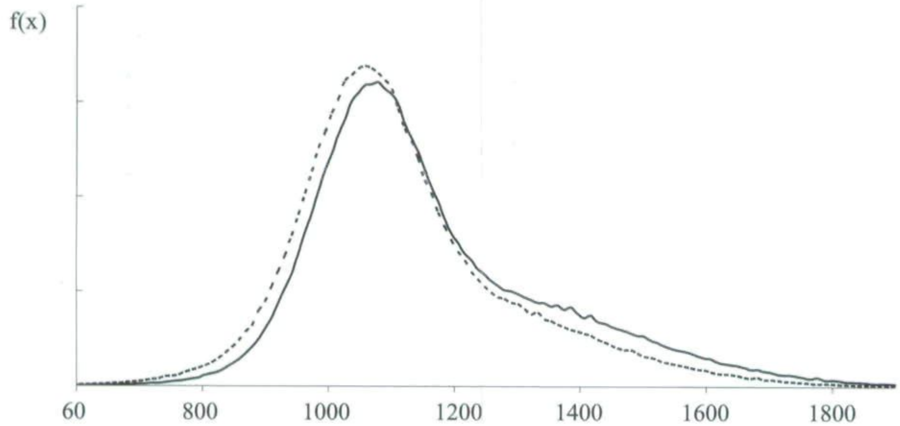
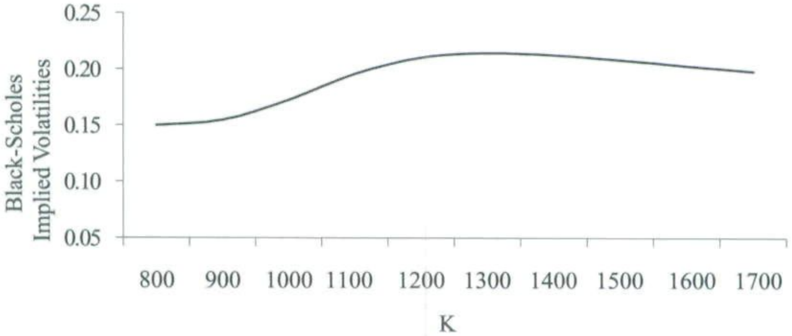


EXHIBIT 7
Black-Scholes Implied Volatility for HDD Options as Derived from Double-*g* Model Prices



Note: Derived from double-g model prices given by Equation (24), with $F = \delta_i = \lambda_i = 1200.20$, $t = 1$, $w = 0.5$, $\sigma_1 = 0.179$, $\sigma_2 = 0.067$, $g_1 = 0.465$, $g_2 = -0.213$.

most important feature of our double-g model is the clear link between the real and the risk neutral distributions constructed based on theory. This allows a clear transition from the real physical measure to the risk neutral measure instead of taking the risk neutral measure as the starting point.

APPENDIX

Proof. (*Equation 13 on RNVR*): Substituting Equation (12) into Equation (11) and changing variables yields

$$F=\int_{-\infty}^{\infty}\left(\delta+\lambda\frac{e^{gz}-1}{g}\right)\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2}\left[z-(\mu+\gamma\rho\sigma_W\sigma)\right]^2}dz$$
$$=\left(\delta-\frac{\lambda}{g}\right)+\frac{\lambda}{g}e^{g\,(\mu+\gamma\rho\sigma_W\sigma)+\frac{1}{2}\sigma^2g^2}$$

Hence

$$\mu+\gamma\rho\sigma_W\sigma=\frac{1}{g}\left[\ln\left(g\frac{F-\delta}{\lambda}+1\right)-\frac{1}{2}g^2\sigma^2\right]$$

Proof. (*Proposition 5 on g-option pricing formula*) From Equation (15), substituting $V_T(S_T) = Max(S_T - K,0)$ and changing variables gives

$$Ce^{rT}=\int_{-\infty}^{\infty}Max\left(\delta+\lambda\frac{e^{gz}-1}{g}-K,0\right)\frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{1}{2\sigma^2T}\left(z-\mu^*\right)^2}dz$$
$$=\int_{\frac{1}{g}\ln\left(g\frac{K-\delta}{\lambda}+1\right)}^{\infty}\frac{\lambda}{g}e^{gz}f(z)dz-\int_{\frac{1}{g}\ln\left(g\frac{K-\delta}{\lambda}+1\right)}^{\infty}\left(K+\frac{\lambda}{g}-\delta\right)f(z)dz$$

where μ^* is defined according to Equation (13).

Using the fact that $z \sim N(\mu^*, \sigma^2T)$, it is possible to write

$$Ce^{rT}=\left(F-\delta+\frac{\lambda}{g}\right)N(d_1)-\left(K+\frac{\lambda}{g}-\delta\right)N(d_2)$$

where $N(d_1)$ is the cumulative normal density and

$$d_1=\frac{1}{g\sigma\sqrt{T}}\ln\left(\frac{gF-g\delta+\lambda}{gK-g\delta+\lambda}\right)+\frac{1}{2}g\sigma\sqrt{T},$$
$$d_2=\frac{1}{g\sigma\sqrt{T}}\ln\left(\frac{gF-g\delta+\lambda}{gK-g\delta+\lambda}\right)-\frac{1}{2}g\sigma\sqrt{T}$$

Discounting both sides of the equation by $\exp(-rT)$ yields Equation (16). The put option formula in Equation (17) can be derived similarly.

Proof. (*Equation 21 on RNVR for double-g*): This proof is similar to the proof of Equation (13) but with two separate changes in variable. Substituting Equation (20) into (19) and changing variables yields

$$F=\sum_{i=1}^{2}w_i\int_{-\infty}^{\infty}\left(\delta_i+\lambda_i\frac{e^{g_iz_i}-1}{g_i}\right)\frac{1}{\sigma_i\sqrt{2\pi}}e^{-\frac{1}{2\sigma_i^2}\left[z_i-(\mu_i+\gamma\rho_i\sigma_W\sigma_i)\right]^2}dz_i$$
$$=\sum_{i=1}^{2}w_i\left[\delta_i+\frac{\lambda_i}{g_i}\left(e^{g_i(\mu_i+\gamma\rho_i\sigma_W\sigma_i)+\frac{1}{2}g_i^2\sigma_i^2}-1\right)\right]$$

which is Equation (21).

Proof. (*Proposition 7 on double-g option pricing formula*) As in the proof of *Proposition5*, substituting $V_T(S_T) = Max(S_T - K, 0)$, the double-$g$ risk adjusted density function (20), and the parameter constraint (22) into Equation (15) and changing variables gives:

$$Ce^{rT}=\sum_{i=1}^{2}w_i\int_{-\infty}^{\infty}Max\left(\delta_i+\lambda_i\frac{e^{g_iz_i}-1}{g_i}-K,0\right)$$
$$\times\frac{1}{\sigma_i\sqrt{2\pi T}}e^{-\frac{1}{2\sigma_i^2T}\left(z_i-\kappa\right)^2}dz_i,$$
$$=\sum_{i=1}^{2}w_i\int_{\frac{1}{g}\ln\left(g_i\frac{K-\delta_i}{\lambda_i}+1\right)}^{\infty}\left[\frac{\lambda_i}{g_i}e^{g_iz_i}-\left(K+\frac{\lambda_i}{g_i}-\delta_1\right)\right]f(z_i)dz_i \quad (26)$$

where $z_i \sim N\left(\kappa,\sigma_i^2T\right)$.

Then using the normal cumulative density

$$Ce^{rT}=\sum_{i=1}^{2}w_i\left\{\frac{\lambda_i}{g_i}e^{\kappa+\frac{1}{2}g_i^2\sigma_i^2T}N\left[\frac{1}{g_i\sigma_i\sqrt{T}}\left(\left(\kappa+g_i^2\sigma_i^2T\right)\right.\right.\right.$$
$$\left.\left.\left.-\ln\left(g_i\frac{K-\delta_i}{\lambda_i}+1\right)\right)\right]-\left(K+\frac{\lambda_i}{g_i}-\delta_i\right)$$
$$\left.\times N\left[\frac{1}{g_i\sigma_i\sqrt{T}}\left(\kappa-\ln\left(g_i\frac{K-\delta_i}{\lambda_i}+1\right)\right)\right]\right\}$$

Next, writing $A = e^k$, expanding and simplifying the above equation gives

$$C = w_1 \frac{\lambda_1}{g_1} e^{\frac{1}{2}g_1^2\sigma_1^2 T - rT} AN\left(d_1 + g_1\sigma_1\sqrt{T}\right)$$
$$- w_1\left(K + \frac{\lambda_1}{g_1} - \delta_1\right) e^{-rT} N(d_1) + w_2 \frac{\lambda_2}{g_2} e^{\frac{1}{2}g_2^2\sigma_2^2 T - rT}$$
$$\times AN\left(d_2 + g_2\sigma_2\sqrt{T}\right) - w_2\left(K + \frac{\lambda_2}{g_2} - \delta_2\right) e^{-rT} N(d_2)$$

where

$$A = \frac{F - w_1\left(\delta_1 - \frac{\lambda_1}{g_1}\right) - w_2\left(\delta_2 - \frac{\lambda_2}{g_2}\right)}{w_1 \frac{\lambda_1}{g_1} e^{\frac{1}{2}g_1^2\sigma_1^2 T} + w_2 \frac{\lambda_2}{g_2} e^{\frac{1}{2}g_2^2\sigma_2^2 T}}$$
$$d_1 = \frac{1}{g_1\sigma_1\sqrt{T}} \ln\left[\frac{\lambda_1 A}{g_1(K - \delta_1) + \lambda_1}\right]$$
$$d_2 = \frac{1}{g_2\sigma_2\sqrt{T}} \ln\left[\frac{\lambda_2 A}{g_2(K - \delta_2) + \lambda_2}\right]$$

ENDNOTES

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[1]In this framework, the risk averse representative investor maximizes her expected utility of terminal wealth, Max $E[U(W_T)]$. The result in Equation (1) follows immediately from the first order condition to a maximum. Considering a complete market setting, the pricing kernel in Equation (1) is unique.

[2]Here we use real probability to refer to the actual probability in contrast to the risk neutral probability used in derivative pricing. The two probability measures are identical if the investor is risk neutral.

[3]This comes from the fact that $Y = [\exp(gz) - 1]/g = z + gz^2/2! + g^2z^3/3! + \cdots$

[4]Note that for a non-satiated, risk averse investor $U'(W_T) > 0$ and $U''(W_T) < 0$.

[5]In practice, the assumption in Equation (22) might narrow the skewness-kurtosis plane covered by the double-g distribution. The impact should be felt immediately on the goodness of fit between model and market prices of option, but considering the benefit of not having to specify the preference parameters, this could be a cost worth paying for. The examples shown in the next section indicate that the goodness of fit of the double-g model does not appear to suffer with this added constraint. This may be due to the fact that the double-g distribution, with the constraint placed by assumption Equation (22), still has many parameters and degrees of freedom to maintain its flexibility.

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