

Closed-Form Approximations for Spread Option Prices and Greeks

MINQIANG LI, SHI-JIE DENG, AND JIEYUN ZHOU

MINQIANG LI

is an assistant professor in the College of Management at Georgia Institute of Technology in Atlanta, GA. minqiang.li@mgt.gatech.edu

SHI-JIE DENG

is an associate professor in the School of Industrial and Systems Engineering at Georgia Institute of Technology in Atlanta, GA. deng@isye.gatech.edu

JIEYUN ZHOU

is a Ph.D student in the School of Industrial and Systems Engineering at Georgia Institute of Technology in Atlanta, GA. jzhou@isye.gatech.edu

We develop a new closed-form approximation method for pricing spread options. Numerical analysis shows that our method is more accurate than existing analytical approximations. Our method is also extremely fast, with computing time more than two orders of magnitude shorter than one-dimensional numerical integration. We also develop closed-form approximations for the greeks of spread options. In addition, we analyze the price sensitivities of spread options and provide lower and upper bounds for digital spread options. Our method enables the accurate pricing of a bulk volume of spread options with different specifications in real time, which offers traders a potential edge in financial markets. The closed-form approximations of greeks serve as valuable tools in financial applications such as dynamic hedging and value-at-risk calculations.

Spread options allow investors to simultaneously take positions in two or more assets and profit from their price difference over some spread. Spread options are prevalent in equity, fixed-income, foreign exchange and commodity markets. For instance, in the fixed-income markets, various instruments are traded on exchanging securities with different maturities (such as Treasury notes and bonds), with different quality levels (such as the Treasury bills and Eurodollars), and with different issuers (such as French and German bonds, or municipal bonds and Treasury bonds). In the agricultural markets, the

CBOT trades the so-called crush spread which exchanges raw soybeans with a combination of soybean oil and soybean meal. In the energy markets, crack spread options, which exchange either crude oil and unleaded gasoline or crude oil and heating oil, are traded on the NYMEX. Electricity spark spread options are also traded over the counter for exchanging a specific fuel for electricity. Many studies have focused on spread options in these markets. For example, Arak et al. [1987]; Jones [1991], and Easterwood and Senchack [1986] study spread options in the fixed-income markets. Johnson et al. [1991] study spread options in the agricultural markets. Girma and Paulson [1998, 1999] and Deng, Johnson, and Sogomonian [2001] study spread options in the energy markets.

In this article, we focus on spread options written on two assets. Spread options written on more than two assets are dealt with in a separate article. The risk-neutral valuation of the two-asset spread option price involves a two-dimensional integration. We introduce a key concept called the *exercise boundary*. It is defined as the minimal log price of asset one, as a function of the log price of asset two, for the option to expire in the money. For convenience, we also standardize these two log prices. Under setups in which the asset returns are jointly normally distributed, a closed-form formula (Margrabe [1978]) exists for pricing exchange options, which are spread options

with zero spreads. The critical reason why such a closed-form formula can be obtained for exchange options is that the exercise boundary of a spread option is linear when the spread is zero, which allows the double integration to be evaluated in closed form. However, for general spread options where the spread is not zero, the

exercise boundary becomes nonlinear, which prevents the exercise boundary of a spread option is linear when the spread is zero, which allows the double integration to be evaluated in closed form. However, for general spread options where the spread is not zero, the exercise boundary becomes nonlinear, which prevents obtaining a closed-form formula. Thus, it is difficult to efficiently and accurately compute spread option prices. No exact closed-form formula exists for spread options with general nonzero spread.

Methods for pricing spread options can be divided into two groups: numerical methods and analytical methods. Numerical methods include binomial, Monte Carlo simulation, and the finite difference method. Analytical methods generally seek a closed-form formula to approximate the spread option price. Analytical methods have been proposed: the Black-Scholes approximation (Wilcox [1990]; Jarrow and Rudd [1982]), one approximates the price of each of the two assets directly as a normal variable and then uses the Bachelier formula for the spread option price. The Bachelier approximation is found to be less accurate than the Black-Scholes approximation. Some attempts (Mbanefo [1997]) have tried to improve the accuracy of the Bachelier approximation by including high-order moments of the price difference or using a Gram-Charlier density function in finance by Jarrow and Rudd [1982]. The Margrabe formula to price spread options by treating the second asset and the fixed spread as a single asset, which is then treated as lognormally distributed. This method is found to be less accurate and thus currently relatively popular among practitioners. Carmona and Durrleman [2000] design a new method to approximate the spread option price by giving the lower and upper price bounds. The Carmona-Durrleman method is generally more accurate than other analytical methods. However, a major weakness is that in this method one needs to solve a system of equations which is computationally not completely trivial. Thus, unlike numerical methods, the Carmona-Durrleman method does not give a closed-form formula for the spread option price.

There are weaknesses in the existing approaches. On the one hand, while

numerical methods are often accurate, their computing times are usually much longer than desirable. On the other hand, analytical approximation methods are generally faster than numerical methods but often lack accuracy and robustness. Thus, it is desirable to have a method that combines the strengths of existing methods while avoiding

their weaknesses, namely, a method that is both accurate and fast.

The purpose of this article is to derive closed-form approximations for the spread option price and greeks which are more accurate and faster than existing methods. We make several important contributions. First, we propose a new closed-form approximation for pricing spread options based on a quadratic approximation of the exercise boundary. Our approximation is extremely accurate, often resulting in relative pricing errors smaller than 10^{-4} . Second, our approach differs from existing analytical approximations in that we approximate separately each term in the spread option price. Approximating individual terms allows us to compute very accurately digital-type spread options. More importantly, it also leads to extremely accurate approximations for the greeks, which are of significant importance in practical applications such as dynamic hedging and value-at-risk calculations. Third, we develop lower and upper bounds for digital spread options. Finally, we provide an analytical study on the price sensitivities of the spread options. In particular, we characterize the signs of the vegas when the correlation coefficient between the two underlyings is negative or positive and large.

Our closed-form approximation formula for spread option prices can offer insights on the design and analysis of real options embedded in financial and real contracts. Spread options with zero spread, or *exchange options*, have been employed extensively by researchers to model real options, partly because of the availability of the Margrabe formula. For example, McDonald and Siegel [1985] use the Margrabe formula to study the investment and valuation of firms when there is an option to shut down. Shelvin [1991] investigates the valuation of research and development (R&D) firms with R&D limited partnerships. Albizzati and Geman [1994] value the surrender option in life insurance policies by extending the Margrabe formula to a Heath-Jarrow-Morton stochastic interest rate framework. Grinblatt and Titman [1989] and Johnson and Tian [2000] apply the Margrabe formula to study the design and effectiveness of performance-based contracts and executive stock options. However, in many

cases, a closed-form formula can be obtained for exchange options when the spread is zero, which allows the double integration to be evaluated in closed form. However, for general spread options where the spread is not zero, the exercise boundary becomes nonlinear, which prevents people from obtaining a closed-form formula. Thus, it is a challenge to efficiently compute spread option prices because no exact closed-form formula exists for spread options with general nonzero spread.

Existing methods for pricing spread options are roughly divided into two groups: numerical methods and analytical methods. Numerical methods include binomial, Monte Carlo simulation, and the finite difference method. Analytical methods generally seek a closed-form formula to approximate the spread option price. Analytical methods have been proposed: the Black-Scholes approximation (Wilcox [1990]; Jarrow and Rudd [1982]), one approximates the price of each of the two assets directly as a normal variable and then uses the Bachelier formula for the spread option price. The Bachelier approximation is found to be less accurate than the Black-Scholes approximation. Some attempts (Mbanefo [1997]) have tried to improve the accuracy of the Bachelier approximation by including high-order moments of the price difference or using a Gram-Charlier density function in finance by Jarrow and Rudd [1982]. The Margrabe formula to price spread options by treating the second asset and the fixed spread as a single asset, which is then treated as lognormally distributed. This method is found to be less accurate and thus currently relatively popular among practitioners. Carmona and Durrleman [2000] design a new method to approximate the spread option price by giving the lower and upper price bounds. The Carmona-Durrleman method is generally more accurate than other analytical methods. However, a major weakness is that in this method one needs to solve a nonlinear system of equations which is computationally costly and time-consuming. Thus, unlike numerical methods, the Carmona-Durrleman method does not give a closed-form formula for the spread option price.

However, the existing methods of both ap

of these applications, it is more natural to assume that we have a spread option instead of an exchange option. The spread K may correspond to the cost or salvage value of shutting down a firm, the cost or salvage value of terminating an R&D partnership, the monetary penalty of surrendering the life insurance policy prematurely, a minimal level ($K > 0$) of performance difference that a manager has to achieve over a benchmark, or a cushion ($K < 0$) to insure the manager that he will not be unfairly penalized because of pure bad luck. The extra degree of freedom arising from a nonzero K could be extremely important in designing and analyzing these real options.

The article is organized as follows. First, we discuss the general framework used; that is, asset returns are jointly normally distributed. We reduce the spread option pricing problem to a one-dimensional problem in Proposition 1, show that it reduces to the Margrabe formula when the exercise boundary is linear in Proposition 2, and discuss the properties of the exercise boundary for general spread options in Proposition 3. Next, we develop lower and upper bounds for the digital spread option in Propositions 4 and 5 based on a tangent line approximation and a chord approximation of the exercise boundary, respectively. We then develop a closed-form approximation for spread options in Proposition 6 based on a quadratic approximation of the exercise boundary. Proposition 6 is the central result of this article. We also analyze the price sensitivities of spread options in our general framework in Proposition 7 and the special geometric Brownian motions case in Proposition 8. We then give closed-form approximations in Proposition 9 for the spread option greeks. The next section compares our method with existing analytical approximations and numerical integration in terms of speed and accuracy and shows that our method is both very accurate and fast. The last section concludes and discusses possible extensions. Proofs are in the appendix.

THE MODEL SETUP

The main purpose of this article is to derive an efficient and accurate method for computing spread option prices. Under the general assumption of jointly normal returns, a closed-form formula exists for exchange options, that is, spread options with the spread K being zero. However, for general spread options with nonzero K , an exact closed-form formula is not available. In this article, we develop an extremely accurate analytical approximation

for spread option prices with general values of K . The relative errors of our approximation are usually smaller than 10^{-4} , well within the observed bid-ask spreads of these options.

We first describe the setup we will use for pricing spread options, that is, the returns of the two assets are jointly normally distributed. Specifically, consider two assets whose prices at time t are denoted by $S_1(t)$ and $S_2(t)$. We are interested in options whose final payoffs are non-negative only when $S_1(T) - S_2(T) - K \geq 0$ at some future time T , where the spread K is a constant. We focus on cash-or-nothing digital spread options with time- T payoff given by $1_{S_1(T) \geq S_2(T) + K}$, and spread options with time- T payoff $[S_1(T) - S_2(T) - K]^+$, where we use f^+ to denote the positive part of the function f . By the martingale pricing approach, the prices of a digital spread option Π^D and a spread option Π are given by

$$\begin{aligned}\Pi^D &= e^{-rT} \mathbb{E}^Q[1_{S_1(T) \geq S_2(T) + K}], \\ \Pi &= e^{-rT} \mathbb{E}^Q[S_1(T) - S_2(T) - K]^+\end{aligned}\quad (1)$$

where Q is the risk-neutral measure under which discounted security prices are martingales.

To compute these option prices, distributional assumptions on $S_1(T)$ and $S_2(T)$ need to be made. We assume that $\log S_1(T)$ and $\log S_2(T)$ are jointly normally distributed. Specifically, let the initial prices of the two assets be $S_1(0) = S_1$, $S_2(0) = S_2$, and

$$\mathbb{E}^Q[\log S_i(T)] = \mu_i, \quad \text{var}^Q[\log S_i(T)] = v_i^2, \quad (i = 1, 2) \quad (2)$$

where the means μ_i 's and the variances v_i 's are all deterministic quantities. Next, we define

$$X = \frac{\log S_1(T) - \mu_1}{v_1}, \quad Y = \frac{\log S_2(T) - \mu_2}{v_2} \quad (3)$$

Notice that X and Y are the standardized log prices of assets one and two, respectively. In our setup, we will assume that X and Y are jointly normal with correlation coefficient ρ and with standard normal marginal densities.

This general setup incorporates two important cases, namely, the geometric Brownian motions (GBMs) case and the mean-reverting log-Ornstein-Uhlenbeck (log-OU) case. Specifically, let $W_1(t)$ and $W_2(t)$ be two

Brownian motions with correlation ρ . In the GBMs case, we have

$$dS_i(t) = (r - q_i)S_i(t)dt + \sigma_i S_i(t)dW_i(t) \quad (4)$$

where r is the risk-free interest rate, σ_i 's are the volatilities, and q_i 's are the dividend rates. A simple application of Ito's lemma tells us that $S_1(T)$ and $S_2(T)$ are jointly normally distributed, with the μ_i 's and v_i 's in Equation (2) given by

$$\mu_i = \log S_i + (r - q_i - \sigma_i^2/2)T, \quad v_i = \sigma_i \sqrt{T}, \quad \rho = \rho \quad (5)$$

The GBMs case can be easily generalized to incorporate seasonality in parameters by allowing σ_i 's, q_i 's and ρ be to deterministic functions of the calendar time t . This is useful since for some spread options, their underlying assets exhibit strong seasonality in price volatilities and in their return correlations. Our general framework incorporates this generalized GBMs case.

In the log-OU case, we have

$$dS_i(t) = -\lambda_i(\log S_i(t) - \eta_i)S_i(t)dt + \sigma_i S_i(t)dW_i(t) \quad (6)$$

where λ_i 's are the mean-reverting strengths and η_i 's are parameters controlling the long-run means. With some algebra, it can be shown that $S_1(T)$ and $S_2(T)$ are jointly normally distributed, with the μ_i 's and v_i 's in Equation (2) and ρ given by

$$\mu_i = \eta_i - \frac{\sigma_i^2}{2\lambda_i} + e^{-\lambda_i T} \left(\log S_i - \eta_i + \frac{\sigma_i^2}{2\lambda_i} \right) \quad (7)$$

$$v_i = \sigma_i \sqrt{\frac{1 - e^{-2\lambda_i T}}{2\lambda_i}}$$

$$\rho = 2\rho \frac{\sqrt{\lambda_1 \lambda_2}}{\lambda_1 + \lambda_2} \frac{1 - e^{-(\lambda_1 + \lambda_2)T}}{\sqrt{1 - e^{-2\lambda_1 T}} \sqrt{1 - e^{-2\lambda_2 T}}} \quad (8)$$

Before introducing our method, we present a thorough analysis of the exercise boundary of the spread option. The exercise boundary is defined to be the minimal standardized log price of asset one for the option to

be in the money as a function of the standardized log price of asset two. As we will see later, the existence of a closed-form formula for exchange options, but not for general spread options, is a direct result of the linearity of the exercise boundary when the spread is zero. Thus a detailed study of the exercise boundary is important.

At time T , the options are in-the-money if $S_1(T) - S_2(T) - K \geq 0$. Let $K \geq 0$. By the definitions of X and Y in Equation (3), this condition is the same as

$$X \geq \frac{\log(e^{v_2 Y + \mu_2} + K) - \mu_1}{v_1} \quad (9)$$

Thus, conditioning on $Y = y$, the option is in-the-money if $X \geq \underline{x}(y)$, where the (conditional) exercise boundary $\underline{x}(y)$ is given by

$$\underline{x}(y) \equiv \frac{\log(e^{v_2 y + \mu_2} + K) - \mu_1}{v_1} \quad (10)$$

When $K < 0$, the condition in Equation (9) is not always well defined because it is possible that $e^{v_2 Y + \mu_2} + K < 0$. However, by making use of the identities

$$1_{S_1(T) \geq S_2(T) + K} = 1 - 1_{S_2(T) \geq S_1(T) - K} \quad (11)$$

$$[S_1(T) - S_2(T) - K]^+ = S_1(T) - S_2(T) - K + (S_2(T) - S_1(T) + K)^+ \quad (12)$$

we can transform the problems of computing Π^D and Π in the $K < 0$ case to the $K > 0$ case. For example, by Equation (11), to compute the price of a digital spread option with final payoff $1_{S_1(T) \geq S_2(T) + K}$ where $K < 0$, we can switch the roles of S_1 and S_2 and compute the price of the digital spread option with final payoff $1_{S_2(T) \geq S_1(T) + |K|}$. Similarly, Equation (12) allows us to only consider spread options with $K \geq 0$. Consequently, throughout this article we assume that $K \geq 0$, so that Equation (9) is always well defined.

The risk-neutral valuation in Equation (1) gives Π^D and Π in terms of two two-dimensional integrations. However, in the following proposition, we utilize a method introduced in Pearson [1995] which reduces the two-dimensional integrations to one-dimensional integrations. Reducing the two-dimensional integration problem in

Equation (1) to a one-dimensional integration is useful not only for our approximation later, but also for numerical methods. For example, if one uses numerical integration, then by Proposition 1, we only need to evaluate one-dimensional integrals, which is considerably faster than evaluating two-dimensional integrals. Also, Proposition 1 is useful for Monte Carlo simulation. Carrying out Monte Carlo directly using Equation (1) is not very efficient because many realizations of $(S_1(T) - S_2(T) - K)^+$ will be zero, especially for out-of-the-money spread options. Also, we need to simulate a bivariate distribution. With Proposition 1, we can simulate a single random variable y using a standard normal density and then compute the I_i 's by taking the sample averages of the three cumulative normal distribution functions. In this way, information in each realization of y is utilized and thus Proposition 1 plays a similar role as importance sampling in variance reduction.

Proposition 1. Let $|\rho| < 1$. Under the jointly normal returns setup, the prices of the spread option and the digital spread option are given by

$$\begin{aligned}\Pi &\equiv e^{\frac{v_1^2}{2} + \mu_1 - rT} I_1 - e^{\frac{v_2^2}{2} + \mu_2 - rT} I_2 - Ke^{-rT} I_3 \quad \text{and} \\ \Pi^D &= e^{-rT} I_3\end{aligned}\quad (13)$$

where the integrals I_i are given by

$$I_1 = \int_{-\infty}^{\infty} N(A(y + \rho v_1) + \sqrt{1 - \rho^2} v_1) n(y) dy \quad (14)$$

$$I_2 = \int_{-\infty}^{\infty} N(A(y + v_2)) n(y) dy \quad (15)$$

$$I_3 = \int_{-\infty}^{\infty} N(A(y)) n(y) dy \quad (16)$$

where $n(\cdot)$ and $N(\cdot)$ are the standard normal density function and the cumulative normal distribution function, respectively, and the function $A(\cdot)$ is given by

$$A(y) = \frac{\rho y - \underline{x}(y)}{\sqrt{1 - \rho^2}} \quad (17)$$

with the exercise boundary $\underline{x}(y)$ given in Equation (10).

Equation (13) in Proposition 1 gives a formula for the spread option price very similar to the Black-Scholes formula (Black and Scholes [1973]). In particular, the

price of the spread option Π consists of three terms. The first term is the present value of the risk-neutral expected future benefit of receiving asset one. The second term is the present value of expected future cost of giving up asset two if the option expires in the money. The last term is the present value of the expected cost of giving up an additional monetary amount K .

The quantities $A(y)$ and I_i have intuitive meanings. We will call $A(y)$ the *conditional moneyness* of the spread option because $A(y)$ plays a role similar to d_2 in the Black-Scholes formula. This can be seen from the proof of Proposition 1 that the quantity $N(A(y))$ is the risk-neutral probability that the spread option expires in the money, on condition that the standardized log price of asset two is y . Written out explicitly, we have

$$N(A(y)) = \text{Prob}^Q \left[S_1(T) \geq S_2(T) + K \mid Y \equiv \frac{\log S_2(T) - \mu_2}{v_2} = y \right] \quad (18)$$

In the Black-Scholes formula, y is a constant, while in the case of spread options, y is distributed as a standard normal random variable. Integrating over y in Equation (16) then gives the unconditional exercise probability I_3 . That is, I_3 is the probability that the spread option will expire in the money under the risk-neutral distribution. The proof of Proposition 1 shows that I_1 and I_2 have meanings similar to I_3 . They are the probabilities that the spread option will expire in the money under the two measures in which asset one and asset two are taken to be the numeraire asset, respectively. This is similar to the case of the Black-Scholes formula, where the term $N(d_1)$ is the probability that the option will expire in the money under the probability measure in which the underlying stock is taken as the numeraire asset. For general change of numeraire technique, see Geman, Karoui, and Rochet [1995]. The quantities I_i are also related to the greeks. As we will see later, in the GBMs case, I_1 and $-I_2$ are also the deltas of the spread option price with respect to the initial prices of asset one and asset two, respectively, while I_3 is related to the price sensitivity of the spread option with respect to the spread K .

In Proposition 1, we have given the spread option price in terms of three one-dimensional integrals I_1 , I_2 , and I_3 . In the rest of the article, we will develop closed-form approximations based on Proposition 1. We carry

this out in a few steps. First, in Proposition 2, we will show that these integrals can be computed in closed form to yield the Margrabe formula if the exercise boundary $\underline{x}(\gamma)$ (and hence the conditional moneyness function $A(\gamma)$) is linear, which happens exactly when the spread K is zero. Second, we study the monotonicity and convexity properties of the exercise boundary and the conditional moneyness function in detail in Proposition 3. Finally, in later sections, we approximate the exercise boundary by making use of these properties. These approximations in turn allow us to derive price bounds for digital options in Propositions 4 and 5, as well as approximate spread option prices in Proposition 6.

We first take a look at the special case when $K = 0$. Proposition 2 derives the Margrabe formula (Margrabe [1978]) for this special case using a new mathematical identity, Equation (19), which will also be very useful for our approximation later on. If we interpret $n(\gamma; \mu, \sigma^2)$ in the proposition as the density of the log price of asset two and $a + b\gamma$ as the conditional moneyness $A(\gamma)$, then this identity says that the unconditional moneyness can be computed in closed form when the conditional moneyness $A(\gamma)$ is a linear function of γ . For spread options, from the expression for the exercise boundary $\underline{x}(\gamma)$ in Equation (10), we see that $\underline{x}(\gamma)$ is linear in γ precisely when $K = 0$. Notice also that $A(\gamma)$ is linear in γ if and only if $\underline{x}(\gamma)$ is linear in γ . Thus, Proposition 2 offers a direct proof of the Margrabe formula which differs from the original partial differential equation approach in Margrabe [1978]. Also, the Margrabe formula we give is more general than the original form, because it applies to all models in which the returns are jointly normally distributed and not just the GBMs case. The proof of Proposition 2 is given in the appendix.

Proposition 2. *Let a and b be real numbers. Then, we have*

$$\int_{-\infty}^{\infty} N(a + b\gamma) n(\gamma; \mu, \sigma^2) d\gamma = N\left(\frac{a + b\mu}{\sqrt{1 + b^2\sigma^2}}\right) \quad (19)$$

For exchange options (spread options with $K = 0$), the conditional moneyness function $A(\gamma)$ is linear in γ . Thus, from Equation (19), the price of an exchange option under the jointly normal returns setup is given by the following Margrabe formula:

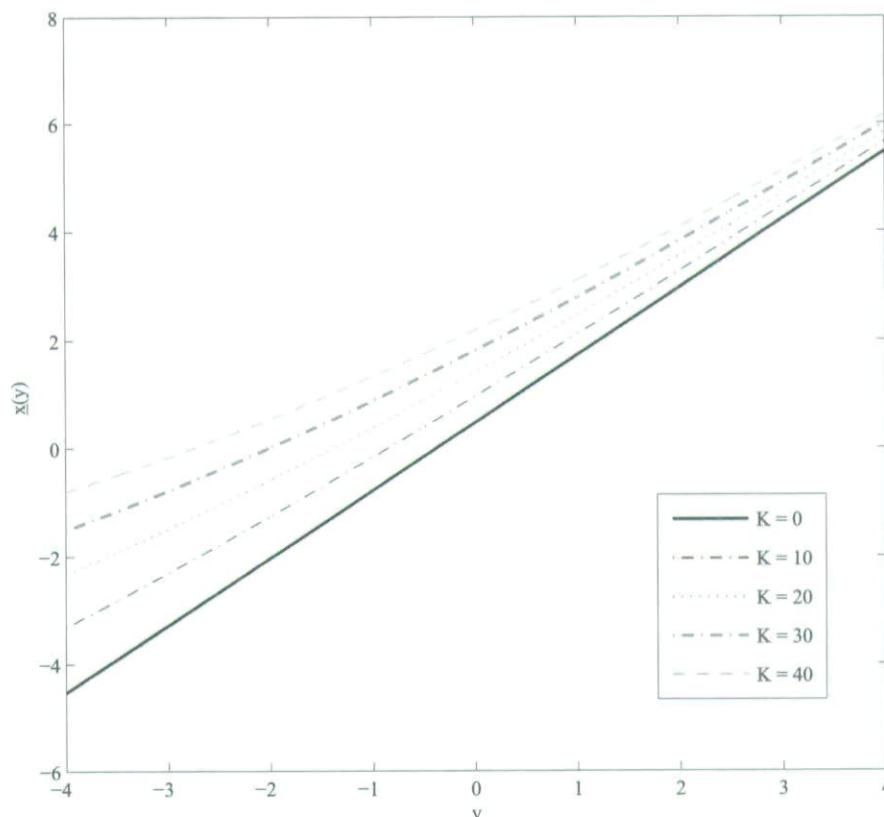
$$\begin{aligned} \Pi = & e^{v_1^2/2 + \mu_1 - rT} N\left(\frac{\mu_1 - \mu_2 + (v_1^2 - \rho v_1 v_2)}{\sqrt{v_1^2 + v_2^2 - 2\rho v_1 v_2}}\right) \\ & - e^{v_2^2/2 + \mu_2 - rT} N\left(\frac{\mu_1 - \mu_2 - (v_2^2 - \rho v_1 v_2)}{\sqrt{v_1^2 + v_2^2 - 2\rho v_1 v_2}}\right) \end{aligned} \quad (20)$$

For general spread options, $K \neq 0$. In this case, an immediate difficulty in applying Proposition 2 is that the arguments for the cumulative normal distributions in the integrals $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_3$, namely, $A(\gamma + \rho v_1) + \sqrt{1 - \rho^2} v_1$, $A(\gamma + v_2)$ and $A(\gamma)$, are not linear functions of γ . From the expression for $A(\gamma)$, we see that this is precisely because the exercise boundary $\underline{x}(\gamma)$ is not linear in γ . However, a closer examination reveals that $\underline{x}(\gamma)$ is quite close to linear locally. When γ is very negative, $\underline{x}(\gamma)$ behaves like a constant function. When γ is very positive, $\underline{x}(\gamma)$ behaves like a linear function. Exhibit 1 plots the function $\underline{x}(\gamma)$ for different values of K . The parameters used are for the GBMs case with $S_1 = 90$, $S_2 = 100$, $r = 5\%$, $q_1 = q_2 = 0$, $\sigma_1 = 0.4$, $\sigma_2 = 0.5$ and $T = 0.25$. The idea of our approximation is to approximate the exercise boundary $\underline{x}(\gamma)$ using a lower-order Taylor expansion, for example, a linear or quadratic function. This has the advantage that we approximate the $\mathbf{I}_i$'s separately. Notice that $\underline{x}(\gamma)$ is exactly linear when either $\sigma_2 = 0$ or $K = 0$. Thus, as we shall see, our approximation is exact when $\sigma_2 = 0$ or $K = 0$. This is appealing because when $\sigma_2 = 0$, our formula collapses to the Black-Scholes formula for an ordinary European call option, and when $K = 0$, our formula collapses to the Margrabe formula for an exchange option.

Because our approximation tries to catch the deviation of the exercise boundary from linearity, a closer look at the regions of monotonicity and convexity of the exercise boundary $\underline{x}(\gamma)$, the risk-neutral conditional moneyness $A(\gamma)$, and the conditional exercise probability $N(A(\gamma))$ is crucial. In addition, by making use of the monotonicity and convexity properties, we are able to look at price bounds for digital spread options as we will do in the next section. The results of this monotonicity and convexity analysis are given in the following proposition. For readers who are less interested in the mathematical details, it suffices to understand the following three points from Proposition 3. First, the exercise boundary $\underline{x}(\gamma)$ is convex in γ while the conditional moneyness function $A(\gamma)$ is concave

EXHIBIT 1

The Exercise Boundary $\underline{x}(y)$



in y . Second, the behavior of $A(y)$ and $N(A(y))$ is influenced by the sign and size of the correlation coefficient ρ . Third, the behavior of $A(y)$ and $N(A(y))$ could be different for different regions of y . These properties will be used later. For example, as we will see, the sign and size of ρ have influence on the signs of spread option vegas. The proof of Proposition 3 is given in the appendix.

Proposition 3. Let $K \geq 0$ and $|\rho| < 1$.

1. The exercise boundary $\underline{x}(y)$ is an increasing and convex function of y .
2. If $\rho \geq v_2/v_1$, the conditional moneyness $A(y)$ and risk-neutral conditional exercise probability $N(A(y))$ are both monotonically increasing in y . If $\rho \leq 0$, $A(y)$ and $N(A(y))$ are both monotonically decreasing in y . If $0 < \rho < v_2/v_1$, both $A(y)$ and $N(A(y))$ have exactly one maximum at

$$\hat{y} = \frac{1}{v_2} \log \left(\frac{e^{-\mu_2} K \rho v_1}{v_2 - \rho v_1} \right) \quad (21)$$

3. The conditional moneyness $A(y)$ is a concave function of y .
4. If $e^{\mu_1 + \rho v_1 y} > e^{\mu_2 + v_2 y} + K$, then the conditional exercise probability $N(A(y))$ is locally concave at y . Furthermore, suppose the solution of $e^{\mu_1 + \rho v_1 y} = e^{\mu_2 + v_2 y} + K$ exists and denote it by $\tilde{y}$. Then, $N(A(y))$ is concave in the region $(\tilde{y}, +\infty)$ if $v_2 < \rho v_1$, and concave in the region $(-\infty, \tilde{y})$ if $\rho < 0$.

Now that we have studied the monotonicity and convexity properties of the conditional moneyness $A(y)$ in Proposition 3, we are ready to develop our approximations. In the next section, we establish lower and upper bounds on the digital spread option with the help of Propositions 2 and 3. Readers who are interested in spread option prices can skip the next section and go directly to the following section where we develop closed-form approximations for spread option prices.

BOUNDS FOR DIGITAL SPREAD OPTION PRICES

The Upper Bound—Tangent Line Approximation

In this section, we derive upper and lower bounds for digital spread options by approximating the exercise boundary. Pricing bounds for spread options can be obtained similarly. Pricing bounds are useful when no closed-form formula exists. Early studies in this area include Perrakis and Ryan [1984], and Lo [1987]. More recently, Nielsen and Sandmann [2003], and Henderson, et al. [2007] derive pricing bounds for Asian options. Pricing bounds for American options are studied in Broadie and Detemple [1996]; Chen and Yeh [2002]; Chung and Chang [2007], among others. Carmona and Durrleman [2006] study pricing bounds for spread options.

By approximating the exercise boundary $x(y)$ more or less favorably, we can establish price bounds for digital spread options. We look at the upper bound first. By Proposition 3, the conditional moneyiness $A(y)$ is concave in y . Thus, if we draw any tangent line of $A(y)$, it will lie completely above the graph of $A(y)$. Pick an arbitrary point y_0 . By the expression of $A(y)$ in Equation (17), the tangent line of $A(y)$ at y_0 is given by $G(y_0) + H(y_0)y$, where

$$G(y_0) = \frac{1}{v_1 \sqrt{1-\rho^2}} \left(\mu_1 - \log(R+K) + \frac{R}{R+K} v_2 y_0 \right) \quad (22)$$

$$H(y_0) = \frac{1}{\sqrt{1-\rho^2}} \left(\rho - \frac{R v_2}{(R+K) v_1} \right) \quad (23)$$

with

$$R = e^{v_2 y_0 + \mu_2} \quad (24)$$

For any tangent position y_0 , the conditional moneyiness approximation $G(y_0) + H(y_0)y$ is more favorable than the actual conditional moneyiness $A(y)$. Propositions 1 and 2 now give us the following inequality:

$$\begin{aligned} I_3 &= \int_{-\infty}^{\infty} N(A(y)) n(y) dy \leq \int_{-\infty}^{\infty} N(G(y_0) + H(y_0)y) n(y) dy \\ &= N \left(\frac{G(y_0)}{\sqrt{1+H^2(y_0)}} \right) \end{aligned} \quad (25)$$

Since the above derivation is valid for any y_0 , we can establish an upper bound for the price of a digital spread option as follows:

Proposition 4. Let $K \geq 0$ and $|\rho| < 1$. Then

$$\Pi^D \leq \inf_{y_0 \in \mathbb{R}} e^{-rT} N \left(\frac{G(y_0)}{\sqrt{1+H^2(y_0)}} \right) \quad (26)$$

In Proposition 4, each candidate in the infimum is the price of a digital option whose exercise boundary is more favorable than the one in our digital spread option. Different values of y_0 match the slopes of the actual and approximating exercise boundaries at different future log prices of asset two. For example, setting $y_0 = 0$ amounts to matching the slope of the exercise boundary exactly when the log price of asset two equals its mean. The two limiting cases $y_0 \rightarrow \pm\infty$ correspond to the prices of an ordinary digital option and a digital exchange option, respectively. For example, when $y_0 \rightarrow -\infty$, from the expressions for $G(y_0)$ and $H(y_0)$, the conditional moneyiness approximation $G(y_0) + H(y_0)y$ for $A(y)$ becomes

$$\begin{aligned} A(y) &\approx G(-\infty) + H(-\infty)y \\ &= \frac{1}{v_1 \sqrt{1-\rho^2}} (\mu_1 - \log K) + \frac{\rho}{\sqrt{1-\rho^2}} y \end{aligned} \quad (27)$$

Thus,

$$e^{-rT} N \left(\frac{G(-\infty)}{\sqrt{1+H^2(-\infty)}} \right) = e^{-rT} N \left(\frac{\mu_1 - \log K}{v_1} \right) \quad (28)$$

which is exactly the price of a plain-vanilla digital option with final payoff $1_{S_1(T) \geq K}$. Similarly, one can show that the upper bound in Equation (26) when $y_0 \rightarrow +\infty$ is the price of the digital exchange option with final payoff $1_{S_1(T) \geq S_2(T)}$. Thus, the infimum in Proposition 4 automatically incorporates the following inequalities:

$$1_{S_1(T) \geq S_2(T)+K} \leq 1_{S_1(T) \geq K} \quad \text{and} \quad 1_{S_1(T) \geq S_2(T)+K} \leq 1_{S_1(T) \geq S_2(T)} \quad (29)$$

This upper bound in Proposition 4 is very tight and can be used as an approximation for the digital spread option price. Numerical analysis shows that the relative

pricing errors, defined as the pricing errors divided by the actual prices, are typically of order 10^{-3} .

In order to make Proposition 4 more useful in practice, we need to have a quick estimate of the optimal value for γ_0 that achieves the infimum in Equation (26). The optimal value for γ_0 is given by the solution of the following equation:

$$\frac{d}{d\gamma_0} N \left(\frac{G(\gamma_0)}{\sqrt{1+H^2(\gamma_0)}} \right) = 0 \quad (30)$$

An analytical solution to the best γ_0 from the above nonlinear equation turns out to be not possible. However, we can linearize Equation (30) around $\gamma_0 = 0$ and then solve for the best γ_0 approximately. This should be a very accurate approximation because the best expansion point for γ_0 ought to be very close to 0 since $n(\gamma)$ peaks at 0. The result is the following choice of γ_0 :

$$\hat{\gamma}_0 = \frac{\xi(\lambda\xi + Kv_2) \log(e^{\mu_1}/\xi)}{\xi^2(v_2^2 + 2\lambda v_2 - v_1^2) - K^2 v_2^2 - 2K\xi\lambda v_2 + e^{\mu_2}\xi\lambda v_2 \log(e^{\mu_1}/\xi)} \quad (31)$$

where

$$\lambda = \rho v_1 - v_2, \quad \xi = e^{\mu_2} + K \quad (32)$$

Although the expression of $\hat{\gamma}_0$ is complicated, it is a simple function of the input parameters and can be computed very quickly. Numerical analysis shows that for all reasonable parameter values, we have

$$\begin{aligned} 0 &\leq N \left(\frac{G(\hat{\gamma}_0)}{\sqrt{1+H^2(\hat{\gamma}_0)}} \right) - \inf_{\gamma_0 \in \mathbb{R}} N \left(\frac{G(\gamma_0)}{\sqrt{1+H^2(\gamma_0)}} \right) \\ &\sim 10^{-4} \cdot \inf_{\gamma_0 \in \mathbb{R}} N \left(\frac{G(\gamma_0)}{\sqrt{1+H^2(\gamma_0)}} \right) \end{aligned} \quad (33)$$

Thus, in practice, we can replace the infimum in Proposition 4 by this particular choice of $\hat{\gamma}_0$ without losing much accuracy.

The Lower Bound—Chord Approximation

A lower bound can also be established for the digital spread option by using the chords of the exercise boundary $\underline{x}(\gamma)$. Pick two points γ_l and γ_r on the real line with $\gamma_r > \gamma_l$. The line passing through points $(\gamma_l, A(\gamma_l))$ and $(\gamma_r, A(\gamma_r))$ is given by $B(\gamma) = P(\gamma_l, \gamma_r) + Q(\gamma_l, \gamma_r)\gamma$, where

$$\begin{aligned} P(\gamma_l, \gamma_r) &= \frac{\gamma_r A(\gamma_l) - \gamma_l A(\gamma_r)}{\gamma_r - \gamma_l}, \\ Q(\gamma_l, \gamma_r) &= \frac{A(\gamma_r) - A(\gamma_l)}{\gamma_r - \gamma_l} \end{aligned} \quad (34)$$

The segment of $B(\gamma)$ between points $(\gamma_l, A(\gamma_l))$ and $(\gamma_r, A(\gamma_r))$ is a chord of the conditional moneyiness function $A(\gamma)$. By Proposition 3, the line $B(\gamma)$ lies below the exercise boundary $\underline{x}(\gamma)$ in the region (γ_l, γ_r) and above the exercise boundary $\underline{x}(\gamma)$ outside this region. Because of this, if we directly use $B(\gamma)$ to approximate the conditional moneyiness $A(\gamma)$, the resulting price could be either an upper bound or a lower bound. However, notice that we always have

$$N(A(\gamma)) \geq N(B(\gamma)) - [N(A(\gamma)) - N(B(\gamma))]^+ \quad (35)$$

This fact can be used to derive a lower bound for the digital spread option given in the following proposition. In the second line of Equation (36), we add back two terms by looking at the behavior of the conditional moneyiness $A(\gamma)$ in more detail with the help of Proposition 3. The proof of Proposition 5 is in the appendix.

Proposition 5. Let $K \geq 0$ and $|\rho| < 1$. Then

$$\begin{aligned} \Pi^D &\geq \sup_{\substack{\gamma_l, \gamma_r \in \mathbb{R} \\ \gamma_r > \gamma_l}} e^{-rT} \left(N \left(\frac{P(\gamma_l, \gamma_r)}{\sqrt{1+Q^2(\gamma_l, \gamma_r)}} \right) \right. \\ &\quad \left. - N(\gamma_l) - N(-\gamma_r) + N(-\gamma_r)N(A(\gamma_r))1_{\rho \geq v_2/v_1} \right. \\ &\quad \left. + N(\gamma_l)N(A(\gamma_l))1_{\rho \leq 0} \right) \end{aligned} \quad (36)$$

In Proposition 5, each candidate in the supremum is the price of a digital option whose exercise boundary

is less favorable than the one in our digital spread option. Since the digital spread option price is greater than each of the candidates, it is also greater than the supremum of them. For given values of y_l and y_r , the lower bound can be computed quickly to give an approximation for Π^D .

This lower bound is not as accurate as the upper bound, yielding relative pricing errors of about 2% in many cases in our numerical analysis. We can also approximate the optimal y_l and y_r as we have done for the upper bound. We choose not to do it and instead introduce a new approximation which is extremely accurate with relative pricing errors being less than 10^{-4} most of the time.

The upper and lower bounds for digital spread options are useful because the deltas and kappa (defined as the option price sensitivity with respect to the spread K) of a spread option can themselves be considered digital spread option prices, as we will see later in Proposition 8. Thus, our bounds for digital spread options give us bounds on the greeks of spread options.

CLOSED-FORM APPROXIMATIONS FOR SPREAD OPTION PRICES AND GREEKS

Approximation for Spread Option Prices

We are now ready to obtain the main result of this article, that is, a fast and accurate approximation for spread options prices. Our improved approximation is based on a quadratic approximation of the exercise boundary and hence a quadratic approximation of the conditional moneyness function $A(y)$. Suppose we approximate $A(y)$ using a parabola by $C^3(y_0) + D^3(y_0)y + \epsilon(y_0)y^2$ around $y = y_0$, then

$$\begin{aligned} I_3 &= \int_{-\infty}^{\infty} N(A(y))n(y)dy \\ &\approx \int_{-\infty}^{\infty} N(C^3 + D^3y + \epsilon y^2)n(y)dy \end{aligned} \quad (37)$$

The expressions for C^3 , D^3 , and ϵ will be given in Proposition 6. The superscripts 3 in C^3 and D^3 indicate that these quantities are for the third term I_3 . They should not be misinterpreted as exponents. We use superscripts instead of subscripts because later we will use subscripts for partial derivatives. The quantities C^3 , D^3 , and ϵ are the intercept, slope, and curvature at $y = 0$, respectively, of the quadratic approximating boundary $C^3(y_0) + D^3(y_0)y + \epsilon(y_0)y^2$.

By Proposition 2, the last integral in the Equation (37) cannot be evaluated in closed form unless we let $\epsilon = 0$. However, if the curvature ϵ is small around the expansion point y_0 , then we can expand the above integral around $\epsilon = 0$. Numerical analysis shows this is indeed the case. Similar observations are made to the integrals I_1 and I_2 . The resulting approximation is given in the following proposition. Proposition 6 approximates I_1 , I_2 , and I_3 with a second-order Taylor expansion in terms of the curvature ϵ of the conditional moneyness function $A(y)$. The proof of Proposition 6 is given in the appendix.

Proposition 6. Let $K \geq 0$ and $|\rho| < 1$. Let y_0 be any real number close to 0. The spread option price Π under the general jointly normal returns setup is given by

$$\Pi = e^{v_1^2/2 + \mu_1 - rT} I_1 - e^{v_2^2/2 + \mu_2 - rT} I_2 - Ke^{-rT} I_3 \quad (38)$$

The integrals I_i are approximated to second order in ϵ as

$$I_i \approx J_0(C^i, D^i) + J_1(C^i, D^i)\epsilon + \frac{1}{2}J_2(C^i, D^i)\epsilon^2 \quad (39)$$

where the functions J_i 's are defined as

$$J_0(u, v) = N\left(\frac{u}{\sqrt{1+v^2}}\right) \quad (40)$$

$$J_1(u, v) = \frac{1 + (1+u^2)v^2}{(1+v^2)^{5/2}} \cdot n\left(\frac{u}{\sqrt{1+v^2}}\right) \quad (41)$$

$$\begin{aligned} J_2(u, v) &= \frac{(6 - 6u^2)v^2 + (21 - 2u^2 - u^4)v^4 + 4(3 + u^2)v^6 - 3}{(1+v^2)^{11/2}} u \\ &\quad \times n\left(\frac{u}{\sqrt{1+v^2}}\right) \end{aligned} \quad (42)$$

and the arguments C^i , D^i , and ϵ are given by

$$C^1 = C^3 + D^3\rho v_1 + \epsilon\rho^2 v_1^2 + \sqrt{1-\rho^2}v_1 \quad (43)$$

$$D^1 = D^3 + 2\rho v_1\epsilon \quad (44)$$

$$C^2 = C^3 + D^3 v_2 + \epsilon v_2^2 \quad (45)$$

$$D^2 = D^3 + 2v_2 \epsilon \quad (46)$$

$$C^3 = \frac{1}{v_1 \sqrt{1-\rho^2}} \left(\mu_1 - \log(R+K) \right. \\ \left. + \frac{v_2 R}{R+K} \gamma_0 - \frac{1}{2} \frac{v_2^2 R K}{(R+K)^2} \gamma_0^2 \right) \quad (47)$$

$$D^3 = \frac{1}{v_1 \sqrt{1-\rho^2}} \left(\rho v_1 - \frac{v_2 R}{R+K} + \frac{v_2^2 R K}{(R+K)^2} \gamma_0 \right) \quad (48)$$

$$\epsilon = -\frac{1}{2v_1 \sqrt{1-\rho^2}} \frac{v_2^2 R K}{(R+K)^2} \gamma_0^2 \quad (49)$$

with $R = e^{v_2 \gamma_0 + \mu_2}$.

Note that Proposition 6 also allows us to approximate the digital spread option price Π^D . Systematic numerical analysis demonstrates that a first-order approximation in ϵ , by setting $\mathbf{J}_2 = 0$, already yields very accurate spread option prices, although its accuracy seems to be consistently dominated by a second-order approximation in ϵ . Moreover, the choice of $\gamma_0 = 0$ works very well and generally produces relative price errors smaller than 10^{-4} . A zero γ_0 also results in simpler expressions for C 's and D 's and makes our approximation faster. Thus, we shall fix $\gamma_0 = 0$ throughout this article. Setting $\gamma_0 = 0$ amounts to matching the slope and curvature of the exercise boundary exactly when the log price of asset two equals its mean.

Our approximation in Proposition 6 has some nice properties. First, it satisfies many boundary conditions. For example, it collapses to the Black-Scholes formula when either μ_1 or μ_2 approaches $-\infty$. Also, as μ_1 goes to infinity, we have $\lim_{\mu_1 \rightarrow \infty} \Pi / e^{v_1^2/2 + \mu_1 - rT} = 1$, and as $\mu_2 \rightarrow \infty$ we have $\lim_{\mu_2 \rightarrow \infty} \Pi = 0$. The approximation also satisfies the terminal boundary condition when $T \rightarrow 0$. Second, our approximation collapses exactly to the Margrabe formula when $K = 0$ and converges to 0 when $K \rightarrow \infty$. Third, our approximation collapses exactly to the Black-Scholes formula when $v_2 \rightarrow 0$. These nice properties add to the attractiveness of our approximation.

In the next section, we will perform a thorough comparison of our method with other methods in terms of computational speed and accuracy. We shall see that our method is extremely fast and accurate. Before we do that, we study the price sensitivity of spread options. We first perform an analysis for the general jointly normal return setup in Proposition 7. We then study the special geometric Brownian motions case and the results are given in Propositions 8 and 9. Readers who are less interested in the analysis on price sensitivity and greeks can proceed directly to the next section.

Price Sensitivity Analysis of Spread Options

The price Π of the spread option is a function of $v_1, v_2, \rho, r, T, \mu_1, \mu_2$, and $\log K$. Sometimes we view Π as a function of e^{μ_1}, e^{μ_2} , and K instead of μ_1, μ_2 , and K . For example, in the geometric Brownian motions case, it is usually more natural to look at price sensitivity with respect to S_i 's instead of $\log S_i$'s. It should be clear from the context which point of view we use. The sensitivities of spread option prices with respect to these parameters are not well understood under the jointly normal returns setup, mainly because of the lack of a closed-form formula. We fill this gap with the following two propositions. Proofs are provided in the appendix.

Proposition 7. *Let $K \geq 0$. Under the jointly normal returns setup, we have*

1. *Holding all other variables constant, Π is increasing and convex in e^{μ_1} .*
2. *Holding all other variables constant, Π is decreasing and convex in e^{μ_2} .*
3. *Holding all other variables constant, Π is decreasing and convex in K .*
4. *Holding all other variables constant, Π is convex in μ_1 .*
5. *Holding all other variables constant, Π is decreasing in ρ .*
6. *Holding all other variables constant, $\partial^2 \Pi / \partial \mu_1 \partial \mu_2 < 0$.*

This proposition generalizes the known price sensitivities of the Black-Scholes formula and the Margrabe formula to the case of spread options. It is easy to understand that Π is increasing in μ_1 , and decreasing in μ_2 and K . For exchange options, from the Margrabe formula, it is well known that the option price is convex in S_1 and S_2 . Proposition 7 generalizes this result to spread options by pointing out that Π is convex in e^{μ_1}, e^{μ_2} , and K . Statement 4 is a new and interesting result.

The fact that Π decreases with ρ agrees with intuition and generalizes the result for exchange options. Intuitively, for larger values of ρ , when the value of asset one increases, the value of asset two tends to increase more, resulting in a smaller value for the spread option.

Notice that under the general jointly normal returns setup, we cannot say too much on the greeks because we have not specified the functional forms for μ_i 's and v_i 's. For example, μ_i 's could be complicated functions of the volatilities, time to maturity, and so forth. Thus, we do not examine the sensitivities of Π with respect to v_i 's in Proposition 7. However, if we assume geometric Brownian motions for the $S_i(t)$'s, the functional forms of μ_i 's and v_i 's are determined and we can obtain more results. For simplicity, we assume that the dividend yields $q_1 = q_2 = 0$ throughout this article. This does not incur any loss of generality since nonzero q_1 and q_2 can be absorbed into a redefinition of S_1 and S_2 . Notice first that in this special case, Equation (5) holds and we have

$$\Pi = S_1 I_1 - S_2 I_2 - Ke^{-rT} I_3 \quad (50)$$

The following proposition expresses the greeks in terms of one-dimensional integrations and points out the signs of the greeks. These one-dimensional integrals are potentially very useful if one computes the greeks using Monte Carlo simulation. Proposition 8 also characterizes the signs of various greeks for the spread option under the geometric Brownian motions assumption.

Proposition 8. Let $K \geq 0$ and assume that $S_i(t)$'s follow geometric Brownian motions with correlation ρ .

1. The sensitivities of the spread option price to initial stock prices S_i and spread K are given by

$$\Delta_1 \equiv \frac{\partial \Pi}{\partial S_1} = I_1 = \int_{-\infty}^{\infty} N(A(\gamma + \rho v_1) + \sqrt{1 - \rho^2} v_1) n(\gamma) d\gamma \quad (51)$$

$$\Delta_2 \equiv \frac{\partial \Pi}{\partial S_2} = -I_2 = -\int_{-\infty}^{\infty} N(A(\gamma + v_2)) n(\gamma) d\gamma \quad (52)$$

$$\kappa \equiv \frac{\partial \Pi}{\partial K} = -e^{-rT} I_3 = -e^{-rT} \int_{-\infty}^{\infty} N(A(\gamma)) n(\gamma) d\gamma \quad (53)$$

Furthermore, we have

$$0 < \Delta_1 < 1, \quad -1 < \Delta_2 < 0, \quad \text{and} \quad -e^{-rT} < \kappa < 0 \quad (54)$$

2. The signs of the gammas are given by

$$\Gamma_{11} \equiv \frac{\partial^2 \Pi}{\partial S_1^2} > 0, \quad \Gamma_{12} \equiv \frac{\partial^2 \Pi}{\partial S_1 \partial S_2} < 0, \quad \Gamma_{22} \equiv \frac{\partial^2 \Pi}{\partial S_2^2} > 0 \quad (55)$$

3. The spread option price is a decreasing function of ρ , and we have

$$\frac{\partial \Pi}{\partial \rho} = S_1 S_2 \sigma_1 \sigma_2 T \Gamma_{12} < 0 \quad (56)$$

4. The vegas of the spread option price are given by

$$\begin{aligned} \mathcal{V}_1 \equiv \frac{\partial \Pi}{\partial \sigma_1} &= e^{-rT} \sqrt{1 - \rho^2} \sqrt{T} \int_{-\infty}^{\infty} n(A(\gamma)) \\ &\quad \times (K + e^{\mu_2 + v_2 \gamma}) n(\gamma) d\gamma \\ &\quad + \rho S_1 \sqrt{T} \int_{-\infty}^{\infty} \gamma N(A(\gamma + \rho v_1)) \\ &\quad + \sqrt{1 - \rho^2} v_1) n(\gamma) d\gamma \end{aligned} \quad (57)$$

$$\mathcal{V}_2 \equiv \frac{\partial \Pi}{\partial \sigma_2} = -S_2 \sqrt{T} \int_{-\infty}^{\infty} \gamma N(A(\gamma + v_2)) n(\gamma) d\gamma \quad (58)$$

Furthermore, if $\rho \leq 0$, we have $\mathcal{V}_1 \geq 0$ and $\mathcal{V}_2 \geq 0$. If $\sigma_2/\sigma_1 \leq \rho \leq 1$, we have $\mathcal{V}_1 \geq 0$ and $\mathcal{V}_2 \leq 0$.

While the signs of the greeks in the first three statements are well-known, Statement 4 gives us some insights on the vegas. It is interesting to note that unlike the Black-Scholes formula, in which both call and put option prices are increasing functions of the volatility, the spread option price can be either increasing or decreasing in the two volatilities. For exchange options, from the Margrabe formula, it can be easily shown the sign of $\mathcal{V}_1$ is the same as that of $v_1 - v_2 \rho$, and the sign of $\mathcal{V}_2$ is the same as that of $v_2 - v_1 \rho$. Proposition 8 is consistent with these results and partially generalizes them to spread options. It is only a partial generalization because the signs of the vegas are still not known when $0 \leq \rho \leq \sigma_2/\sigma_1$. The difficulty lies in Proposition 3, which points out that for values of ρ in this range, the conditional moneyness $A(\gamma)$ is no longer monotone in γ . Besides expressing the greeks in simple

one-dimensional integrals, Proposition 8 is useful for other purposes. In particular, bounds on the vegas can be established with the help of Proposition 8. For example, by Equation (58), it is easy to show that when $K \geq 0$, we have $|\mathcal{V}_2| \leq S_2 \sqrt{T} / \sqrt{2\pi}$.

Finally, in Propositions 7 and 8, we have assumed that $K \geq 0$. For the case $K \leq 0$, we need to use Equation (12) to reduce to the $K \geq 0$ case. The modifications are as follows. For Proposition 7, all statements are still true when $K \leq 0$ except Statement 4. Most expressions in Proposition 8 can no longer be directly used when $K \leq 0$. The correct expressions in this case can be worked out easily with the help of Equation (12). Because this process is not completely trivial, we use κ as an illustration. Writing $\Pi = \Pi(S_1, S_2, \sigma_1, \sigma_2, \rho, K)$, we have by Equation (12) that

$$\Pi(S_1, S_2, \sigma_1, \sigma_2, \rho, K) = S_1 - S_2 - Ke^{-rT} + \Pi(S_2, S_1, \sigma_2, \sigma_1, \rho, -K) \quad (59)$$

We will call Equation (59) the generalized put-call parity for spread options. Indeed, if $S_2 = 0$ and $\sigma_2 = 0$, then this relation reduces to the put-call parity for plain-vanilla European options in the Black-Scholes formula. If $K = 0$, then Equation (59) reduces to the put-call parity for exchange options in the Margrabe formula. Now let $K \leq 0$ in Equation (59). To compute κ , we have by the generalized put-call parity and Proposition 8 that

$$\begin{aligned} \kappa &= \frac{\partial}{\partial K} \Pi(S_1, S_2, \sigma_1, \sigma_2, \rho, K) \\ &= -e^{-rT} + \frac{\partial}{\partial K} \Pi(S_2, S_1, \sigma_2, \sigma_1, \rho, -K) \\ &= -e^{-rT} + e^{-rT} \int_{-\infty}^{\infty} N(A^\dagger(x)) n(x) dx \\ &= -e^{-rT} \int_{-\infty}^{\infty} N(-A^\dagger(x)) n(x) dx \end{aligned} \quad (60)$$

where $A^\dagger(x)$ is the mirror image of $A(y)$ given as

$$A^\dagger(x) = \frac{\rho x - \underline{y}(x)}{\sqrt{1 - \rho^2}} \quad (61)$$

with

$$\underline{y}(x) = \frac{\log(e^{v_1 x + \mu_1} + |K|) - \mu_2}{v_2} \quad (62)$$

Equations (53) and (60) seem to give two different expressions for κ when $K = 0$. They actually lead to exactly the same result if we evaluate them using Proposition 2. The reason that the function $A^\dagger(x)$ now gets involved is that by using the put-call parity, the roles of asset one and asset two get switched. Other greeks can be modified similarly when $K \leq 0$ and are omitted here. It is worthwhile pointing out that when $K \leq 0$, the expression for $\mathcal{V}_1$ now involves only one integral while the expression for $\mathcal{V}_2$ now involves two integrals. Also, Equations (54), (55), and (56) in Proposition 8 are still correct. The last sentence in Statement 4 when $K \leq 0$ should be changed to: If $\sigma_1/\sigma_2 \leq \rho \leq 1$, we have $\mathcal{V}_1 \leq 0$ and $\mathcal{V}_2 \geq 0$. Again, it is easy to check that this new statement is consistent with the known behavior of $\mathcal{V}_1$ and $\mathcal{V}_2$ for the exchange option case.

Approximations for the Greeks in the GBMs Case

Proposition 6 gives the approximate price for the spread option in closed form. However, in practice, one often needs to compute the greeks efficiently. This can be done by differentiating the approximate price Π in Proposition 6. However, the algebra is quite complicated. Instead, we design separate approximations for the greeks in the special geometric Brownian motions case in Proposition 9. This proposition allows for fast and accurate computation of the greeks for the spread option in this important special case. The proof relies on Proposition 8 and is again based heavily on the boundary approximation. It is given in the appendix.

Proposition 9. Let $K \geq 0$ and assume that $S_i(t)$'s follow geometric Brownian motions with correlation ρ .

1. The deltas and kappa can be approximated as

$$\Delta_1 \approx J_0(C^1, D^1) + J_1(C^1, D^1)\epsilon + \frac{1}{2} J_2(C^1, D^1)\epsilon^2 \quad (63)$$

$$\Delta_2 \approx -J_0(C^2, D^2) - J_1(C^2, D^2)\epsilon - \frac{1}{2} J_2(C^2, D^2)\epsilon^2 \quad (64)$$

$$\kappa \approx -e^{-rT} (J_0(C^3, D^3) + J_1(C^3, D^3)\epsilon + \frac{1}{2} J_2(C^3, D^3)\epsilon^2) \quad (65)$$

2. The gammas can be approximated as

$$\Gamma_{11} \approx \Phi(C^1, D^1, C_{S_1}^1, D_{S_1}^1, \epsilon_{S_1}) \quad (66)$$

$$\Gamma_{12} \approx \Phi(C^1, D^1, C_{S_2}^1, D_{S_2}^1, \epsilon_{S_2}) \quad (67)$$

$$\Gamma_{22} \approx -\Phi(C^2, D^2, C_{S_2}^2, D_{S_2}^2, \epsilon_{S_2}) \quad (68)$$

where the function Φ is defined by

$$\Phi(x, y, u, v, w) = n \left(\frac{x}{\sqrt{1+y^2}} \right) \times \frac{(1+y^2)^2 u - x(y+y^3)v + (1+(1+x^2)y^2)w}{(1+y^2)^{5/2}} \quad (69)$$

and the partial derivatives are given by

$$C_{S_1}^1 = \frac{1}{v_1 S_1 \sqrt{1-\rho^2}}, \quad D_{S_1}^1 = 0, \quad \epsilon_{S_1} = 0 \quad (70)$$

and

$$C_{S_2}^1 = \frac{\epsilon}{S_2} \left(\frac{2(R+K)}{v_2^2 K} + \frac{2\rho v_1}{v_2} - \left(1 - \frac{2R}{R+K} \right) \rho^2 v_1^2 \right) \quad (71)$$

$$D_{S_2}^1 = \frac{\epsilon}{S_2} \left(\frac{2}{v_2} + 2\rho v_1 \left(1 - \frac{2R}{R+K} \right) \right) \quad (72)$$

$$C_{S_2}^2 = \frac{\epsilon}{S_2} \left(\frac{2(R+K)}{v_2^2 K} + 2 + \left(1 - \frac{2R}{R+K} \right) v_2^2 \right) \quad (73)$$

$$D_{S_2}^2 = \frac{\epsilon}{S_2} \left(\frac{2}{v_2} + 2v_2 \left(1 - \frac{2R}{R+K} \right) \right) \quad (74)$$

$$\epsilon_{S_2} = \frac{\epsilon}{S_2} \left(1 - \frac{2R}{R+K} \right) \quad (75)$$

3. The approximation for vega with respect to σ_1 is given by

$$\begin{aligned} \mathcal{V}_1 \approx & S_1 \Phi(C^1, D^1, C_{\sigma_1}^1, D_{\sigma_1}^1, \epsilon_{\sigma_1}) \\ & - S_2 \Phi(C^2, D^2, C_{\sigma_1}^2, D_{\sigma_1}^2, \epsilon_{\sigma_1}) \\ & - Ke^{-rT} \Phi(C^3, D^3, C_{\sigma_1}^3, D_{\sigma_1}^3, \epsilon_{\sigma_1}) \end{aligned} \quad (76)$$

where

$$C_{\sigma_1}^1 = -\frac{C^3}{\sigma_1} + \epsilon \rho^2 \sigma_1 T, \quad D_{\sigma_1}^1 = D_{\sigma_1}^3 \quad (77)$$

$$C_{\sigma_1}^2 = C_{\sigma_1}^3 + D_{\sigma_1}^3 v_2 + \epsilon_{\sigma_1} v_2^2, \quad D_{\sigma_1}^2 = D_{\sigma_1}^3 + 2v_2 \epsilon_{\sigma_1} \quad (78)$$

$$C_{\sigma_1}^3 = -\frac{\sqrt{T}}{\sqrt{1-\rho^2}} - \frac{C^3}{\sigma_1}, \quad D_{\sigma_1}^3 = -\frac{\sigma_2}{\sigma_1^2 \sqrt{1-\rho^2}} \frac{R}{R+K} \quad (79)$$

$$\epsilon_{\sigma_1} = -\frac{\epsilon}{\sigma_1} \quad (80)$$

The approximation for vega with respect to σ_2 is given by

$$\mathcal{V}_2 \approx S_2 \sqrt{T} \Upsilon(C^2, D^2, \epsilon) \quad (81)$$

where the function Υ is given by

$$\Upsilon(u, v, w) = n \left(\frac{u}{\sqrt{1+v^2}} \right) v \frac{(1+v^2)^3 + uw(3+(3+u^2)v^2)}{(1+v^2)^{7/2}} \quad (82)$$

Although we do not discuss them here, higher-order greeks such as the vommas and vannas (defined as the sensitivities of option vegas with respect to volatilities and spot prices, respectively) can be easily approximated as well using our method. Our approximations for the greeks involve only simple arithmetic and are very fast to compute. Also, they are derived using the same idea as for the price approximation and are thus extremely accurate, often giving relative errors well within 0.1%.

COMPARISON OF ACCURACY AND SPREAD WITH EXISTING METHODS

Existing Approximation Methods

Various methods have been proposed for pricing spread options and can be broadly divided into two groups. The first group follows the numerical approach and includes Pearson [1995]'s one-dimensional numerical integration method, fast Fourier transform methods as in Dempster and Hong [2000], numerical solutions to partial differential equations, and Monte Carlo methods. The second group tries to analytically approximate the spread

option price and greeks, and includes Bachelier approximation, Bachelier approximation with Gram-Charlier adjustment, Kirk's approximation, Carmona-Durrleman approximation, and others. Eydeland and Wolyniec [2003] also present a closed-form approximation for spread options which is not very accurate. In this section, we will compare the speed and accuracy of our method with alternative methods. We first discuss some of the alternative methods, namely, the Bachelier approximation, Bachelier approximation with Gram-Charlier adjustment, Kirk's approximation, and the Carmona-Durrleman approximation. In the next subsection, we then perform a comparison of speed and accuracy of existing methods with our approximation.

In the Bachelier approximation, the quantity $B \equiv e^{-rT}(S_1(T) - S_2(T))$ is approximated as a normal random variable, with mean μ_B and standard deviation σ_B given by

$$\begin{aligned}\mu_B &= S_1 - S_2 \\ \sigma_B &= \sqrt{S_1^2(\theta_1 - 1) - 2S_1S_2(\theta_2 - 1) + S_2^2(\theta_3 - 1)}\end{aligned}\quad (83)$$

where

$$\theta_1 = e^{\sigma_1^2 T}, \quad \theta_2 = e^{\sigma_2^2 T}, \quad \theta_3 = e^{\rho\sigma_1\sigma_2 T} \quad (84)$$

The spread option price is then approximated using the Bachelier formula for a plain-vanilla European call option (Bachelier [1900]) as

$$\Pi_B = (\mu_B - Ke^{-rT})N(d_B) + \sigma_B n(d_B) \quad (85)$$

where

$$d_B = \frac{\mu_B - Ke^{-rT}}{\sigma_B} \quad (86)$$

The assumption that B is normal is very crude. Mbafeno [1997] proposes a skewness and kurtosis adjustment based on the Gram-Charlier approximation in Jarrow and Rudd [1982]. However, the formula in Mbafeno contains a typo. Also, he does not give explicit expressions for the skewness and kurtosis under the GBMs setup. Thus, we present the result for the Gram-Charlier approximation in the following proposition, which takes into account the nonzero skewness γ_B and excess kurtosis κ_B explicitly. The proof is given in the appendix.

Proposition 10. Suppose that the asset prices $S_1(t)$ and $S_2(t)$ follow geometric Brownian motions with initial prices S_1 and S_2 . Assume that $B \equiv e^{-rT}(S_1(T) - S_2(T))$ follows a Gram-Charlier approximation which includes skewness and kurtosis adjustments. Then, the approximate spread option price is

$$\begin{aligned}\Pi_{GC} &= (\mu_B - Ke^{-rT})N(d_B) \\ &+ \sigma_B n(d_B) \left(1 - \frac{d_B}{6} \gamma_B + \frac{d_B^2 - 1}{24} \kappa_B \right)\end{aligned}\quad (87)$$

with skewness γ_B and excess kurtosis κ_B given by

$$\gamma_B = \frac{\mathbb{E}^Q(B - \mu_B)^3}{\sigma_B^3}, \quad \kappa_B = \frac{\mathbb{E}^Q(B - \mu_B)^4}{\sigma_B^4} - 3 \quad (88)$$

where

$$\begin{aligned}\mathbb{E}^Q(B - \mu_B)^3 &= (\theta_1 - 1)^2(\theta_1 + 2)S_1^3 \\ &- (\theta_2 - 1)^2(\theta_2 + 2)S_2^3 \\ &- 3(\theta_3 - 1)(\theta_1 + \theta_1\theta_3 - 2)S_1^2S_2 \\ &+ 3(\theta_3 - 1)(\theta_2 + \theta_2\theta_3 - 2)S_1S_2^2\end{aligned}\quad (89)$$

and

$$\begin{aligned}\mathbb{E}^Q(B - \mu_B)^4 &= (6\theta_1 - 4\theta_1^3 + \theta_1^6 - 3)S_1^4 \\ &+ (6\theta_2 - 4\theta_2^3 + \theta_2^6 - 3)S_2^4 \\ &+ 4(3 - 3\theta_1 + \theta_1^3 - 3\theta_3 - \theta_1^3\theta_3 \\ &+ 3\theta_1\theta_3^2)S_1^3S_2 + 4(3 - 3\theta_2 \\ &+ \theta_2^3 - 3\theta_3 - \theta_2^3\theta_3 + 3\theta_2\theta_3^2)S_1S_2^3 \\ &+ 6(\theta_1 + \theta_2 + 3\theta_3 - 2\theta_2\theta_3^2 \\ &- 2\theta_1\theta_3^2 + \theta_1\theta_2\theta_3^4 - 3)S_1^2S_2^2\end{aligned}\quad (90)$$

Except for the fact that the expressions for the skewness γ_B and excess kurtosis κ_B are complicated, Equation (87) in Proposition 10 is a standard result. For example, it is well known that the skewness adjustment is proportional to the moneyness d_B , while the excess kurtosis adjustment is quadratic in d_B . Unlike the Bachelier approximation, in the Gram-Charlier approximation the spread option price can become negative, especially when the skewness and excess kurtosis are large. Whenever this happens, we will assume the approximated spread option

price is zero. Also, as many researchers have noticed, it is possible for the Gram-Charlier approximation to perform worse than the Bachelier approximation if the deviation of B from normality is significant. We found that this is indeed the case.

In Kirk's approximation, $Z(T) = S_2(T) + K$ is considered as a lognormal random variable. The initial value of Z is given by $S_2 + Ke^{-rT}$. The volatility for $Z(T)$ is value weighted using the relative proportions of the stock and bond positions and is given by

$$\sigma_Z = \frac{S_2}{S_2 + Ke^{-rT}} \sigma_2 \quad (91)$$

The correlation of $S_1(T)$ and $Z(T)$ is still ρ . Kirk then uses the Margrabe formula to price the spread option (now viewed as a simple exchange option), as follows:

$$\begin{aligned} \Pi_K &= e^{-rT} \mathbb{E}^Q [S_1(T) - S_2(T) - K]^+ \\ &\approx e^{-rT} \mathbb{E}^Q [S_1(T) - Z(T)]^+ \end{aligned} \quad (92)$$

$$\begin{aligned} &= S_1 N(d_K + \sigma_K \sqrt{T}/2) \\ &\quad - (S_2 + Ke^{-rT}) N(d_K - \sigma_K \sqrt{T}/2) \end{aligned} \quad (93)$$

where

$$\begin{aligned} d_K &= \frac{\log(S_1/(S_2 + Ke^{-rT}))}{\sigma_K \sqrt{T}} \\ \sigma_K &= \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 \frac{S_2}{S_2 + Ke^{-rT}} + \sigma_2^2 \left(\frac{S_2}{S_2 + Ke^{-rT}} \right)^2} \end{aligned} \quad (94)$$

Although not obvious, Kirk's approximation can be thought of as a rough version of our approximation, in which the exercise boundary is approximated using three straight lines very close to it. Details are available upon request from the authors. Kirk's approximation is relatively accurate given its simple form, with relative price errors usually within a few percentages.

More recently, a new approximation formula was proposed in Carmona and Durrleman [2003a, 2003b]. In this method, one first solves the following equation to get an optimal θ^* :

$$\begin{aligned} &\frac{1}{\sigma_2 \sqrt{T} \cos \theta} \log \left(-\frac{\sigma_1 Ke^{-rT} \sin(\theta + \phi)}{S_2(\sigma_1 \sin(\theta + \phi) - \sigma_2 \sin \theta)} \right) \\ &+ \frac{\sigma_2 \sqrt{T} \cos \theta}{2} = \frac{1}{\sigma_1 \sqrt{T} \cos(\theta + \phi)} \\ &\times \log \left(-\frac{\sigma_2 Ke^{-rT} \sin \theta}{S_1(\sigma_1 \sin(\theta + \phi) - \sigma_2 \sin \theta)} \right) \\ &+ \frac{\sigma_1 \sqrt{T} \cos(\theta + \phi)}{2} \end{aligned} \quad (95)$$

where $\phi = \arccos(\rho)$. Now, let

$$\begin{aligned} d^* &= \frac{1}{\sigma \sqrt{T} \cos(\theta^* - \psi)} \log \left(\frac{S_1 \sigma_1 \sin(\theta^* + \phi)}{S_2 \sigma_2 \sin \theta^*} \right) \\ &+ \frac{1}{2} (\sigma_1 \cos(\theta^* + \phi) + \sigma_2 \cos \theta^*) \sqrt{T} \end{aligned} \quad (96)$$

where

$$\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}, \quad \psi = \arccos\left(\frac{\sigma_2 - \rho\sigma_1}{\sigma}\right) \quad (97)$$

In Equations (95), (96), and (97), we have corrected a few typos in Carmona and Durrleman [2003a, 2003b], ranging from missing minus signs and switching of cos and sin functions. Carmona and Durrleman show that the spread option price is always larger than the following lower bound

$$\begin{aligned} \Pi_{CD} &= S_1 N(d^* + \sigma_1 \sqrt{T} \cos(\theta^* + \phi)) \\ &\quad - S_2 N(d^* + \sigma_2 \sqrt{T} \cos \theta^*) - Ke^{-rT} N(d^*) \end{aligned} \quad (98)$$

In addition, they show that the lower bound is very tight and can be used as a good approximation for the actual price. Our numerical analysis confirms their claim. It is also noteworthy to point out that Carmona and Durrleman [2006] have recently extended their results to multiple assets cases.

Our method improves upon existing approximations in many aspects. First, previous approximations focused on the spread option as a single identity and developed formulas for the whole spread option, while in our method

we approximate the three individual terms separately. Each individual term in our method is approximated extremely accurately. This contrasts with many previous methods. For example, the accuracy of Kirk's method relies on a delicate cancellation of errors among the three terms in Equation (93). The individual terms in Kirk's approximation are very inaccurate, often having relative pricing errors reaching 200%. Approximating individual terms also allows us to compute digital-type spread options, such as asset-or-nothing and cash-or-nothing spread options. Although we do not develop formulas in this article, our boundary approximation method also allows us to compute more exotic spread options, for example, options with quadratic payoff $S_1(T)^2 1_{S_1(T) \geq S_2(T) + K}$.

Second, and perhaps the most significant advantage of our method, is that we provide extremely accurate and fast approximations for the greeks. For example, deltas in our method often have relative errors in the order of 0.1%.

Third, our method achieves a good balance between speed and accuracy. It is more accurate than the Bachelier approximation and Kirk's method. While the Carmona-Durrleman approximation is more accurate than Kirk's approximation, it is still about one or two orders of magnitude less accurate than our method. Furthermore, like most analytical methods, our method is very fast. The computing times of Bachelier approximation, Kirk's approximation, and ours are roughly of the same order of magnitude. However, the Carmona-Durrleman approximation is slower than most analytical approximations since one needs to solve a complicated trigonometrical equation to get the optimal θ^* . This is time consuming even if one utilizes fast algorithms such as the Newton-Raphson method.

Finally, our approximation is very straightforward to implement and is very robust. We find that it is not trivial to find θ^* for Equation (95) in the Carmona-Durrleman approximation because sometimes the objective function is not smooth in θ^* and there is little guidance on the initial value for θ^* . Thus, it is hard to use the Newton-Raphson method. The lack of guidance on the initial value for θ^* also prevents us from systematically comparing the performance of the Carmona-Durrleman approximation with ours. The important case $K = 0$ also poses some problems in Carmona-Durrleman because Equation (95) breaks down. Another issue is that whenever θ^* is the solution to Equation (95), so is $\theta^* + \pi$. Only one of the two solutions will correspond to the lower bound. Lacking any guidance on how to select θ^* and $\theta^* + \pi$, one would

have to compute option prices for both solutions and then compare. Thus, the Carmona-Durrleman approximation is not as straightforward as our method.

Comparison of Accuracy and Speed with Existing Methods

For definiteness, we shall assume the special GBMs case and compare our method with Bachelier approximation, Bachelier approximation with Gram-Charlier adjustment, Kirk's approximation, and the one-dimensional numerical integration method. The one-dimensional numerical integration is based on Proposition 1.

Notice that the spread option price is a function of many variables, namely, $S_1, S_2, T, r, q_1, q_2, \sigma_1, \sigma_2, K$, and ρ . We will set $q_1 = q_2 = 0$ as nonzero q_i 's can be absorbed into S_1 and S_2 . Also, since σ_i 's and $\sqrt{T}$ always go together, we set $T = 1$. Because the spread option price is homogeneous of degree 1 in S_1, S_2 , and K , we fix $S_1 = 100$. The interest rate r usually does not play a major role, so we fix it at 5%. In order to perform a systematic accuracy comparison, we vary $S_2/S_1, K/S_1, \sigma_1, \sigma_2$, and ρ . We set the range of S_2/S_1 to be $[0.7, 1.2]$, the range of K/S_1 to be $[0, 0.4]$, the range of σ_1 and σ_2 to be both $[0.1, 0.8]$, and the range of ρ to be $[-0.75, 0.75]$. In addition, we impose the restriction $S_1 - S_2 - Ke^{-rT} \geq -30$ so that we exclude deeply out-of-the-money options. Parameters are uniformly generated from these ranges. We generate altogether 123,783 individual options. Generating options uniformly will tend to weigh extreme scenarios more heavily and thus tend to exaggerate an approximation method's pricing errors.

We then compute the spread option prices for those 123,783 options using Bachelier approximation, Bachelier approximation with Gram-Charlier adjustment, Kirk's approximation and the approximation in this article. We also compute the prices using one-dimensional numerical integration with the recursive adaptive Simpson quadrature method. Two error tolerance levels are used: 10^{-6} and 10^{-8} . The prices computed from the latter tolerance level are then used as the actual option prices. All methods are implemented in MATLAB 7.0 on an IBM computer with 1.60 GHz Intel Pentium CPU and 768M memory and some other computers. The computing times of different methods do not seem to vary with the specification of the computer on which they are run. For each option and for each approximation method, we first compute the pricing error $\Delta\Pi = \Pi_{\text{Approximation}} - \Pi_{\text{Actual}}$

and then compute the relative pricing error zero $\Delta\Pi/\Pi_{\text{Actual}}$. Because the actual prices vary greatly from zero to around 65, we focus on the relative pricing errors.

The results are reported in Exhibit 2. For the numerical methods, we do not list their accuracy since it has been specified. Looking at $\Delta\Pi/\Pi_{\text{Actual}}$, Bachelier approximation gives quite inaccurate spread option prices. Also, Bachelier approximation with Gram-Charlier adjustment seems to perform worse than no adjustment. Detailed analysis shows that the deviations of B from normality are often quite significant, with mean excess kurtosis for those 123,783 options being around 10. In addition, Bachelier approximation and Kirk's approximation seem to be biased upward. Most of the time, they give a price larger than the actual price. The bias in our method seems to be extremely small. Also, we see that our method is extremely accurate, with a median $|\Delta\Pi/\Pi_{\text{Actual}}|$ of 3.8×10^{-6} . In terms of speed, Bachelier approximation and Kirk's approximation are the fastest while our method is not too far behind. In particular, we see that our method is capable of computing one million spread option prices within 10 seconds. The numerical integration methods

are much slower, with computing times roughly two orders of magnitude larger than those of the analytical methods.

CONCLUSION

In this article, we first obtain lower and upper bounds for digital spread options by analyzing the exercise boundary. We then develop a new closed-form approximation for pricing spread options. Numerical analysis demonstrates that our method is more accurate than existing analytical approximations. It is also extremely fast, capable of computing one million spread options within 10 seconds. Thus, our method enables the accurate pricing of a bulk volume of spread options with different specifications in real time which offers traders a potential edge in financial markets. The availability of a closed-form formula for spread options also helps us design and analyze real and financial contracts with embedded spread-option-like features.

We also derive closed-form approximations for the greeks of spread options. The closed-form approximations

EXHIBIT 2

Performance Comparison of Various Methods in Computing Spread Option Prices

This exhibit reports the speed and accuracy of Bachelier approximation, Bachelier approximation with Gram-Charlier adjustment, Kirk's approximation, the approximation in this article and one-dimensional numerical integration. Total number of options is 123,783. NI stands for numerical integration.

	Methods					
	Ours	Bachelier	Gram-Charlier	Kirk	NI- 10^{-6}	NI- 10^{-8}
$\Delta\Pi/\Pi_{\text{Actual}}$						
max	0.027	2.558	0.411	0.460		
min	-0.030	-0.621	-1.000	-0.018		
mean	-5.1×10^{-5}	0.139	-0.337	0.007		
median	-1.3×10^{-7}	0.117	-0.209	9.1×10^{-4}		
std. deviation	7.6×10^{-4}	0.142	0.338	0.017		
$ \Delta\Pi/\Pi_{\text{Actual}} $						
max	0.030	2.558	1.000	0.460		
min	$\sim 10^{-15}$	2.5×10^{-6}	1.0×10^{-7}	$\sim 10^{-15}$		
mean	1.7×10^{-4}	0.147	0.338	0.008		
median	3.8×10^{-6}	0.119	0.209	0.002		
std. deviation	7.4×10^{-4}	0.133	0.337	0.016		
Time (seconds)	1.02	0.22	1.12	0.25	363.57	891.60

of greeks serve as valuable tools in financial applications. For instance, they can be used for calculating value-at-risk for a portfolio containing spread options. As byproducts, we analyze the price sensitivities of spread options. In particular, we point out the signs of vegas when the correlation is negative and when the correlation is positive and large. The analysis of the price sensitivities leads to improved understanding of the price behavior of spread options and is useful in formulating effective dynamic hedging strategies.

There are a few directions that one can take to extend and improve the results in this article. First, Deng, Li, and Zhou [2008] show that one can extend the boundary approximation method for two-asset spread options to the multiple-asset case. The extended approximation is shown to be extremely fast and accurate. Second, our approximation is useful even if one wants to incorporate jumps in the price processes of the assets. Carmona and Durrleman [2003b] discuss in detail how the approximated price can be used in such cases. Third, Li [2008] studies the correction to the exchange option price when the asset returns deviate from the jointly normal distribution using a bivariate Gram-Charlier approximation. Li's method can be extended to the spread option case if we couple it with our exercise boundary approximation. This will allow the pricing of spread options under arbitrary distributions that are close to jointly normal. Finally, it is possible to further improve the accuracy of our method. For example, in our spread option approximation, we have set the expansion point to be $\gamma_0 = 0$; that is, we have expanded around the point where the future log price of asset two equals its mean. This is a degree of freedom which we have not utilized. Preliminary numerical analysis suggests that the optimal expansion point γ_0 seems to depend on the correlation coefficient ρ . Thus, relaxing γ_0 may make our approximation even more accurate, especially when $|\rho|$ is large. We leave these issues to future research.

APPENDIX

Proof of Proposition 1

The random variables X and Y are jointly normally distributed. Denote this density $n(x, y; \rho)$. The conditional density of X given $Y = y$ is $n(x; \rho y, 1 - \rho^2)$ (i.e., a normal density with mean ρy and variance $1 - \rho^2$). Thus, we can now compute the price of the digital spread option as follows:

$$\begin{aligned}\Pi^D &= e^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 1_{x \geq \underline{x}(y)} n(x, y; \rho) dx dy \\ &= e^{-rT} \int_{-\infty}^{\infty} n(y) dy \int_{\underline{x}}^{\infty} n(x; \rho y, 1 - \rho^2) dx\end{aligned}\quad (99)$$

$$= e^{-rT} \int_{-\infty}^{\infty} n(y) N(A(y)) dy \quad (100)$$

Similarly, the spread option price can be computed as

$$\Pi = e^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{V_1 x + \mu_1} - e^{V_2 y + \mu_2} - K)^+ n(x, y; \rho) dx dy \quad (101)$$

$$= e^{-rT} \int_{-\infty}^{\infty} n(y) dy \int_{\underline{x}}^{\infty} (e^{V_1 x + \mu_1} - e^{V_2 y + \mu_2} - K) n(x; \rho y, 1 - \rho^2) dx \quad (102)$$

By virtue of the identity

$$\int_{x_0}^{\infty} e^{bx} n(x; \mu, \sigma^2) dx = e^{\mu + \sigma^2 x^2 / 2} N\left(\frac{\mu - x_0}{\sigma} + \sigma t\right) \quad (103)$$

the inner integral can be performed to yield

$$\Pi = e^{-rT} \int_{-\infty}^{\infty} n(y) e^{\rho y V_1 + (1 - \rho^2) V_1^2 / 2 + \mu_1} N(A(y) + \sqrt{1 - \rho^2} V_1) dy \quad (104)$$

$$- e^{-rT} \int_{-\infty}^{\infty} n(y) e^{V_2 y + \mu_2} N(A(y)) dy \quad (105)$$

$$- e^{-rT} \int_{-\infty}^{\infty} n(y) K N(A(y)) dy \quad (106)$$

It is very useful to perform a change of variable to the above equation to obtain

$$\Pi = e^{V_1^2 / 2 + \mu_1 - rT} \int_{-\infty}^{\infty} n(y) N(A(y + \rho V_1) + \sqrt{1 - \rho^2} V_1) dy \quad (107)$$

$$- e^{V_2^2 / 2 + \mu_2 - rT} \int_{-\infty}^{\infty} n(y) N(A(y + V_2)) dy \quad (108)$$

$$- K e^{-rT} \int_{-\infty}^{\infty} n(y) N(A(y)) dy \quad (109)$$

$$\equiv e^{V_1^2 / 2 + \mu_1 - rT} I_1 - e^{V_2^2 / 2 + \mu_2 - rT} I_2 - K e^{-rT} I_3 \quad (110)$$

Mathematically, this is similar to a change of numeraire.

Proof of Proposition 2

Define

$$F(a) \equiv \int_{-\infty}^{\infty} N(a + y) n(y) dy, \quad G(b) \equiv \int_{-\infty}^{\infty} N(a + by) n(y) dy \quad (111)$$

Notice that

$$F(0) = \int_{-\infty}^{\infty} N(y)n(y)dy = \int_{-\infty}^{\infty} N(y)dN(y) = \frac{1}{2} \quad (112)$$

and

$$\begin{aligned} F'(a) &= \int_{-\infty}^{\infty} n(a+y)n(y)dy = \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{a^2}{4}\right) \\ &= \frac{1}{\sqrt{2}} n\left(\frac{a}{\sqrt{2}}\right) \end{aligned} \quad (113)$$

Thus,

$$F(a) = F(0) + \int_0^a F'(a)da = N\left(\frac{a}{\sqrt{2}}\right) \quad (114)$$

We can compute

$$\begin{aligned} G(1) &= F(a) = N\left(\frac{a}{\sqrt{2}}\right), \text{ and} \\ G'(b) &= -\frac{ab}{(1+b^2)^{3/2}} n\left(\frac{a}{\sqrt{1+b^2}}\right) \end{aligned} \quad (115)$$

Thus,

$$\begin{aligned} \int_{-\infty}^{\infty} N(a+by)n(y)dy &= G(b) \\ &= G(1) + \int_1^b G'(b)db = N\left(\frac{a}{\sqrt{1+b^2}}\right) \end{aligned} \quad (116)$$

The first integral in the lemma now follows immediately:

$$\begin{aligned} \int_{-\infty}^{\infty} N(a+by)n(y;\mu,\sigma^2)dy &= \int_{-\infty}^{\infty} N(a+b\mu+b\sigma z)n(z)dz \\ &= N\left(\frac{a+b\mu}{\sqrt{1+b^2\sigma^2}}\right) \end{aligned} \quad (117)$$

To prove the Margrabe formula, notice that when $K = 0$, we have

$$\begin{aligned} \underline{x}(y) &= \frac{v_2 y + \mu_2 - \mu_1}{v_1}, \quad A(y) = \frac{\mu_1 - \mu_2}{v_1 \sqrt{1 - \rho^2}} + \frac{(\rho v_1 - v_2)}{v_1 \sqrt{1 - \rho^2}} y \\ &\equiv a + by \end{aligned} \quad (118)$$

By Proposition 1, we need to compute I_1 and I_2 . Since the conditional moneyness $A(y)$ is linear, we can use Equation (19). For example, to compute I_2 , we have

$$I_2 = \int_{-\infty}^{\infty} N(A(y+v_2))n(y)dy = \int_{-\infty}^{\infty} N(A(y))n(y; -v_2, 1)dy \quad (119)$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} N(a+by)n(y; -v_2, 1)dy \\ &= N\left(\frac{a-bv_2}{\sqrt{1+b^2}}\right) = N\left(\frac{\mu_1 - \mu_2 - (v_2^2 - \rho v_1 v_2)}{\sqrt{v_1^2 + v_2^2 - 2\rho v_1 v_2}}\right) \end{aligned} \quad (120)$$

Similarly for I_1 .

Proof of Proposition 3

This proposition follows from pure algebraic manipulation. In particular, we need to examine the first- and second-order derivatives of $\underline{x}(y)$, $A(y)$, and $N(A(y))$ with respect to y .

Proof of Proposition 4

This proposition follows immediately from Equation (26) and Proposition 1.

Proof of Proposition 5

With $B(y)$ defined in the text, we have

$$\begin{aligned} I_3 &= \int_{-\infty}^{\infty} n(y)N(A(y))dy = \int_{-\infty}^{\infty} n(y)N(B(y))dy \\ &\quad + \int_{-\infty}^{\infty} n(y)(N(A(y)) - N(B(y)))dy \end{aligned} \quad (121)$$

$$\geq \int_{-\infty}^{\infty} n(y)N(B(y))dy - \int_{-\infty}^{\infty} n(y)(N(A(y)) - N(B(y)))^+ dy \quad (122)$$

$$\begin{aligned} &= N\left(\frac{P}{\sqrt{1+Q^2}}\right) - \int_{-\infty}^{y_1} n(y)(N(A(y)) - N(B(y)))dy \\ &\quad - \int_{y_1}^{\infty} n(y)(N(A(y)) - N(B(y)))dy \end{aligned} \quad (123)$$

The upper bound is approximated by replacing $N(B(y))$ with 1. When $\rho \leq 0$ or $\rho \geq v_2/v_1$, we can use Proposition 3 to tighten up the bound.

Proof of Proposition 6

We will consider the last term for the spread option first.

$$I_3 = \int_{-\infty}^{\infty} n(y)(N(A(y)))dy \approx \int_{-\infty}^{\infty} n(y)N(C^3 + D^3 y + \epsilon y^2)dy \quad (124)$$

This integral cannot be performed. However, if the curvature ϵ is small around the expansion point $y = y_0$, then we can expand the integral around $\epsilon = 0$. Since,

$$\frac{dN(C^3 + D^3\gamma + \epsilon\gamma^2)}{d\epsilon} = n(C^3 + D^3\gamma + \epsilon\gamma^2)\gamma^2 \quad (125)$$

$$\frac{d^2N(C^3 + D^3\gamma + \epsilon\gamma^2)}{d\epsilon^2} = -(C^3 + D^3\gamma + \epsilon\gamma^2) \times n(C^3 + D^3\gamma + \epsilon\gamma^2)\gamma^4 \quad (126)$$

we have

$$\mathbf{I}_3 \approx \mathbf{J}_0 + \mathbf{J}_1\epsilon + \frac{1}{2}\mathbf{J}_2\epsilon^2 \quad (127)$$

Where

$$\mathbf{J}_0 = \int_{-\infty}^{\infty} n(\gamma)N(C^3 + D^3\gamma)d\gamma \quad (128)$$

$$\mathbf{J}_1 = \int_{-\infty}^{\infty} n(\gamma)n(C^3 + D^3\gamma)\gamma^2 d\gamma \quad (129)$$

$$\mathbf{J}_2 = -\int_{-\infty}^{\infty} n(\gamma)(C^3 + D^3\gamma) \times n(C^3 + D^3\gamma)\gamma^4 d\gamma \quad (130)$$

The $\mathbf{J}_i$'s can be computed to give the expressions in the Proposition. In particular, $\mathbf{J}_0$ can be computed using Proposition 2. The integrals $\mathbf{I}_1$ and $\mathbf{I}_2$ can be treated similarly. However, in Proposition 1, the expansion points for $\mathbf{I}_1$ and $\mathbf{I}_2$ are chosen to be $\gamma_0 - \rho v_1$ and $\gamma_0 - v_2$, respectively. This amounts to using the same expansion point γ_0 for all three terms in Equation (102).

Proof of Proposition 7

The first four statements can be proven by differentiating either Equations (104) to (106) or the expression for Π in Proposition 1. The algebra is tedious and omitted here. Statements 5 and 6 are shown in Carmona and Durrleman [2003a].

Proof of Proposition 8

Notice that in the geometric Brownian motions cases, we have Equation (5) for the μ_i 's. The first statement follows from directly differentiating the expression of Π in Proposition 1. Notice that, by definition, when $x = \underline{x}(\gamma)$, we have $e^{\sigma_1\sqrt{T}x+\mu_1} - e^{\sigma_2\sqrt{T}\gamma+\mu_2} - K = 0$. Statement 2 follows directly from Proposition 7. A proof of Statement 3 is contained in Carmona and Durrleman [2003a]. Finally, Statement 4 follows from directly differentiating the expression of Π in Proposition 1 and simplifying. We will take $\mathcal{V}_2$ as an example. Notice that for the geometric Brownian motions case, Equation (102) becomes

$$\Pi = e^{-rT} \int_{-\infty}^{\infty} n(\gamma)d\gamma \int_{\underline{x}}^{\infty} (e^{v_1x+\mu_1} - e^{\sigma_2\sqrt{T}\gamma+(r-q-\sigma_2^2/2)T} - K) \times n(x; \rho\gamma, 1-\rho^2)dx \quad (131)$$

The derivative of Π with respect to σ_2 has two terms: one arising from the dependence of the lower inner integration limit $\underline{x}(\gamma)$ on σ_2 and another arising from the term $e^{\sigma_2\sqrt{T}\gamma+(r-q-\sigma_2^2/2)T}$. Differentiating on the integration limit gives 0 since by definition of the exercise boundary

$$e^{v_1\underline{x}(\gamma)+\mu_1} - e^{\sigma_2\sqrt{T}\gamma+(r-q-\sigma_2^2/2)T} - K = 0 \quad (132)$$

Thus,

$$\begin{aligned} \frac{\partial \Pi}{\partial \sigma_2} &= -e^{-rT} \int_{-\infty}^{\infty} n(\gamma)\sqrt{T}(\gamma - \sigma_2\sqrt{T})d\gamma \\ &\quad \times \int_{\underline{x}}^{\infty} e^{\sigma_2\sqrt{T}\gamma+(r-q-\sigma_2^2/2)T} n(x; \rho\gamma, 1-\rho^2)dx \quad (133) \\ &= -e^{-rT} \int_{-\infty}^{\infty} n(\gamma)\sqrt{T}(\gamma - \sigma_2\sqrt{T}) \\ &\quad \times e^{\sigma_2\sqrt{T}\gamma+(r-q-\sigma_2^2/2)T} N(A(\gamma))d\gamma \quad (134) \end{aligned}$$

Completing the square like we did in the proof of Proposition 1 now gives

$$\frac{\partial \Pi}{\partial \sigma_2} = -S_2\sqrt{T} \int_{-\infty}^{\infty} n(\gamma; \sigma_2\sqrt{T}, 1)(\gamma - \sigma_2\sqrt{T})N(A(\gamma))d\gamma \quad (135)$$

Finally, a change of variable gives the final expression for $\mathcal{V}_2$ in the proposition

$$\frac{\partial \Pi}{\partial \sigma_2} = -S_2\sqrt{T} \int_{-\infty}^{\infty} n(\gamma)\gamma N(A(\gamma + \sigma_2\sqrt{T}))d\gamma \quad (136)$$

The expression for $\mathcal{V}_1$ is more complicated because differentiating Π in Equation (131) with respect to σ_1 will give

$$\frac{\partial \Pi}{\partial \sigma_1} = e^{-rT} \int_{-\infty}^{\infty} n(\gamma)d\gamma \int_{\underline{x}}^{\infty} \sqrt{T}(x - v_1)e^{v_1x+\mu_1} n(x; \rho\gamma, 1-\rho^2)dx \quad (137)$$

and the factor $x - v_1$ contributes to the inner integration. If we now let $z \equiv (x - \rho\gamma) / \sqrt{1-\rho^2}$, then we have

$$\begin{aligned} \frac{\partial \Pi}{\partial \sigma_1} &= \sqrt{T}e^{-rT} \int_{-\infty}^{\infty} n(\gamma)e^{\rho v_1\gamma+\mu_1}d\gamma \\ &\quad \times \int_{-A(\gamma)}^{\infty} (\sqrt{1-\rho^2}z + \rho\gamma - v_1)e^{\sqrt{1-\rho^2}v_1z} n(z)dz \quad (138) \end{aligned}$$

By virtue of Equation (103),

$$\int_{-A(\gamma)}^{\infty} e^{\sqrt{1-\rho^2}v_1z} n(z)dz = e^{v_1^2(1-\rho^2)/2} N(A(\gamma) + v_1\sqrt{1-\rho^2}) \quad (139)$$

Integration by parts using $n'(z) = -zn(z)$ gives

$$\int_{-A(y)}^{\infty} ze^{\sqrt{1-\rho^2}v_1 z} n(z) dz = e^{v_1(\mathbb{E}(y)-\rho y)} n(A(y)) + v_1 \sqrt{1-\rho^2} e^{v_1^2(1-\rho^2)/2} N(A(y) + v_1 \sqrt{1-\rho^2}) \quad (140)$$

Substituting the last two equations into Equation (138), simplifying, and performing a last change of variable, $w = y - \rho v_1$ gives the expression for v_1 in the proposition.

For the signs of vegas, notice that v_1 is positive whenever

$$\begin{aligned} & \rho \int_{-\infty}^{\infty} n(y) y N(A(y + \rho v_1) + \sqrt{1-\rho^2} v_1) dy \\ &= \rho \int_0^{\infty} n(y) y (N(A(y + \rho v_1) + \sqrt{1-\rho^2} v_1) \\ &\quad - N(A(-y + \rho v_1) + \sqrt{1-\rho^2} v_1)) dy \end{aligned} \quad (141)$$

is positive. By Proposition 3, $\rho N(A(y + \rho v_1) + \sqrt{1-\rho^2} v_1)$ is an increasing function when ρ

≤ 0 or when $\rho \geq \sigma_2/\sigma_1$. From the preceding equation, $v_1 \geq 0$ when $\rho \leq 0$ or when $\rho \geq \sigma_2/\sigma_1$. Similarly,

$$v_2 \equiv \frac{\partial \Pi}{\partial \sigma_2} = -S_2 \sqrt{T} \int_{-\infty}^{\infty} n(y) y N(A(y + v_2)) dy \quad (142)$$

$$= -S_2 \sqrt{T} \int_0^{\infty} n(y) y (N(A(y + v_2)) - N(A(-y + v_2))) dy \quad (143)$$

This integral is positive when $\rho \geq 0$ and negative when $\rho \leq \sigma_2/\sigma_1$. Alternatively, one could apply Chebyshev's algebraic inequality on Equation (142). See, for example, Chapter IX of Mitrinovic, Pecaric, and Fink [1992].

Proof of Proposition 9

Statement 1 follows from Proposition 8. For Statement 2, we will only derive the approximation for Γ_{11} since the other gammas are similar. We have

$$\Gamma_{11} \equiv \frac{\partial^2 \Pi}{\partial S_1^2} = \frac{\partial \Delta_1}{\partial S_1} = \frac{\partial}{\partial S_1} \int_{-\infty}^{\infty} n(y) N(A(y + \rho v_1) + \sqrt{1-\rho^2} v_1) dy \quad (144)$$

$$\approx \frac{\partial}{\partial S_1} \int_{-\infty}^{\infty} n(y) N(C^1 + D^1 y + \epsilon y^2) dy \quad (145)$$

$$\approx \int_{-\infty}^{\infty} n(y) n(C^1 + D^1 y + \epsilon y^2) (C_{S_1}^1 + D_{S_1}^1 y + \epsilon_{S_1} y^2) dy \quad (146)$$

$$\begin{aligned} & \approx \int_{-\infty}^{\infty} n(y) n(C^1 + D^1 y) (C_{S_1}^1 + D_{S_1}^1 y + \epsilon_{S_1} y^2) dy \\ &= \Phi(C^1, D^1, C_{S_1}^1, D_{S_1}^1, \epsilon_{S_1}) \end{aligned} \quad (147)$$

where the function Φ can be computed analytically to give the expression in the proposition. The greek $\mathcal{V}_1$ can be computed similarly. The approximation for $\mathcal{V}_2$ is developed similarly using the last statement in Proposition 8.

Proof of Proposition 10

This proposition follows from brute force calculation. We also verified the statements using both Mathematica and MATLAB. We sketch the proof as follows. Let $Z = (B - \mu_B)/\sigma_B$ be the standardized random variable of B . Then, the Gram-Charlier density for Z is given by

$$f_Z^{GC}(z) = n(z) \left(1 + \frac{\gamma_B}{3!} h_3(z) + \frac{\kappa_B}{4!} h_4(z) \right) \quad (148)$$

where $h_3(\cdot)$ and $h_4(\cdot)$ are Hermite polynomials of order 3 and 4, respectively. The option price under this Gram-Charlier density can be computed by direct integration to give Equation (87). Finally, the following fact is useful in the computation of γ_B and κ_B . Let X and Y be jointly normal with means μ_X and μ_Y , variances σ_X^2 and σ_Y^2 , and correlation coefficient ρ . For any real numbers t and s , the joint moment generating function is given by

$$\mathbb{E}[e^{tX+sY}] = \exp \left(t\mu_X + s\mu_Y + \frac{1}{2} t^2 \sigma_X^2 + \frac{1}{2} s^2 \sigma_Y^2 + \rho st \sigma_X \sigma_Y \right) \quad (149)$$

ENDNOTES

The authors thank Stephen Figlewski (the editor) for valuable comments and suggestions. We also thank seminar participants at the 2007 Financial Management Association Annual Meeting and the Georgia Tech Quantitative and Computational Finance seminar for useful discussions. Feedback from academic scholars, especially Neil Pearson at UIUC and Rene Carmona at Princeton, and practitioners, especially Krzysztof Wolyniec at Sampra Commodities, Sandeep Jain at UBS, Alexander Eydeland at Morgan Stanley, Ying Shao at Goldman Sachs, and Gary Kennedy at Calypso are greatly appreciated. Kyuseok Lee provided valuable assistance during this project. All remaining errors are ours.

REFERENCES

- Albizzati, M.-O., and H. Geman. "Interest Rate Risk Management and Valuation of the Surrender Option in Life Insurance Policies." *Journal of Risk and Insurance*, Vol. 61, No. 4 (1994), pp. 616-637.
- Arak, M., P. Fisher, L. Goodman, and R. Daryanani. "The Municipal-Treasury Futures Spread." *Journal of Futures Markets*, 7 (1987), pp. 355-371.
- Black, F., and M. Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, 81 (1973), pp. 637-659.

- Broadie, M., and J. Detemple. "American Option Valuation: New Bounds, Approximations, and a Comparison of Existing Methods." *Review of Financial Studies*, Vol. 9, No. 4 (1996), pp. 1211–1250.
- Carmona, R., and V. Durrleman. "Pricing and Hedging Spread Options." *SIAM Review*, Vol. 45, No. 4 (2003a), pp. 627–685.
- . "Pricing and Hedging Spread Options in a Lognormal Model." Working paper, 2003b.
- . "Generalizing the Black-Scholes Formula to Multivariate Contingent Claims." *Journal of Computational Finance*, Vol. 9, No. 2 (2006), pp. 43–67.
- Chen, R.R., and S.-K. Yeh. "Analytical Upper Bounds for American Option Prices." *Journal of Financial and Quantitative Analysis*, Vol. 37, No. 1 (2002), pp. 117–136.
- Chung, S.L., and H.-C. Chang. "Generalized Analytical Upper Bounds for American Option Prices." *Journal of Financial and Quantitative Analysis*, Vol. 42, No. 1 (March 2007), pp. 209–228.
- Dempster, M., and S. Hong. "Pricing Spread Options with the Fast Fourier Transform." First World Congress of the Bachelier Finance Society, Paris, 2000.
- Deng, S.J., B. Johnson, and A. Sogomonian. "Exotic Electricity Options and the Valuation of Electricity Generation and Transmission Assets." *Decision Support Systems*, Vol. 30, No. 3 (2001), pp. 383–392.
- Deng, S.J., M. Li, and J. Zhou. "Multi-Asset Spread Option Pricing and Hedging." Working paper, 2008.
- Easterwood, J.C., and A.J. Senchack, Jr. "Arbitrage Opportunities with T-Bills/T-Bonds Combinations." *Journal of Futures Markets*, 6 (1986), pp. 433–442.
- Eydeland, A., and K. Wolyniec. *Energy and Power Risk Management: New Developments in Modeling, Pricing and Hedging*. New York, NY: John Wiley & Sons, 2003.
- Geman, H., N.E. Karoui, and J.C. Rochet. "Changes of Numeraire, Changes of Probability Measure and Option Pricing." *Journal of Applied Probability*, 32 (1995), pp. 443–458.
- Girma, P.B., and A.S. Paulson. "Seasonality in Petroleum Futures Spreads." *Journal of Futures Markets*, 18 (1998), pp. 581–598.
- . "Risk Arbitrage Opportunities in Petroleum Futures Spreads." *Journal of Futures Markets*, 18 (1999), pp. 931–955.
- Grinblatt, M., and S. Titman. "Adverse Risk Incentives and the Design of Performance-Based Contracts." *Management Science*, Vol. 35, No. 7 (July 1989), pp. 807–822.
- Henderson, V., D. Hobson, W.T. Shaw, and R. Wojakowski. "Bounds for In-Progress Floating Strike Asian Options Using Symmetry." *Annals of Operations Research*, Vol. 151, No. 1 (April 2007), pp. 81–98.
- Jarrow, R., and A. Rudd. "Approximate Option Valuation for Arbitrary Stochastic Processes." *Journal of Financial Economics*, 10 (1982), pp. 347–369.
- Johnson, S.A., and Y.S. Tian. "Indexed Executive Stock Options." *Journal of Financial Economics*, 57 (2000), pp. 35–64.
- Johnson, R.L., C.R. Zulauf, S.H. Irwin and M.E. Gerlow. "The Soy-Bean Complex Spread: An Examination of Market Efficiency from the Viewpoint of a Production Process." *Journal of Futures Markets*, 11 (1991), pp. 25–37.
- Jones, F.J. "Spreads: Tails, Turtles and All That." *Journal of Futures Markets*, 11 (1991), pp. 565–596.
- Kirk, E. "Correlations in the Energy Markets, in Managing Energy Price Risk." Risk Publications and Enron, 1995.
- Li, M. "The Impact of Return Nonnormality on Exchange Options." *Journal of Futures Markets*, Forthcoming, 2008.
- Lo, A. "Semiparametric Upper Bounds for Option Prices and Expected Payoffs." *Journal of Financial Economics*, 19 (1987), pp. 373–388.
- Margrabe, W. "The Value of an Option to Exchange One Asset for Another." *Journal of Finance*, 33 (1978), pp. 177–186.
- Mbafeno, A. "Co-movement Term Structure and the Valuation of Energy Spread Options." In M. Dempster and S. Pliska, eds., *Mathematics of Derivative Securities*. Cambridge, UK: Cambridge University Press, 1997.
- McDonald, R. L., and D. R. Siegel. "Investment and the Valuation of Firms When there is an Option to Shut Down." *International Economic Review*, Vol. 26, No. 2 (June 1985), pp. 331–349.
- Mitrinovic, D.S., J. Pecaric, and A.M. Fink. *Classical and New Inequalities in Analysis*. New York, NY: Springer, 1992.
- Nielsen, J.A., and K. Sandmann. "Pricing Bounds on Asian Options." *Journal of Financial and Quantitative Analysis*, Vol. 38, No. 2 (2003), pp. 449–473.
- Pearson, N. "An Efficient Approach for Pricing Spread Options." *Journal of Derivatives* (Fall 1995), pp. 76–91.
- Perrakis, S., and P.J. Ryan. "Option Pricing Bounds in Discrete Time." *Journal of Finance*, 39 (1984), pp. 519–525.
- Poitras, G. "Spread Options, Exchange Options, and Arithmetic Brownian Motion." *Journal of Futures Markets*, 18 (1998), pp. 487–517.
- Shevlin, T. "The Valuation of R&D Firms with R&D Limited Partnerships." *The Accounting Review*, Vol. 66, No. 1 (January 1991), pp. 1–21.
- Shimko, D. "Options on Future Spreads: Hedging, Speculating, and Valuation." *Journal of Futures Markets*, Vol. 14, No. 2 (1997), pp. 183–213.
- Wilcox, D. "Energy Futures and Options: Spread Options in Energy Markets." Goldman Sachs & Co., New York, 1990.

To order reprints of this article, please contact Dewey Palmieri at dpalmieri@ijjournals.com or 212-224-3675

©Euromoney Institutional Investor PLC. This material must be used for the customer's internal business use only and a maximum of ten (10) hard copy print-outs may be made. No further copying or transmission of this material is allowed without the express permission of Euromoney Institutional Investor PLC. Source: Journal of Derivatives and
<http://www.iijournals.com/JOD/Default.asp>