A Generalized Single Common Factor Model of Portfolio Credit Risk

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The Vasicek portfolio credit loss model is generalized to include stochastic exposures (EADs) and loss rates (LGDs). Revolving exposures draw against committed lines of credit. Dependence among EADs and LGDs are modeled using a common Gaussian factor. Distribution assumptions for LGDs and EADs are unrestricted. The model produces a closed-form expression for an asymptotic portfolio's conditional loss rate distribution. Correlation among EAD and LGD realizations increases portfolio systematic risk, producing increased skewness in portfolio credit loss distributions and larger unexpected credit loss estimates. The results are important for issues related to capital allocation and subordination levels on structured credit products.

The Gaussian asymptotic single factor model of portfolio credit losses (ASFM), developed by Vasicek [1987]; Finger [1999]; Schönbucher [2001]; Gordy [2003]; Kupiec [2007], and others provides an approximation for the loss rate distribution on a credit portfolio in which the idiosyncratic credit risks are fully diversified and dependence among individual defaults is driven by a single common factor. The ASFM assumes that the unconditional probability of default on an individual credit (PD) is fixed and known. In addition, all obligors' exposures at default (EADs) and all loss rates in default (LGDs) are assumed to be fixed in magnitude.

The ASFM has been widely applied in the financial industry. It has been used to estimate economic capital allocations and is the model that underlies the Basel II Advanced Internal Ratings–based approach (Basel AIRB) for bank minimum regulatory capital requirements. In addition to capital allocation, the ASFM has been adapted to estimate market price and potential loss distributions for structured credit products (e.g., Li [2000]; Andersen, Sidenius, and Basu [2003]; Gibson [2004]; Gordy and Jones [2002], and others). Notwithstanding widespread use, empirical findings suggest that the ASFM omits important systematic factors that determine the characteristics of a portfolio's true underlying credit loss distribution.

Many studies find significant time variability among realized *LGD*s for a given credit facility or ratings class. The stylized facts hold that default losses increase in periods when default rates are elevated. Studies by Frye [2000b]; Hu and Perraudin [2002]; Schuermann [2004]; Araten, Jacobs, and Varshney [2004]; Altman, Brady, Resti, and Sironi [2004]; Hamilton, Varma, Ou, and Cantor [2004]; Carey and Gordy [2004]; Emery, Cantor, and Arnet [2004], and others show pronounced decreases in the recovery rates during periods when default rates are elevated. These results highlight the existence of a systematic relationship between default frequency and recovery rates that is not captured in the Vasicek ASFM framework.

In addition to issues related to stochastic *LGD*, the Vasicek ASFM is often used to estimate capital needs for portfolios of revolving credits even though the model is based on the assumption that individual credit *EAD*s are fixed. For example, the Basel AIRB framework uses a modified Vasicek model to estimate minimum regulatory capital requirements for a bank's revolving retail and wholesale credits. The AIRB treatment requires a bank to estimate the *EAD* that applies to each credit facility, but the capital modeling framework treats *EAD* as a fixed quantity.

The Basel AIRB treatment of *EAD* is at odds with the behavior of revolving account balances. Studies by Allen and Saunders [2003]; Asarnow and Marker [1995]; Araten and Jacobs [2001], and Jiménez, Lopez, and Saurina [2006] suggest that obligors draw on committed lines of credit as their credit quality deteriorates. Analysis of creditors' draw rate behavior shows that revolving account *EAD*s are positively correlated with default rates—a correlation suggesting that there is at least one common factor that simultaneously determines portfolio *EAD*, *LGD*, and default rate realization.

The empirical evidence suggests that the ASFM assumption of fixed *LGD* and *EAD* excludes two important sources of systematic credit risk that are present in historical loss rate data. A number of existing models, including those by Frye [2000a]; Pykhtin [2003]; Tasche [2004], and Andersen and Sidenius [2005], have extended the Vasicek ASFM framework to include stochastic *LGD* rates. These extensions require restrictive assumptions on individual credit *LGD* distributions to produce tractable expressions for an asymptotic portfolio's loss rate distribution.[1] This article is the first to extend the ASFM framework to include both stochastic *EAD* and *LGD* and produce a closed-form expression for a portfolio loss rate distribution. In addition, this study generalizes existing models for nonrevolving credits by allowing *LGD* to be modeled using any probability distribution.

In the remainder of this article, the Gaussian ASFM is extended to incorporate obligors with *EAD*s and *LGD*s that are correlated random variables. In this extension, default is a random event driven by a compound latent factor as in the standard ASFM. Two additional compound latent factors are used to drive correlations among individual credits' *EAD*s and *LGD*s through a probability integral transformation. A closed-form expression for the inverse of the portfolio's conditional credit loss distribution is derived for cases when *LGD* and *EAD* probability density functions are continuous.

When *LGD* and *EAD* probability density functions are discrete, the probability integral transformation is inadmissible. In this case, a closed-form expression for the inverse of the portfolio's conditional loss rate distribution is constructed using a step function to approximate the underlying *LGD* and *EAD* distributions. The characteristics of the *LGD* and *EAD* distributions that can be modeled using this approach are unrestricted. Approximation error can be made arbitrarily small by increasing the number of steps in the approximation. The step-function approach can also be used to approximate the inverse of a portfolio's conditional loss rate distribution for cases when *LGD* and *EAD* distributions are continuous.

Selected examples of portfolio *EAD*, *LGD*, and overall loss rate distributions are calculated using alternative *EAD* and *LGD* distribution and correlation assumptions that are consistent with stylized representations of selected corporate and retail portfolios. The results show that the additional systematic risk in *EAD* and *LGD* realizations increases the skewness of portfolio credit loss distributions. The increase in skewness is reflected in significantly larger loss rates associated with upper-tail critical values of the portfolio credit loss distribution. These features have potentially important implications for issues related to economic capital allocation and the support levels and pricing of structured credit securities.

THE GAUSSIAN ASFM MODEL

The Vasicek single common factor model of portfolio credit risk assumes that uncertainty on credit i is driven by a latent unobserved factor $\tilde{V}_i$ with the following properties

$$\begin{aligned} &\tilde{V}_i = \sqrt{\rho_V}\,\tilde{e}_M + \sqrt{1-\rho_V}\,\tilde{e}_{id} \\ &\tilde{e}_M \sim \phi(e_M) \\ &e_{id} \sim \phi(e_{id}) \\ &E(\tilde{e}_{id}\,\tilde{e}_{jd}) = E(\tilde{e}_M\,\tilde{e}_{jd}) = 0, \quad \forall\, i, j \end{aligned} \tag{1}$$

where $\phi(\cdot)$ represents the standard normal density function. The factor $\tilde{V}_i$ is distributed standard normal, $E(\tilde{V}_i) = 0$, and $\sigma^2(\tilde{V}_i) = E(\tilde{V}_i^2) - E(\tilde{V}_i)^2 = 1$. The common factor $\tilde{e}_M$ induces correlation between individual credits' latent factors, $\rho_V = \frac{Cov(\tilde{V}_i,\tilde{V}_j)}{\sigma(\tilde{V}_i)\,\sigma(\tilde{V}_j)}$. Often $\tilde{V}_i$ is interpreted as a proxy for the market value of the firm that issued credit i.

Credit i is assumed to default when its latent factor takes on a value less than a credit-specific threshold, $\tilde{V}_i < D_i$. The unconditional probability that credit i defaults is $PD = \Phi(D_i)$, where $\Phi(\cdot)$ represents the cumulative standard normal density function. Time is not an independent factor in the ASFM, but is implicitly recognized through the calibration of input values for PD.

REVOLVING CREDITS WITH CORRELATED EXPOSURES AND LOSS RATES

A Model of Stochastic EAD

Assume that a generic revolving credit account, i, has a maximum line of credit, M_i, upon which it may draw. The account utilization rate $\tilde{X}_i \in [0,1]$ is a random variable that determines the end-of-period account exposure, $\tilde{X}_i M_i$. Basel II conventions require that EAD is at least as large as initial exposure, and so account-level EAD is modeled as an initial outstanding exposure and a random draw rate, $\tilde{\delta}_i$, on a remaining line of credit.

Assume that an individual account begins the period with a drawn exposure $G_{i0} M_i$, where $G_{i0} \in [0,1]$ is the initial utilization of the account line of credit. Let $\tilde{\delta}_i \in [0,1]$ represent the share of the remaining line of credit that is borrowed over the period, and let $\Omega(\tilde{\delta}_i)$ represent the cumulative density function for $\tilde{\delta}_i$. The account's end-of-period exposure is[2]

$$M_i \tilde{X}_i = M_i (G_{i0} + (1 - G_{i0}) \tilde{\delta}_i), \quad \tilde{\delta}_i \sim \Omega(\delta_i), \quad \delta_i \in [0,1] \quad (2)$$

Systematic dependence among individual accounts' draw rates is incorporated by assuming that $\tilde{\delta}_i$'s are driven by a latent Gaussian factor, $\tilde{Z}_i$, with the following properties:

$$\begin{aligned}
&\tilde{Z}_i = \sqrt{\rho_Z}\, \tilde{e}_M + \sqrt{1-\rho_Z}\, \tilde{e}_{iZ} \\
&\tilde{e}_M \sim \phi(e_M) \\
&e_{iZ} \sim \phi(e_{iZ}) \\
&E(\tilde{e}_{iZ}\, \tilde{e}_{jZ}) = E(\tilde{e}_M\, \tilde{e}_{jZ}) = E(\tilde{e}_{iZ}\, \tilde{e}_{jd}) = 0 \quad \forall\, i, j
\end{aligned} \quad (3)$$

where $\tilde{Z}_i$ has a standard normal distribution. The correlation between the latent variables that determine each account's draw rate is $\rho_Z = \frac{Cov(\tilde{Z}_i, \tilde{Z}_j)}{\sigma(\tilde{Z}_i)\sigma(\tilde{Z}_j)}$, and the correlation between the latent factors that drive account EADs and defaults is $\sqrt{\rho_Z \rho_V} = \frac{Cov(\tilde{V}_i, \tilde{Z}_j)}{\sigma(\tilde{V}_i)\sigma(\tilde{Z}_j)}$. Under the normalization convention that higher account draw rates are associated with smaller realizations of $\tilde{Z}_i$, there is a positive correlation between EAD and the frequency of default ($\sqrt{\rho_V} > 0$ and $\sqrt{\rho_Z} > 0$).

The latent variable $\tilde{Z}_i$ drives account exposure through the probability integral transformation.[3] Correlation among the draw rates on individual credit facilities is introduced by equating the probability integral transformations for the physical draw rate $\tilde{\delta}_i$ and the latent variable, $\tilde{Z}_i$, $\Omega(\delta_i) = 1 - \Phi(Z_i)$, which implies a unique mapping between $\tilde{Z}_i$ and $\tilde{\delta}_i$ given by

$$\tilde{\delta}_i = \Omega^{-1}(1 - \Phi(\tilde{Z}_i)) \quad (4)$$

A Model of Stochastic LGD

Let $\tilde{\lambda}_i \in [0,1]$ represent the loss rate on account i's outstanding balance in the event of default. Let $\Theta(\tilde{\lambda}_i)$ represent the cumulative density function for $\tilde{\lambda}_i$. Systematic dependence among the loss rates on individual credits is introduced by assuming that $\tilde{\lambda}_i$ is driven by a latent Gaussian factor, $\tilde{Y}_i$, with the following properties:

$$\begin{aligned}
&\tilde{Y}_i = \sqrt{\rho_Y}\, \tilde{e}_M + \sqrt{1-\rho_Y}\, \tilde{e}_{iY} \\
&\tilde{e}_M \sim \phi(e_M) \\
&e_{iY} \sim \phi(e_{iY}) \\
&E(\tilde{e}_{iY}\, \tilde{e}_{jY}) = E(\tilde{e}_M\, \tilde{e}_{jY}) = E(\tilde{e}_{iY}\, \tilde{e}_{jZ}) \\
&\qquad = E(\tilde{e}_{iY}\, \tilde{e}_{id}) = 0 \quad \forall\, i, j
\end{aligned} \quad (5)$$

where $\tilde{Y}_i$ has a standard normal distribution.

When higher account loss rates are associated with smaller realizations of the latent variable $\tilde{Y}_i$ there is positive correlation between a portfolio's default rate and its realized LGD. The correlation between the latent factors that determine default and LGD is $\sqrt{\rho_V \rho_Y} > 0$. The correlation between the Gaussian drivers of default and EAD is $\sqrt{\rho_V \rho_Z} > 0$. Using the probability integral transformation to introduce a correlation structure, the mapping between $\tilde{\lambda}_i$ and $\tilde{Y}_i$ is given by

$$\tilde{\lambda}_i = \Theta^{-1}(1 - \Phi(\tilde{Y}_i)) \quad (6)$$

The Loss Rate for an Individual Account

To derive the portfolio credit loss distribution, first define a function over $\tilde{V}_i$ that indicates default status

$$1_{Di}(\tilde{V}_i)=\begin{cases}1 & \text{if} \quad \tilde{V}_i < D_i \\ 0 & \text{otherwise}\end{cases} \tag{7}$$

The indicator function, $1_{D_i}(\tilde{V}_i)$, is a binomial random variable that has an expected value of $\Phi(D_i)$.

Let $\Lambda_i(\tilde{V}_i,\tilde{Z}_i,\tilde{Y}_i)$ represent the loss rate for account i measured relative to the account's maximum credit limit, M_i, where $\Lambda_i(\tilde{V}_i,\tilde{Z}_i,\tilde{Y}_i)$ is defined as

$$\begin{aligned}\Lambda_i(\tilde{V}_i,\tilde{Z}_i,\tilde{Y}_i) &= 1_{D_i}(\tilde{V}_i)(G_{i0}+(1-G_{i0})\tilde{\delta}_i)\tilde{\lambda}_i \\ &= 1_{D_i}(\tilde{V}_i)(G_{i0}+(1-G_{i0}) \\ &\quad \times \Omega^{-1}(1-\Phi(\tilde{Z}_i)))\Theta^{-1}(1-\Phi(\tilde{Y}_i))\end{aligned} \tag{8}$$

The Conditional Loss Rate for an Individual Account

Let $1_{D_i}(\tilde{V}_i|e_M)$ represent the value of the default indicator function conditional on a realized value for e_M, the common latent factor. The conditional expected value of the indicator function is

$$E(1_{D_i}(\tilde{V}_i\,|\,e_M))=\Phi\left(\frac{D_i-\sqrt{\rho_V}\,e_M}{\sqrt{1-\rho_V}}\right) \tag{9}$$

Similarly, let $\tilde{Z}_i|e_M$ and $\tilde{Y}_i|e_M$ represent, respectively, the values of the latent variables that drive account i's draw rate and LGD, conditional on a realized value for e_M. These conditional random variables are normally distributed with means $E(\tilde{Z}_i|e_m) = \sqrt{\rho_Z}e_M$ and $E(\tilde{Y}_i|e_m) = \sqrt{\rho_Y}e_M$, and variances $\sigma^2(\tilde{Z}_i|e_M) = 1-\rho_Z$ and $\sigma^2(\tilde{Y}_i|e_M) = 1-\rho_Y$.

The probability that $\tilde{Z}_i \le Z_i$, conditional on $\tilde{e}_M = e_M$, is equivalent to the probability $\tilde{e}_i \le \frac{Z_i-\sqrt{\rho_Z}e_M}{\sqrt{1-\rho_Z}}$ or $\Phi\left(\frac{Z_i-\sqrt{\rho_Z}e_M}{\sqrt{1-\rho_Z}}\right)$. A similar relationship defines the conditional distribution function for $\tilde{Y}_i|e_M$. It follows that the cumulative conditional distribution functions for the latent variables are

$$\begin{aligned}\tilde{Z}_i\,|\,e_M &\sim \Phi(Z_i\,|\,\tilde{e}_M=e_M) \\ &= \Phi\left(\frac{Z_i-\sqrt{\rho_Z}\,e_M}{\sqrt{1-\rho_Z}}\right) \quad \forall\, Z_i \in(-\infty,\,\infty)\end{aligned} \tag{10}$$

$$\begin{aligned}\tilde{Y}_i\,|\,e_M &\sim \Phi(Y_i\,|\,\tilde{e}_M=e_M) \\ &= \Phi\left(\frac{Y_i-\sqrt{\rho_Z}\,e_M}{\sqrt{1-\rho_Z}}\right) \quad \forall\, Y_i \in(-\infty,\infty)\end{aligned} \tag{11}$$

and the loss rate on an individual account, conditional on a value for e_M, is

$$\begin{aligned}\Lambda_i(\tilde{V}_i,\tilde{Z}_i,\tilde{Y}_i\,|\,e_M) &= 1_{D_i}(\tilde{V}_i\,|\,e_M)(G_{i0}+(1-G_{i0}) \\ &\quad \times\Omega^{-1}(1-\Phi(\tilde{Z}_i\,|\,e_M))) \\ &\quad \times\Theta^{-1}(1-\Phi(\tilde{Y}_i\,|\,e_{Mi}))\end{aligned} \tag{12}$$

The Loss Rate on an Asymptotic Portfolio of Revolving Credits

In a so-called asymptotic portfolio, all idiosyncratic risks are diversified and portfolio loss or return realizations depend only on the realizations of the systematic risk factors that impact individual accounts' performance. Asymptotic portfolios are important for capital allocation modeling as they yield portfolio-invariant capital allocation rules. A portfolio- invariant capital allocation is one in which the economic capital required on an investment is independent of the composition of the portfolio to which the investment is added.[4] The asymptotic portfolio framework is used to define economic capital allocations for individual assets or credit facilities that are completely independent of the other assets or exposures that are held by the bank.

Consider a portfolio with N accounts that have identical credit limits, $M_i = M$; identical initial drawn balances, $G_{i0}M_i = G_0M$; identical latent factor correlations, $\{\rho_V, \rho_X, \rho_Y\}$; and identical default thresholds, $D_i = D$. Assume that all accounts' end-of-period draw rates ($\tilde{\delta}_i$) and LGD ($\tilde{\lambda}_i$) are taken, respectively, from the unconditional distributions which are identical across credits (the distributions for $\tilde{\delta}_i$ and $\tilde{\lambda}_i$ are generally different). Under these assumptions, $1_{D_i}(\tilde{V}_i) = 1_D(\tilde{V}_i)$ for all i, and the loss rate for an individual credit depends on the identity of the credit only because the realizations of idiosyncratic risk factors differ across credits. Taken together, these features imply that the subscript on the loss rate can be eliminated, $\Lambda_i(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i) = \Lambda(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i)$.

Let $\vec{\tilde{V}} = (\tilde{V}_1, \tilde{V}_2, \ldots, \tilde{V}_N)$ represent the vector of N latent variables that determine account defaults. Define $\vec{\tilde{Y}}$ and $\vec{\tilde{Z}}$ analogously. Let $\Lambda_p(\vec{\tilde{V}}, \vec{\tilde{Z}}, \vec{\tilde{Y}} \mid e_M)$ represent the loss rate on the portfolio of N accounts conditional on a realization of e_M,

$$\Lambda_p(\vec{\tilde{V}}, \vec{\tilde{Z}}, \vec{\tilde{Y}} \mid e_M) = \left(\frac{\sum_{i=1}^{N} (M \cdot \Lambda(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i \mid e_M))}{N \cdot M} \right) = \left(\frac{\sum_{i=1}^{N} (\Lambda(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i \mid e_M))}{N} \right) \quad (13)$$

Recall that $\Lambda(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i \mid e_M)$ is independent of $\Lambda(\tilde{V}_j, \tilde{Z}_j, \tilde{Y}_j \mid e_M)$ for all $i \neq j$, and the conditional loss rates for individual credits are identically distributed. Under these conditions, the Strong Law of Large Numbers requires, for any admissible value of e_M,

$$\lim_{N\to\infty} \Lambda_p(\vec{\tilde{V}}, \vec{\tilde{Z}}, \vec{\tilde{Y}} \mid e_M) = \lim_{N\to\infty} \left(\frac{\sum_{i=1}^{N} \Lambda(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i \mid e_M)}{N} \right) \xrightarrow[a.s.]{} E(\Lambda(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i \mid e_M)) \quad (14)$$

where *a.s.* indicates convergence with probability one. Independence among the conditional latent factors and the indicator function implies

$$\lim_{N\to\infty} \Lambda_p(\vec{\tilde{V}}, \vec{\tilde{Z}}, \vec{\tilde{Y}} \mid e_M) \xrightarrow[a.s]{} E(1_D(\tilde{V}_i \mid e_M)) \times \left[G_0 + (1-G_0) E(\Omega^{-1}(1-\Phi(\tilde{Z}_i \mid e_M))) \times E(\Theta^{-1}(1-\Phi(\tilde{Y}_i \mid e_M))) \right] \quad (15)$$

Expression (15) is the inverse of the conditional distribution function for an asymptotic portfolio's loss rate evaluated at $e_M \in (-\infty, +\infty)$. The only random factor driving the unconditional portfolio loss rate distribution is the common latent factor $\tilde{e}_M$. All idiosyncratic risks have been diversified. As a consequence, an asymptotic portfolio's loss rate, $\tilde{\Lambda}_p$, has a cumulative probability distribution defined by the implicit function,

$$\tilde{\Lambda}_p \sim \left\{ \Phi\left(\frac{D - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}} \right) \cdot \left[G_0 + (1-G_0) \times E(\Omega^{-1}(1-\Phi(\tilde{Z}_i \mid e_M))) \right] \times E(\Theta^{-1}(1-\Phi(\tilde{Y}_i \mid e_M))), 1-\Phi(e_M) \right\} \quad (16)$$

for $e_M \in (-\infty, +\infty)$.

Calculation of the Critical Values of a Portfolio's Loss Rate Distribution

Many risk management functions and structured finance calculations require portfolio loss rate estimates for particular cumulative probability thresholds of the portfolio loss rate distribution. Consider, for example, the portfolio loss rate that exceeds a proportion, α, of all potential portfolio credit losses or, alternatively, a loss rate exceeded by, at most, $1-\alpha$ of all potential portfolio losses. Because the portfolio loss rate function is decreasing in e_M, Expression (15) evaluated at $e_M = \Phi^{-1}(1-\alpha)$ is the loss rate consistent with a cumulative probability of α. Using the identities $\Phi^{-1}(1-\alpha) = -\Phi^{-1}(\alpha)$ and $D = \Phi^{-1}(PD)$, the portfolio loss rate consistent with a cumulative probability of α is

$$\Lambda_p(\vec{\tilde{V}}, \vec{\tilde{Z}}, \vec{\tilde{Y}} \mid e_M = \Phi^{-1}(1-\alpha)) \xrightarrow[a.s.]{} \Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho_V}\,\Phi^{-1}(\alpha)}{\sqrt{1-\rho_V}} \right) \times \left[G_0 + (1-G_0) \times E(\Omega^{-1}(1-\Phi(\tilde{Z}_i \mid e_M = -\Phi^{-1}(\alpha)))) \right] \times E(\Theta^{-1}(1-\Phi(\tilde{Y}_i \mid e_M = -\Phi^{-1}(\alpha)))) \quad \text{for } \alpha \in [0,1] \quad (17)$$

Discussion

The first term in Expression (17), $\Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho_V}\,\Phi^{-1}(\alpha)}{\sqrt{1-\rho_V}} \right)$, is the inverse of an asymptotic portfolio's cumulative default rate distribution evaluated at a probability of α.[5]

The remaining terms in Expression (17) are the α-level critical values for the asymptotic portfolio's *EAD* distribution, $G_0 + (1-G_0)E(\Omega^{-1}(1-\Phi(\tilde{Z}_i \mid e_M = -\Phi^{-1}(\alpha))))$, and the asymptotic portfolio's *LGD* distribution, $E(\Theta^{-1}(1-\Phi(\tilde{Y}_i \mid e_M = -\Phi^{-1}(\alpha))))$.

In general, the critical values of the asymptotic portfolio *EAD* and *LGD* distributions must be calculated using numerical techniques. Although quadrature methods of estimation are preferred on efficiency grounds (i.e., smaller estimation error for a given number of calculations), a simple Monte Carlo estimator of $E(\Theta^{-1}(1-\Phi(\tilde{Y}_i|e_M=-\Phi^{-1}(\alpha))))$ provides an example that aids in understanding some of the portfolio's distribution properties.

To construct a simple Monte Carlo estimator for a given value for α, generate a random sample of size J of standard normal deviates, $\tilde{e}_j \sim \phi(0,1), j=1,2,3,\ldots,J$. For each observation, e_j, calculate $\hat{\lambda}_j=\Theta^{-1}(1-\Phi(\sqrt{1-\rho_Y}e_j-\sqrt{\rho_Y}\Phi^{-1}(\alpha)))$. The Monte Carlo estimate of $E(\Theta^{-1}(1-\Phi(\tilde{Y}_i|e_M=-\Phi^{-1}(\alpha))))$ is $\frac{1}{J}\Sigma_{j=1}^{J}\hat{\lambda}_j$. The precision of this estimator improves as the Monte Carlo sample size, J, increases:[6]

$$\lim_{J\to\infty}\frac{1}{J}\sum_{j=1}^{J}\hat{\lambda}\xrightarrow[a.s]{} E(\Theta^{-1}(1-\Phi(\tilde{Y}_i|e_M=-\Phi^{-1}(\alpha)))) \tag{18}$$

A similar approach can be used to estimate the critical value of the asymptotic portfolio's draw rate distribution, $E(\Omega^{-1}(1-\Phi(\tilde{Z}_i|e_M=-\Phi^{-1}(\alpha))))$.

The Monte Carlo estimators highlight two characteristics of the asymptotic portfolio's draw and *LGD* distributions. First, as the correlations in their latent factors converge to zero, the asymptotic portfolio draw rate and *LGD* distributions converge to their unconditional expected values,[7]

$$\begin{aligned}&\lim_{\rho_Y\to 0}E(\Theta^{-1}(1-\Phi(\tilde{Y}_i|e_M=-\Phi^{-1}(\alpha))))\\&\quad=E(\tilde{\lambda})\quad\forall\,\alpha\in[0,1]\\&\lim_{\rho_Y\to 0}E(\Omega^{-1}(1-\Phi(\tilde{Z}_i|e_M=-\Phi^{-1}(\alpha))))\\&\quad=E(\tilde{\delta})\quad\forall\,\alpha\in[0,1]\end{aligned} \tag{19}$$

Second, as the correlations in these distributions' latent factors approach 1, there is no diversification in the portfolio-level distributions; that is, the distributions of the portfolio draw and *LGD* rate distributions converge to distributions that characterize the loss or exposure behavior of a single credit,[8]

$$\begin{aligned}&\lim_{\rho_Y\to 1}E(\Theta^{-1}(1-\Phi(\tilde{Y}_i|e_M=-\Phi^{-1}(\alpha))))\\&\quad=\Theta^{-1}(\alpha)\quad\forall\alpha\in[0,1]\\&\lim_{\rho_Z\to 1}E(\Omega^{-1}(1-\Phi(\tilde{Z}_i|e_M=-\Phi^{-1}(\alpha))))\\&\quad=\Omega^{-1}(\alpha)\quad\forall\alpha\in[0,1]\end{aligned} \tag{20}$$

A STEP-FUNCTION APPROXIMATION FOR THE PORTFOLIO LOSS DISTRIBUTION

The derivation of Expression (17) is not fully general as it requires use of the probability integral transform which is applicable only when individual account *LGD* and draw rates have continuous distributions. This section discusses an approximation for the asymptotic portfolio loss distribution that is applicable in cases when the individual credit *LGD* and draw rate distributions are discrete. In this approximation, a random variable's range of support is divided into equal increments, and these increments are used in conjunction with latent Gaussian factors to construct a step-function approximation for individual correlated *EAD* and *LGD* distributions.

Conceptually, the step function approach is similar to a digital signal-processing algorithm. It divides the entire range of the *EAD* and *LGD* distribution into discrete segments that are sequentially activated by digital switches that are turned on (or off) by the latent Gaussian factors that drive *EAD* and *LGD* realizations. An exceptionally strong negative signal from an *EAD* or *LGD* latent factor leaves all switches off, and the realized *EAD* or *LGD* will be the maximum possible. A slightly less negative signal from the latent Gaussian factor will flip on a few digital switches and thereby reduce the magnitude of the account's realized *EAD* or *LGD*. A very strong positive signal from the latent Gaussian factors will flip on all switches and EAD or LGD will be reduced to zero.

The step function approximation is necessary when *LGD* and the draw rate distributions are discrete. It can also be used to approximate the portfolio loss distribution when the *LGD* and draw rate distributions are continuous. Because it avoids the need to use numerical methods to calculate expectations, depending on the intended use and accuracy requirements, the step-function formulation of the model may be preferred in some applications even when the *LGD* and draw rate distributions are continuous.

Approximating Individual Account EADs

Divide the [0,1] range of support for account draw rates into $n+1$ overlapping regions, and define $n+1$ corresponding events: $\{\xi(\tilde{\delta}, j, n)\}$ is the set of events $\tilde{\delta}_i \in [0, \frac{j}{n}]$ for $j = 0, 1, 2, \ldots, n$.

Approximate the draw rate probability distribution using a uniform-size step function defined on $\tilde{Z}_i$,

$$\tilde{\delta}_i = \begin{bmatrix} 0 & \text{for } \tilde{Z}_i \geq A_{i1} \\ \left(\frac{1}{n}\right) & \text{for } A_{i2} \leq \tilde{Z}_i < A_{i1} \\ \left(\frac{2}{n}\right) & \text{for } A_{i3} \leq \tilde{Z}_i < A_{i2} \\ \vdots & \vdots \\ \left(\frac{j}{n}\right) & \text{for } A_{ij+1} \leq \tilde{Z}_i < A_{ij} \\ \vdots & \vdots \\ \left(\frac{n-1}{n}\right) & \text{or } A_{in} \leq \tilde{Z}_i < A_{in-1} \\ 1 & \text{for } \tilde{Z}_i < A_{in} \end{bmatrix} \tag{21}$$

where $A_{in} < A_{in-1} < \cdots A_{i2} < A_{i1}$. Expression (21) models the draw rate as a monotonically decreasing function of $\tilde{Z}_i$ with $n + 1$ distinct draw rates with uniform increments of size $\frac{1}{n}$ beginning at $\delta_i = 1$.

Calibrate the thresholds $\{A_{i1}, A_{i2}, \ldots, A_{in}\}$ by equating the Gaussian probabilities for the latent variable thresholds to the probability that the corresponding events occur under $\Omega(\tilde{\delta}_i)$. For example, the equality $1 - \Phi(A_{i1}) = \Omega(\tilde{\delta}_i = 0)$ is used to define $A_{i1} = \Phi^{-1}(1 - \Omega(0))$. Similarly, $1 - \Phi(A_{i2}) = \Omega(\tilde{\delta} = \frac{1}{n})$ is used to define $A_{i2} = \Phi^{-1}(1 - \Omega\,(\frac{1}{n}))$, and so on. The threshold values for the unconditional draw rate distribution approximation are given in Exhibit 1.

Approximating Individual Account LGDs

The methodology used to approximate the *LGD* distribution is analogous to the approach used to approximate the draw rate distribution. Divide the interval [0,1] into $n+1$ overlapping regions and define a corresponding set of events, $\{\xi(\tilde{\lambda}, j, n) \ni \tilde{\lambda}_i \in [0, \frac{j}{n}]\}$ for $j = 0, 1, 2, \ldots, n$.

The model is normalized so that higher realized loss rates are associated with smaller realized values of $\tilde{Y}_i$. Approximate $\Theta(\tilde{\lambda}_i)$ as

EXHIBIT 1

Step-Function Approximation for Individual Credit's Draw Rate Distribution

Draw Rate	Event	Cumulative Probability of Draw Rate	Threshold Value for Latent Variable $\tilde{Z}_i$
0	$\xi(\tilde{\delta}, 0, n)$	$\Omega(\tilde{\delta}_i = 0)$	$A_{i1} = \Phi^{-1}(1 - \Omega(0))$
$\frac{1}{n}$	$\xi(\tilde{\delta}, 1, n)$	$\Omega\left(\tilde{\delta}_i = \frac{1}{n}\right)$	$A_{i2} = \Phi^{-1}\left(1 - \Omega\left(\frac{1}{n}\right)\right)$
$\frac{2}{n}$	$\xi(\tilde{\delta}, 2, n)$	$\Omega\left(\tilde{\delta}_i = \frac{2}{n}\right)$	$A_{i3} = \Phi^{-1}\left(1 - \Omega\left(\frac{2}{n}\right)\right)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{n-1}{n}$	$\xi(\tilde{\delta}, n-1, n)$	$\Omega\left(\tilde{\delta}_i = \frac{n-1}{n}\right)$	$A_{in} = \Phi^{-1}\left(1 - \Omega\left(\frac{n-1}{n}\right)\right)$
1	$\xi(\tilde{\delta}, n, n)$	1	

$$\tilde{\lambda}_i = \begin{cases} 0 & \text{for } \tilde{Y}_i \geq B_{i1} \\ \left(\frac{1}{n}\right) & \text{for } B_{i2} \leq \tilde{Y}_i < B_{i1} \\ \left(\frac{2}{n}\right) & \text{for } B_{i3} \leq \tilde{Y}_i < B_{i2} \\ \vdots & \vdots \\ \vdots & \vdots \\ \left(\frac{n-1}{n}\right) & \text{for } B_{in} \leq \tilde{Y}_i < B_{in-1} \\ 1 & \text{for } \tilde{Y}_i < B_{in} \end{cases} \tag{22}$$

for $B_{in} < B_{in-1} < \cdots < B_{i2} < B_{i1}$. The latent variable thresholds are defined in Exhibit 2.

The Loss Rate for an Individual Account

The loss rate distribution for an individual account can be modeled using $2n + 1$ indicator functions defined over the latent variables $\tilde{V}_i, \tilde{Z}_i$, and $\tilde{Y}_i$. One indicator function indicates default status; n indicator functions are used to approximate the cumulative *EAD* distribution, $\Omega(\tilde{\delta})$; and n indicator functions are used to approximate the cumulative *LGD* distribution, $\Theta(\tilde{\lambda})$ as follows:

$$1_{D_i}(\tilde{V}_i) = \begin{cases} 1 & \text{if } \tilde{V}_i < D_i \\ 0 & \text{otherwise} \end{cases}$$
$$1_{A_{ij}}(\tilde{Z}_i) = \begin{cases} 1 & \text{if } \tilde{Z}_i < A_{ij} \\ 0 & \text{otherwise} \end{cases} \quad \text{for } j = 1,2,3,\ldots,n \tag{23}$$
$$1_{B_{ik}}(\tilde{Y}_i) = \begin{cases} 1 & \text{if } \tilde{Y}_i < B_{ik} \\ 0 & \text{otherwise} \end{cases} \quad \text{for } k = 1,2,3,\ldots,n$$

Each indicator function defines a binomial random variable.

Let $\Lambda_i^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n)$ represent the approximate loss rate for account i measured relative to the account's maximum credit limit, M_i. Define $\Lambda_i^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n)$ as

$$\Lambda_i^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n) = 1_{D_i}(\tilde{V}_i)\left(\left(G_{i0} + (1 - G_{i0})\left(\frac{1}{n}\right) \times \sum_{k=1}^{n} 1_{A_{ik}}(\tilde{Z}_i)\right)\left(\left(\frac{1}{n}\right)\sum_{j=1}^{n} 1_{B_{ij}}(\tilde{Y}_i)\right)\right) \tag{24}$$

Exhibit 2
Step-Function Approximation for Individual Credit's Loss Rate Distribution

Loss Rate	Event	Cumulative Probability of Loss Rate	Threshold Value for Latent Variable $\tilde{Y}_i$
0	$\xi(\tilde{\lambda}, 0, n)$	$\Theta(\tilde{\lambda}_i = 0)$	$B_{i1} = \Phi^{-1}(1 - \Theta(0))$
$\frac{1}{n}$	$\xi(\tilde{\lambda}, 1, n)$	$\Theta\left(\tilde{\lambda}_i = \frac{1}{n}\right)$	$B_{i2} = \Phi^{-1}\left(1 - \Theta\left(\frac{1}{n}\right)\right)$
$\frac{2}{n}$	$\xi(\tilde{\lambda}, 2, n)$	$\Theta\left(\tilde{\lambda}_i = \frac{2}{n}\right)$	$B_{i3} = \Phi^{-1}\left(1 - \Theta\left(\frac{2}{n}\right)\right)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{n-1}{n}$	$\xi(\tilde{\lambda}, n-1, n)$	$\Theta\left(\tilde{\lambda}_i = \frac{n-1}{n}\right)$	$B_{in-1} = \Phi^{-1}\left(1 - \Theta\left(\frac{n-2}{n}\right)\right)$
1	$\xi(\tilde{\lambda}, n, n)$	1	$B_{in} = \Phi^{-1}\left(1 - \Theta\left(\frac{n-1}{n}\right)\right)$

where the notation indicates that the approximation depends on n, the number of increments used to model the account's LGD and EAD cumulative distribution functions.

The Conditional Loss Rate for an Individual Account

Let $1_{A_{ij}}(\tilde{Z}_i | e_M)$ and $1_{B_{ij}}(\tilde{Y}_i | e_M)$ represent the values of the LGD and draw rate indicator functions ($j = 1, 2, 3, \ldots, n$), conditional on $\tilde{e}_M = e_M$. These conditional indicator functions define independent binomial random variables with expected values

$$E(1_{A_{ij}}(\tilde{Z}_i | e_M)) = \Phi\left(\frac{A_{ij} - \sqrt{\rho_Z}\, e_M}{\sqrt{1-\rho_Z}}\right) \quad j = 1,2,3,\ldots n \quad (25)$$

$$E(1_{B_{ij}}(\tilde{Y}_i | e_M)) = \Phi\left(\frac{B_{ij} - \sqrt{\rho_Y}\, e_M}{\sqrt{1-\rho_Y}}\right) \quad j = 1,2,3,\ldots n \quad (26)$$

An individual account's conditional loss rate is approximated as

$$\Lambda_i^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n | e_M) = 1_{D_i}(\tilde{V}_i | e_M) \times \left(\left(G_{i0} + (1-G_{i0})\left(\frac{1}{n}\right)\sum_{k=1}^{n} 1_{A_{ik}}(\tilde{Z}_i | e_M)\right) \times \left(\left(\frac{1}{n}\right)\sum_{j=1}^{n} 1_{B_{ij}}(\tilde{Y}_i | e_M)\right)\right) \quad (27)$$

The Loss Rate on an Asymptotic Portfolio of Revolving Credits

We considered a portfolio composed of N accounts that are identical except for their idiosyncratic risk factors.[9] Let $\Lambda_p^A(\vec{\tilde{V}}, \vec{\tilde{Z}}, \vec{\tilde{Y}}, n | e_M)$ represent the approximate loss rate on this portfolio of these N accounts conditional on a realization of e_M, and n increments in the step-function approximation,

$$\Lambda_p^A(\vec{\tilde{V}}, \vec{\tilde{Z}}, \vec{\tilde{Y}}, n | e_M) = \left(\frac{\sum_{i=1}^{N}(M \cdot \Lambda^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n | e_M))}{N \cdot M}\right) = \left(\frac{\sum_{i=1}^{N}(\Lambda^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n | e_M))}{N}\right) \quad (28)$$

Because the individual account conditional loss rates are independent and identically distributed, the Strong Law of Large Numbers requires

$$\begin{aligned}
&\lim_{N\to\infty} \Lambda_p^A(\vec{\tilde{V}}, \vec{\tilde{Z}}, \vec{\tilde{Y}}, n | e_M) \\
&= \lim_{N\to\infty}\left(\frac{\sum_{i=1}^{N} \Lambda^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n | e_M)}{N}\right) \\
&\xrightarrow[a.s.]{} E(\Lambda^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n | e_M)) \\
&\approx E(1_D(\tilde{V}_i | e_M))\left(G_0 + (1-G_0)\left(\frac{1}{n}\right) \times \sum_{j=1}^{n} E(1_{A_j}(\tilde{Z}_i | e_M))\right) \\
&\times \left(\left(\frac{1}{n}\right)\sum_{j=1}^{n} E(1_{B_j}(\tilde{Y}_i | e_M))\right) = \Phi\left(\frac{D - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right) \\
&\times \left(G_0 + (1-G_0)\left(\frac{1}{n}\right)\sum_{j=1}^{n}\Phi\left(\frac{A_j - \sqrt{\rho_Z}\, e_M}{\sqrt{1-\rho_Z}}\right) \times \left(\left(\frac{1}{n}\right)\sum_{j=1}^{n}\Phi\left(\frac{B_j - \sqrt{\rho_Y}\, e_M}{\sqrt{1-\rho_Y}}\right)\right)\right)
\end{aligned} \quad (29)$$

Expression (29) is an approximation for the inverse of the conditional distribution function for an asymptotic portfolio's loss rate evaluated at $e_M \in (-\infty, +\infty)$. Propositions 1 and 2 in the appendix can be used to show that, in the limit, as $n \to \infty$, the approximation converges to the true underlying asymptotic portfolio conditional loss rate. An asymptotic portfolio's loss rate cumulative distribution function can be approximated by the implicit function

$$\begin{aligned}
\tilde{\Lambda}_p^A \sim \Bigg\{ &\Phi\left(\frac{D - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right) \cdot \left(G_0 + (1-G_0)\left(\frac{1}{n}\right) \times \sum_{j=1}^{n}\Phi\left(\frac{A_j - \sqrt{\rho_Z}\, e_M}{\sqrt{1-\rho_Z}}\right) \times \left(\left(\frac{1}{n}\right)\sum_{j=1}^{n}\Phi\left(\frac{B_j - \sqrt{\rho_Y}\, e_M}{\sqrt{1-\rho_Y}}\right)\right)\right), 1 - \Phi(e_M) \Bigg\}
\end{aligned}$$

for $e_M\ (-\infty, \infty)$ (30)

EXAMPLES OF PORTFOLIO DRAW RATE AND *LGD* DISTRIBUTIONS

In this section, three alternative unconditional account-level *LGD* and draw rate distributions are used to generate portfolio level distributions. These account-level distributions could represent individual account draw rates or *LGD* rates depending on the specific application. The three distributions considered are all members of the Beta family. The Beta family of distributions is ideal for these applications because it has a very flexible shape and its domain can be restricted to the [0,1] range.

In the examples that follow, the Beta distribution parameters are selected so that one account-level unconditional distribution is skewed right, one is symmetric, and one is skewed left. The analysis demonstrates that skewness and correlation among individual *LGD* and *EAD* distributions are important determinants of the shape of the asymptotic portfolio *LGD* and draw rate distributions. Although the examples could represent either individual account draw rate or *LGD* rate distributions, to simplify the discussion they are described as if they represent *LGD* distributions.

Positively Skewed Distribution

The density function for the Beta distribution with the first parameter (alpha) equal to 1.6 and the second parameter (beta) equal to 7, plotted in Exhibit 3, is

$$\begin{aligned}&\tilde{\lambda} \sim Beta(1.6,\, 7,\, \tilde{\lambda})\\ &Beta(1.6,\, 7,\, \lambda) = \frac{\Gamma(8.6)}{\Gamma(1.6)\Gamma(7)}\\ &\quad \times \lambda^{0.6}(1-\lambda)^{6} \quad \text{for } 0 < \lambda < 1\end{aligned} \tag{31}$$

where $\Gamma(b) = \int_0^\infty y^{b-1}e^{-y}dy,\ b > 0,$ is the mathematical gamma function. This unconditional distribution is skewed right and could, for example, represent the random draw rates on revolving corporate credits or the loss given default rates on wholesale bank loans or alt-A mortgages.

Exhibit 4 plots the asymptotic portfolio's *LGD* distribution for alternative correlation assumptions assuming that individual *LGD*s are distributed *Beta*(1.6, 7) and that *LGD* correlation is driven by the single common factor structure previously described. When individual credit loss rates are uncorrelated, the portfolio's unconditional *LGD* distribution converges to $E(Beta(1.6, 7)) = 0.1862$. When correlation among account-level *LGD* realizations increases,

EXHIBIT 3

***Beta* (1.6,7) Probability Density**

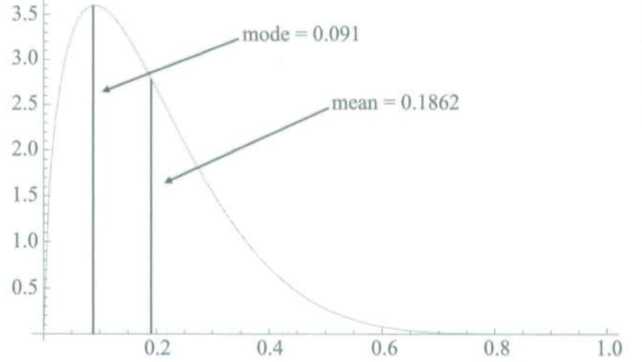
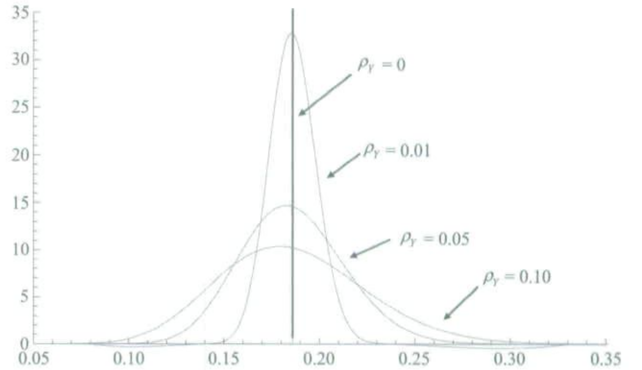


EXHIBIT 4

Asymptotic Portfolio Unconditional LGD Distribution for Alternative Correlations when Individual Account LGDs are Distributed *Beta* (1.6,7)



the range of the portfolio *LGD* distribution increases and the distribution becomes increasingly positively skewed. As correlation approaches 1, the ability to diversify *LGD* risk within the portfolio diminishes. When correlation reaches 1, the asymptotic portfolio *LGD* distribution converges to the *Beta*(1.6, 7) distribution (not shown).

Negatively Skewed Distribution

The second unconditional distribution example considered is the Beta distribution with parameters alpha = 4 and beta = 1.1:

$$\begin{aligned}&\tilde{\lambda} \sim Beta(4,\, 1.1,\, \tilde{\lambda})\\ &Beta(1.6,\, 7,\, \pi) = \frac{\Gamma(5.1)}{\Gamma(4)\Gamma(1.1)}\lambda^{3}(1-\lambda)^{0.1}\\ &\quad \text{for } 0 < \lambda < 1\end{aligned} \tag{32}$$

This negatively skewed distribution, plotted in Exhibit 5, could represent individual draw rates or *LGD* rates on subprime credit card accounts or other revolving retail credits.

Exhibit 6 plots the asymptotic portfolio *LGD* distribution generated under different correlation assumptions when individual account *LGDs* are distributed $\tilde{\lambda} \sim Beta$ (4,1.1). When account-level *LGDs* are uncorrelated, portfolio-level *LGD* risk is completely diversified, and the asymptotic portfolio's *LGD* distribution converges to $E(Beta(4,1.1)) = 0.784$. As the correlation among account level *LGDs* increases, the portfolio-level *LGD* distribution becomes increasingly negatively skewed; it converges to *Beta*(4,1.1) for $\rho_Y = 1$.

Symmetric Distribution

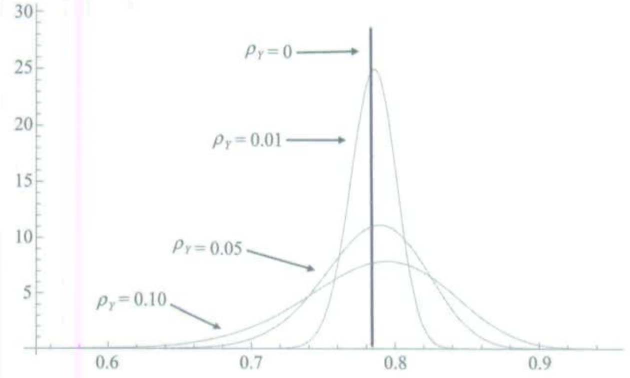
The final example is the Beta distribution with parameters alpha = 7 and beta = 7,

$$\tilde{\lambda} \sim Beta(7,7,\tilde{\lambda})$$
$$Beta(7,7,\lambda) = \frac{\Gamma(14)}{\Gamma(7)\Gamma(7)} \times \lambda^6 (1-\lambda)^6 \quad \text{for } 0 < \lambda < 1 \tag{33}$$

This distribution, plotted in Exhibit 7, is symmetric and could represent *LGD* rates on investment-grade corporate debt.

Exhibit 8 plots, for different correlation assumptions, the unconditional asymptotic portfolio *LGD* distribution that is generated when *LGDs* are distributed $\tilde{\lambda} \sim Beta(7,7)$. The portfolio *LGD* distribution converges to $E(Beta(7,7)) = 0.5$ when individual *LGD* realizations are uncorrelated. As the correlation increases, the range of the asymptotic portfolio *LGD* distribution increases, and the portfolio *LGD* distribution converges to *Beta*(7,7) as $\rho_Y \to 1$.

EXHIBIT 6
Asymptotic Portfolio Unconditional LGD Distribution for Alternative Correlations when Individual Account LGDs are Distributed *Beta* (4,1.1)



Sample Asymptotic Portfolio Unconditional Loss Rate Distributions

This section illustrates the calculation of unconditional loss rate distributions for alternative asymptotic portfolios. The examples represent hypothetical portfolios that are broadly consistent with the stylized facts associated with selected fixed-term loans and revolving credit facilities for both wholesale and retail credits.

Published evidence on the shape and correlations of account-level *LGD* and *EAD* distributions is limited.

EXHIBIT 5
***Beta* (4,1.1) Probability Density**

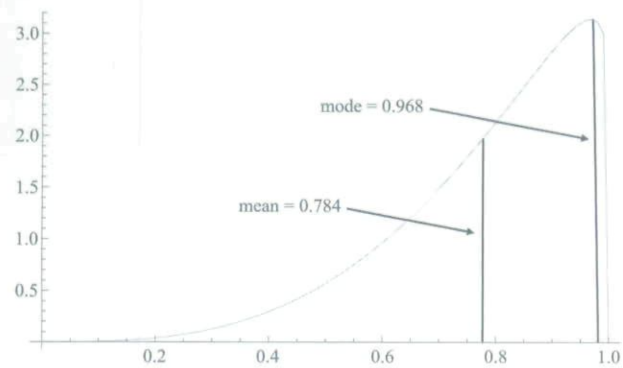


EXHIBIT 7
***Beta* (7,7) Probability Density**

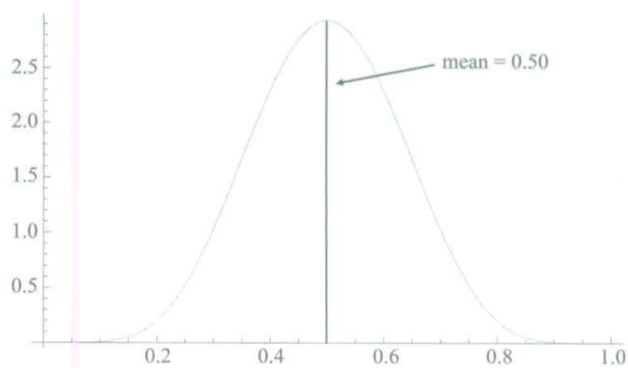
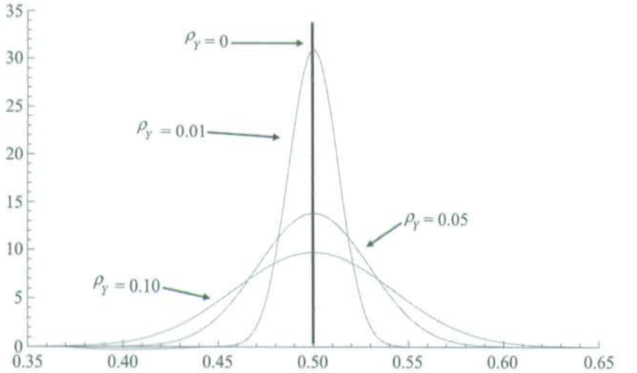


EXHIBIT 8

Asymptotic Portfolio Unconditional LGD Distribution for Alternative Correlations when Individual Account LGDs are Distributed *Beta* (7,7)



Few studies characterize the shape of individual account *EAD* distributions, and no study has attempted to estimate the strength of *EAD* correlation in a structural model.[10] A larger number of studies focus on the distribution of *LGD* rates, but the evidence is sparse and much of it is specialized to default rates for agency-rated credits.

Most studies investigating *LGD* correlation behavior investigate time series correlation estimates between observed default frequencies and default recovery rates. Only one study estimates a structural model *LGD* correlation parameter. Frye [2000b] estimates ρ_Y to be about 20% for agency-rated bonds, but his estimate is based on a structural model that assumes that *LGD* distributions are symmetric. It seems likely that alternative specifications for *LGD* that include significant skew in account-level *LGD* distributions will produce more modest estimates of correlation, but such issues have yet to be studied. Also, as noted by Carey and Gordy [2004], most *LGD* correlation estimates have been derived from rating-agency bond data, and the correlations for different liability classes are likely to differ according to firm capital structure characteristics and the identity of important stakeholders, including the presence (or absence) of significant banking interests.

A review of the publicly available literature suggests that the shapes of individual unconditional *LGD* and *EAD* distributions, as well as the magnitudes of their correlations, are largely open issues. This study will not contribute on the issue of model calibration, but will illustrate the asymptotic portfolio loss rate distributions that are generated under alternative structural model parameterizations.

Hypothetical Portfolio of Term Loans

The first example is chosen to represent the portfolio loss rate distribution on a portfolio of senior secured term loans to non-investment-grade borrowers. Exhibit 9 plots the distribution of projected *LGD* rates on loans that receive a recovery rating by Fitch Ratings.[11] Fitch Ratings reports that a large share of the loans that are referenced in Exhibit 9 are secured first-lien loans which explains the favorable projected recovery rate distribution. Exhibit 9 is based on a forward-looking *LGD* rate distribution that is not conditioned on any realized state of the economy, so it proxies for an unconditional *LGD* distribution.

The distribution in Exhibit 9 is not very granular. While this discrete distribution could be used directly in Expression (30), it is broadly similar to the continuous *Beta*(1.6,7) distribution, so we will use the latter continuous distribution to represent the *LGD* distribution for individual secured first-lien loans. To construct the asymptotic portfolio loss rate distribution for this class of exposures, we assume that all loans are fully drawn ($EAD = 1$) and that individual credits have an unconditional probability of default of one-half of a percent. The default correlation parameter is set at 20% ($\rho_V = 0.20$) to reflect both the wholesale nature of these credits and the average corporate correlation used in the Basel AIRB capital framework.

The asymptotic portfolio loss distribution for this class of credits is plotted for different *LGD* correlation assumptions (ρ_Y) in Exhibit 10. As the correlation between account *LGD* rates increases, the skewness of the asymptotic portfolio's loss rate distribution increases. As correlation increases from zero to 10%, the 99.5% critical value of the portfolio loss rate distribution increases by almost 60%. When individual *LGD* correlations are 20%, the portfolio 99.5% critical value is almost double the estimate

EXHIBIT 9

Projected LGD Distribution for Loans Rated by *FitchRatings*

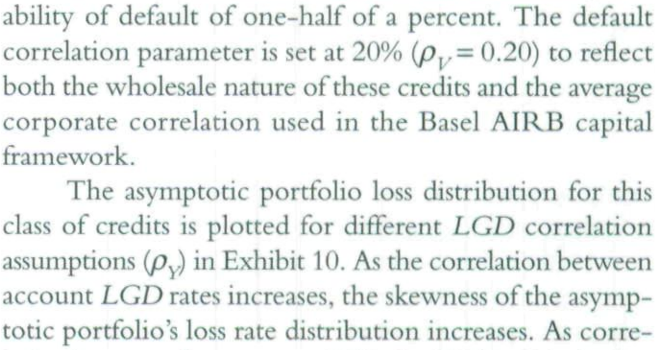
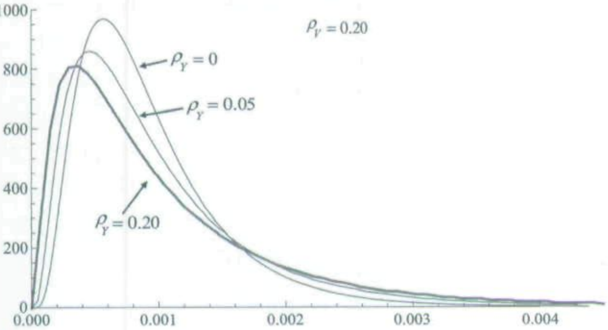


EXHIBIT 10
Asymptotic Portfolio Loss Rate Distribution for Alternative LGD Correlation Assumptions



Individual Accounts have PD = 0.5%, Default Correlation of 20%, Fixed EAD and Individual LGD ~ Beta (1.6,7).

produced by the simple Vasicek ASRF formula that assumes uncorrelated *LGD*s.

Hypothetical Portfolio of Revolving Senior Unsecured Credits

A second example is chosen to represent the loss rate distribution of an asymptotic portfolio of revolving senior unsecured bank loans made to investment-grade obligors. The example assumes that portfolio obligors begin with a 30% facility utilization rate and draw on their remaining credit line over the subsequent period. Because these are wholesale credits, we use a default correlation assumption consistent with Basel II ($\rho_V = 0.20$). We examine the shape of an asymptotic portfolio loss rate distribution under alternative correlation assumptions for *LGD* and *EAD*.

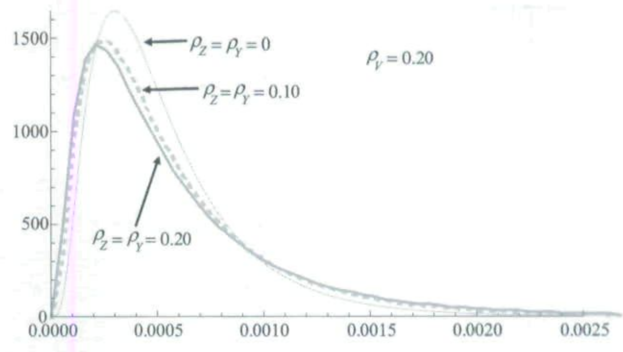
Data reported by Altman [2006, Exhibit 2] suggests that historical loss rates on senior unsecured bank loans are nearly symmetric, with an average loss rate of about 50% and a standard deviation of about 25%. For purposes of this example, the *Beta*(7,7) distribution provides a reasonable approximation to this *LGD* distribution. Araten and Jacobs [2001, Exhibit 1] estimate that, for historical Chase data, a BBB+/BBB credit has, on average, about a 55% loan-equivalent value a year before default. No published study provides any additional information on the *EAD* distributions for such facilities. The assumption, however, of an initial utilization rate of 30% and a *Beta*(1.6,7) draw rate distribution matches both the mean of the Araten and Jacobs *EAD* data, as well as conventional wisdom which suggests that bankers are at least partially successful at limiting credit line takedowns by distressed obligors. We assume an unconditional default rate of 0.25%.

Exhibit 11 plots estimates of the asymptotic portfolio loss rate distribution under alternative correlation assumptions for *LGD* (ρ_Y) and *EAD* (ρ_z). Correlation among account *LGD* and *EAD* realizations reduces the potential for credit loss diversification. As correlation in *EAD*s and *LGD*s increases from zero to 10% (zero to 20%), the portfolio loss distribution suggests that the credit loss rate associated with a 99.5% cumulative probability increases by 43% (64%).

Hypothetical Portfolio of Subprime Retail Credits

The final example is designed to mimic the behavior of credit losses on a subprime credit card portfolio. Unlike earlier examples, we are unable to reference any published study to anchor our choice of distributional assumptions. Individual accounts are assumed to have an unconditional probability of default of 4% and, consistent with the Basel AIRB treatment of qualified retail exposures, default correlations are assumed to be 4%. Customers are assumed to begin the period with 20% utilization of their credit limits and are assumed to draw on the remaining 80% of their credit limit with a draw rate modeled using the *Beta*(4,1.1) distribution. Because these are unsecured

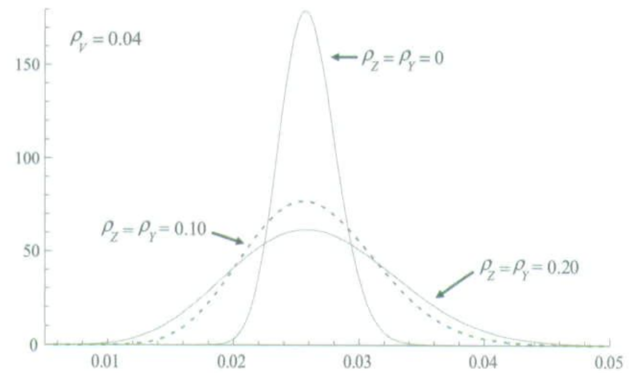
EXHIBIT 11
Asymptotic Portfolio Loss Rate Distribution for Alternative LGD and EAD Correlation Assumptions



Individual Accounts have PD = 0.25%, Default Correlation of 20%, Individual Account LGD ~ Beta (7,7), 30% Initial Drawn Balance, 70% Revolving Balance, and an Individual Draw Rate ~ Beta (1,6,7).

EXHIBIT 12

Asymptotic Portfolio Loss Rate Distribution for Alternative LGD and EAD Correlation Assumptions



Individual Accounts have PD = 4%, Default Correlation of 4%, Individual Account LGD ~ (4,1.1), 20% Initial Drawn Balance, 80% Revolving Balance, and an Individual Draw Rate ~ Beta (4,1.1).

credits, recovery rates are low and account *LGD*s are assumed to follow the *Beta*(4,1.1) distribution.

Exhibit 12 plots estimates of the asymptotic portfolio's loss rate distribution under alternative assumptions for *LGD* and *EAD* correlations. The plots show that correlation in account-level *LGD* and *EAD* distributions has only a modest effect on the tails of the portfolio's credit loss distribution. As correlation in *EAD*s and *LGD*s increases from zero to 10% (zero to 20%), the loss value associated with a 99.5% cumulative probability increases by 26% (35%). The relatively modest correlation impact is driven by the extreme negative skew in the assumed *LGD* and *EAD* distributions; most of the probability mass is associated with high account draw and loss rates.

CONCLUSION

This article has developed a tractable generalization of the single common factor portfolio credit loss model that includes correlated stochastic exposures and loss rates. The model yields an exact closed-form representation when individual account *LGD* and *EAD* distributions are continuous and a closed-form approximation when *LGD* and *EAD* distributions are discrete. The model does not restrict *EAD* or *LGD* distributions or their correlations. The closed-form representation of an asymptotic portfolio's inverse cumulative conditional loss rate can be used to calculate the unconditional portfolio loss rate distribution, economic capital allocations, and support level for securitization structures. Portfolio loss rate distributions are illustrated for representative wholesale and retail portfolios. The results show that the additional systematic risk created by positive correlation among individual account *EAD* and *LGD* realizations increases the skewness of an asymptotic portfolio's loss rate distribution. The additional skewness will increase potential loss projections for lower tranches of structured products and securitizations, and mandate the need for additional economic capital compared with Vasicek model-based estimates.

APPENDIX

DISCRETE APPROXIMATION FOR THE PORTFOLIO LOSS DISTRIBUTION

Let $\Xi(\tilde{a})$ represent the cumulative density function for $\tilde{a} \in [0,1]$. $\Xi(\tilde{a})$ is monotonic and non-decreasing in $\tilde{a}$. Divide the range of support for $\tilde{a}$ into n equal increments of size $\frac{1}{n}$. Define a set of overlapping events that span the support: $\{\xi(\tilde{a}, j, n)\}$, $j = 0, 1, 2, \ldots, n$, where $\xi(\tilde{a}, j, n)$ is the event, $\tilde{a} \in [0, \frac{j}{n}]$, for $j = 0, 1, 2, 3, \ldots, n$. By construction, $\Xi\left(\frac{j}{n}\right)$ is the probability that event $\xi(\tilde{a}, j, n)$ occurs.

Define $1_{\xi(\tilde{a},j,n)}$ as the indicator function for the event $\xi(\tilde{a}, j, n)$,

$$1_{\xi(\tilde{a},j,n)} = \begin{cases} 1 & \text{if } a_i \in \xi(\tilde{a}, j, n) \\ 0 & \text{otherwise} \end{cases} \tag{A1}$$

The expected value of the indicator function is the probability of occurrence of the event,

$$E(1_{\xi(\tilde{a},j,n)}) = \Xi\left(\frac{j}{n}\right), \quad j = 0, 1, 2, 3, \ldots, n \tag{A2}$$

Consider the probability associated with the intersection of two adjacent events, $\xi(\tilde{a}, j, -1, n) \cap \xi(\tilde{a}, j, n)$, as $n \to \infty$. In the limit as $n \to \infty$, $j \to \infty$, and $\frac{1}{n} \to 0$ the ratio $\frac{j}{n}$ remains unchanged, and the intersection $\xi(\tilde{a}, j-1, n) \cap \xi(\tilde{a}, j, n)$ converges to the point $\frac{j}{n} \in [0,1]$. In the limit as $n \to \infty$, the difference in the expected values of indicator functions $\frac{j-1}{n}$ and $\frac{j}{n}$ converges to the value of the probability density of $\tilde{a}$ evaluated at the point $\frac{j}{n}$, $\lim_{n\to\infty} E(1_{\xi(\tilde{a},j,n)} - E(1_{\xi(\tilde{a},j-1,n)}) = \Xi'\left(\frac{j}{n}\right)$.

If $a_i \in [0,1]$ is rational, then $a_i = \frac{j}{n}$ for some integers, j and n. When $\Xi(\tilde{a})$ is discrete, if n is sufficiently large, each rational point in the support of $\tilde{a}$ can be associated with a unique compound event, $\xi(\tilde{a}, i, n) \cap \xi(\tilde{a}, j, n)$, for some integers i and j. Lagrange established that if $a_i \in [0,1]$ is irrational and it is approximated by a rational number, then the absolute value of the approximation error is bounded and decreasing in n, $|a_i - \frac{j}{n}| < \frac{1}{\sqrt{5}n^2}$. Thus, irrational points in the support of $\tilde{a}$ can be approximated to any desired degree of accuracy by $a_i \approx \frac{j}{n}$ for some integers, j and n. Taken together, these results imply that a discrete distribution $\Xi(\tilde{a})$ can be approximated to any desired level of accuracy by choosing an appropriately large value for n.

Proposition 1: $E(\tilde{a}) = 1 - \lim_{n\to\infty}(\frac{1}{n})\Sigma_{j=0}^{n-1} E(1_{\mathrm{E}(\tilde{a},j,n)})$

Using $\lim_{n\to\infty} E(1_{\mathrm{E}(\tilde{a},j,n)}) - E(1_{\mathrm{E}(\tilde{a},j-1,n)}) = \Xi'(\frac{j}{n})$, the mathematical expectation of $\tilde{a}$ can be written $E(\tilde{a}) = \lim_{n\to\infty} \Sigma_{j=1}^{n}(\frac{j}{n})$ $(E(1_{E(\tilde{a},j,n)}) - E(1_{E(\tilde{a},j-1,n)})) = 1 - \lim_{n\to\infty} \left(\frac{1}{n}\right)\Sigma_{j=0}^{n-1} E(1_{E(\tilde{a},j,n)})$.

Proposition 2: $\lim_{n\to\infty}(\frac{1}{n})\Sigma_{j=1}^{n} \Phi\left(\frac{B_j - \sqrt{\rho_Y}\, e_M}{\sqrt{1-\rho_Y}}\right) = E(\tilde{\lambda} \mid e_M)$

Proposition 1 and Expression (29) imply $E(\tilde{\lambda} \mid e_m) = \lim_{n\to\infty}$ $[1 - (\frac{1}{n})\Sigma_{j=0}^{n-1} E(1_{B_j}(\tilde{Y}_i \mid e_m))] = \lim_{n\to\infty} [1 - \left(\frac{1}{n}\right) \Sigma_{j=0}^{n-1} \Phi\,(B_j - \sqrt{\rho_Y}\, e_M/\sqrt{1-\rho_Y})]$ Substitution of the definitions of the B_j, $\{B_j = \Phi^{-1}(1 - \Theta\,((\frac{j}{n})),\ j = 0, 1, 2, \ldots, n-1\}$ (from Exhibit 2) establishes the result. A similar logic can be used to show $\lim_{n\to\infty}$ $\left(\frac{1}{n}\right)\Sigma_{j=1}^{n} \Phi\left(\frac{A_j - \sqrt{\rho_Z}\, e_M}{\sqrt{1-\rho_Z}}\right) = E(\tilde{\delta} \mid e_m)$.

ENDNOTES

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[1]A common convention used to facilitate computation is to assume LGDs are log normally distributed. This assumption is not without costs as lognormal loss rates are unbounded. The framework in this article does not require this strong assumption. Loss rate distributions can have realistic distributions with bounded supports and the model still produces a closed-form expression for the conditional portfolio loss rate.

[2]This specification accommodates the Basel II convention that exposure estimated at the end of the one-year capital horizon is at least as large as the initial level of extended credit, $G_{i0} M_i$. The model can be generalized to recognize creditors' ability to reduce or eliminate their outstanding balances by setting $G_{i0} = 0$ and directly modeling an account's end-of-period utilization rate $\tilde{X}_i \in [1,0]$ instead of modeling an account's draw rate $\tilde{\delta}_i$.

[3]For any random variable $\tilde{s}$ with continuous density function $f(\tilde{s})$, the probability integral transformation requires that the random variable $\tilde{S}$ be distributed uniformly over the interval [0,1], when the random variable $\tilde{S}$ is defined by the probability integral transformation, $S_i = \int_{-\infty}^{s_i} f(s)\,ds$.

[4]See Gordy [2003] for further discussion.

[5]When *EAD* and *LGD* are both constant, as they are in the Vasicek ASFM framework, the formula used to estimate a capital allocation with a coverage rate of α is $\Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho_V}\,\Phi^{-1}(\alpha)}{\sqrt{1-\rho_V}}\right) \times$ $LGD \cdot EAD$. For example, this loss rate formula (with $\alpha = 0.999$) is used to calculate minimum regulatory capital requirements in the Basel II AIRB approach (see the Basel Committee on Banking Supervision [2006]).

[6]The convergence rate of the Monte Carlo estimator is $O(J^{\frac{-1}{2}})$.

[7]When $\rho_Y \to 0, \hat{\lambda}_j = \Theta^{-1}(1 - \Phi(e_j))$ and $\frac{1}{J}\Sigma_{j=1}^{J}\ \hat{\lambda}_j = \xrightarrow[a.s]{}$ $E(\hat{\lambda})$. A similar argument holds for the asymptotic portfolio draw rate.

[8]When $\rho_Y \to 1,\ \hat{\lambda}_j = \Theta^{-1}(1 - \Phi\,(-\Phi^{-1}(\alpha))) = \Phi^{-1}(\alpha)$ and $\frac{1}{J}\Sigma_{j=1}^{J} \hat{\lambda}_j = \Phi^{-1}(\alpha)$. A similar argument holds for the asymptotic portfolio draw rate.

[9]The threshold values for the account level *LGD* and draw rate distributions are identical across credits.

[10]Araten and Jacobs [2001] provide simple descriptive statistics on a sample of Chase revolving facilities. Jiménez, Lopez, and Saurina [2006] provide an aggregated histogram of all corporate *EAD*s derived from the Spanish credit registry for all credit institution loans in excess of €6,000 over a period spanning 1984–2005.

[11]Exhibit 7 was constructed by the author from information provided in Fitch Ratings [2006].

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