Cross-Currency Equity Swaps in the BGM Model

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Under the arbitrage-free framework of the HJM model (Heath, Jarrow and Morton [1992]), this article simultaneously extends the BGM model (Brace, Gatarek and Musiela [1997]) from a single-currency economy to a cross-currency case and incorporates the stock price dynamics under the martingale measure. The resulting model is very general for pricing almost every kind of cross-currency equity swap traded in over-the-counter markets. Pricing formulas for equity swaps with hedged or unhedged exchange rate risk are derived using either a constant or a variable notional principal. The calibration procedure, hedging strategies, and numerical examples are also provided.

An equity swap is an agreement that designates two counterparties to exchange periodically two streams of payments over a prespecified period. One party agrees to pay the return on a predetermined stock or on a stock index on a predetermined notional principal, while the other party agrees to pay a predetermined fixed rate, a floating rate, or the return on another stock or stock index on the same notional principal. There is no principal exchange involved in an equity swap and the associated assets can be either domestic or foreign. The cash flow streams can be denominated in the same currency or in a different currency.

The use of equity swaps has been rapidly growing since their introduction in the late 1980s. As a powerful hedging tool, equity swaps have integrated money and equity markets. For example, equity swaps have been used by fund managers to transform their exposure from a stock market to a money market. Equity swaps not only allow investors to mimic investments in the stock market, but also are cheaper and more convenient than direct investment.

Equity swaps have been adopted to simplify portfolio management by avoiding cumbersome administrative processes. They can also be used to circumvent investment restrictions and engage in tax arbitrage, with taxes such as the transfer tax, withholding tax on dividends, capital gains tax, and so on. Equity swaps make it easier to invest in emerging markets, reducing liquidity risk and credit risk. Moreover, investors can convert the returns of their stock holdings to returns of other assets without losing control over the underlying stocks, so they retain their votes on new business plans and their ability to serve on the board of directors.

Equity swaps have various structures, differing, for example, according to whether their notional principal is variable or fixed, whether the streams of payments being swapped are denominated in a single-currency or cross-currency, and whether the exchange rate risk is hedged or unhedged. With a high degree of flexibility, equity swaps can meet almost all investors' needs.

There is some early research on the pricing of equity swaps. Marshall et al. [1992] provided the pricing model for equity swaps, but the model included expectation operators and was not a complete arbitrage-free valuation formula. Rich [1995] employed a forward-start forward contract approach to price equity swaps. Jarrow and Turnbull [1996] provided a preference-free formula for basic equity-for-fixed swaps, but the formula was applicable only at the initiation of the underlying swap and did not cover valuation during its subsequent lifetime. Thus, at first glance, the formula appears to imply that the stock price doesn't affect the equity swap rate. Chance and Rich [1998] contributed to the literature on equity swaps by providing the valuation formulas for several equity swap variants in the framework of arbitrage-free replicating portfolios. In these formulas, the value of equity swaps is affected not only by the term structure of interest rates, but also by the stock price if it is priced between the reset dates. In a stochastic interest rate economy, Kijima and Muromachi [2001] provided the pricing model of equity-for-fixed equity swaps with constant and variable notional principal structures. Under the Amin and Jarrow [1991] cross-currency model setting, Wang and Liao [2003] adopted the risk-neutral valuation method to price several types of cross-currency two-way equity swaps.

Although many pricing models of the various types of equity swaps have been presented, there are few articles on the floating-for-equity type. One possible reason for this is that the interest rate model used in Kijima and Muromachi [2001] and Wang and Liao [2003] is the Gaussian HJM framework. HJM modeled the term structure of interest rates by specifying the dynamics of instantaneous forward rates. The instantaneous forward rates are not appropriate for pricing equity-for-floating swaps with a paid-in-arrears feature because they are continuously compounded rather than simply compounded. Furthermore, the instantaneous forward rates are not observable in the market, so the recovery of the model parameters from the market-observed data is a difficult task. In addition, the Gaussian HJM forward rates can become negative with a positive probability, which may cause some pricing error.

BGM [1997] developed a continuous-time model of simple forward LIBOR rates that are market-observable quantities. Since the forward LIBOR rates in the BGM model (the LIBOR market model) are simply compounded, they are suitable for pricing swaps with a floating leg associated with a paid-in-arrears feature. Because the forward LIBOR rates are market observable, the BGM model circumvents the difficulty of transforming the traded quantities observed in the market into the model parameters. Avoiding the explosion problem in the HJM framework, BGM assumes that forward LIBOR rates have a lognormal volatility structure, which prevents the forward rates from becoming negative with a positive probability.

The purpose of this article is simultaneously to extend the BGM model from a single-currency economy to a cross-currency case and to incorporate the stock prices and exchange rate processes. The resulting model is very general for pricing almost every kind of cross-currency equity swap traded in over-the-counter markets. Unlike the formulas derived under the Gaussian HJM model in the previous articles, the parameters in our formulas can be acquired easily from the market quantities. Because the LIBOR rates are positive, there is no pricing error arising from possible negative rates in the Gaussian HJM model.

This article is organized as follows: Under the arbitrage-free relation between the drift and volatility terms in the HJM model, an arbitrage-free extended BGM model is derived in the second section. The third section develops the pricing formulas of three common variants of cross-currency equity swaps. The fourth section provides the calibration procedure for practical operation and a numerical example showing how the model actually works. A conclusion is given in the last section.

ARBITRAGE-FREE EXTENDED BGM MODEL

In this section, we first briefly review the result in Amin and Jarrow [1991] and then employ the result to extend the BGM model.

Review of AJ Model

We assume that trading takes place continuously in time over an interval $[0, \tau]$, $0 < \tau < \infty$. The uncertainty is described by the filtered probability space $(\Omega\, F,\ Q,\ \{F\}_{t\in[0,\tau]})$, where the filtration is generated by independent standard Brownian motions $W(t) = (W_1(t), W_2(t), \ldots, W_m(t))$. Q represents the domestic spot martingale probability measure. We list the notations with "d" for domestic and "f" for foreign as follows:

$f_k(t, T)$ = The kth country's forward interest rate contracted at time t (or today) for instantaneous borrowing and lending effective at time T with $0 \le t \le T \le \tau$, where τ is the investment horizon and $k \in \{d, f\}$.

$P_k(t, T)$ = The time t price of the kth country's zero coupon bond (ZCB) paying one currency unit at maturity T.
$S_k(t)$ = The time t price of the kth country's asset (or stock).
$r_k(t)$ = The kth country's risk-free short rate at time t.
$\beta_k(t) = \exp\left[\int_0^t r_k(u)du\right]$, the kth country's money market account at time t with initial value $\beta_k(0) = 1$.
$X(t)$ = The spot exchange rate at $t \in [0, \tau]$ for one unit of the foreign currency expressed in terms of units of the domestic currency.

By incorporating the asset price dynamics into the AJ model, the extended AJ model under the domestic martingale measure Q is given in Proposition 1.[1]

Proposition 1. The dynamics under the domestic martingale measure.

Under the domestic martingale measure Q, for any $T \in [0, \tau]$, the dynamics of the forward rates, the ZCB prices, the stock prices, and the exchange rate are

$$df_d(t,T) = \sigma_{fd}(t,T)\cdot\sigma_{Pd}(t,T)dt + \sigma_{fd}(t,T)\cdot dW(t)$$

$$df_f(t,T) = \sigma_{ff}(t,T)\cdot\left[\sigma_{Pf}(t,T) - \sigma_X(t)\right]dt + \sigma_{ff}(t,T)\cdot dW(t)$$

$$\frac{dP_d(t,T)}{P_d(t,T)} = r_d(t)dt - \sigma_{Pd}(t,T)\cdot dW(t)$$

$$\frac{dP_f(t,T)}{P_f(t,T)} = \left[r_f(t) + \sigma_X(t)\cdot\sigma_{Pf}(t,T)\right]dt - \sigma_{Pf}(t,T)\cdot dW(t)$$

$$\frac{dS_d(t)}{S_d(t)} = r_d(t)dt + \sigma_{Sd}(t)\cdot dW(t)$$

$$\frac{dS_f(t)}{S_f(t)} = \left[r_f(t) - \sigma_X(t)\cdot\sigma_{Sf}(t)\right]dt + \sigma_{Sf}(t)\cdot dW(t)$$

$$\frac{dX(t)}{X(t)} = \left(r_d(t) - r_f(t)\right) + \sigma_X(t)\cdot dW(t)$$

where $\sigma_{fk}(t, T)$ denotes the forward rate volatility of the domestic ($k = d$) or the foreign ($k = f$) country. Other double-subscript notations can be explained accordingly.

It is worth emphasizing that even if the more general AJ model, as given in Proposition 1, is considered, *the drift restriction of the domestic forward rate for no-arbitrage* is still not changed. However, for the foreign forward rate dynamics, the drift appears to have one additional term, $\sigma_{ff}(t, T)\cdot\sigma_X(t)$, which specifies the instantaneous correlation between the exchange rate and the foreign forward rate. It is also observed that the drift terms of the foreign assets are augmented by the instantaneous correlations between the exchange rate and the assets.

The Extended BGM Model

This subsection employs the arbitrage-free relations between the drift and volatility terms in Proposition 1 to establish the extended BGM model under the martingale measure.

Fixing some accrual period $\delta > 0$, we define the forward LIBOR rate process $\{L_k(t, T); 0 \le t \le T\}$ as

$$1+\delta L_k(t,T) = \frac{P_k(t,T)}{P_k(t,T+\delta)} = \exp\left(\int_T^{T+\delta} f_k(t,u)du\right) \tag{1}$$

where δ is the length of the compounding period $[T, T + \delta]$. The conditions which follow, regarding the stochastic processes of the forward LIBOR rates, the asset prices, and the exchange rate, specify volatility terms with a lognormal structure. The drift term of each process is defined later.

Condition 1. A family of LIBOR rate processes.

We assume that $L_k(t, T)$ has a lognormal volatility structure and its dynamic process is given by

$$dL_k(t, T) = \mu_{Lk}(t, T)dt + L(t, T)\gamma_{Lk}(t, T)\cdot dW(t) \tag{2}$$

where $\gamma_{Lk}(\cdot, T)$ is a deterministic volatility vector function satisfying the standard regularity conditions and $\mu_{Lk}(t, T)$ is some drift function.

Condition 2. The asset price dynamics.

The dynamics of the asset price can be expressed as

$$dS_k(t) = S_k(t)\mu_{Sk}(t) + S_k(t)\sigma_{Sk}(t)\cdot dW(t) \tag{3}$$

where $\sigma_{Sk}(t)$ is a deterministic volatility vector function satisfying the standard regularity conditions and $\mu_{Sk}(t)$ is some drift function.

Condition 3. The spot exchange rate dynamics.

The dynamics of the spot exchange rate $X(t)$ can be expressed as

$$dX(t) = X(t)\mu_X(t) + X(t)\sigma_X(t) \cdot dW(t) \tag{4}$$

where $\sigma_X(t)$ is a deterministic volatility vector function satisfying the standard regularity conditions and $\mu_X(t)$ is some drift function.

It is important to emphasize that the drift terms of the preceding stochastic processes are not yet determined. The specific forms of the drift terms must be determined to make the economy arbitrage-free. We will use the arbitrage-free relation between the drift and the volatility terms in Proposition 1 to determine the drift terms in (2), (3) and (4).

First, let us determine the drift term $\mu_{Lf}(t, T)$ in the foreign forward LIBOR rate process. Assume that $Y(t) = \int_T^{T+\delta} f_f(t,u)du$ and $L_f(t, T) = (1/\delta)(\exp(Y(t)) - 1)$. Making use of Itô's lemma, we can derive the dynamics of $L_f(t, T)$ and $Y(t)$ as follows:

$$\begin{aligned} dL_f(t,T) &= \frac{1}{\delta}\exp\left(\int_T^{T+\delta} f_f(t,u)du\right) \\ &\quad \times \left\{ dY(t) + \frac{1}{2}dY(t)dY(t) \right\} \end{aligned} \tag{5}$$

and

$$\begin{aligned} dY(t) &= \left(\frac{1}{2}\left\|\sigma_{Pf}(t,T+\delta)\right\|^2 - \frac{1}{2}\left\|\sigma_{Pf}(t,T)\right\|^2\right)dt \\ &\quad - \sigma_X(t)\cdot\left(\sigma_{Pf}(t,T+\delta) - \sigma_{Pf}(t,T)\right)dt \\ &\quad + \left(\sigma_{Pf}(t,T+\delta) - \sigma_{Pf}(t,T)\right)dW(t) \end{aligned} \tag{6}$$

Combining (5) and (6), we have

$$\begin{aligned} dL_f(t,T) &= \frac{1}{\delta}\left(1+\delta L_f(t,T)\right)\left(\sigma_{Pf}(t,T+\delta) - \sigma_{Pf}(t,T)\right) \\ &\quad \times \left(\sigma_{Pf}(t,T+\delta) - \sigma_X(t)\right)dt \\ &\quad + \frac{1}{\delta}\left(1+\delta L_f(t,T)\right) \\ &\quad \times \left(\sigma_{Pf}(t,T+\delta) - \sigma_{Pf}(t,T)\right)\cdot dW(t) \end{aligned} \tag{7}$$

Equation (7) indicates that the relation, under the martingale measure, between the drift and the diffusion terms of the foreign forward LIBOR rate processes can be used to determine $\mu L_f(t, T)$. By using Condition 1 and assuming that the foreign forward LIBOR rate's volatility structure is lognormal, we may compare the diffusion coefficients of (2) and (7) to obtain (8), which provides the arbitrage-free definition for the drift term,

$$\begin{aligned} &\frac{1}{\delta}\left(1+\delta L_f(t,T)\right)\left(\sigma_{Pf}(t,T+\delta) - \sigma_{Pf}(t,T)\right) \\ &\quad = L_f(t,T)\gamma_{Lf}(t,T) \end{aligned} \tag{8}$$

By substituting (8) into the drift term in (7), the foreign forward LIBOR rate dynamics under the martingale measure can be expressed as

$$\begin{aligned} \frac{dL_f(t,T)}{L_f(t,T)} &= \gamma_{Lf}(t,T)\cdot\left(\sigma_{Pf}(t,T+\delta) - \sigma_X(t)\right)dt \\ &\quad + \gamma_{Lf}(t,T)\cdot dW(t) \end{aligned}$$

where the foreign drift term is now formally defined under the domestic spot martingale measure Q.

Similarly, the domestic forward LIBOR process with the formally defined drift term under the measure Q can be derived as

$$\frac{dL_d(t,T)}{L_d(t,T)} = \gamma_{Ld}(t,T)\cdot\sigma_{Pd}(t,T+\delta)dt + \gamma_{Ld}(t,T)\cdot dW(t)$$

The bond volatility vector $\sigma_{Pk}(t,T)$ is not yet specified in the model. In order to make use of the HJM arbitrage-free structure, we must specify $\sigma_{Pk}(t,T)$. By using the recurrence relation of (8), the bond price volatility $\sigma_{Pk}(t,T)$ can be represented as

$$\sigma_{Pk}(t,T) = \begin{cases} \displaystyle\sum_{j=1}^{\lfloor \delta^{-1}(T-t) \rfloor} \frac{\delta L_k(t,T-j\delta)}{1+\delta L_k(t,T-j\delta)}\gamma_{Lk}(t,T-j\delta) & t\in[0,T-\delta]\;\&\;T-\delta>0 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

where $\lfloor \delta^{-1}(T-t) \rfloor$ denotes the greatest integer that is less than $\delta^{-1}(T-t)$.

Finally, we take advantage of the drift-volatility relations of the asset prices and the exchange rate processes in Proposition 1 to determine the drift functions, namely, $\mu_{Sd}(t)$, $\mu_{Sf}(t)$ and $\mu_X(t)$, under the martingale measure.

Because of the same volatility structure in Proposition 1 and Conditions 2 and 3, we have the following results:

$$\mu_{Sd}(t) = r_d(t)$$
$$\mu_{Sf}(t) = r_f(t) - \sigma_X(t) \cdot \sigma_{Sf}(t)$$
$$\mu_X(t) = r_d(t) - r_f(t)$$

These results lead to an extended LIBOR market model, given in Proposition 2, that can be used to price cross-currency derivatives.

Proposition 2. The extended LIBOR market model under the measure Q.

Under the domestic martingale measure, the processes of the forward LIBOR rates, the ZCBs, the stock prices, and the exchange rate are

$$\frac{dL_d(t,T)}{L_d(t,T)} = \gamma_{Ld}(t,T) \cdot \sigma_{Pd}(t,T+\delta)dt + \gamma_{Ld}(t,T) \cdot dW(t) \quad (10)$$

$$\frac{dL_f(t,T)}{L_f(t,T)} = \gamma_{Lf}(t,T) \cdot \left(\sigma_{Pf}(t,T+\delta) - \sigma_X(t)\right)dt + \gamma_{Lf}(t,T) \cdot dW(t) \quad (11)$$

$$\frac{dP_d(t,T)}{P_d(t,T)} = r_d(t)dt - \sigma_{Pd}(t,T) \cdot dW(t) \quad (12)$$

$$\frac{dP_f(t,T)}{P_f(t,T)} = \left[r_f(t) + \sigma_X(t) \cdot \sigma_{Pf}(t,T)\right]dt - \sigma_{Pf}(t,T) \cdot dW(t) \quad (13)$$

$$\frac{dS_d(t)}{S_d(t)} = r_d(t)dt + \sigma_{Sd}(t) \cdot dW(t) \quad (14)$$

$$\frac{dS_f(t)}{S_f(t)} = \left[r_f(t) - \sigma_X(t) \cdot \sigma_{Sf}(t)\right]dt + \sigma_{Sf}(t) \cdot dW(t) \quad (15)$$

$$\frac{dX(t)}{X(t)} = \left[r_d(t) - r_f(t)\right] dt + \sigma_X(t) \cdot dW(t) \quad (16)$$

where $t \in [0, T]$ and $\sigma_{Pk}(t, T)$ is defined in (9). If a continuous dividend is paid, $r_d(t)$ and $r_f(t)$ are, replaced by $r_d(t) - q_d(t)$ and $r_f(t) - q_f(t)$ respectively, in (14) and (15), where $q_d(t)$ and $q_f(t)$ denote, the deterministic domestic and foreign dividend processes respectively.

Unlike the instantaneous forward rates in the HJM model, the forward LIBOR rates are market observable. Therefore, the volatility $\gamma_{Lk}(t, T)$ can be extracted from the interest rate derivatives traded in the market and $\sigma_{Pk}(t, T)$ can be calculated from (9). Because of the lognormal volatility structure, the forward LIBOR rates are positive and prevent the negative rate problem in the Gaussian HJM model.

The extended LIBOR market model is very general. It is useful for pricing many kinds of swaps, such as differential swaps, equity swap, and cross-currency equity swaps. In the third section, examples of variants of the cross-currency equity swap are priced.

PRICING CROSS-CURRENCY EQUITY SWAPS

In this section, we will employ the model derived in the previous section to price three variants of cross-currency swaps.

Pricing Hedged Cross-Currency Equity Swaps with a Constant Notional Principal (HCESC)

An HCESC is defined as follows: The contract starts at time t_0 with the reset dates $t_0 \le t_1 \le \cdots \le t_{n-1}$ and the payment dates $t_1 \le t_2 \le \cdots \le t_n$. For simplicity, we define $\delta = t_j - t_{j-1}$ as the accrual period and q_f^j as the accrual dividend during the period $[t_{j-1}, t_j]$, for $j = 1,\ldots,n$.[2] During the tenor of the swap, the notional principal V_d of the contract is fixed and we assume that $V_d = 1$ in this case. At each payment date t_j, for $j = 1,\ldots, n$, one party receives in domestic currency the total return on the underlying foreign asset, $(S_f(t_j) + q_f^j)/S_f(t_{j-1}) - 1$, from the counterparty and pays the counterparty a domestic floating payment, $\delta(L_d(t_{j-1}, t_{j-1}) + K)$, where $L_d(t_{j-1}, t_{j-1})$ is the domestic LIBOR rate for the period $[t_{j-1}, t_j]$ that is applied to the notional principal as a simple rate and K is the spread in basis points.

To the party who pays the domestic floating rate and receives in domestic currency the foreign total asset return, the cash flow stream is

$$\begin{aligned}
\text{At time } t_1: &\left[\left(S_f(t_1)+q_f^1\right)/S_f(t_0)-1\right] \\
&-\delta\left[L_d(t_0,t_0)+K\right] \\
\text{At time } t_2: &\left[\left(S_f(t_2)+q_f^2\right)/S_f(t_1)-1\right] \\
&-\delta\left[L_d(t_1,t_1)+K\right] \\
\vdots \qquad & \qquad \vdots \\
\text{At time } t_n: &\left[\left(S_f(t_n)+q_f^n\right)/S_f(t_{n-1})-1\right] \\
&-\delta\left[L_d(t_{n-1},t_{n-1})+K\right]
\end{aligned}$$

The pricing formula of an HCESC is presented in the following theorem and the proof is given in Appendix A.

Theorem 1. The pricing formula of an HCESC. *Under the extended BGM model, the price of an* HCESC *at time* s, $t_0 \le s \le t_1$, is

$$\begin{aligned}
\text{HCESC} = &\frac{S_f(s)}{S_f(t_0)} QA_1(s,t_1) - \left(1+\delta L_d(t_0,t_0)\right) \\
&\times P_d(s,t_1) - \delta K \sum_{j=1}^{n} P_d(s,t_j) \\
&+ \sum_{j=2}^{n} P_f(s,t_{j-1}) QA_1(s,t_j) - \sum_{j=2}^{n} P_d(s,t_{j-1}) \\
&+ \frac{q_f^1}{S_f(t_0)} P_d(s,t_1) \\
&+ \sum_{j=2}^{n} P_d(s,t_j) \frac{q_f^j}{\hat{S}_f(s,t_{j-1})} \xi(s,t_{j-1})
\end{aligned} \tag{17}$$

where

$$\hat{S}_f(s,t_j) = \frac{S_f(s)}{P_f(s,t_j)} \tag{18}$$

$$QA_1(s,t_j) = \frac{P_d(s,t_j)}{P_f(s,t_j)} \rho_1(s,t_j) \tag{19}$$

$$\begin{aligned}
\rho_1(s,t_1) = \exp\Bigg(&\int_s^{t_1} \left(\bar{\sigma}_{Pf}^s(t,t_1) + \sigma_{Sf}(t)\right) \\
&\times \left(\bar{\sigma}_{Pf}^s(t,t_1) - \sigma_X(t) - \bar{\sigma}_{Pd}^s(t,t_1)\right) dt \Bigg)
\end{aligned}$$

and for $j = 2, \ldots, n$

$$\begin{aligned}
\rho_1(s,t_j) = \exp\Bigg(&\int_s^{t_{j-1}} \left(\bar{\sigma}_{Pf}^s(t,t_j) - \bar{\sigma}_{Pf}^s(t,t_{j-1})\right) \\
&\times \left(\bar{\sigma}_{Pf}^s(t,t_j) - \sigma_X(t) - \bar{\sigma}_{Pd}^s(t,t_j)\right) dt \\
&+ \int_{t_{j-1}}^{t_j} \left(\bar{\sigma}_{Pf}^s(t,t_j) + \sigma_{Sf}(t)\right) \\
&\times \left(\bar{\sigma}_{Pf}^s(t,t_j) - \sigma_X(t) - \bar{\sigma}_{Pd}^s(t,t_j) dt\right) \Bigg)
\end{aligned}$$

$$\begin{aligned}
\xi_1(s,t_j) = \exp\Bigg(&\int_s^{t_{j-1}} (\bar{\sigma}_{Pf}^s(t,t_{j-1}) + \sigma_{Sf}(t)) \\
&\times (\sigma_{Sf}(t) + \sigma_X(t) + \bar{\sigma}_{Pd}^s(t,t_j)) dt \Bigg)
\end{aligned}$$

and $\bar{\sigma}_{Pf}^s$ is defined in (A.8) in Appendix A.

The term $QA_1(s,\cdot)$ denotes the quanto adjustment due to the hedged risk of the exchange rate. It contains the exchange rate adjustment, which appears because the expected foreign cash flow is computed under the domestic martingale measure, and the compound correlations between all the involved factors, such as the exchange rate and the domestic and foreign forward LIBOR rates.[3] The term $q_j^i/\hat{S}_f(s,t_{i-1})$ is the forward dividend return on the foreign stock.

The advantage of adopting the BGM model over the Gaussian HJM model is that the forward LIBOR rates in (17) are the market-observable (simply compounded) forward LIBOR rates, while those under the Gaussian HJM model are market-unobservable instantaneous forward rates. Therefore, the compound correlations can be derived directly from the market quantities. As mentioned previously, the extended LIBOR model has a lognormal volatility structure, leading to positive LIBOR rates, which eliminates the pricing error arising from negative rates with a positive probability in the Gaussian HJM model.

Theorem 1 not only provides the pricing formula for the HCESC, but also reveals a clue to the construction of the hedging (replicating) portfolio. For the cash flow at t_1, the hedging portfolio is

$$H_1^{(1)} = \Delta_{11}^{(1)} S_f(s) + \Delta_{12}^{(1)} P_d(s,t_1)$$

where

$$\Delta_{11}^{(1)} = \frac{QA_1(s,t_1)}{S_f(t_0)}$$

$$\Delta_{12}^{(1)} = \frac{q_f^1}{S_f(t_0)} - (1+\delta L_d(t_0,t_0)+\delta K)$$

Thus, the hedging portfolio for the cash flow at t_1 requires holding $\Delta_{11}^{(1)}$ units of the foreign stock and $\Delta_{12}^{(1)}$ units of the domestic ZCB with maturity t_1.

For the cash flow at t_j, for $j = 2, 3, \ldots, n$, the hedging portfolio is given as follows:

$$H_j^{(1)} = \Delta_{j1}^{(1)} P_f(s,t_{j-1}) + \Delta_{j2}^{(1)} P_d(s,t_{j-1}) + \Delta_{j3}^{(1)} P_d(s,t_j)$$

where

$$\Delta_{j1}^{(1)} = QA_1(s,t_j)$$

$$\Delta_{j2}^{(1)} = -1$$

$$\Delta_{j3}^{(1)} = \frac{q_f^j}{\hat{S}_f(s,t_{j-1})}\xi(s,t_{j-1}) - \delta K$$

Thus, the hedging portfolio for the cash flow at t_j, $2 \leq j \leq n$, consists of holding $\Delta_{j1}^{(1)}$ units of the foreign ZCB with maturity t_{j-1} and $\Delta_{j3}^{(1)}$ units of the domestic ZCB with maturity t_j and selling short one unit of the domestic ZCB with maturity t_{j-1}.

By adjusting the spread K, the HCESC price at initiation is set to zero so that the trading is a fair game. This fair rate K may be called the HCESC swap rate, and is given by

$$K = \frac{R_1}{\delta \Sigma_{j=1}^{n} P_d(s,t_j)}$$

where

$$R_1 = \sum_{j=1}^{n} P_f(t_0,t_{j-1}) QA_1(t_0,t_j) - \sum_{j=1}^{n} P_d(t_0,t_{j-1}) + \sum_{j=1}^{n} P_d(s,t_j) \frac{q_f^j}{\hat{S}_f(s,t_{j-1})}\xi(s,t_{j-1})$$

The HCESC swap rate is affected by the dividend structure of the underlying foreign stock, the current term structures of interest rates of both the domestic and foreign countries, and the correlations between the involved factors. However, it is unrelated to the asset price, a finding which is contrary to the results of Chance and Rich [1998] and Kijima and Muromachi [2001] in the case of a single-currency equity swap with a constant notional principal.

In practice, as the exchange rate risk is hedged, there will be an additional adjustment, either up or down, to the cost of the swaps. The direction of the adjustment depends on the relative level of the term structures of the interest rates between the two currencies. Basically, as the yield rate differentials between the two currencies become larger with respect to the maturities, the investor who receives a lower-interest currency and pays higher-interest currency obtains a compensation in the spread. This phenomenon is reflected by the quanto adjustment in (19).

Pricing Hedged Cross-Currency Equity Swaps with a Variable Notional Principal (HCESV)

Consider the equity swap in the previous subsection, but with a variable notional principal. Denote the domestic and the foreign notional principals at t_j by V_{d,t_j} and V_{f,t_j}, respectively. In the variable notional principal case, V_{d,t_j} and V_{f,t_j} have the following relations:

$$V_{f,t_{j+1}} = V_{f,t_j} \frac{S_f(t_{j+1})}{S_f(t_j)} \qquad (20)$$

$$V_{d,t_j} = V_{f,t_j} \bar{X}$$

where $\bar{X} = X(t_0)$ and $j = 0, \ldots, n-1$.

The cash flow stream of an HCESV is

$$\begin{aligned}
\text{At time } t_1 &: V_{f,t_0}\left[\left(S_f(t_1)+q_f^1\right)/S_f(t_0)-1\right] \\
&\quad \times \bar{X} - \delta V_{d,t_0}\left[L_d(t_0,t_0)+K\right] \\
\text{At time } t_2 &: V_{f,t_1}\left[\left(S_f(t_2)+q_f^2\right)/S_f(t_1)-1\right] \\
&\quad \times \bar{X} - \delta V_{d,t_1}\left[L_d(t_1,t_1)+K\right] \\
\text{At time } t_3 &: V_{f,t_2}\left[\left(S_f(t_3)+q_f^3\right)/S_f(t_2)-1\right] \\
&\quad \times \bar{X} - \delta V_{d,t_2}\left[L_d(t_2,t_2)+K\right] \\
&\vdots \qquad\qquad \vdots \\
\text{At time } t_n &: V_{f,t_{n-1}}\left[\left(S_f(t_n)+q_f^n\right)/S_f(t_{n-1})-1\right] \\
&\quad \times \bar{X} - \delta V_{d,t_{n-1}}\left[L_d(t_{n-1},t_{n-1})+K\right]
\end{aligned}$$

This cash flow stream indicates that, at each time t_j, one party receives in domestic currency the total return on the foreign stock on the variable foreign notional principal and pays the counterparty the domestic floating interest rate plus a spread on the domestic variable notional principal.

The pricing formula of an HCESV is presented in Theorem 2; the proof is similar to Theorem 1.[4]

Theorem 2. The pricing formula of an HCESV.

Under the extended BGM model, the price of an HCESV at time s, $t_0 \le s \le t_1$, *is*

$$V_{d,t_0}\left[\frac{S_f(s)}{S_f(t_0)}QA_2(s,t_1)+\frac{q_f^1}{S_f(t_0)}P_d(s,t_1)\right.$$
$$\left.-(1+\delta K+\delta L_d(t_0,t_0))P_d(s,t_1)\right]$$
$$+\sum_{j=2}^{n}\hat{V}_{d,f,s,t_{j-1}}\left[P_f(s,t_{j-1})QA_2(s,t_j)\right.$$
$$\left.+P_d(s,t_j)\frac{q_f^j}{\hat{S}_f(s,t_{j-1})}-P_d(s,t_j)\eta_2(s,t_{j-1})\right]$$
$$-\delta\sum_{j=2}^{n}\hat{V}_{d,f,s,t_{j-1}}P_d(s,t_j)\eta_2(s,t_{j-1})$$
$$\times[L_d(s,t_{j-1})\phi_2(s,t_{j-1})+K] \tag{21}$$

where $\hat{S}_f(s, t_j)$ is defined in (18) and

$$QA_2(s,t_j)=\frac{P_d(s,t_j)}{P_f(s,t_j)}\rho_2(s,t_j)$$

$$\hat{V}_{d,k,s,t_j}=\frac{V_{d,s}}{P_k(s,t_j)},\ k\in\{d,f\}$$

$$V_{d,s}=\frac{S_f(s)}{S_f(t_0)}V_{f,t_0}\bar{X}$$

$$\rho_2(s,t_j)=\exp\left(\int_s^{t_j}\left(\bar{\sigma}_{Pf}^s(t,t_j)+\sigma_{Sf}(t)\right)\right.$$
$$\left.\times(\bar{\sigma}_{Pf}^s(t,t_j)-\sigma_X(t)-\bar{\sigma}_{Pd}^s(t,t_j))dt\right)$$

$$\eta_2(s,t_{j-1})=\exp\left(\int_s^{t_{j-1}}\left(\bar{\sigma}_{Pf}^s(t,t_{j-1})+\sigma_{Sf}(t)\right)\right.$$
$$\left.\times(\bar{\sigma}_{Pf}^s(t,t_{j-1})-\sigma_X(t)-\bar{\sigma}_{Pd}^s(t,t_j))dt\right)$$

$$\phi_2(s,t_{j-1})=\exp\left(\int_s^{t_{j-1}}(\bar{\sigma}_{Pf}^s(t,t_{j-1})+\sigma_{Sf}(t))\cdot\gamma_d(t,t_{j-1})dt\right)$$

As given in $QA_1(s,\cdot)$, the quanto adjustment due to the hedged risk of the exchange rate is also reflected in $QA_2(s,\cdot)$, and its implication is similar to $QA_1(s,\cdot)$. $\hat{V}_{d,k,s,t_j}$ may be viewed as the forward notional principal applied to the period $[t_j, t_{j+1}]$. The variables $\eta_2(s,\cdot)$ and $\phi_2(s,\cdot)$ represent the compound correlations associated with the variable notional principal and the LIBOR rate, respectively.

As explained previously Equation (21) also reveals a clue to the hedging portfolio. The hedging portfolio can be accomplished by a linear combination of the foreign stock, the domestic ZCBs, and the foreign ZCBs. For the cash flow at t_1, the hedge portfolio is given by

$$H_1^{(2)}=\Delta_{11}^{(2)}S_f(s)+\Delta_{12}^{(2)}P_d(s,t_1)$$

where

$$\Delta_{11}^{(2)}=V_{d,t_0}\frac{QA_2(s,t_1)}{S_f(t_0)}$$
$$\Delta_{12}^{(2)}=V_{d,t_0}\left[\frac{q_f^1}{S_f(t_0)}-(1+\delta K+\delta L_d(t_0,t_0))\right]$$

Thus, the hedging portfolio for the cash flow at t_1 consists of holding $\Delta_{11}^{(2)}$ units of the foreign stock and $\Delta_{12}^{(2)}$ units of the domestic ZCB with maturity t_1.

For the cash flow at t_j, $j = 2, 3, \ldots, n$, the hedging portfolio is

$$H_j^{(2)}=\Delta_{j1}^{(2)}P_f(s,t_{j-1})+\Delta_{j2}^{(2)}P_d(s,t_j)$$

where

$$\Delta_{j1}^{(2)}=\hat{V}_{d,f,s,t_{j-1}}QA_2(s,t_j)$$
$$\Delta_{j2}^{(2)}=\hat{V}_{d,f,s,t_{j-1}}\left[\frac{q_f^j}{\hat{S}_f(s,t_{j-1})}-\eta_2(s,t_{j-1})\right.$$
$$\left.\times[1+\delta(L_d(s,t_{j-1})\phi_2(s,t_{j-1})+K)]\right]$$

Thus, the hedging portfolio for the cash flow at t_j, $j = 2, 3, \ldots, n$, contains holding $\Delta_{j1}^{(2)}$ units of the foreign ZCB with maturity t_{j-1} and $\Delta_{j2}^{(2)}$ units of the domestic ZCB with maturity t_j (or selling short $\Delta_{j2}^{(2)}$ units if $\Delta_{j2}^{(2)}$ is negative).

The HCESV swap rate at initiation is

$$K = \frac{R_2}{\delta \Sigma_{j=1}^{n} \hat{V}_{d,f,t_0,t_{j-1}} P_d(t_0,t_j)\eta_2(t_0,t_{j-1})} \tag{22}$$

where

$$R_2 = \sum_{j=1}^{n} \hat{V}_{d,f,t_0,t_{j-1}} \left[P_f(t_0,t_{j-1}) QA_2(t_0,t_j) + P_d(t_0,t_j) \frac{q_f^j}{\hat{S}_f(t_0,t_{j-1})} - P_d(t_0,t_j)\eta_2(t_0,t_{j-1}) \right] - \delta \sum_{i=1}^{n} \hat{V}_{d,f,t_0,t_{j-1}} P_d(t_0,t_j) L_d(t_0,t_{j-1}) \eta_2(t_0,t_{j-1}) \phi_2(t_0,t_{j-1})$$

Similar to the HCESC swap rate, the compensation for the hedged exchange rate risk is also reflected through $QA_2(t_0, \cdot)$.

Pricing Unhedged Cross-Currency Equity Swaps with a Variable Notional Principal (UHCESV)

A UHCESV is similar to the HCESV but with unhedged currency risk (i.e., $\bar{X}$ is replaced by the floating exchange rate $X(t_j)$ for each time t_j, $j = 1, 2, \ldots, n$.) The structure of a UHCESV is slightly different from that of a HCESV in that there is an additional stream of payments due to a change in the exchange rate.

The normal cash flow stream of an UHCESV received by the party is

$$\begin{aligned}
\text{At time } t_1 &: V_{f,t_0}\left[\left(S_f(t_1)+q_f^1\right)/S_f(t_0)-1\right] \\
&\quad \times X(t_1) - \delta V_{d,t_0}\left[L_d(t_0,t_0)+K\right] \\
\text{At time } t_2 &: V_{f,t_1}\left[\left(S_f(t_2)+q_f^2\right)/S_f(t_1)-1\right] \\
&\quad \times X(t_2) - \delta V_{d,t_1}\left[L_d(t_1,t_1)+K\right] \\
\text{At time } t_3 &: V_{f,t_2}\left[\left(S_f(t_3)+q_f^3\right)/S_f(t_2)-1\right] \\
&\quad \times X(t_3) - \delta V_{d,t_2}\left[L_d(t_2,t_2+K\right] \\
\vdots & \qquad\qquad \vdots \\
\text{At time } t_n &: V_{f,t_{n-1}}\left[\left(S_f(t_n)+q_f^n\right)/S_f(t_{n-1})-1\right] \\
&\quad \times X(t_n) - \delta V_{d,t_{n-1}}\left[L_d(t_{n-1},t_{n-1})+K\right]
\end{aligned}$$

The additional stream of payments received by the party, due to a change in the exchange rate, is

$$\begin{aligned}
\text{At time } t_1 &: \left[V_{f,t_1} X(t_1) - V_{f,t_0} X(t_0)\right] \\
\text{At time } t_2 &: \left[V_{f,t_2} X(t_2) - V_{f,t_1} X(t_1)\right] \\
\vdots & \qquad\qquad \vdots \\
\text{At time } t_n &: \left[V_{f,t_n} X(t_n) - V_{f,t_{n-1}} X(t_{n-1})\right]
\end{aligned}$$

Note that a negative payment represents the payment that must be paid from the party to the counterparty. The pricing formula of a UHCESV is presented in the following theorem; the proof is similar to Theorem 1.[5]

Theorem 3. The Pricing Formula of a UHCESV *Under the extended BGM model, the price of a UHCESV at time s, $t_0 \le s \le t_1$ is*

$$\begin{aligned}
&\left[\frac{S_f(s)}{S_f(t_0)} + \frac{q_f^1}{S_f(t_0)} P_f(s,t_1) - P_f(s,t_1)\right] \\
&\times V_{f,t_0} X(s) - \left[V_{d,t_0} P_d(s,t_1)\delta\left(L_d(t_0,t_0)+K\right)\right] \\
&+ \left[V_{d,s} - V_{d,t_0} P_d(s,t_1)\right] + \sum_{j=2}^{n} \hat{V}_{d,f,s,t_{j-1}} \\
&\times \left[P_f(s,t_{j-1}) + \frac{q_f^j}{\hat{S}_f(s,T_{j-1})} P_f(s,t_j) - P_f(s,t_j)\rho_3(s,t_{j-1})\right] \\
&- \delta \sum_{j=2}^{n} \hat{V}_{d,d,s,t_{j-1}} P_d(s,t_j)\eta_3(s,t_{j-1})(L_d(s,t_{j-1})\phi_3(s,t_{j-1})+K) \\
&+ \sum_{j=2}^{n} \hat{V}_{d,d,s,t_{j-1}} \left[P_d(s,t_{j-1}) - P_d(s,t_j)\eta_3(s,t_{j-1})\right]
\end{aligned} \tag{23}$$

where

$$\hat{V}_{d,k,s,t_j} = \frac{V_{d,s}}{P_k(s,t_j)}, \quad k \in \{d, f\}$$

$$\rho_3(s,t_{j-1}) = \exp\left(\int_s^{t_{j-1}} (\bar{\sigma}_{Pf}^s(t,t_{j-1}) + \sigma_{Sf}(t)) \times (\bar{\sigma}_{Pf}^s(t,t_{j-1}) - \bar{\sigma}_{Pf}^s(t,t_j))dt\right)$$

$$\eta_3(s,t_{j-1}) = \exp\left(\int_s^{t_{j-1}} (\sigma_{Sf}(t) + \sigma_X(t) + \bar{\sigma}_{Pd}^S(t,t_{j-1})) \times (\bar{\sigma}_{Pd}^s(t,t_{j-1}) - \bar{\sigma}_{Pd}^s(t,t_j))dt\right)$$

$$\phi_3(s,t_{j-1}) = \exp\left(\int_s^{t_{j-1}} (\sigma_{Sf}(t) + \sigma_X(t) + \bar{\sigma}_{Pd}^s(t,t_{j-1})) \times \gamma_d(t,t_{j-1})dt\right)$$

Since the exchange rate risk in the UHCESV is unhedged, the quanto adjustment term disappears in (23). However, there is an additional stream of cash flows reflecting the profit or loss in the notional principal due to the unanticipated change in the exchange rate.

The UHCESV swap rate is

$$K_3 = \frac{R_3}{\delta \Sigma_{j=1}^n \hat{V}_{d,d,t_0,t_{j-1}} P_d(t_0,t_j)\eta_3(t_0,t_{j-1})}$$

where

$$\begin{aligned} R_3 = & \sum_{j=1}^n \hat{V}_{d,f,t_0,t_{j-1}} \left[P_f(t_0,t_{j-1}) + \frac{q_f^j}{\hat{S}_f(t_0,T_{j-1})} P_f(t_0,t_j) \right. \\ & \left. - P_f(t_0,t_j)\rho_3(t_0,t_{j-1}) \right] \\ & - \delta \sum_{j=1}^n \hat{V}_{d,d,t_0,t_{j-1}} P_d(t_0,t_j)\eta_3(t_0,t_{j-1}) L_d(t_0,t_{j-1})\phi_3(t_0,t_{j-1}) \\ & + \sum_{j=1}^n \hat{V}_{d,d,t_0,t_{j-1}} \left[P_d(t_0,t_{j-1}) - P_d(t_0,t_j)\eta_3(t_0,t_{j-1}) \right] \end{aligned}$$

CALIBRATION PROCEDURE AND NUMERICAL EXAMPLES

This section provides a calibration procedure and an implementing example for practitioners.

Calibration Procedure

Based on the HJM framework, the rates described in the extended LIBOR market model are the market-observable forward LIBOR rates rather than the abstract instantaneous forward rates. With the advantage of pricing interest rate caps and floors consistent with the popular Black formula [1976], the extended LIBOR market model is easier for calibration.

We employ the mechanism presented by Rebonato [1999] to engage in a simultaneous calibration of the extended LIBOR market model to the percentage volatilities and the correlation matrix of the underlying forward LIBOR rates and the stock index. Assume that there are $n - 2$ forward LIBOR rates, a stock index, and the exchange rate in an m-factor framework. Without loss of generality, the foreign stock index and domestic forward LIBOR rates are employed. The steps to calibrate the model parameters are as follows:

First, as given in Brigo and Mercurio [2001], we assume that $L_d(t,\cdot)$ has a piecewise-constant instantaneous total volatility structure depending only on the time-to-maturity (i.e., $v_{ij} = v_{i-j}$). The elements in Exhibit 1, which specify the instantaneous total volatility applied to each period for each rate, can be stripped from the market data. A detailed computational process is presented in Hull [2003]. In addition, we assume that the foreign stock index $S_f(t)$ and the exchange rate $X(t)$ have piecewise-constant instantaneous total volatility structures. The elements in Exhibit 2, which represent the instantaneous total volatility applied to each period for the stock index and the exchange rate, can be calculated from the prices of the on-the-run options in the market. However, the durations of the stock options and the exchange rate options are usually shorter than one year, so the market-obtainable elements in Exhibit 2 are usually not sufficient for pricing equity swaps. This problem may be resolved

EXHIBIT 1

Instantaneous Volatilities of $L(t, \cdot)$

Instant.Total Vol.	Time $t \in (t_0, t_1]$	$(t_1, t_2]$	$(t_2, t_3]$	...	$(t_{n-2}, t_{n-1}]$
Fwd. Rate: $L(t, t_1)$	$v_{1,1} = v_0$	Dead	Dead	...	Dead
$L(t, t_2)$	$v_{2,1} = v_1$	$v_{2,2} = v_0$	Dead	...	Dead
$\vdots$	...	...	...	...	...
$L(t, t_{n-1})$	$v_{n-1,1} = v_{n-2}$	$v_{n-1,2} = v_{v-3}$	$v_{n-1,3} = v_{n-4}$	...	$v_{n-1,n-1} = v_0$

EXHIBIT 2

Instantaneous Volatilities of the Stock Index and Exchange Rate

Instant Total Vol.	Time $t \in (t_0, t_1]$	$(t_1, t_2]$	$(t_2, t_3]$	$\cdots$	$(t_{n-1}, t_n]$
Fwd. Rate: $S_f(t)$	ν_{Sf1}	ν_{Sf2}	ν_{Sf3}	$\cdots$	ν_{Sfn}
Fwd. Rate: $X(t)$	ν_{X1}	ν_{X2}	ν_{X3}	$\cdots$	ν_{Xn}

by using the implied (or historical) volatility of the underlying stock index and the underlying exchange rate, and assuming the term structure of volatilities is flat (i.e., $\zeta_k(t) = \zeta_k$ for $t \in (t_0, t_n]$ and $k \in \{S, X\}$).

Second, we use historical price data for domestic forward LIBOR rates, the underlying foreign stock index, and the exchange rate to derive a full-rank $n \times n$ instantaneous-correlation matrix Γ. Thus, Γ is a positive-definite symmetric matrix and can be written as

$$\Gamma = H\Lambda H'$$

where H is a real orthogonal matrix and Λ is a diagonal matrix. Let $A \equiv H\Lambda^{1/2}$ and, thus, $AA' = \Gamma$.

In this way, we can find a suitable m-rank matrix B such that the m-rank matrix $\Gamma^B = BB'$ can be used to mimic the market correlation matrix Γ, where $m \leq n$.

The purpose of the second step is to replace the n-dimensional original Brownian motions $dW(t)$ with $BdZ(t)$, where $dZ(t)$ is a vector of m-dimensional Brownian motions. In other words, we change the market correlation structure

$$dW(t)dW(t)' = \Gamma dt$$

to a modeled correlation structure

$$BdZ(t)(BdZ(t))' = BdZ(t)dZ(t)'B' = BB'dt = \Gamma^B dt$$

The remaining problem is how to choose a suitable matrix B. Rebonato [1999] proposed the following form for the ith row of B:

$$b_{i,k} = \begin{cases} \cos\theta_{i,k}\Pi_{j=1}^{k-1}\sin\theta_{i,j} & \text{if } k = 1,2,\ldots,m-1 \\ \Pi_{j=1}^{k-1}\sin\theta_{i,j} & \text{if } k = m \end{cases}$$

for $i = 1, 2, \ldots, n$. By finding a $\hat{\theta}$ that solves the optimization problem

$$\min_{\theta} \sum_{i,j=1}^{n} |\Gamma_{i,j}^B - \Gamma_{i,j}|^2$$

and substituting $\hat{\theta}$ into B, we obtain a suitable matrix $\hat{B}$ such that $\Gamma^B(=\hat{B}\hat{B}')$ is an approximate correlation matrix for Γ.

Third, $\hat{B}$ can be used to distribute the instantaneous total volatility to each Brownian motion at each period for the stock index, the exchange rate, and each LIBOR rate, without changing the amount of the instantaneous total volatility. That is,

$$\nu_{i,j}\hat{B}(i,:) = (\gamma_{Ld1}(t,t_i), \gamma_{Ld2}(t,t_i), \ldots, \gamma_{Ldm}(t,t_i))$$

$$\nu_{S_f}\hat{B}(n-1,:) = (\sigma_{S_f1}(t), \sigma_{S_f2}(t), \ldots, \sigma_{S_fm}(t))$$

$$\nu_X\hat{B}(n,:) = (\sigma_{X1}(t), \sigma_{X2}(t), \ldots, \sigma_{Xm}(t))$$

where $i = 1, 2, \ldots, n-2$ and $t \in (t_{j-1}, t_j]$, for each $j = 1, 2, \ldots, n$.

Under the assumption that the instantaneous total volatility structures are piecewise-constant, the previous procedure represents a general calibration method without a constraint on choosing the number of factors. Via the distributing matrix $\hat{B}$, the individual instantaneous volatility applied to each Brownian motion at each period for each process can be derived. With these data calibrated from the market correlation matrix and volatilities, we can employ a Monte Carlo simulation to price any associated interest rate derivatives. The data can also be used to calculate the prices of the HCESC, the HCESV, and the UHCESV, as derived in Theorems 1, 2, and 3, respectively.

Numerical Analysis

This subsection offers practical examples that examine the accuracy of the approximate pricing formulas as derived in the previous section and compare the results with Monte Carlo simulations. Based on actual 2-year market data, shown in Exhibits 6 to 11 in Appendix B, the 3-year cross-currency equity swaps are priced at different quarterly dates, and the results are listed in Exhibits 3, 4, and 5. The flat volatility of the exchange rate and the FTSE-100 is assumed to be 15% and 30%, respectively. The notional value is assumed to be \$1 and the spread K is 50 bp. The simulation is based on 50,000 paths.

Note that in the examples, the domestic country is the U.S. and the foreign country is the U.K. By comparison to Monte Carlo simulations, the approximation formulas are shown to be accurate and robust for the recent 2-year market data.

CONCLUSION

We have contributed to the literature on the pricing of cross-currency equity swaps by providing a general pricing model, the cross-currency BGM model, and applied it to the pricing of three variants of cross-currency equity swaps: HCESC, HCESV, and UHCESV.

By assuming that the LIBOR rate, the exchange rate, and the stock price processes follow a lognormal volatility structure and by applying the drift term restriction for no-arbitrage in the extended HJM model, we have derived the extended BGM model under the martingale measure.

Because forward LIBOR rates have a lognormal volatility structure, they are positive and avoid the negative rate problem. Since LIBOR rates are market observable and their related derivatives, such as caps, floors, and swaptions, are widely traded in the market, we can extract these market-available quantities to calibrate the parameters associated with the cross-currency BGM model. Moreover, the cross-currency BGM model is suitable for pricing swaps with a paid-in-arrears feature associated with simply compounded LIBOR rates.

The cross-currency LIBOR model is a general model that can be used to price several variants of equity swaps; for example, equity swaps with single-currency or cross-currency cash flow streams, with hedged or unhedged currency risk, and constant or variable notional principal. We have derived the pricing models of three popular variants—HCESC, HCESV and UHCESV—and analyzed their properties and hedging characteristics. In addition, we can also apply the cross-currency BGM model to the pricing of other types of swaps, such as differential swaps, two-way equity swaps, and capped equity swaps, which we will examine in the near future.

EXHIBIT 3
Theorem 1

Date	**9/30/05**	**12/30/05**	**3/31/06**	**6/30/06**
Thm. 1	-0.1171	-0.1228	-0.1246	-0.1265
M.C.	-0.1184	-0.1221	-0.1259	-0.1278
s.e.	0.0022	0.0022	0.0022	0.0022
Date	**9/29/06**	**12/29/06**	**3/30/07**	**6/29/07**
Thm. 1	-0.1111	-0.1026	-0.0922	-0.0889
M.C.	-0.1103	-0.1013	-0.0935	-0.0893
s.e.	0.0022	0.0022	0.0022	0.0022

The prices of a 3-year HCESC with semiannual accrual periods are presented in this exhibit. The abbreviations M.C. and s.e. represent the results of Monte Carlo simulations and their standard errors, respectively.

EXHIBIT 4
Theorem 2

Date	**9/30/05**	**12/30/05**	**3/31/06**	**6/30/06**
Thm. 2	-0.2056	-0.2091	-0.2172	-0.2342
M.C.	-0.2071	-0.2110	-0.2155	-0.2379
s.e.	0.0040	0.0039	0.0038	0.0041
Date	**9/29/06**	**12/29/06**	**3/30/07**	**6/29/07**
Thm. 2	-0.2084	-0.2015	-0.1834	-0.1785
M.C.	-0.2091	-0.2033	-0.1868	-0.1761
s.e.	0.0042	0.0044	0.0045	0.0046

The prices of a 3-year HCESV with semiannual accrual periods are presented in this exhibit.

EXHIBIT 5
Theorem 3

Date	**9/30/05**	**12/30/05**	**3/31/06**	**6/30/06**
Thm 3	0.2215	0.2181	0.2347	0.2672
M.C.	0.2280	0.2149	0.2310	0.2659
s.e.	0.0119	0.0116	0.0117	0.0129
Date	**9/29/06**	**12/29/06**	**3/30/07**	**6/29/07**
Thm. 3	0.2706	0.3028	0.3168	0.3604
M.C.	0.2749	0.3026	0.3189	0.3639
s.e.	0.0130	0.0133	0.0135	0.0138

The prices of a 3-year UHCESV with semiannual accrual periods are presented in this exhibit.

Appendix A

Proof of Theorem 1

By applying the martingale pricing method, the price of an HCESC at time s, t_0 ″ s ″ t_1, is the sum of the discounted value of the expected future cash flows,

$$
\begin{aligned}
&V_{\text{HCESC}}\\
&=\sum_{j=1}^{n}\beta_d(s)E^{Q}\left(\left.\frac{\frac{S_f(t_j)+q_f^j}{S_f(t_{j-1})}-(1+\delta L_d(t_{j-1},t_{j-1})+\delta k)}{\beta_d(t_j)}\right|\mathcal{F}_s\right)\\
&=\frac{1}{S_f(t_0)}\underbrace{\beta_d(s)E^{Q}\left(\left.\frac{S_f(t_1)}{\beta_d(t_j)}\right|\mathcal{F}_s\right)}_{(A.1)}+\frac{q_f^j}{S_f(t_0)}P_d(s,t_1)\\
&\quad-(1+\delta L_d(t_0,t_0)+\delta K)+P_d(s,t_1)\\
&\quad+\underbrace{\sum_{j=2}^{n}\beta_d(s)E^{Q}\left(\left.\frac{\frac{S_f(t_j)}{S_f(t_{j-1})}}{\beta_d(t_j)}\right|\mathcal{F}_s\right)}_{(A.2)}-(1+\delta K)\sum_{j=2}^{n}P_d(s,t_j)\\
&\quad-\underbrace{\sum_{j=2}^{n}\delta\beta_d(s)E^{Q}\left(\left.\frac{L_d(t_{j-1},t_{j-1})}{\beta_d(t_j)}\right|\mathcal{F}_s\right)}_{(A.3)}\\
&\quad+\underbrace{\sum_{j=2}^{n}\beta_d(s)E^{Q}\left(\left.\frac{\frac{q_f^j}{S_f(t_{j-1})}}{\beta_d(t_j)}\right|\mathcal{F}_s\right)}_{(A.4)}
\end{aligned}
$$

We solve, parts (A.1), (A.2), (A.3) and (A.4) respectively, below:

$$
(A.2)=\beta_d(s)E^{Q}\left(\left.\frac{1}{S_f(t_{j-1})}\underbrace{E^{Q}\left(\left.\frac{S_f(t_j)}{\beta_d(t_j)}\right|\mathcal{F}_{t_{j-1}}\right)}_{(A.5)}\right|\mathcal{F}_s\right)
$$

$$
\begin{aligned}
(A.5)&=E^{Q}\left(\left.\frac{S_f(t_j)}{\beta_d(t_j)}\right|\mathcal{F}_{t_{j-1}}\right)\\
&=E^{Q}\left(\left.\frac{S_f(t_j)X(t_j)}{\beta_d(t_j)}\frac{P_d(t_j,t_j)/\beta_d(t_j)}{X(t_j)P_f(t_j,t_j)/\beta_d(t_j)}\right|\mathcal{F}_{t_{j-1}}\right)\\
&=E^{Q}\left(\left.f(t_j)\frac{g(t_j)}{h(t_j)}\right|\mathcal{F}_{t_{j-1}}\right)
\end{aligned}
$$

where $f(t)=\frac{S_f(t)X(t)}{\beta_d(t)}$, $g(t)=\frac{P_d(t,t_j)}{\beta_d(t)}$ and $h(t)=\frac{P_f(t,t_j)X(t)}{\beta_d(t)}$, which are all martingales under the domestic martingale measure. By Itô's lemma, the stochastic processes are derived as follows:

$$
\begin{aligned}
\frac{df(t)}{f(t)}&=(\sigma_X(t)+\sigma_{Sf}(t))\cdot dW(t)\\
\frac{dg(t)}{g(t)}&=-\sigma_{Pd}(t,t_j)\cdot dW(t)\\
\frac{dh(t)}{h(t)}&=(\sigma_X(t)-\sigma_{Pf}(t,t_j))\cdot dW(t)
\end{aligned}
$$

Then, defining $V(t)=f(t)\frac{g(t)}{h(t)}$ and applying Itô's lemma, we can derive

$$
\begin{aligned}
\frac{dV(t)}{V(t)}&=(\sigma_{Pf}(t,t_j)+\sigma_{Sf}(t))\cdot(\sigma_{Pf}(t,t_j)-\sigma_X(t)\\
&\quad-\sigma_{Pd}(t,t_j))dt+(\sigma_{Sf}(t)+\sigma_{Pf}(t,t_j)\\
&\quad-\sigma_{Pf}(t,t_j))\cdot dW(t)
\end{aligned}
\tag{A.7}
$$

According to the definition of the bond volatility process $\sigma_{Pk}(t, t_j)$ in (9), $\sigma_{Pk}(t, t_j)$ is not deterministic. Thus, the stochastic differential equation (A.7) is not allowed to solve the distribution of $V(t_j)$. However, given any fixed initial time s, we can approximate $\sigma_{Pk}(t, T)$ by $\bar{\sigma}^{s}_{Pk}(t, T)$, which is defined by

$$
\bar{\sigma}^{s}_{Pk}(t,T)=\begin{cases}\displaystyle\sum_{i=1}^{\lfloor\delta^{-1}(T-t)\rfloor}\frac{\delta L_k(s,T-i\delta)\gamma_{Lk}(t,T-i\delta)}{1+\delta L_k(s,T-i\delta)}, & t\in[0,T-\delta]\,\&\,T-\delta>0,\\ 0 & \text{otherwise}\end{cases}
\tag{A.8}
$$

where 0 ″ s ″ t ″ T. It means that the calendar time of the process $\{L_k(t, T)\}_{t\in[s,T]}$ in (A.8) is frozen at its initial time s and thus the process $\{\bar{\sigma}^{s}_{Pk}(t, T)\}_{t\in[s,T]}$ becomes deterministic. This is the Wiener chaos order 0 approximation, which is used by BGM for pricing swaptions.

Substituting $\bar{\sigma}^{s}_{Pk}(t, T)$ for $\sigma_{Pk}(t, T)$ in the drift term of (A.7), we have

$$
\begin{aligned}
\frac{dV(t)}{V(t)}&=(\bar{\sigma}^{s}_{Pf}(t,t_j)+\sigma_{Sf}(t))(\bar{\sigma}^{s}_{Pf}(t,t_j)-\sigma_X(t)-\bar{\sigma}^{s}_{Pd}(t,t_j))dt\\
&\quad+(\sigma_{Sf}(t)+\bar{\sigma}^{s}_{Pf}(t,t_j)-\bar{\sigma}^{s}_{Pd}(t,t_j))\cdot dW(t)
\end{aligned}
\tag{A.9}
$$

In this way, the drift and volatility terms in (A.9) are deterministic, so we can solve (A.9) and find the approximate distribution of $V(t_j)$. Taking $V(t_j)$ into (A.6), we can derive (A.5) as follows:

$$(\mathrm{A.5}) = V(t_{j-1})\phi_1(t_{j-1}, t_j)$$

where

$$\phi_1(u,v) = \exp\left(\int_u^v (\bar{\sigma}_{Pf}^s(t,t_j) + \sigma_{Sf}(t))(\bar{\sigma}_{Pf}^s(t,t_j) - \sigma_X(t) - \bar{\sigma}_{Pd}^s(t,t_j))dt\right)$$

Then, by substituting (A.5) into (A.2), it can be rewritten as

$$\begin{aligned}(\mathrm{A.2}) &= \beta_d(s)E^Q\left(\frac{P_d(t_{j-1},t_j)}{\beta_d(t_{j-1})}\frac{1}{P_f(t_{j-1},t_j)}\middle|\mathcal{F}_s\right)\phi_1(t_{j-1},t_j)\\ &= \beta_d(s)E^Q\left(\frac{P_d(t_{j-1},t_j)}{\beta_d(t_{j-1})}\frac{\frac{P_f(t_{j-1},t_{j-1})X(t_{j-1})}{\beta_d(t_{j-1})}}{\frac{P_f(t_{j-1},t_j)X(t_{j-1})}{\beta_d(t_{j-1})}}\middle|\mathcal{F}_s\right)\phi_1(t_{j-1},t_j)\end{aligned}$$

Similarly, (A.2) can be shown as follows:

$$\begin{aligned}(\mathrm{A.2}) &= P_d(s,t_j)\frac{P_f(s,t_{j-1})}{P_f(s,t_j)}\rho_1(s,t_j)\\ &= QA_1(s,t_j)P_f(s,t_{j-1}) \qquad (\mathrm{A.10})\end{aligned}$$

where, for $j = 2, 3, \ldots, n$,

$$\begin{aligned}QA_1(s,t_j) &= \frac{P_d(s,t_j)}{P_f(s,t_j)}\rho_1(s,t_j)\\ \rho_1(s,t_j) &= \phi_1(t_{j-1},t_j)\eta_1(s,t_{j-1})\\ \eta_1(u,v) &= \exp\left(\int_u^v (\bar{\sigma}_{Pf}^s(t,t_j) + \bar{\sigma}_{Pf}^s(t,t_{j-1}))(\bar{\sigma}_{Pf}^s(t,t_j) - \sigma_X(t) - \bar{\sigma}_{Pd}^s(t,t_j))dt\right)\end{aligned}$$

Similar to the pricing process of (A.5), (A.1) and (A.4) can be shown to be

$$(\mathrm{A.1}) = \frac{S_f(s)}{S_f(t_0)}QA_1(s,t_1) \qquad (\mathrm{A.11})$$

where

$$\begin{aligned}QA_1(s,t_1) &= \frac{P_d(s,t_1)}{P_f(s,t_1)}\rho_1(s,t_1)\\ \rho_1(s,t_1) &= \exp\left(\int_s^{t_1} (\bar{\sigma}_{Pf}^s(t,t_1) + \sigma_{Sf}(t))\cdot(\bar{\sigma}_{Pf}^s(t,t_1) - \sigma_X(t) - \bar{\sigma}_{Pd}^s(t,t_1))dt\right)\end{aligned}$$

$$(\mathrm{A.4}) = \frac{q_f^j}{\frac{S_f(s)}{P_f(s,t_{j-1})}} + P_d(s,t_j)\xi(s,t_j) \qquad (\mathrm{A.12})$$

where

$$\xi_1(s,t_{j-1}) = \exp\left(\int_s^{t_{j-1}} (\bar{\sigma}_{Pf}^s(t,t_{j-1}) + \sigma_{Sf}(t)) \times(\sigma_{Sf}(t) + \sigma_X(t) + \bar{\sigma}_{Pd}^s(t,t_j))dt\right)$$

Then, (A.3) is derived as follows:

$$\begin{aligned}(\mathrm{A.3}) &= \delta\beta_d(s)E^Q\left(\frac{L_d(t_{j-1},t_{j-1})}{\beta_d(t_j)}\middle|\mathcal{F}_s\right)\\ &= \delta P_d(s,t_j)E^{Q_{t_j}}\left(L_d(t_{j-1},t_{j-1})\middle|\mathcal{F}_s\right) \qquad (\mathrm{A.13})\end{aligned}$$

where Q_{t_j} is a forward martingale measure with respect to the numèraire $P(t, t_j)$.

Via the change of the numèraire mechanism, the dynamics of $L_d(t, t_{j-1})$ in (10), can be transformed from the measure Q to the measure Q_{t_j} as follows:

$$\frac{dL_d(t,t_{j-1})}{L_d(t,t_{j-1})} = \gamma_{Ld}(t,t_{j-1})\cdot dW(t).$$

Obviously, $L_d(t, t_{j-1})$ is a martingale under Q_{t_j}, so (A.13) is solved as follows:

$$\begin{aligned}(\mathrm{A.3}) &= \delta P_d(s,t_j)L_d(s,t_{j-1})\\ &= P_d(s,t_{j-1}) - P_d(s,t_j)\end{aligned} \qquad (\mathrm{A.14})$$

With (A.10), (A.11), (A.12) and (A.14), the pricing formula of the HCESC can then be derived.

APPENDIX B

The Market Data

Exhibits 6 to 11 are drawn and computed from the DataStream database and used for the numerical example in the fourth section.

EXHIBIT 6

The Foreign Stock Index and the Exchange Rate

Date	9/30/05	12/30/05	3/31/06	6/30/06	9/29/06	12/29/06	3/30/07	6/29/07
UK/US	1.765	1.718	1.738	1.847	1.872	1.957	1.970	2.007
FTSE-100	5477.71	5618.76	5964.57	5833.42	5960.81	6220.81	6308.03	6607.90

The U.K./U.S. exchange rate and the foreign stock index FTSE-100 are presented quarterly for the past 2 years.

EXHIBIT 7

Domestic Cap Volatilities Quoted in the U.S. Market

Date	9/30/05	12/30/05	3/31/06	6/30/06	9/29/06	12/29/06	3/30/07	6/29/07
1	13.93	11.3	9.93	10.36	11.77	11.19	12.41	8.51
2	20.13	18.98	16.47	15.20	18.93	17.60	18.70	15.25
3	22.33	21.53	18.00	16.94	19.95	18.22	18.19	15.93

The quoted volatilities of the caps in the U.S. market are presented quarterly for the past 2 years. The semiannual LIBOR rate volatility can be obtained by interpolation.

EXHIBIT 8

Foreign Cap Volatilities Quoted in the U.K. Market

Date	9/30/05	12/30/05	3/31/06	6/30/06	9/29/06	12/29/06	3/30/07	6/29/07
1	10.65	11.58	7.49	8.67	8.60	8.02	7.21	8.17
2	15.91	16.16	12.46	12.90	13.65	12.75	10.42	12.01
3	16.49	16.04	14.14	13.80	14.20	12.96	11.14	12.73

The quoted volatilities of the caps in the U.K. market are presented quarterly for the past 2 years. The semiannual LIBOR rate volatility can be obtained by interpolation.

EXHIBIT 9
Initial Domestic Forward LIBOR Rates

Date	9/30/05	12/30/05	3/31/06	6/30/06	9/29/06	12/29/06	3/30/07	6/29/07
1	4.350	4.838	5.317	5.809	5.574	5.564	5.499	5.592
$\frac{1}{2}$	4.672	4.994	5.492	5.855	5.350	5.412	5.241	5.541
1	4.759	4.990	5.393	5.769	5.130	5.210	5.011	5.521
$1\frac{1}{2}$	4.922	5.033	5.373	5.720	4.897	5.001	4.762	5.490
2	4.743	4.953	5.383	5.781	5.109	5.128	4.993	5.639
$2\frac{1}{2}$	4.761	4.940	5.370	5.767	5.049	5.045	4.940	5.678

The domestic initial forward LIBOR rates in the U.S. market are presented quarterly for the past 2 years. The rates are obtained from the associated bond prices derived from the zero curves obtained in DataStream.

EXHIBIT 10
Initial Foreign Forward LIBOR Ratcs

Date	9/30/05	12/30/05	3/31/06	6/30/06	9/29/06	12/29/06	3/30/07	6/29/07
0	4.675	4.699	4.763	4.928	5.313	5.577	5.896	6.329
$\frac{1}{2}$	4.510	4.562	4.876	5.171	5.407	5.693	6.001	6.547
1	4.531	4.628	5.018	5.363	5.319	5.675	5.912	6.583
$1\frac{1}{2}$	4.610	4.602	5.077	5.415	5.255	5.708	5.874	6.541
2	4.647	4.649	5.019	5.374	5.221	5.599	5.793	6.483
$2\frac{1}{2}$	4.664	4.731	5.056	5.438	5.178	5.573	5.746	6.478

The foreign initial forward LIBOR rates in the U.K. market are presented quarterly for the past 2 years. The rates are obtained from the associated bond prices derived from the zero curves obtained in DataStream.

EXHIBIT 11
The Four-Factor B Matrix

	Factor 1	Factor 2	Factor 3	Factor 4
$D\frac{1}{2}$	0.7526	0.5321	0.2429	0.3023
D1	0.6792	0.7017	0.1386	0.1646
$D1\frac{1}{2}$	0.4318	0.8995	-0.0444	-0.0492
D2	0.6447	0.7578	0.0848	0.0537
$D2\frac{1}{2}$	0.579	0.8139	0.0222	-0.0424
$F\frac{1}{2}$	0.9426	-0.2758	0.1848	-0.0363
F1	0.9728	-0.1662	0.1495	-0.0606
$F1\frac{1}{2}$	0.9788	-0.1122	0.1125	-0.1292
F2	0.9766	-0.1238	0.1038	-0.1419
$F2\frac{1}{2}$	0.9795	-0.0625	0.0961	-0.1657
Sf	0.7902	-0.4234	0.4305	-0.1048
X	0.9341	-0.2362	-0.075	0.2568

The matrix B is computed based on the correlation matrix of the relevant variables calculated from the 2-year data July 1, 2005/June 29, 2007.

ENDNOTES

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[1]The derivation of Proposition 1 is available upon request from the authors.

[2]Since the dividend process is assumed to be deterministic, $q_f^j = \int_{t_{j-1}}^{t_j} q_f(u)du$ is a constant.

[3]The compound correlations mean that $\rho_1(s, t_1)$ and $\rho_1(s, t_j)$ in the quanto adjustment $QA_1(s, t_j)$ are the exponential functions of the correlations between the exchange rate and the domestic and foreign forward LIBOR rates.

[4]The proof of Theorem 2 is available upon request from the authors.

[5]The proof of Theorem 3 is available upon request from the authors.

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