Properties of the Chooser Flexible Cap

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In this article, we present properties of a chooser flexible cap, which is used for hedging interest rate risk. The chooser flexible cap is a financial instrument written on an underlying market interest rate index, LIBOR. While the chooser flexible cap is more flexible than a standard cap, its flexibility makes pricing more complicated and slower. We present two properties of the chooser flexible cap. The first is that when the number of permitted exercise opportunities is increased, exercise remains optimal at all of the same points as in the more restricted case. The second property is the set of conditions under which the option holder does not exercise a particular caplet. We can use the first property to reduce the calculation time. We can also utilize the second property to make profitable exercise strategies based on observable LIBORs.

A holder of a chooser flexible cap has the right to cap some, but not all, of the floating period rates at an exercise rate in a contract period. Fewer caplets can be exercised than the number of contract periods, so the option holder must decide at each payment date whether to exercise a caplet or to save it for use at a later time. This decision cannot be changed at a later date. The chooser flexible cap offers the holder more flexibility than a cap in that the option holder chooses exercise periods.

To the best of our knowledge, there is only one article, Pedersen and Sidenius [1998], about the chooser flexible cap, in which a pricing method is proposed utilizing the dynamic programming approach. This method is appropriate for pricing exotic derivatives because we can solve the problem by backward induction in a recombining tree. Pedersen and Sidenius [1998] made an important contribution in terms of introducing the pricing method utilizing the optimality equation. They used the Black–Karasinski model(see Black and Karasinski [1991]) and the Black, Derman, and Toy model (see Black, et al. [1990]) in which the short rate is the underlying interest rate for the tree construction. They also use the short rate for the payoff calculation of the chooser flexible cap, considering that the spot London Inter-Bank Offered Rate (LIBOR) is approximately equal to the short rate. They also present numerical examples of pricing methods for the chooser flexible cap in some typical market scenarios under the above model setting.

We extended the pricing method for the chooser flexible cap in Ito, Ohnishi, and Tamba [2004] under the Hull–White one factor model (Hull and White [1994]). In Ito, Ohnishi, and Tamba [2004], we focus on the pricing problem and the efficient calibration of the chooser flexible cap and the chooser flexible floor, based on the observed market prices of relatively simple interest rate derivatives, caplets and floorlets, etc. For the calibration, it is appropriate to adopt similar instruments in order to maintain consistency with market data. The most important contribution of this article is that we derive BGM-like equations (Brace, et al. [1997]) from the Hull–White model; that is, each forward LIBOR with an additive constant follows a log-normal martingale under the corresponding probability measure. Using this result, we can easily derive the theoretical prices of the caplet and the floorlet utilizing the Black–Scholes formula (Black and Scholes [1973]) and use these theoretical prices for the calibration. We can make calibration very easy and fast with this method. This is an important point for complicated instruments, such as exotic options, in practice. Calculating the spot LIBOR from the short rate, we use the spot LIBOR for the payoff calculation of the chooser flexible cap. The short rate, or Hull–White, model has some convincing properties such as mean reversion that matches interest rate movements in the real world. Using the Hull–White model, we can set parameters so that the model fits real market data. We also show numerical examples for pricing the chooser flexible cap and comparative statics for each model parameter.

The chooser flexible cap is not popular, although it is traded in all the major markets. Possible reasons for its unpopularity are the complexity of the price calculation and the difficulty of effective utilization in practice. But there are no articles discussing properties and exercise strategies of the chooser flexible cap. Pedersen and Sidenius [1998] and Ito, Ohnishi, and Tamba [2004] describe only the pricing methods for the chooser flexible cap. We found that the price calculation of the chooser flexible cap in the one-factor model is very fast on the numerical examples in Ito, Ohnishi, and Tamba [2004]. But under more complex interest rate models, calculations may take much longer. On the other hand, although the chooser flexible cap is a useful financial instrument, it is difficult to decide when the option holder should exercise caplets because he/she needs to forecast future changes in the LIBOR.

In this article, we discuss certain properties of the chooser flexible cap to reduce its complexity. We prove two theoretical properties of the chooser flexible cap. First, we show that when the number of permitted exercise opportunities is increased, exercise remains optimal at all of the same points as in the more restricted case. For example, if the option holder exercises a caplet out of three remaining exercise opportunities at a particular LIBOR and in a period, he/she should also exercise a caplet at the same LIBOR and in the period when he/she has four exercise opportunities. Second, we derive the theoretical conditions under which the option holder does not exercise a caplet. We can use the first property to reduce the calculation time of the chooser flexible cap price because we can decide whether we should exercise options without numerical calculations at nodes satisfying specific conditions. At the same time, the option holder can use the second property to develop theoretically profitable exercise strategies using current LIBOR data. These properties can be applied to many interest rate models because we can use the properties under the BGM framework, that is, each forward LIBOR follows a log-normal martingale process under the corresponding probability measure.

This article is organized as follows. The next section shows the pricing method of the chooser flexible cap using the dynamic programming approach. The following section shows the property that when the number of permitted exercise opportunities is increased, exercise remains optimal at all of the same points as in the more restricted case. In the section after, we derive the conditions under which the option holder does not exercise a caplet. We end with our conclusions.

PRICING THE CHOOSER FLEXIBLE CAP

For $N \in \mathbb{Z}_+$ and $T^* \in \mathbb{R}_{++}$, let

$$0 \le T_0 < T_1 \cdots < T_i < T_{i+1} < \cdots < T_{N-1} < T_{N-1} \le T^* \quad (1)$$

be the sequence of setting times and payment times of floating interest rates, that is, for $i = 0, \ldots, N-1$, the floating interest rate that covers time interval $(T_i, T_{i+1}]$ is set at time T_i and paid at time T_{i+1}. For convenience, we let

$$T_{i+1} - T_i = \delta\ (= \text{costant} \in \mathbb{R}_{++}),\ i = 0, \ldots, N-1 \quad (2)$$

Let $D(t,T)$ $0 \le t \le T \le T^*$ be the time t price of the discount bond (or zero-coupon bond) with maturity T, a "T-bond" for short, which pays one unit of money at maturity T (where $D(T,T) = 1$ for any $T \in \mathbb{T}^* := [0, T^*]$). For $0 \le t < T \le T^*$,

$$r(t) = -\frac{\partial}{\partial T} \ln D(t,T)\bigg|_{T=t} \quad (3)$$

is the short rate at time t. For $0 \le t \le T \le T^*$,

$$B(t,T) := \exp\left\{\int_t^T r(s)ds\right\} \quad (4)$$

is the risk-free bank account at time T with unit investment capital at time t (where $B(t, t) = 1$). For $i = 0, \ldots, N-1$, we define a spot LIBOR that covers time interval (T_i, T_{i+1}) by

$$L_{T_i}(T_i) := \frac{1}{\delta}\left\{\frac{1}{D(T_i, T_{i+1})} - 1\right\} \quad (5)$$

This amount is set at time T_i, paid at time T_{i+1}. For $i = 0, \ldots, N-1$,

$$L_{T_i}(t) := \frac{1}{\delta}\left\{\frac{D(t,T_i)}{D(t,T_{i+1})} - 1\right\} \quad (6)$$

is a forward LIBOR prevailing at time t ($\in [0, T_i]$) that covers time interval $(T_i, T_{i+1}]$. For $i = 0, \ldots, N-1$,

$$L(t; T_i, T_{i+k}) := \frac{1}{k\delta}\left\{\frac{D(t,T_i)}{D(t,T_{i+k})} - 1\right\} \quad (7)$$

is a forward LIBOR for a multiple period prevailing at time t ($\in [0, T_i]$) that covers time interval (T_i, T_{i+k}).

An i-caplet is a financial contract in which, at time T_{i+1}, the seller pays to the buyer the amount of money corresponding to the difference between the spot LIBOR $L_{T_i}(T_i)$ and a predetermined upper-limit exercise rate K ($\in \mathbb{R}$) if the former exceeds the latter:

$$\delta[L_{T_i}(T_i) - K] + (= \delta \max\{L_{T_i}(T_i) - K, 0\}) \quad (8)$$

It could be considered as a call option written on the underlying LIBOR for hedging its upside risk. Exhibit 1 shows an example of 2-Caplet's cash flow.

The Chooser Flexible Cap

An interest rate cap is a financial contract that effectively fixes a maximum level of LIBOR for all contracted periods. A cap consists of N caplets. A cap holder has the right to receive payments as in Equation (8) for all periods covered by the contract. Let us consider the case that an investor owes a floating interest rate debt. Exhibit 2 shows an example of the total interest the investor pays with a two-year cap contract (thick line), and without the contract (thin line). In this example, the cap holder can cap the interest rate at the exercise rate, 3%, for the contract period of two years, because payoffs from the cap contract and the floating interest rate expenses from the debt cancel each other. Exhibit 3 shows the cash flow that the cap holder receives at each period in the case of Exhibit 2, equal to the difference between LIBOR and the exercise rate if LIBOR exceeds the exercise rate.

A chooser flexible cap allows a buyer the right to exercise at most l ($1 \le l \le N$) out of N caplets whose i-th one ($i = 0, \ldots, N-1$) is written at the LIBOR whose setting time and payment time are T_i and T_{i+1}, respectively. On each reset date, the option holder decides whether or not to exercise the particular caplet. If he/she exercises it, it is counted as part of l. This decision, once made, cannot be changed at a later time. The option holder receives a payment equal to the payment from a standard caplet for each exercise.

Consider the case of an investor with floating interest rate debt who has a two-year chooser flexible cap that can be exercised at a rate $K = 3\%$ in two out of the eight quarterly LIBOR fixings. In Exhibit 4, the thick line shows the effective interest rate paid if the option holder

EXHIBIT 1

An Example of a 2-Caplet

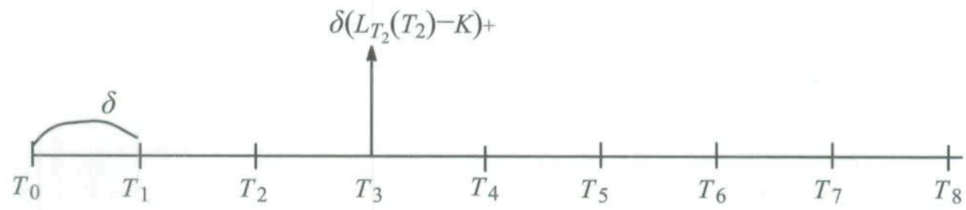
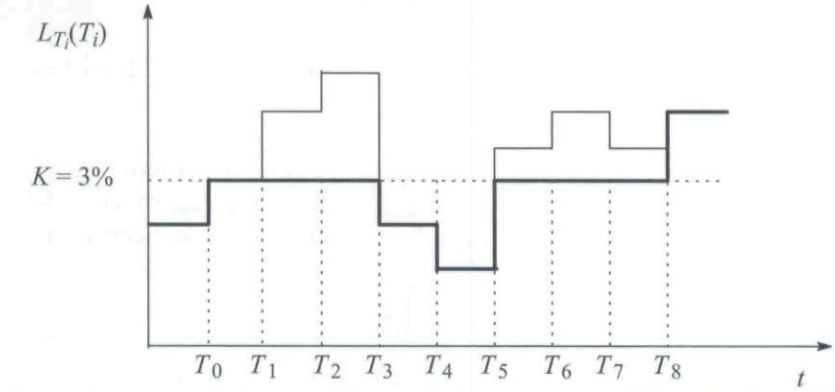


EXHIBIT 2
An Example of Interest Rate Payment for the Cap Holder



exercises the options at T_2 and T_5. The thin line shows payments by the investor without the option. If the first LIBOR setting is 3%, equal to the exercise rate at T_0, the holder does not exercise the 0-caplet. If at the next period T_1, LIBOR is 4%, 1% above the exercise rate and seven exercisable periods remain, the holder may decide not to use the caplet but leave it for later periods. If at the next period T_2, LIBOR is 4.5%, 1.5% above the exercise rate and six exercisable periods remain, the holder would choose to exercise the caplet. Then the holder gets a payment for the 1.5% and he/she has one exercise opportunity remaining. The holder exercises the remaining one exercise opportunity at T_5, considering only three exercisable periods remain. Exhibit 5 shows the cash flow that the chooser flexible cap holder receives at each period in the case of Exhibit 4. The chooser flexible cap holder can get the payments only for the two periods at which the holder exercises the caplets. Exhibit 4 shows that the holder can cap the payments only for the two periods. He/she cannot always exercise the caplets at the best periods because he/she dynamically exercises the caplets watching a present LIBOR and cannot know when are the best exercise periods.

There is another type of flexible cap, an auto flexible cap. The auto flexible cap is the same instrument in that it has fewer exercise opportunities than the number of exercisable periods. A holder of an auto flexible cap automatically gets payments from the first in-the-money caplets until there are no more caplets to exercise. The two types of flexible caps are traded in all major markets but the trade volume is small. They are most frequently traded in Europe, where the auto flexible cap is more common than the chooser flexible cap.

There are two main attractive points of the chooser flexible cap compared to the normal cap. First, it is cheaper, because the chooser flexible cap has fewer exercise opportunities. Therefore, the option holder can flexibly hedge the interest rate risk at lower cost than with the cap. The chooser flexible cap is preferred by investors who do not need to hedge all periods of the contract but hedge some periods with low cost. Second, it is flexible. The holder of the chooser flexible cap can reflect his/her

EXHIBIT 3
Cash Flow from the Cap Contract

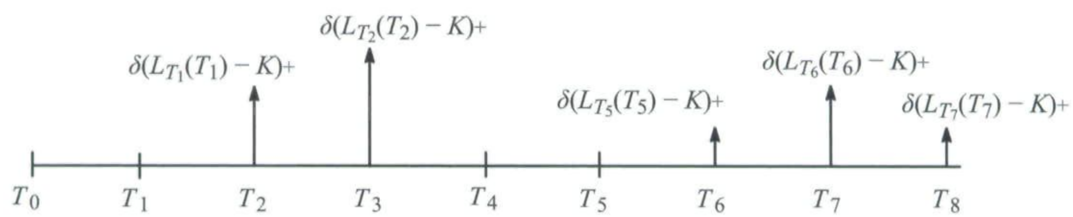
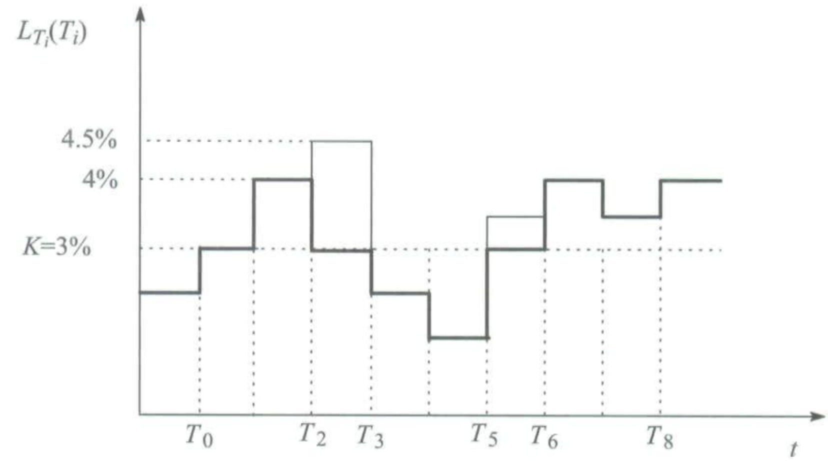


EXHIBIT 4

An Example of Interest Rate Payment for the Chooser Flexible Cap Holder



expectation of the future interest rate change in his/her hedging strategies. For example, if he/she has the expectation that the floating interest rate will increase next year, he/she should follow a strategy of exercising caplets mainly in that year. And, he/she can change the exercise strategy if he/she realizes that his/her expectation is wrong. Thus, the chooser flexible cap is a suitable alternative to the normal cap, especially for investors who are willing to take an active role in risk management.

Pricing the Chooser Flexible Cap Under the Risk-neutral Probability $\mathbb{P}^*$

We consider a continuous trading economy with a finite time horizon given by $\mathbb{T}^* := [0, T^*]$ ($T^* \in \mathbb{R}_{++}$). The uncertainty is modeled by a filtered probability space (Ω, $\mathcal{F}$, $\mathbb{P}$; $\mathbb{F}$). In this notation, Ω denotes a sample space with elements $\omega \in \Omega$; $\mathcal{F}$ denotes a σ-algebra on Ω; and $\mathbb{P}$ denotes a probability measure on (Ω,F). The uncertainty is resolved over $\mathbb{T}^*$ according to an $N-1$-dimensional Brownian motion filtration $\mathbb{F} := (\mathcal{F}(t) : t \in \mathbb{T}^*)$ satisfying the usual conditions. $W := (W(t) : t \in \mathbb{T}^*)$ denotes an $N-1$-dimensional standard ($\mathbb{P}$; $\mathbb{F}$)-Brownian motion. Consistent with the no-arbitrage and complete market paradigm, we assume the existence of the risk-neutral equivalent martingale measure $\mathbb{P}^*$ with a bank account as a numéraire in this economy. We assume that in the BGM framework, the forward LIBOR of each period follows a geometric Brownian motion under the corresponding forward-neutral probability.

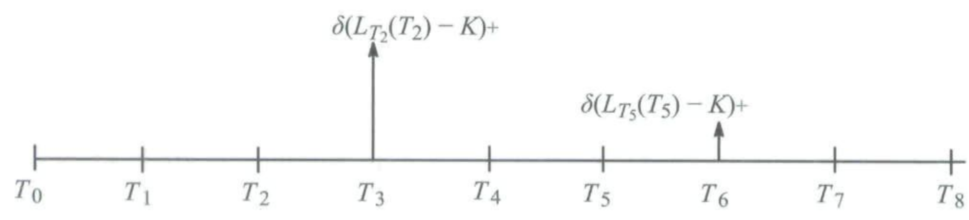
Let $W(T_i, L_{T_i}(T_i), l)$, $i = 0, \ldots, N-1$, $l = 1, \ldots, M$ be the fair (no-arbitrage) price of the chooser flexible cap when, at time T_i, at most l exercises remain to the buyer. The optimality equation can be derived by using the Bellman principle. In this subsection, we derive the optimality equation under the risk-neutral probability measure $\mathbb{P}^*$ with a bank account as a numéraire.

Optimality Equation:

i. For $i = N-1$ (Terminal Condition):

EXHIBIT 5

Cash Flow from the Chooser Flexible Cap with Two Exercise Opportunities



$$W(T_{N-1},L_{T_{N-1}}(T_{N-1}),l) \\ = D(T_{N-1},T_N)\delta[L_{T_{N-1}}(T_{N-1})-K]_+, \quad l=1,\ldots,M \tag{9}$$

ii. For $i = N-2, \ldots, 0$:

$$\begin{aligned} &W(T_i,L_{T_i}(T_i),l) \\ &= \max\Big\{D(T_i,T_{i+1})\delta(L_{T_i}(T_i)-K)_+ \\ &\quad + \mathbb{E}^*\left[\frac{W(T_{i+1},L_{T_{i+1}})(T_{i+1}),l-1)}{B(T_i,T_{i+1})}\bigg| L_{T_i}(T_i)\right], \\ &\mathbb{E}^*\left[\frac{W(T_{i+1},L_{T_{i+1}})(T_{i+1}),l}{B(T_i,T_{i+1})}\bigg| L_{T_i}(T_i)\right]\Big\}, \quad l=1,\ldots,M \end{aligned} \tag{10}$$

where

$$W(T_i, L_{T_i}(T_i),0) = 0, \; i = 0, \ldots, N-1 \tag{11}$$

Pricing the Chooser Flexible Cap under the Forward-Neutral Probability $\mathbb{P}^{T_N}$

We can also derive the optimality equation under the forward-neutral probability $\mathbb{P}^{T_N}$ with a T_N-bond as a numéraire.

Optimality Equation:

i. For $i = N-1$ (Terminal Condition):

$$W(T_{N-1},L_{T_{N-1}}(T_{N-1}),l) \\ = D(T_{N-1},T_N)\delta[L_{T_{N-1}}(T_{N-1})-K]_+, \quad l=1,\ldots,M \tag{12}$$

ii. For $i = N-2,\ldots,0$:

$$\begin{aligned} &W(T_i,L_{T_i}(T_i),l) \\ &= \max\Big\{D(T_i,T_{i+1})\delta(L_{T_i}(T_i)-K)_+ \\ &\quad +D(T_i,T_N)\mathbb{E}^{T_N}\left[\frac{W(T_{i+1},L_{T_{i+1}}(T_{i+1}),l-1)}{D(T_{i+1},T_N)}\bigg| L_{T_i}(T_i)\right], \\ &D(T_i,T_N)\mathbb{E}^{T_N}\left[\frac{W(T_{i+1},L_{T_{i+1}}(T_{i+1}),l)}{D(T_{i+1},T_N)}\bigg| L_{T_i}(T_i)\right]\Big\}, \\ &l=1,\ldots,M \end{aligned} \tag{13}$$

where

$$W(T_i, L_{T_i}(T_i), 0) = 0, \; i = 0,\ldots,N-1 \tag{14}$$

Pricing the Chooser Flexible Cap under Varying Forward-Neutral Probabilities $\mathbb{P}^{T_i}(1 \le i \le N)$

In this subsection, we write the optimality equation under forward-neutral probabilities $\mathbb{P}^{T_i}$ varying at each period with a T_i-bond as a numéraire. This optimality equation is different from both equations in the prior two subsections, which have fixed probability measures at all periods. We will use this method for proofs of later propositions and corollary.

Optimality Equation:

i. For $i = N-1$ (Terminal Condition):

$$W(T_{N-1},L_{T_{N-1}}(T_{N-1}),l) \\ = D(T_{N-1},T_N)\delta[L_{T_{N-1}}(T_{N-1})-K]_+, \; l=1,\ldots,M \tag{15}$$

ii. For $i = N-2, \ldots, 0$:

$$\begin{aligned} &W(T_i,L_{T_i}(T_i),l) \\ &= \max\{D(T_i,T_{i+1})\delta(L_{T_i}(T_i)-K)_+ \\ &\quad +D(T_i,T_{i+1})\mathbb{E}^{T_{i+1}}[W(T_{i+1},L_{T_{i+1}}(T_{i+1}),l-1) \mid L_{T_i}(T_i)] \\ &\quad D(T_i,T_{i+1})\mathbb{E}^{T_{i+1}}[W(T_{i+1},L_{T_{i+1}}(T_{i+1}),l) \mid L_{T_i}(T_i)]\}, \\ &l=1,\ldots,M \end{aligned} \tag{16}$$

where

$$W(T_i, L_{T_i}(T_i), 0) = 0, \; i = 0, \ldots, N-1 \tag{17}$$

OPTIMAL EXERCISE OF THE *i*-CAPLET WITH MORE EXERCISE OPPORTUNITIES AT T_i

In this section, we theoretically prove that when the number of permitted exercise opportunities is increased, exercise remains optimal at all of the same points as in the more restricted case with fewer exercise opportunities utilizing the evaluation under varying forward-neutral probabilities.

Proposition 1. *The holder of the chooser flexible cap should exercise the option, i-caplet, with the exercise opportu-*

nity l at T_i if he/she exercises it with $l-1$ at the same state, the spot LIBOR, and T_i for $i=0, \ldots, N-1$.

Corollary 1. *The holder of the chooser flexible cap should not exercise the option, i-caplet, with the exercise opportunity $l-1$ at T_i if he/she does not exercise it with l at the same state, the spot LIBOR, and T_i for $i=0, \ldots, N-1$.*

Proof: See Appendix A.

Proposition 1. indicates that when he/she has fewer exercise opportunities, the exercise opportunities are more valuable. We use Proposition 1. to reduce the calculation time of the chooser flexible cap price. We present an example for the time reduction. In the example, using a trinomial tree of the short rate $r(t)$ that follows the Hull–White model, we construct the trinomial tree of the short rate and the chooser flexible cap value in the discrete time setting under the risk-neutral probability $\mathbb{P}^*$. For the simplicity of the calculation, we set $\Delta t = \delta = T_{i+1} - T_i$ as constant for all periods.

The Hull–White model is one of the most popular short rate models in practice with the affine term structure feature, because it has desirable interest rate characteristics such as mean reversion. It can also be fitted with an observable initial term structure. The Hull–White model assumes that, under the risk-neutral probability measure $\mathbb{P}^*$, the short rate process $r = (r(t) : t \in \mathbb{T}^*)$ satisfies the following special form of the stochastic differential equation,

$$dr(t) = \{\alpha(t) - \beta r(t)\}dt + \sigma dW^*(t),\ t \in \mathbb{T}^* \quad (18)$$

We try three algorithms. The first algorithm is a normal calculation, shown as "Normal" in Exhibit 6. In this algorithm, we compare the two option values that the holder can obtain when exercising the i-caplet or not exercising it. The option value at the node takes the higher value of these. We calculate the values at each l in the order from $l=1$ to $l=M$. This is because, in order to decide the value of l, we need the value of $l-1$ as we can see in Equation (10). The second algorithm is an improved calculation, shown as "Refinement 1" in Exhibit 6. In this algorithm, we skip the value-comparing process for nodes, at which the option holder exercises the i-caplet with one fewer exercise opportunity. This is because the option holder should exercise the i-caplet with more exercise opportunity if the state and the period are the same. Then, for the remaining nodes, we do the same comparison process as "Normal" for each l. According to Exhibit 6, the calculation times of "Refinement 1" are improved slightly. This is because, in this algorithm, we renew the exercise area at each l. This renewal process requires close to the same calculation time as the "Normal" calculation. The third algorithm is an even more improved calculation, shown as "Refinement 2" in Exhibit 6. In this algorithm, we skip the value-comparing process for nodes, at which the option holder exercises the

EXHIBIT 6
Calculation Time (Second)

This Exhibit shows the calculation times of the chooser flexible cap prices for each T (option maturity) and l (exercise opportunity) with parameters β (parameter of HW model) $= 0.3$, σ (parameter of HW model) $= 0.01$, K (exercise rate) $= 70\%$, δ (time interval) $= 0.25$, and R (initial yield curve): increasing 0.0025 in each period starting from 0.1.

	Normal	Refinement1	Refinement2
$T=5$ ($N=20$) $l=10$	0.3210	0.2700	0.1910
$T=10$ ($N=40$) $l=20$	0.9210	0.8310	0.5400
$T=20$ ($N=80$) $l=40$	3.2550	2.9040	1.7320
$T=30$ ($N=120$) $l=60$	7.3910	6.4300	3.6550
$T=40$ ($N=160$) $l=80$	12.9780	11.6470	6.1090

i-caplet with $l = 1$. Then, for the remaining nodes we do the same comparison process as "Normal" at each l. According to Exhibit 6, the calculation times are cut to about half of "Normal." This is because, in this algorithm, we do not need to renew the exercise area for each l. The third algorithm is very effective, especially when the exercise area does not expand rapidly as l gets bigger. We show the case where l takes a value equal to half the value of N. For other values of l, we obtain results with close to the same improvement. In this calculation, we use MAT-LAB6.1 (Release 12.1). The computer environment is a CPU:Intel Pentium M processor 1000MHz, Memory:256MB.

CONDITIONS UNDER WHICH THE OPTION HOLDER DOES NOT EXERCISE THE CHOOSER FLEXIBLE CAP

In this section, we derive the theoretical conditions under which the option holder does not exercise the i-caplet at T_i.

Conditions of the Exercise Opportunity $l = 1$

In this subsection, we derive the theoretical conditions under which we should not exercise an $N-2$-caplet of the chooser flexible cap with $l = 1$ at $t = T_{N-2}$ in Proposition 2, utilizing the evaluation under the forward-neutral probability. Using the induction, we will derive the conditions under which we should not exercise the chooser flexible cap with $l = 1$ at T_i for $i = 0, \ldots, N-2$ in Proposition 3.

Proposition 2 *The holder of the chooser flexible cap with the $l = 1$ exercise opportunity should not exercise the $N-2$-caplet at $t = T_{N-2}$ under the condition:*

$$L_{T_{N-2}}(T_{N-2}) < \frac{L_{T_{N-1}}(T_{N-2})}{1+\delta L_{T_{N-1}}(T_{N-2})} \tag{19}$$

Proposition 3 *The holder of the chooser flexible cap with the $l = 1$ exercise opportunity should not exercise the i-caplet at $t = T_i$ under the condition:*

$$L_{T_i}(T_i) < \frac{L_{T_{i+1}}(T_i)}{1+\delta L_{T_{i+1}}(T_i)}, \quad i = 0, \ldots, N-2 \tag{20}$$

Proof. See Appendix A.

Conditions of the Exercise Oppurtunity $2 \le l < N-i$

In this subsection, we derive the theoretical conditions under which we should not exercise an $N-3$-caplet of the chooser flexible cap with the exercise opportunity $l = 2$ at $t = T_{N-3}$ in Proposition 4 utilizing evaluation under varying forward-neutral probabilities. Using induction, we will derive the conditions under which we should not exercise the chooser flexible cap with l exercise opportunities $2 \le l < N-i$ at T_i for $i = 0, \ldots, N-3$ in Proposition 5.

Proposition 4 *The holder of the chooser flexible cap with $l = 2$ exercise opportunities should not exercise the $N-3$-caplet at $t = T_{N-3}$ under the conditions:*

$$L_{T_{N-3}}(T_{N-3}) < \frac{L_{T_{N-2}}(T_{N-3})}{1+\delta L_{T_{N-2}}(T_{N-3})} \tag{21}$$

$$L_{T_{N-3}}(T_{N-3}) < \frac{L_{T_{N-1}}(T_{N-3})}{1+2\delta L(T_{N-3};T_{N-2},T_N)} \tag{22}$$

Proposition 5 *The holder of the chooser flexible cap with $2 \le l < N-i$ should not exercise the i-caplet at $t = T_i$ under the conditions:*

$$L_{T_i}(T_i) < \frac{L_{T_{i+1}}(T_i)}{1+\delta L_{T_{i+1}}(T_i)} \tag{23}$$

$$L_{T_i}(T_i) < \frac{L_{T_{i+2}}(T_i)}{1+2\delta L(T_i;T_{i+1},T_{i+3})}, \quad i = 0, \ldots, N-3 \tag{24}$$

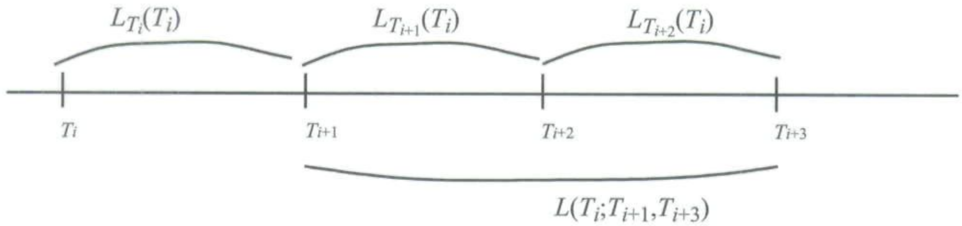
Proof: See Appendix A.

We show some examples for Proposition 5. Exhibit 7 represents a relation of the spot LIBOR, the forward LIBOR, and the forward LIBOR for the multiple period used in conditions Equation (23) and Equation (24).

Suppose that the option holder can observe the following LIBORs at T_i:

EXHIBIT 7

An Example of Proposition 5



i. Flat forward LIBOR curve

$$L_{T_i}(T_i) = 0.1,\quad L_{T_{i+1}}(T_i) = 0.1,\quad L_{T_{i+2}}(T_i) = 0.1,$$
$$L(T_i;T_{i+1},\ T_{i+3}) = 0.2,\ \delta = 0.25$$

Equation (23) $0.1 \nless 0.0976$, Equation (24) $0.1 \nless 0.091$

$\Rightarrow$ The data do not satisfy the conditions.

ii. Increasing forward LIBOR curve

$$L_{T_i}(T_i) = 0.1,\ L_{T_{i+1}}(T_i) = 0.2,\ L_{T_{i+2}}(T_i) = 0.3,$$
$$L(T_i, T_{i+1}, T_{i+3}) = 0.5,\ \delta = 0.25$$

Equation (23) $0.1 < 0.190$, Equation (24) $0.1 < 0.240$

$\Rightarrow$ The data satisfy the conditions. $\Rightarrow$ The option holder should not exercise the i-caplet at T_i.

iii. Hump forward LIBOR curve

$$L_{T_i}(T_i) = 0.1,\quad L_{T_{i+1}}(T_i) = 0.3,\quad L_{T_{i+2}}(T_i) = 0.1,$$
$$L(T_i;\ T_{i+1};\ T_{i+3}) = 0.4,\ \delta = 0.25$$

Equation (23) $0.1 < 0.279$, Equation (24) $0.1 \nless 0.083$

$\Rightarrow$ The data do not satisfy the conditions.

In the example shown above, the option holder should not exercise the i-caplet in the case of the increasing forward LIBOR curve. Propositions 2–5 indicate that if the interest rate is expected to increase in the future, the option holder should not exercise the i-caplet because he/she can expect to get a higher payment by saving the caplet for use at a later time.

CONCLUSION

We present two properties of the chooser flexible cap in this article. The first property is that when the number of permitted exercise opportunities is increased, exercise remains optimal at all of the same points as in the more restricted case with fewer exercise opportunities. The second property is that we derive the conditions under which the option holder does not exercise the i-caplet. We can use the first property to reduce the computational time of the chooser flexible cap price. We can also use the second property to develop profitable exercise strategies based on observable LIBOR data in the market. We derive the properties only assuming that the LIBOR of each period follows a geometric Brownian motion process. Thus, the properties are valid not only under the Hull–White model as we see in Ito, Ohnishi, and Tamba [2004], but also under the BGM framework.

APPENDIX A

Proof of Proposition 1

Proof. We prove that the holder of the chooser flexible cap should exercise the option, i-caplet, with l exercise opportunities at T_i, if he/she exercises it with $l-1$ at T_i for $i = 0, \ldots, N-1$. We define $W(T_i, j, l)$ as the value of the chooser flexible cap with l exercise opportunities at a state j and a period T_i. $W(T_i, j, l; E)$ represents the condition that an option holder exercises the i-caplet of the chooser flexible cap with l exercise opportunities at a state j and a period T_i. Under the assumption, $W(T_i, j, l-1; E)$ at T_i, we want to prove $W(T_i, j, l; E)$. $W(T_i, j, l-1; E)$ means:

$$D(T_i,T_{i+1})\delta(L_{T_i}(T_i)-K)_+ + D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}\left[\frac{W(T_{i+1},j,l-2)}{D(T_{i+1},T_{i+2})}\,\middle|\, L(T_i)\right] > D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}\left[\frac{W(T_{i+1},j,l-1)}{D(T_{i+1},T_{i+2})}\,\middle|\, L(T_i)\right] \tag{25}$$

This inequality can be rewritten as:

$$D(T_i,T_{i+1})\delta(L_{T_i}(T_i)-K)_+ > D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}\left[\frac{W(T_{i+1},j,l-1)-W(T_{i+1},j,l-2)}{D(T_{i+1},T_{i+2})}\,\middle|\, L(T_i)\right] \tag{26}$$

In the same way, $W(T_i, j, l; E)$ means:

$$D(T_i,T_{i+1})\delta(L_{T_i}(T_i)-K)_+ > D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}\left[\frac{W(T_{i+1},j,l)-W(T_{i+1},j,l-1)}{D(T_{i+1},T_{i+2})}\,\middle|\, L(T_i)\right] \tag{27}$$

Hence, from Equation (26) to prove Equation (27), we should prove:

$$W(T_{i+1},j,l-1)-W(T_{i+1},j,l-2) \geq W(T_{i+1},j,l)-W(T_{i+1},j,l-1) \tag{28}$$

$W(T_{i+1}, j, l-1) - W(T_{i+1}, j, l-2)$, the LHS of Equation (28), is considered as the option holder buys the chooser flexible cap with $l-1$ exercise opportunities and sells the chooser flexible cap with $l-2$ exercise opportunities at T_{i+1}. The option holder of $W(T_{i+1}, j, l-1)$ can mimic the exercise strategy of $W(T_{i+1}, j, l-2)$. After finishing the mimic of the strategy of $W(T_{i+1}, j, l-2)$, the option holder of $W(T_{i+1}, j, l-1)$ can exercise a caplet at one of the remaining time periods. The logic is the same as above in the case of the RHS of Equation (28). $W(T_{i+1}, j, l) - W(T_{i+1}, j, l-1)$ is considered as the option holder buys the chooser flexible cap with l exercise opportunities and sells the chooser flexible cap with $l-1$ exercise opportunities at T_{i+1}. The option holder of $W(T_{i+1}, j, l)$ can mimic the exercise strategy of $W(T_{i+1}, j, l-1)$. After finishing the mimic of the strategy of $W(T_{i+1}, j, l-1)$, the option holder of $W(T_{i+1}, j, l)$ can exercise a caplet at one of the remaining time periods. Comparing both sides of Equation (28), $W(T_{i+1}, j, l-1) - W(T_{i+1}, j, l-2)$ is at least worth more than $W(T_{i+1}, j, l) - W(T_{i+1}, j, l-1)$ because more periods remain to exercise the remaining caplet.

Proof of Corollary 1

Proof. We prove that the holder of the chooser flexible cap should not exercise the option, i-caplet, with $l-1$ exercise opportunities at T_i if he/she does not exercise it with l at T_i for $i = 0, \ldots, N-1$. $W(T_i, j, l, X)$ represents the condition that the option holder does not exercise the i-caplet of the chooser flexible cap with l exercise opportunities at a state j and $t = T_i$. Under the assumption, $W(T_i, j, l; X)$, we want to prove $W(T_i, j, l-1; X)$ at T_i. The condition of $W(T_i, j, l; X)$ can be written as:

$$D(T_i,T_{i+1})\delta(L_{T_i}(T_i)-K)_+ < D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}\left[\frac{W(T_{i+1},j,l)-W(T_{i+1},j,l-1)}{D(T_{i+1},T_{i+2})}\middle| L(T_i)\right] \tag{29}$$

$W(T_i, j, l-1; X)$ means:

$$D(T_i,T_{i+1})\delta(L_{T_i}(T_i)-K)_+ < D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}\left[\frac{W(T_{i+1},j,l-1)-W(T_{i+1},j,l-2)}{D(T_{i+1},T_{i+2})}\middle| L(T_i)\right] \tag{30}$$

Hence, from Equation (29) in order to prove Equation (30), we should prove:

$$W(T_{i+1},j,l-1)-W(T_{i+1},j,l-2) \ \geq\ W(T_{i+1},j,l)-W(T_{i+1},j,l-1) \tag{31}$$

This inequality is satisfied for the same reason as we see in the proof of Proposition 1.

Proof of Proposition 2

Proof. We derive the theoretical conditions under which the option holder should not exercise the $N-2$-caplet at T_{N-2}. Because the holder exercises the option at the terminal period T_{N-1}, the condition is:

$$D(T_{N-2},T_N)\delta(L_{T_{N-2}}(T_{N-2})-K)_+ < D(T_{N-2},T_N)\mathbb{E}^{T_N}\left[\delta(L_{T_{N-1}}(T_{N-1})-K)_+ \,\middle|\, L(T_{N-2})\right] \tag{32}$$

From the convexity of the function $G(L_{T_{N-1}}(T_{N-1})) := (L_{T_{N-1}}(T_{N-1})-K)_+$, $G(0)=0$ and the log normality of $L_{T_{N-1}}(T_{N-1})$ under $\mathbb{P}^{T_N}$, we have:

$$\text{RHS of Equation (32)} \geq D(T_{N-2},T_{N-1})\delta\left(\frac{L_{T_{N-1}}(T_{N-2})}{1+\delta L_{T_{N-1}}(T_{N-2})}-K\right)_+ \tag{33}$$

Because $D(T_{N-2}, T_N)/D(T_{N-2}, T_{N-1}) < 1$, the sufficient condition to satisfy this proposition is:

$$L_{T_{N-2}}(T_{N-2}) < \frac{L_{T_{N-1}}(T_{N-2})}{1+\delta L_{T_{N-1}}(T_{N-2})} \tag{34}$$

Proof of Proposition 3

Proof. In the case $l=1$ and $t=T_{N-2}$, we prove that we should not exercise the option under Equation (34) in Proposition 2. In the case of $l=1$ and $t=T_{i+1}$, we suppose that we should not exercise the option under the condition:

$$L_{T_{i+1}}(T_{i+1}) < \frac{L_{T_{i+2}}(T_{i+1})}{1+\delta L_{T_{i+2}}(T_{i+1})} \tag{35}$$

that is,

$$D(T_{i+1},T_{i+2})\delta(L_{T_{i+1}}(T_{i+1})-K)_+ < D(T_{i+1},T_{i+3})\mathbb{E}^{T_{i+3}}\left[\frac{W(T_{i+2},L_{T_{i+2}}(T_{i+2}),1)}{D(T_{i+2},T_{i+3})}\middle| L(T_{i+1})\right] = W(T_{i+1},L_{T_{i+1}}(T_{i+1}),1) \tag{36}$$

In the case of $l = 1$ and $t = T_i$, we would like to show that under the condition:

$$L_{T_i}(T_i) < \frac{L_{T_{i+1}}(T_i)}{1+\delta L_{T_{i+1}}(T_i)} \tag{37}$$

we should not exercise the option at $t = T_i$ by the induction. The optimality equation at $t = T_i$ is:

$$W(T_i,L_{T_i}(T_i),1) = \max\left\{D(T_i,T_{i+1})\delta\left(L_{T_i}(T_i)-K\right)_+ \, D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}\left[\frac{W(T_{i+1},L_{T_{i+1}},(T_{i+1}),1)}{D(T_{i+1},T_{i+2})}\middle| L(T_i)\right]\right. \tag{38}$$

From the hypothesis, Equation (36), substituting the LHS of Equation (36) for the second term of Equation (38), we obtain:

$$D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}\left[\frac{W(T_{i+1},L_{T_{i+1}}(T_{i+1}),1)}{D(T_{i+1},T_{i+2})}\middle| L(T_i)\right] > D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}\left[\delta(L_{T_{i+1}}(T_{i+1})-K)_+ \middle| L(T_i)\right] \tag{39}$$

Utilizing the relation:

$$D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}\left[\delta(L_{T_{i+1}}(T_{i+1})-K)_+ \middle| L(T_i)\right] \geq D(T_i,T_{i+1})\delta\left(\frac{L_{T_{i+1}}(T_i)}{1+\delta(L_{T_{i+1}}(T_i)}-K\right)_+ \tag{40}$$

and comparing the first term of the RHS of Equation (38) and Equation (39), we obtain the objective condition:

$$D(T_i,T_{i+1})\delta(L_{T_i}(T_i)-K)_+ \leq D(T_i,T_{i+1})\delta\left(\frac{L_{T_{i+1}}(T_i)}{1+\delta(L_{T_{i+1}}(T_i)}-K\right)_+ \tag{41}$$

Then the sufficient condition under which the option holder should not exercise the i-caplet at T_i is:

$$L_{T_i}(T_i) < \frac{L_{T_{i+1}}(T_i)}{1+\delta L_{T_{i+1}}(T_i)} \tag{42}$$

This condition is satisfied from hypothesis Equation (37).

Proof of Proposition 4

Proof. We consider the case $2 \leq l < N - i$ at $t = T_{N-3}$. In the case $3 \leq l$, we exercise the options at $t = T_{N-3}$, T_{N-2} and T_{N-1}. In the case $l = 2$, we want to derive the conditions under which the option holder should not exercise the $N - 3$-caplet at $t = T_{N-3}$. From the forward-neutral evaluation, the value of the chooser flexible cap with $l = 2$ at $t = T_{N-3}$ is:

$$W(T_{N-3},\ L_{T_{N-3}}(T_{N-3}),\ 2)\ =\ \max\{A,\ B\} \tag{43}$$

where

$$\begin{aligned}A = {} & D(T_{N-3},T_{N-2})\delta(L_{T_{N-3}}(T_{N-3})-K)_+ + D(T_{N-3},\ T_{N-1})\mathbb{E}^{T_{N-1}} \\ & \left[\max\left\{\delta\left(L_{T_{N-2}}(T_{N-2})-K\right)_+, \frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N-1})}\mathbb{E}^{T_N}\left[\delta\left(L_{T_{N-1}}(T_{N-1})-K\right)_+ \middle| L(T_{N-2})\right]\right\}\middle| L(T_{N-3})\right]\end{aligned} \tag{44}$$

On the other hand, because B is the value when the option holder exercises options at T_{N-2} and T_{N-1}, we have:

$$\begin{aligned}B = {} & D(T_{N-3},\ T_{N-1})\mathbb{E}^{T_{N-1}}\left[\delta(L_{T_{N-2}}(T_{N-2})-K)_+ \middle| L(T_{N-3})\right] \\ & + D(T_{N-3},\ T_{N-1})\mathbb{E}^{T_{N-1}}\left[\frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N-1})}\mathbb{E}^{T_N}\left[\delta(L_{T_{N-1}}(T_{N-1})-K)_+ \middle| L(T_{N-2})\right]\middle| L(T_{N-3})\right]\end{aligned} \tag{45}$$

The sufficient conditions under which the option holder should not exercise the $N-3$-caplet at $t = T_{N-3}$, is $A < B$. Using Jensen's inequality,

$$\begin{aligned}A \geq {} & D(T_{N-3},T_{N-2})\delta\left(L_{T_{N-3}}(T_{N-3})-K\right)_+ \\ & + D(T_{N-3},T_{N-1})\max\left\{\mathbb{E}^{T_{N-1}}\left[\delta(L_{T_{N-2}}(T_{N-2})-K)_+ \mid L(T_{N-3})\right],\right. \\ & \left.\mathbb{E}^{T_{N-1}}\left[\frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N-1})}\mathbb{E}^{T_N}\left[\delta\left(L_{T_{N-1}}(T_{N-1})-K\right)_+ \mid L(T_{N-2})\right]\middle| L(T_{N-3})\right]\right\} =: C\end{aligned} \tag{46}$$

we derive the sufficient conditions under which $C < B$ is satisfied. These are the sufficient conditions under which the option holder does not exercise the option at T_{N-3}. We have two cases in Equation (46). The first case is:

$$\mathbb{E}^{T_{N-1}}\left[\delta\left(L_{T_{N-2}}(T_{N-2})-K\right)_+ \mid L(T_{N-3})\right] < \mathbb{E}^{T_{N-1}}\left[\frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N-1})}\mathbb{E}^{T_N}\left[\delta(L_{T_{N-1}}(T_{N-1})-K)_+ \mid L(T_{N-2})\right]\middle| L(T_{N-3})\right] \tag{47}$$

In this case, we have:

$$C = D(T_{N-3},T_{N-2})\delta\left(L_{T_{N-3}}(T_{N-3})-K\right)_+ + D(T_{N-3},T_{N-1})\mathbb{E}^{T_{N-1}}\left[\frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N-1})}\mathbb{E}^{T_N}\left[\delta\left(L_{T_{N-1}}(T_{N-1})-K\right)_+ \middle| L(T_{N-2})\right]\middle| L(T_{N-3})\right] \tag{48}$$

Thus, in order to satisfy $C < B$, we need:

$$D(T_{N-3},T_{N-2})\delta(L_{T_{N-3}}(T_{N-3})-K)_+ < D(T_{N-3},T_{N-1})\mathbb{E}^{T_{N-1}}\left[\delta(L_{T_{N-2}}(T_{N-2})-K)_+ \mid L(T_{N-3})\right] \tag{49}$$

Utilizing the relation:

$$D(T_{N-3},T_{N-1})\mathbb{E}^{T_{N-1}}\left[\delta\left(L_{T_{N-2}}(T_{N-2})-K\right)_+ \middle| L(T_{N-3})\right] \geq D(T_{N-3},T_{N-2})\delta\left(\frac{L_{T_{N-2}}(T_{N-3})}{1+\delta L_{T_{N-2}}(T_{N-3})}-K\right)_+ \tag{50}$$

we find that the sufficient condition for $C < B$ is:

$$L_{T_{N-3}}(T_{N-3}) < \frac{L_{T_{N-2}}(T_{N-3})}{1+\delta L_{T_{N-2}}(T_{N-3})} \tag{51}$$

The second case is:

$$\mathbb{E}^{T_{N-1}}\left[\frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N-1})}\mathbb{E}^{T_N}\left[\delta(L_{T_{N-1}}(T_{N-1})-K)_+ \,\middle|\, L(T_{N-2})\right]\middle| L(T_{N-3})\right] < D(T_{N-3},T_{N-1})\mathbb{E}^{T_{N-1}}\left[\delta\left(L_{T_{N-2}}(T_{N-2})-K\right)_+ \,\middle|\, L(T_{N-3})\right] \tag{52}$$

In this case, we have:

$$C = D(T_{N-3},T_{N-2})\delta\left(L_{T_{N-3}}(T_{N-3})-K\right)_+ + D(T_{N-3},T_{N-1})\mathbb{E}^{T_{N-1}}\left[\delta\left(L_{T_{N-2}}(T_{N-2})-K\right)_+ \,\middle|\, L(T_{N-3})\right] \tag{53}$$

Thus, in order to satisfy C < B, we need:

$$D(T_{N-3},T_{N-2})\delta\left(L_{T_{N-3}}(T_{N-3})-K\right)_+ < D(T_{N-3},T_{N-1})\mathbb{E}^{T_{N-1}}\left[\frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N-1})}\mathbb{E}^{T_N}\left[\delta\left(L_{T_{N-1}}(T_{N-1})-K\right)_+ \,\middle|\, L(T_{N-2})\right]\middle| L(T_{N-3})\right] \tag{54}$$

From the forward-neutral evaluation method, we have the relation:

$$\begin{aligned} &D(T_{N-3},T_{N-1})\mathbb{E}^{T_{N-1}}\left[\frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N-1})}\mathbb{E}^{T_N}\left[\delta\left(L_{T_{N-1}}(T_{N-1})-K\right)_+ \,\middle|\, L(T_{N-2})\right]\middle| L(T_{N-3})\right] \\ &\qquad = D(T_{N-3},T_N)\mathbb{E}^{T_N}\left[\mathbb{E}^{T_N}\left[\delta(L_{T_{N-1}}(T_{N-1})-K)_+ \,\middle|\, L(T_{N-2})\right]\middle| L(T_{N-3})\right] \end{aligned} \tag{55}$$

RHS of Equation (55)

$$= D(T_{N-3},T_{N-2})\frac{1}{1+2\delta L(T_{N-3};T_{N-2},T_N)}\mathbb{E}^{T_N}\left[\delta\left(L_{T_{N-1}}(T_{N-1})-K\right)_+ \mid L(T_{N-3})\right] \geq D(T_{N-3},T_{N-2})\delta\left(\frac{L_{T_{N-1}}(T_{N-3})}{1+2\delta L(T_{N-3};T_{N-2},T_N)}-K\right)_+ \tag{56}$$

Thus, in the second case, we can derive the following condition:

$$L_{T_{N-3}}(T_{N-3}) < \frac{L_{T_{N-1}}(T_{N-3})}{1+2\delta L(T_{N-3};T_{N-2},T_N)} \tag{57}$$

Proof of Proposition 5

Proof. We consider the case $2 \leq l < N-i$ at $t = T_i$. In the case $t = T_{N-3}$, from the result of Proposition 4, we found that we do not exercise the option under conditions Equation (51) and Equation (57). We suppose that we do not exercise the option under the conditions:

$$L_{T_{i+1}}(T_{i+1}) < \frac{L_{T_{i+2}}(T_{i+1})}{1+\delta L_{T_{i+2}}(T_{i+1})} \tag{58}$$

$$L_{T_{i+1}}(T_{i+1}) < \frac{L_{T_{i+3}}(T_{i+1})}{1+2\delta L(T_{i+1};T_{i+2},T_{i+4})} \tag{59}$$

at $t = T_{i+1}$. That is,

$$W(T_{i+1}, L_{T_{i+1}}(T_{i+1}), l) = \max\{H, I\} \tag{60}$$

satisfies $H < I$, where

$$H = D(T_{i+1},T_{i+2})\delta\left(L_{T_{i+1}}(T_{i+1}) - K\right)_{+} + D(T_{i+1},T_{i+3})\mathbb{E}^{T_{i+3}}\left[\frac{W(T_{i+2}, L_{T_{i+2}}(T_{i+2}), l-1)}{D(T_{i+2},T_{i+3})}\middle| L(T_{i+1})\right] \tag{61}$$

$$I = D(T_{i+1},T_{i+3})\mathbb{E}^{T_{i+3}}\left[\frac{W(T_{i+2}, L_{T_{i+2}}(T_{i+2}), l)}{D(T_{i+2},T_{i+3})}\middle| L(T_{i+1})\right] \tag{62}$$

We would like to show that, under the conditions:

$$L_{T_i}(T_i) < \frac{L_{T_{i+1}}(T_i)}{1+\delta L_{T_{i+1}}(T_i)} \tag{63}$$

$$L_{T_i}(T_i) < \frac{L_{T_{i+2}}(T_i)}{1+2\delta L(T_i;T_{i+1},T_{i+3})} \tag{64}$$

we do not exercise the option at $t = T_i$. The optimality equation is:

$$W(T_i, L_{T_i}(T_i), l) = \max\{P, Q\} \tag{65}$$

where

$$P = D(T_i,T_{i+1})\delta\left(L_{T_i}(T_i) - K\right)_{+} + D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}\left[\frac{W(T_{i+1}, L_{T_{i+1}}(T_{i+1}), l-1)}{D(T_{i+1},T_{i+2})}\middle| L(T_i)\right] \tag{66}$$

$$Q = D(T_i,T_{i+2})\mathbb{E}^{T_{i+2}}\left[\frac{W(T_{i+1}, L_{T_{i+1}}(T_{i+1}), l)}{D(T_{i+1},T_{i+2})}\middle| L(T_i)\right] \tag{67}$$

From the hypothesis, H < I, substituting H for Q, we obtain:

$$Q > D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}\left[\delta\left(L_{T_{i+1}}(T_{i+1}) - K\right)_+ \mid L(T_i)\right] + D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}\left[\frac{D(T_{i+1}, T_{i+3})}{D(T_{i+1}, T_{i+2})}\mathbb{E}^{T_{i+3}}\left[\frac{W(T_{i+2}, L_{T_{i+2}}(T_{i+2}), l-1)}{D(T_{i+2}, T_{i+3})}\middle| L(T_{i+1})\right]\middle| L(T_i)\right] \tag{68}$$

Utilizing the relation:

$$D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}\left[\delta\left(L_{T_{i+1}}(T_{i+1}) - K\right)_+ \mid L(T_i)\right] \geq D(T_i, T_{i+1})\delta\left(\frac{L_{T_{i+1}}(T_i)}{1 + \delta L_{T_{i+1}}(T_i)} - K\right)_+ \tag{69}$$

and comparing the first terms of P and Equation (68), we obtain the sufficient condition:

$$D(T_i, T_{i+1})\delta\left(L_{T_i}(T_i) - K\right)_+ \leq D(T_i, T_{i+1})\delta\left(\frac{L_{T_{i+1}}(T_i)}{1 + \delta L_{T_{i+1}}(T_i)} - K\right)_+ \tag{70}$$

Then, the sufficient condition is:

$$L_{T_i}(T_i) < \frac{L_{T_{i+1}}(T_i)}{1 + \delta L_{T_{i+1}}(T_i)} \tag{71}$$

This condition is satisfied from hypothesis Equation (63). In order to compare the second terms of P and Equation (68), we use Corollary 1. From the hypothesis, $H < I$ with l exercise opportunities, we know that the option holder does not exercise the option with $l-1$ at T_{i+1} with the same state. Thus, we have:

$$W\left(T_{i+1}, L_{T_{i+1}}(T_{i+1}), l-1\right) = D(T_{i+1}, T_{i+3})\mathbb{E}^{T_{i+3}}\left[\frac{W(T_{i+2}, L_{T_{i+2}}(T_{i+2}), l-1)}{D(T_{i+2}, T_{i+3})}\middle| L(T_{i+1})\right] \tag{72}$$

We substitute $W(T_{i+1}, L_{T_{i+1}}(T_{i+1}), l-1)$ for the second term of P. We obtain:

$$D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}\left[\frac{W(T_{i+1}, L_{T_{i+1}}(T_{i+1}), l-1)}{D(T_{i+1}, T_{i+2})}\middle| L(T_i)\right] = D(T_i, T_{i+2})\mathbb{E}^{T_{i+2}}\left[\frac{D(T_{i+1}, T_{i+3})}{D(T_{i+1}, T_{i+2})}\mathbb{E}^{T_{i+3}}\left[\frac{W(T_{i+2}, L_{T_{i+2}}(T_{i+2}), l-1)}{D(T_{i+2}, T_{i+3})}\middle| L(T_{i+1})\right]\middle| L(T_i)\right] \tag{73}$$

Thus, the second term of P and the second term of Equation (68) are equal. Then under conditions Equation (63) and Equation (64) we do not exercise the option at $t = T_i$.

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