A New Approach for Computing Option Prices of the Hull-White Type with Stepwise Reversion and Volatility Functions

HUI JIN, JUN-YA GOTOH, AND USHIO SUMITA

HUI JIN
is an assistant professor at School of Finance and Economics, Hangzhou Dianzi University in Hangzhou, China.
jinhui@hdu.edu.cn

JUN-YA GOTOH
is an associate professor at Department of Industrial and Systems Engineering, Chuo University in Tokyo, Japan.
jgoto@indsys.chuo-u.ac.jp

USHIO SUMITA
is a professor at Graduate School of Systems and Information Engineering, University of Tsukuba in Tsukuba, Japan.
sumita@sk.tsukuba.ac.jp

The Hull-White model is a one factor Markov model well known for its capability to capture the current term structure of interest rates. Analytical results are available for pricing zero-coupon discount bonds and associated European options when both reversion and volatility functions are constant. In reality, however, these functions do vary over time. It is then of practical interest to develop efficient computational algorithms that can deal with time dependent reversion and volatility functions. The purpose of this article is to achieve this goal via the Ehrenfest approximation of the underlying O-U process, where the time dependent structure is represented by step functions. Based on the convergence theorem by Sumita, Gotoh and Jin [2006] and the uniformization procedure of Keilson [1979], a novel approach is proposed to evaluate the prices of zero-coupon discount bonds and associated European options for stepwise reversion and volatility functions. The ordinary Hull-White trinomial tree approach is also modified to cope with this case for comparison purposes. However, it is shown that the modified trinomial tree approach is not applicable for certain step functions, while the Ehrenfest approach can always be used for any step functions. Numerical results are given, demonstrating the excellent speed and accuracy of the Ehrenfest approach.

The Hull-White model is a one factor term structure model characterized by a stochastic differential equation of the form

$$\mathrm{d}R(t) = (\phi(t) - \alpha(t)R(t))\mathrm{d}t + \sigma(t)\mathrm{d}W(t) \quad (1)$$

where $R(t)$ is a random short rate, $W(t)$ is the standard Wiener process, $\phi(t)$ is a market fitting function, $\alpha(t) \geq 0$ is a reversion function and $\sigma(t) \geq 0$ is a volatility function.

A basic function describing this process is the conditional transition probability density $g(r_0, r, t) \stackrel{\text{def.}}{=} \frac{\mathrm{d}}{\mathrm{d}r}\mathrm{P}[R(t) \leq r \mid R(0) = r_0]$ given by

$$g(r_0, r, t) = \frac{1}{\sqrt{2\pi\sigma^2(t \mid r_0)}} \exp\left\{\frac{(r - \mu(t \mid r_0))^2}{2\sigma(t \mid r_0)}\right\} \quad (2)$$

where

$$\begin{cases} \mu(t \mid r_0) = \mathrm{E}[R(t) \mid R(0) = r_0] \\ \qquad = r_0 \exp\{-\int_0^t \alpha(\gamma)\mathrm{d}\gamma\} + \int_0^t \phi(\tau) \\ \qquad \exp\{-\int_\tau^t \alpha(\gamma)\mathrm{d}\gamma\}\mathrm{d}\tau \\ \sigma^2(t \mid r_0) = \mathrm{V}[R(t) \mid R(0) = r_0] \\ \qquad = \int_0^t \sigma(\tau)^2 \exp\{-2\int_\tau^t \alpha(\gamma)\mathrm{d}\gamma\}\mathrm{d}\tau \end{cases} \quad (3)$$

The Hull-White model is well known for its capability to capture the current term structure of interest rates, e.g., the current yield curve when the yield curve is flat. Because of

this reason, the Hull-White model is considered to be one of the most reasonable models for pricing interest rate options. When both the reversion function $\alpha(t)$ and the volatility function $\sigma(t)$ are constant, Hull and White [1990] show explicit formulas for evaluating the prices of zero-coupon discount bonds and associated European options. However, when $\alpha(t)$ and $\sigma(t)$ are time-dependent, $R(t)$ cannot be evaluated explicitly and it becomes impossible to obtain the desired prices directly from those formulas.

In reality, the reversion function $\alpha(t)$ and the volatility function $\sigma(t)$ are dependent on time t and are often nonlinear. Since no analytical results are available for dealing with such reversion and volatility functions, one would often resort to Monte Carlo simulation in practice. As in the theory of integrals, however, it is possible to approximate such nonlinear functions by step functions. One may then expect to overcome the step functions by applying a computational approach for the case of constant reversion and volatility functions repeatedly. Based on this idea, the purpose of this article is to develop a new computational algorithm for evaluating the prices of zero-coupon discount bonds and associated European options when both $\alpha(t)$ and $\sigma(t)$ are step functions. The parameter $\phi(t)$ relates to the yield curve. The time dependent structure of $\phi(t)$, however, does not affect the proposed approach in any essential way. Accordingly, in order to focus on the stepwise reversion and volatility functions, and avoid the yield curve fitting, only the case of constant $\phi(t)$ is considered in this article. A remark will be provided, regarding how the described numerical procedures should be altered when $\phi(t)$ is not constant.

For the case of constant $\phi(t)$, $\alpha(t)$, and $\sigma(t)$, the Hull-White model is reduced to the Vasicek model [1977] specified by the following stochastic differential equation:

$$\mathrm{d}\hat{X}_{\mathrm{OU}}(t) = (\phi - \alpha \hat{X}_{\mathrm{OU}}(t))\mathrm{d}t + \sigma \mathrm{d}W(t) \qquad (4)$$

The associated conditional transition probability density $\hat{g}(\hat{x}_0, x, t) \overset{\text{def.}}{=} \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{P}[\hat{X}_{\mathrm{OU}}(t) \le x \mid \hat{X}_{\mathrm{OU}}(0) = \hat{x}_0]$ is given by

$$\hat{g}(\hat{x}_0, x, t) = \frac{1}{\sqrt{2\pi\hat{\sigma}^2(t \mid \hat{x}_0)}} \exp\left\{ \frac{(x - \hat{\mu}(t \mid \hat{x}_0))^2}{2\hat{\sigma}(t \mid \hat{x}_0)} \right\} \qquad (5)$$

where

$$\begin{cases} \hat{\mu}(t \mid \hat{x}_0) = \mathrm{E}[\hat{X}_{\mathrm{OU}}(t) \mid \hat{X}_{\mathrm{OU}}(0) = \hat{x}_0] = \dfrac{\phi}{\alpha}(1 - e^{-\alpha t}) + \hat{x}_0 e^{-\alpha t} \\ \hat{\sigma}(t \mid \hat{x}_0) = \mathrm{V}[\hat{X}_{\mathrm{OU}}(t) \mid \hat{X}_{\mathrm{OU}}(0) = \hat{x}_0] = \dfrac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}) \end{cases} \qquad (6)$$

If $\alpha(t)$ and $\sigma(t)$ are step functions, the process $R(t)$ can be expressed by patching a sequence of $\hat{X}_{\mathrm{OU}}(t)$'s. More specifically, a single process $R(t)$ with time varying parameters $\alpha(t)$ and $\sigma(t)$ is replaced by a sequence of distinct processes $\hat{X}_{\mathrm{OU}}(t)$'s, each defined in a time interval in which both $\alpha(t)$ and $\sigma(t)$ are constant, that are connected together at the end points of the time intervals. Exhibit 1 illustrates a stepwise reversion function and a stepwise volatility function treated in this article. The idea is to define the time intervals in an exclusive and exhaustive manner so that both the reversion function and the volatility function remain constant within each interval. For the example given in Exhibit 1, the reversion function takes three different values, α_i $(i = 1, 2, 3)$ while the volatility function assumes another three different values, σ_j $(j = 1, 2, 3)$, resulting in four time intervals. Consequently, the process $R(t)$ can be constructed by patching four different $\hat{X}^i_{\mathrm{OU}}(t)$ $(i = 1, 2, 3, 4)$. More specifically, one has

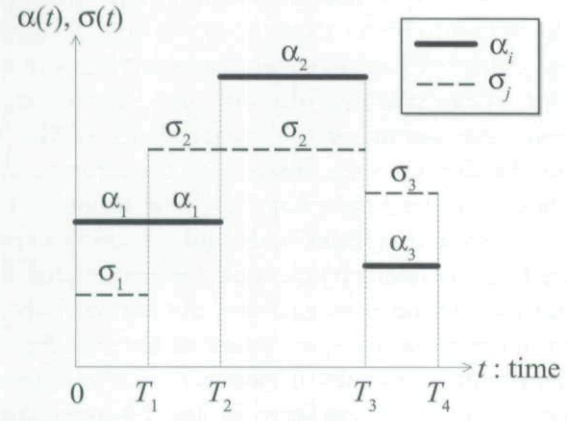
$$R(t) = \begin{cases} \hat{X}^1_{\mathrm{OU}}(t) : \mathrm{d}\hat{X}^1_{\mathrm{OU}}(t) = (\phi - \alpha_1 \hat{X}^1_{\mathrm{OU}}(t))\mathrm{d}t \\ \qquad\qquad + \sigma_1 \mathrm{d}W(t), \ 0 \le t < T_1 \\ \hat{X}^2_{\mathrm{OU}}(t) : \mathrm{d}\hat{X}^2_{\mathrm{OU}}(t) = (\phi - \alpha_1 \hat{X}^2_{\mathrm{OU}}(t))\mathrm{d}t \\ \qquad\qquad + \sigma_2 \mathrm{d}W(t), T_1 \le t < T_2 \\ \hat{X}^3_{\mathrm{OU}}(t) : \mathrm{d}\hat{X}^3_{\mathrm{OU}}(t) = (\phi - \alpha_2 \hat{X}^3_{\mathrm{OU}}(t))\mathrm{d}t \\ \qquad\qquad + \sigma_2 \mathrm{d}W(t), T_2 \le t < T_3 \\ \hat{X}^4_{\mathrm{OU}}(t) : \mathrm{d}\hat{X}^4_{\mathrm{OU}}(t) = (\phi - \alpha_3 \hat{X}^4_{\mathrm{OU}}(t))\mathrm{d}t \\ \qquad\qquad + \sigma_3 \mathrm{d}W(t), T_3 \le t < T_4 \end{cases} \qquad (7)$$

where the conditional time dependent probability density at the end of one interval becomes the initial probability density of the following interval.

In order to facilitate the idea above, it is essential to develop efficient numerical procedures for evaluating the dynamic behavior of $\hat{X}_{\mathrm{OU}}(t)$ of the Vasicek model with speed and accuracy. In addition, all of the computational procedures ought to be automated because of repeated patching. While the trinomial tree procedure of Hull and

EXHIBIT 1

Illustration of Stepwise Reversion and Volatility Functions



White [1993, 1994] is originally designed for the Vasicek model, it is possible to modify it so as to deal with stepwise reversion and volatility functions, which we call the modified trinomial tree approach. (A succinct summary of the Hull-White trinomial tree approach and its modification is given in Appendix A.) However, the modified trinomial tree approach has two essential pitfalls described below.

1. For the modified trinomial tree approach, it is necessary to discretize the time axis, say with a unit of Δt. The state transitions in Δt occur in a lattice continuous manner and involve only adjacent states. Accordingly, when the volatility is rather high, the modified trinomial tree approach may require much smaller Δt, possibly causing a substantial computational burden.
2. The modified trinomial tree approach requires the level of discretization of the state space to be determined by satisfying a certain inequality so that transition probabilities to adjacent states remain positive. As discussed above, when both $\alpha(t)$ and $\sigma(t)$ are step functions, the conditional time dependent probability density at the end of one interval becomes the initial probability density of the following interval. Hence, the level of discretization of the state space has to be the same throughout all the intervals. This may hinder the modified trinomial tree approach substantially since there is no guarantee that the necessary inequalities are satisfied simultaneously throughout all the intervals, while making the level of discretization the same. This limitation makes it difficult to automate all the numerical procedures based on the modified trinomial tree approach.

A new approach proposed in this article relies upon the Ehrenfest process approximation based on Sumita, Gotoh and Jin [2006] (hereafter SGJ [2006]), combined-with the uniformization procedure of Keilson [1979]. As we discuss below, the Ehrenfest process captures the basic characteristics of the Vasicek model $\hat{X}_{OU}(t)$ in Equation (4). In order to see this, we first note that the Vasicek model $\hat{X}_{OU}(t)$ is a diffusion process called an *Ornstein-Uhlenbeck (O-U) process*, which is characterized by its Markov property, normal distribution for the increment between any two points in time, and exponential covariance function. Furthermore, the O-U process is known to have a property called the mean-reversion, which implies that, when the process diverts away from the mean, it tends to be driven back to the mean. The Ehrenfest process is a birth-death process which can satisfy these properties in the limit with appropriate scaling and shifting of the underlying state space. The reason for this can be found in that the Ehrenfest process is a sum of K independent and identical Markov chains, each defined on $\{0,1\}$ governed by $\nu_{0,1} = \nu_{1,0} = \bar{\nu}$. Because of this independent sum, the Markov property is automatically satisfied and the normality emerges in the limit. Furthermore, the Ehrenfest process has the state space $\{0,1, \ldots, S\}$ and the upward and downward transition rates given by

$$\nu_{n,n+1} = \lambda_n = (S-n)\bar{\nu},\ 0 \le n \le S-1;$$
$$\nu_{n,n-1} = \mu_n = n\bar{\nu},\ 1 \le n \le S \qquad (8)$$

Consequently, the local growth rate of the variance is given by $\nu_{n,n+1} + \nu_{n,n-1} = S\bar{\nu}$, which is independent of n, and the local velocity is given by $\nu_{n,n+1} - \nu_{n,n-1} = (S-2n)\bar{\nu}$. This transition structure yields the desired exponential covariance function when the Ehrenfest process starts with the stationary distribution. In addition, the probability of bringing the Ehrenfest process to the middle point $S/2$ becomes higher as the process diverts away from the middle point, demonstrating the desired mean-reversion property.

In SGJ [2006], it is shown that a sequence of Ehrenfest processes converges in law to an O-U process as the density of the discretized state space goes to infinity. Since an Ehrenfest process is a Markov chain in continuous time, the state transition during Δt could occur from any

state to any other state with positive probability, no matter how small Δt may be. Accordingly, the Ehrenfest approximation approach enables one to overcome the first pitfall in 1) above. Furthermore, in the proposed approach, the level of discretization of the state space can be chosen arbitrarily. While more accurate discretization does increase the computational burden, it does not affect the automated mechanism of the computational procedures in any way, thereby avoiding the second pitfall of 2) above.

As we will see, numerical experiments reveal that the proposed approach provides a better accuracy in a consistent manner than the trinomial tree approach for the case of constant $\alpha(t)$ and $\sigma(t)$ for which the prices of zero-coupon discount bonds and associated European options are known. For the case of stepwise reversion and volatility functions, the proposed approach also seems to be more attractive for practical use than the modified trinomial tree approach because of the two reasons stated above. Indeed, concerning the second pitfall of the modified trinomial tree approach, an example will be given for which the modified trinomial tree approach is not applicable while the proposed approach can handle it without any difficulty.

The structure of this article is as follows. In the next section, the O-U process $\hat{X}_{OU}(t)$ associated with the Vasicek model is first expressed using another O-U process $X_{OU}(t)$ and a deterministic shift function $\theta(t)$. Based on the result of SGJ [2006], it is shown that a sequence of certain Ehrenfest processes converges in law to the process $X_{OU}(t)$ with appropriate shifting and scaling as the state space size of the Ehrenfest processes goes to infinity. Using the uniformization procedure of Keilson [1979], a numerical algorithm is then developed to compute the time dependent transition probability matrix of the Ehrenfest process, which in turn enables one to approximate the conditional transition probability density of $\hat{X}_{OU}(t)$ systematically.

The following section is devoted to development of computational procedures for evaluating the prices of zero-coupon discount bonds and associated European options for the Vasicek model with $\alpha(t) = \alpha$ and $\sigma(t) = \sigma$ based on the new approach. Computational results are then compared with the numerical results of the trinomial tree approach and the analytically known results. It is found that the computational results by both the Ehrenfest approach and the Hull-White trinomial tree approach converge to the analytical solution monotonically in K where the discount bond maturity is $T = K\Delta t = 5$. For the prices of zero-coupon discount bonds, the accuracy of the absolute relative errors for both the Ehrenfest approach and the Hull-White trinomial tree approach is around 0.3% at $K = 120$. The former performs slightly but consistently better than the latter, and this performance difference is enlarged when α and σ become large. In case of the prices of the associated European call options, the performance characteristics of the two approaches remain to be similar with the accuracy of the absolute relative errors at $K = 120$ being around 0.4%.

The section after that deals with the case of stepwise reversion and volatility functions. Computational algorithms are developed by patching the numerical procedures of the previous section based on the idea discussed in this section. The modified trinomial tree approach is also constructed for accommodating this case. Numerical results of the two different approaches are presented, where the absolute relative errors of the two approaches are contained within 0.4% for the range of parameter values, as well as different levels of discretization of time axis and the state space considered in this article. While the exact solutions are not available, the fact that the relative errors are bounded by 0.4% indicates the trustworthiness of the new proposed approach and the modified trinomial tree approach. A numerical example is given for which the modified trinomial tree approach collapses but that our proposed approach can handle without any difficulty. Because of this and the ease of automating the patching procedure, the proposed approach seems to be more attractive for practical use than the modified trinomial tree approach. We end with some concluding remarks. Some mathematical details are deferred to appendices. Appendix A describes the Hull-White trinomial tree approach and its modification for dealing with stepwise reversion and volatility functions. The uniformization procedure of Keilson [1979] is summarized in Appendix B and a succinct summary of the basic computational algorithms developed in this article is provided in Appendix C.

For notational convenience, throughout the article, we denote a vector by attaching a single underline as $\underline{x}$, and a matrix by attaching double underlines as $\underline{\underline{a}}$. Moreover, $\underline{1}$ means a vector whose elements are all 1 and the vector $\underline{u}_m$ means that its element corresponding to state m is 1 and all other elements are 0. In addition, $\lceil x \rceil$ means the minimum integer which is greater than or equal to x, and $\lfloor x \rfloor$ means the maximum integer which is smaller than or equal to x.

DEVELOPMENT OF COMPUTATIONAL ALGORITHMS FOR ANALYZING DYNAMIC BEHAVIORS OF O-U PROCESSES ASSOCIATED WITH THE VASICEK MODEL

In order to develop computational algorithms for analyzing dynamic behaviors of O-U processes $\hat{X}_{OU}(t)$ of Equation (4) associated with the Vasicek model, our first step is to link a standard O-U process $X_{OU}(t)$ obtained from SGJ [2006] with time scaling to $\hat{X}_{OU}(t)$. More formally, for a positive integer V, let $N_{2V}(t)$ be the Ehrenfest process defined on $N_V = \{0,1, ..., 2V\}$ governed by the upward transition rates λ_n and the downward transition rates μ_n as given in Equation (8) with $\bar{v} = \alpha/2$. For $\alpha > 0$ and $\sigma > 0$, we also define $X_V(t)$ by

$$Xv(t) \stackrel{\text{def}}{=} \frac{\sigma}{\sqrt{\alpha V}} N_{2V}(t) - \sigma\sqrt{\frac{V}{\alpha}} \tag{9}$$

It should be noted that the process $X_V(t)$ has a discrete support on $\{x_V(0), ..., x_V(2V)\}$ where

$$x_V(n) = \frac{\sigma}{\alpha V} n - \sigma\sqrt{\frac{V}{\alpha}}, n = 0,1,2,\ldots,2V \tag{10}$$

From Theorem 4.1 of SGJ [2006], with some initial conditions appropriately set, it can be shown that $X_V(t)$ converges in law to $X_{OU}(t)$ as $V \to \infty$, where $X_{OU}(t)$ is the O-U process associated with the Vasicek model with $\phi(t) = 0$, which satisfies from Equation (4),

$$dX_{OU}(t) = -\alpha X_{OU}(t)dt + \sigma dW(t) \tag{11}$$

The relationship between $\hat{X}_{OU}(t)$ with $\phi(t) = \phi \neq 0$ and $X_{OU}(t)$ with $\phi(t) = 0$ can be given as

$$\hat{X}_{OU}(t) = X_{OU}(t) + \theta(t), \text{ with } X_{OU}(0) = 0 \tag{12}$$

where

$$\theta(t) \stackrel{\text{def}}{=} \frac{\phi}{\alpha}(1 - e^{-\alpha t}) + \hat{X}_{OU}(0)e^{-\alpha t} \tag{13}$$

The relationship in Equation (12) is exploited by Kijima and Nagayama [1994] where a better version of the Hull-White trinomial tree approach is proposed by using the shift function $\theta(t)$ explicitly.

When $\theta(t)$ is a one-to-one mapping, from Equation (12), the transition probabilities of $\hat{X}_{OU}(t)$ can be constructed from those of $X_{OU}(t)$, which in turn can be approximated by those of $X_V(t)$ from Equation (12), and hence by those of $N_{2V}(t)$ from Equation (9). The transition probabilities of $N_{2V}(t)$, denoted by $\underline{\underline{P}}_{2V}(t) = [p_{2V;mn}(t)]$, can be computed based on the uniformization procedure of Keilson [1979] for general Markov chains in continuous time, which can be interpreted in the following manner. Let $N(t)$ be a general Markov chain on $\mathcal{N}$ governed by transition rate matrix $\underline{\underline{v}} = [v_{ij}]$ and define $E(i,j)$ to be the exponential variate with mean v_{ij}^{-1}. Then the dwell time of $N(t)$ at state i is given by $E(i) = \min_{j \in \mathcal{N}} \{E(i,j)\}$. Upon expiration of $E(i)$, $N(t)$ moves to state j with probability v_{ij}/v_i where $v_i = \Sigma_{j \in \mathcal{N}} v_{ij}$. While keeping all the underlying probabilistic properties intact, Keilson [1979] has shown that the dwell time of $N(t)$ at any state can be uniformized to a common exponential variate with mean v^{-1} where $v \geq \max_{i \in \mathbb{N}}\{v_i\}$, provided that the next transition to state $j (j \neq i)$ occurs with probability v_{ij}/v and a possible self-transition to state i is introduced newly with probability $1-(v_i/v)$. In this manner, the uniformization procedure allows one to capture $N(t)$ in terms of a common Poisson process with intensity v and the associated discrete time Markov chain. A strong advantage is that all the computations can be done involving only nonnegative numbers, enabling one to deal with large scale Markov chains with speed and accuracy. Formally, the uniformization procedure is summarized in Appendix B. The computational algorithm for evaluating the transition probabilities of $N_{2V}(t)$ is given in Appendix C as Algorithm C1.

Remark 1

It should be noted from Equations (12) and (13) that the market fitting function $\phi(t)$ appears only in the shift function $\theta(t)$. Hence, whether or not $\phi(t)$ is constant does not affect the above computational procedure for evaluating the transition probabilities of $\hat{X}_{OU}(t)$ in any essential way, provided that $\theta(t)$ remains a one-to-one mapping. Accordingly, Algorithm C1 remains intact even when $\phi(t)$ is not constant.

DEVELOPMENT OF COMPUTATIONAL PROCEDURES FOR EVALUATING PRICES OF DISCOUNT BONDS AND EUROPEAN OPTIONS FOR THE VASICEK MODEL

Using Algorithm C1, we are now in a position to develop computational procedures for evaluating the prices

of zero-coupon discount bonds and the prices of European options on the zero-coupon discount bonds for the Vasicek model, where both the reversion function $\alpha(t)$ and the volatility function $\sigma(t)$ are constant. While the analytical results are available in a closed form for this case, the algorithms described in this section provide a computational vehicle for dealing with the Hull-White model with stepwise reversion and volatility functions which is analytically intractable. The numerical results of the proposed algorithms are compared with the analytical results as well as those of the Hull-White trinomial tree approach, demonstrating excellent accuracy of the Ehrenfest approximation involved in the algorithms.

In order to evaluate the prices of the zero-coupon discount bonds, a discrete time backward recursive formula is employed together with the shift function $\theta(t)$ of Equation (13). Let Δt be the length of each time step with the maturity time $T = K\Delta t$. Let $D(k, m \mid K)$ $(0 \le k \le K)$ be the discount bond price at time $\tau = k\Delta t$ at a state corresponding to state m of the underlying Ehrenfest process. For simplicity, this state is called m from now on. One then sees that, for $0 \le k < K$,

$$D(k, m \mid K) = e^{-r(k,m)\Delta t} \sum_{n \in \mathcal{N}_V} p_{2V:mn}(\Delta t) D(k+1, n \mid K) \quad (14)$$

where

$$D(K, m \mid K) = d_m \text{ for } m \in \mathcal{N}_V$$

and

$$r(k, m) = x_V(m) + \theta(k\Delta t) \quad (15)$$

with $x_V(m)$ as in Equation (10). Normally, one has $\underline{d} = [d_0, \ldots, d_{2V}]^T = \underline{1}$. In order to prepare for patching different O-U processes over different time intervals to be discussed later, the procedure in Equation (14) is defined for an arbitrary positive vector $\underline{d}$. Since $\underline{\underline{P}}_{2V}(\Delta t) = [p_{2V:mn}(\Delta t)]$ can be computed using Algorithm C1, the discount bond price $D(0, m \mid K)$ of interest can be readily evaluated based on Equation (14). This algorithm is summarized in a pseudo-code form as Algorithm C2 in Appendix C.

We next turn our attention to the prices of European options on the zero-coupon discount bond. Since the prices of put options can be derived from those of call options as shown later, we focus on call options only. For discrete economies, it is well known that the price of any security with known payoffs at time T can be viewed as a portfolio of Arrow-Debreu securities and can be priced as the payoff-weighted sum over all states of the prices of the Arrow-Debreu securities (see e.g., Pelsser [2000]). Accordingly, we first evaluate the price of an Arrow-Debreu security. Let $Q_0(k, m)$ be the present value at time $t = 0$ of the Arrow-Debreu security with maturity price of 1 at time $\tau = k\Delta t$ $(k \ge 0)$ and state $m \in \mathcal{N}_V$. Then, given an initial state $m_0 \in \mathcal{N}_V$, one has

$$Q_0(k+1, m) = \sum_{n \in \mathcal{N}_V} e^{-r(k,n)\Delta t} p_{2V:nm}(\Delta t) Q_0(k, n) \quad (16)$$

starting with $Q_0(0, m) = \delta_{m_0 m}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. We note from Equations (13) and (15) that $\theta(0) = \hat{X}_{OU}(0) = r(0, m_0)$ and hence $m_0 = V$ because $x_V(V) = 0$ and $x_V(m) \ne 0$ for $m \ne V$ from Equation (10). When different O-U processes are patched over different time intervals, Equation (16) has to be modified since the transition probability matrix $\underline{\underline{P}}_{2V}(\Delta t)$ would change from one time interval to another. In order to accommodate this modification, we define $Q_0(k, m \mid k_0)$ for $0 < k_0 \le k$ by

$$Q_0(k+1, m \mid k_0) = \sum_{n \in \mathcal{N}_V} e^{-r(k,n)\Delta t} p_{2V:nm}(\Delta t) Q_0(k, n \mid k_0) \quad (17)$$

starting with $[Q_0(k_0, m \mid k_0)]_{m \in \mathcal{N}_V} = \underline{s} = [s_0, \ldots, s_{2V}]^T$. This algorithm is summarized in a pseudo-code form as Algorithm C3 in Appendix C.

Remark 2

It may be worthwhile to note that the discount bond price $D(0, V \mid K)$ at time $t = 0$ and state $m = V$ can be expressed in terms of $Q_0(K, m)$ as $D(0, V \mid K) = \sum_{m \in \mathcal{N}_V} Q_0(K, m)$. However, one needs to employ Equation (14) for computing $D(0, m \mid K)$ for $m \ne V$.

Let $\pi_c(0 \mid M)$ be the present value at time $t = 0$ of a European call option with maturity at time $\tau_M = M\Delta t$ and the strike price K_S on the zero-coupon discount bond maturing at time $T = K\Delta t$ $(K > M)$. The corresponding payoff function $h_c(D)$ at time τ_M and state $m \in \mathcal{N}_V$ can be written as

$$h_c(D(M, m \mid K)) = [D(M, m \mid K) - K_S]^+ \quad (18)$$

where $[x]^+ \stackrel{\text{def}}{=} \max\{x, 0\}$. It then follows that

$$\pi_c(0 \mid M) = \sum_{m \in \mathcal{N}_V} h_c(D(M, m \mid K)) Q_0(M, m) \quad (19)$$

Using Equation (14) and Algorithm C1, $D(M, m \mid K)$ can be readily computed as in Algorithm C2. $Q_0(M, m)$ can also be obtained from Equation (16) as shown in Algorithm C3. Consequently, one can evaluate the price of interest, $\pi_c(0 \mid M)$, from Equations (18) and (19). This algorithm is described in a pseudo-code form as Algorithm C4 in Appendix C. It should be noted that the backward formula for $D(k, m \mid K)$ in Equation (14) and the forward formula for $Q_0(k+1, m \mid k_0)$ in Equation (17) can be computed starting from an arbitrary terminal value vector $\underline{d}$ and an arbitrary beginning value vector $\underline{s}$ in Algorithms C2 and C3, respectively.

For the associated European put option, the payoff function $h_p(D)$ is given by

$$h_p(D(M, m \mid K)) = [K_S - D(M, m \mid K)]^+ \tag{20}$$

Its present value $\pi_p(0 \mid M)$ can be obtained from Equation (19) by substituting $h_p(D(M, m \mid K))$ in place of $h_c(D(M, m \mid K))$. Alternatively, $\pi_c(0 \mid M)$ and $\pi_p(0 \mid M)$ are related to each other by

$$\pi_c(0 \mid M) - \pi_p(0 \mid M) = D(0, V \mid K) - K_S D(0, V \mid M) \tag{21}$$

This relationship is known as "put-call parity."

Hull and White [1990] show that, with $\sigma(t) = \sigma$ and $\alpha(t) = \alpha$, the price at time t of a zero-coupon discount bond maturing at time T is given by

$$D(t \mid T) = H_1(t, T) e^{-H_2(t,T) R(t)}, \quad 0 \le t < T \tag{22}$$

where $H_2(t, T) = \frac{1}{\alpha}(1 - e^{-\alpha(T-t)})$ and $\log H_1(t, T) = \frac{1}{2}\int_t^T \sigma^2 H_2^2(u, T)\, du - \int_t^T \phi(u) H_2(u, T) du$. The explicit formula for the price of the associated European call option can then be expressed as

$$\pi_c(0 \mid \tau) = D(0 \mid T)\Phi(d) - KD(0 \mid \tau)\Phi(d - \sigma_F), \quad 0 \le \tau < T \tag{23}$$

where

$$d = \frac{\ln D(0|T) - \ln(KD(0|\tau))}{\sigma_F} + \frac{\sigma_F}{2},$$

$$\sigma_F^2 = \frac{\sigma^2}{2\alpha}\left(\frac{e^{-\alpha T} - e^{-\alpha\tau}}{\alpha}\right)^2 (e^{2\alpha\tau} - 1), \text{ and } \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

Accuracy of the computational procedures developed in this section can be tested by comparing numerical results with the exact known solutions of Equations (22) and (23) where $\phi(t) = \phi$. Numerical results based on the Hull-White trinomial tree approach are also obtained, since it will be modified later so as to cope with the case of stepwise reversion and volatility functions for comparison purposes (see Appendix A).

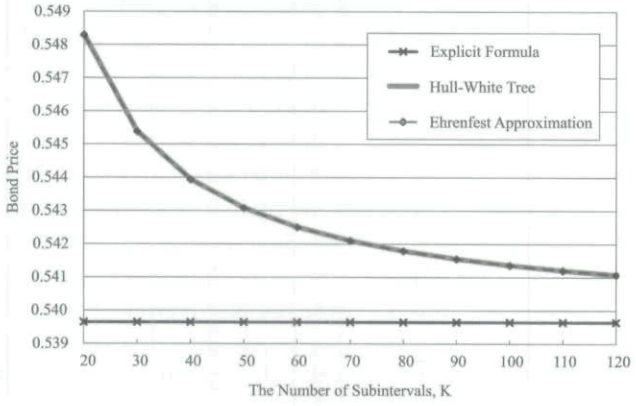
Numerical experiments are implemented for a parameter range of $\alpha = 0.05$ to 0.5 and $\sigma = 0.005$ to 0.05 with $\phi = 0.05$ and $R(0) = 0.05$. Exhibits 2(a) and (b) demonstrate the convergence of zero-coupon discount bond prices as a function of K where $K = T/\Delta t$, with maturity $T = 5$, $\alpha = 0.2$, and $\sigma = 0.01$ for the former and $\sigma = 0.05$ for the latter, respectively. In the exhibit, "Explicit formula" indicates that the prices are calculated using the explicit formula in Equation (22), "Hull-White Tree" based on the Hull-White trinomial procedure, and "Ehrenfest Approximation" employing the new proposed approach. Similarly, Exhibits 3(a) and (b) show the convergence of the prices of the associated European call options with strike price of $K_S = 0.7$ and maturity at time $\tau_M = M\Delta t = 4$.

From these exhibits, one sees that the computational results by both the Ehrenfest approach and the Hull-White trinomial tree approach converge to the analytical solution monotonically. The accuracy of the absolute relative error for the Ehrenfest approach is 0.263% for Exhibit 2(a) and 0.238% for Exhibit 2(b) at $K = 120$, and these numbers for the Hull-White trinomial tree approach are 0.264% and 0.253%, respectively. While the former performs slightly, but consistently, better than the latter, and this performance difference is enlarged when α and σ become large, the performance difference is minimal in that the absolute relative errors for the discount bond prices are contained within 2.664% for the Ehrenfest approach and 2.665% for the Hull-White trinomial tree approach throughout the parameter range of $K = 20$ to 120, $\alpha = 0.05$ to 0.5 and $\sigma = 0.005$ to 0.05 with $\phi = 0.05$ and $R(0) = 0.05$. In case of the prices of the associated European call options, the best accuracy of the Ehrenfest approach is attained at $K = 120$, with accuracy of the absolute relative error of 0.426% for Exhibit 3(a) and 0.364% for Exhibit 3(b). The absolute relative errors within the parameter range mentioned above are bounded by 6.629%. These exhibits for the Hull-White trinomial tree approach are 0.428%, 0.406% and 6.631% respectively.

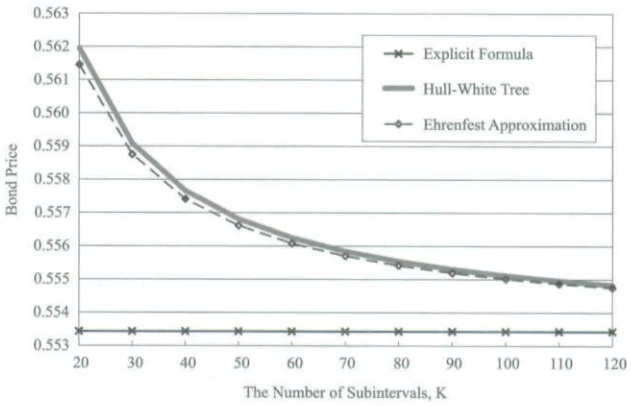
EXHIBIT 2

Convergence Behavior of Discount Bond Prices

These exhibits show zero-coupon discount bond prices as a function of the number K of subintervals on the time axis under different σs: (a) corresponds to $\sigma = 0.01$ while (b) corresponds to $\sigma = 0.05$. "Ehrenfest Approximation" in each exhibit indicates the prices by the proposed Ehrenfest approch, while "Hull-White Tree" indicates the trinomial tree approach and "Explicit Formula" indicates the analytical formula.



(a) $\phi = 0.05, \alpha = 0.2, \sigma = 0.01$



(b) $\phi = 0.05, \alpha = 0.2, \sigma = 0.05$

PRICING DISCOUNT BONDS AND ASSOCIATED EUROPEAN OPTIONS WITH STEPWISE REVERSION AND VOLATILITY FUNCTIONS

When the reversion and volatility functions are dependent on time t, it is necessary to find the short rate $R(t)$ for computing the price at time t of a zero-coupon discount bond based on Equation (22). However, $R(t)$ is not known except at the initial time $t = 0$ and the explicit formula in Equation (22) cannot be used. It is possible to modify the Hull-White trinomial tree approach so as to deal with stepwise reversion and volatility functions, which we call the modified trinomial tree approach. A succinct summary of how to construct the modified version is given in Appendix A. Because of the two reasons stated earlier, however, the modified trinomial tree approach is rather limited. The purpose of this section is to establish a new computational approach for computing the prices of discount bonds and associated European options with stepwise reversion and volatility functions based on the Ehrenfest approach described in the two prior sections. For descriptive simplicity, we illustrate the new approach using the example given in Exhibit 1.

The stochastic differential equation corresponding to the example in Exhibit 1 can be written as

$$dR(t) = (\phi - \alpha(t)R(t))dt + \sigma(t)dW(t) \qquad (24)$$

where

$$\alpha(t) = \begin{cases} \alpha_1, & 0 \le t < T_2 \\ \alpha_2, & T_2 \le t < T_3 \\ \alpha_3, & T_3 \le t \le T_4 \end{cases} \quad \sigma(t) = \begin{cases} \sigma_1, & 0 \le t < T_1 \\ \sigma_2, & T_1 \le t < T_3 \\ \sigma_3, & T_3 \le t \le T_4 \end{cases} \qquad (25)$$

for $0 < T_1 < T_2 < T_3 < T_4$. As discussed in the first section, the stochastic process $R(t)$ describing the short rate can then be constructed as in Equation (7) by patching O-U processes with constant reversion and volatility functions defined on individual time intervals separately.

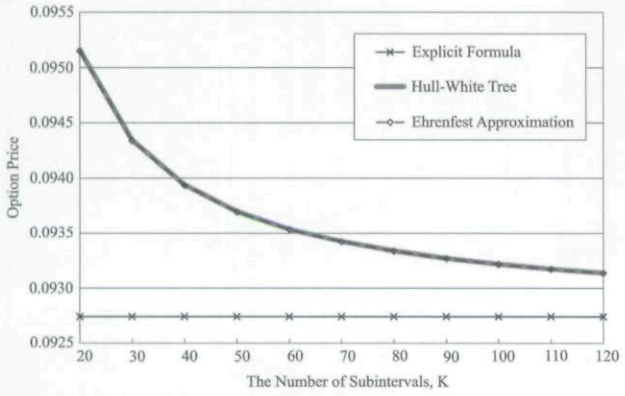
Based on the relationship between $\hat{X}_{OU}(t)$ and $X_{OU}(t)$ given in Equation (12) combined with the convergence of $X_V(t)$ to $X_{OU}(t)$ as $V \to \infty$, $\hat{X}^i_{OU}(t)$ in each time interval can be approximated by $\hat{X}^i_{OU}(t) \approx X^i_V(t) + \theta^i(t)$, $i = 1, 2, 3, 4$. For the processes $X^i_V(t)(i = 1, 2, 3, 4)$, the size of the state space is fixed as $2V + 1$ which is independent of the underlying parameters α_j and σ_j $(j = 1, 2, 3)$. Accordingly, the patching process can be repeated without any modification. Because of this, it is possible to automate all of the procedures with speed and accuracy. This is one advantage of the Ehrenfest approach to the modified trinomial tree approach.

The numerical algorithm based on the Ehrenfest approach can now be described in general terms for evaluating the price of a zero-coupon discount bond with stepwise reversion and volatility functions. Let $\alpha(t)$ be a stepwise reversion function defined by

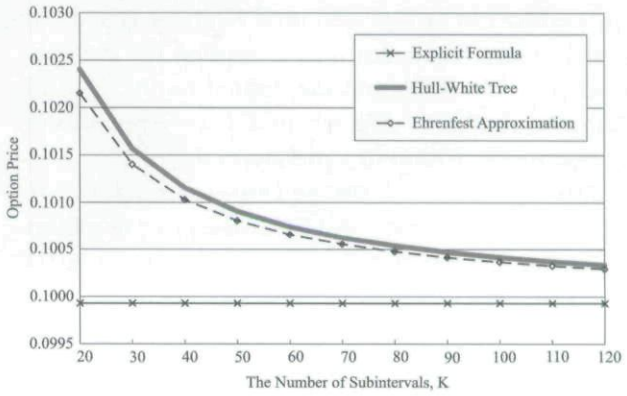
EXHIBIT 3

Convergence Behavior of European Call Option Prices

These exhibits show prices of a European call option on a discount bond as a function of the number K of subintervals on the time axis under different σs: (a) corresponds to $\sigma = 0.01$ while (b) corresponds to $\sigma = 0.05$. "Ehrenfest Approximation" in each exhibit indicates the prices by the proposed Ehrenfest approximation, while "Hull-White Tree" indicates the trinomial tree approach and "Explicit Formula" indicates the analytical formula.



(a) $\phi = 0.05, \alpha = 0.2, \sigma = 0.01$



(b) $\phi = 0.05, \alpha = 0.2, \sigma = 0.05$

$$\alpha(t) = \alpha_l, t \in [T_{l-1}^{\alpha}, T_l^{\alpha}], 1 \le l \le L \tag{26}$$

where $T_0^{\alpha} = 0 < T_1^{\alpha} < \cdots < T_L^{\alpha} = T$. A stepwise volatility function $\sigma(t)$ is defined similarly by

$$\sigma(t) = \sigma_j, t \in [T_{j-1}^{\sigma}, T_j^{\sigma}], 1 \le j \le J \tag{27}$$

where $T_0^{\sigma} = 0 < T_1^{\sigma} < \cdots < T_J^{\sigma} = T$. We suppose that T_I^{α}s and T_J^{α}s are sorted together in ascending order to form a combined and renumbered sequence $T_0 = 0 < T_1 < \cdots < T_I = T$ for some I, $0 \le I \le L + J$. For notational convenience, we write this operation as

$$I \leftarrow \text{JOIN}[L, J] \tag{28}$$

Throughout this section, we assume that Δt is chosen in such a way that, for some positive integers K_i, $0 \le i \le I$ and M, one has

$$T_i = K_i \Delta t, 0 \le i \le I; \ \tau_M = M\Delta t \tag{29}$$

Let i_0 be such that

$$M \in [K_{i_0-1}, K_{i_0}] \tag{30}$$

The two stepwise functions $\alpha(t)$ and $\sigma(t)$ can be rewritten as

$$\alpha(k\Delta t) = \alpha_{l_i} \text{ and } \sigma(k\Delta t) = \sigma_{j_i} \quad \text{for } k \in [K_{i-1}, K_i], 1 \le i \le I \tag{31}$$

Of interest is $D_{\text{SW}}(M, m \mid K_I)$, the discount bond price at time $\tau_M = M\Delta t$ with maturity at time $T = K_I \Delta t$, when $\alpha(t)$ and $\sigma(t)$ are stepwise functions as in Equation (31). Starting with the discount bond price of 1 at every state at time $T = T_I$, the discount bond prices $D(K_{I-1}, m \mid K_I)$ for $m \in \mathcal{N}_V$ at time T_{I-1} are computed using Algorithm C2 with $\underline{d} = \underline{1}$, α_{l_I} and σ_{j_I}. Starting with $D(K_{I-1}, m \mid K_I)$ for $m \in \mathcal{N}_V$ as the terminal values, one can produce $D(K_{I-2}, m \mid K_I)$ for $m \in \mathcal{N}_V$ in a similar manner. As depicted in Exhibit 4(a), this procedure is repeated until K_{i_0} is reached. For the final cycle, Algorithm C2 is applied for the time interval $[M\Delta t, K_{i_0}\Delta t]$, yielding the desired results $D_{\text{SW}}(M, m \mid K_I)$ for $m \in \mathcal{N}_V$. This algorithm is summarized in a pseudo-code form as Algorithm C5 in Appendix C.

We next turn our attention to computation of Arrow-Debreu prices. What we want is $Q_{\text{sw:0}}(M, m)$, which is the present value of an Arrow-Debreu price at node (M, m), when $\alpha(t)$ and $\sigma(t)$ are stepwise functions as in Equation (31). For finding $D_{\text{sw}}(M, m \mid K_I)$, Algorithm C2 is used repeatedly in a backward manner. The

idea behind how to compute $Q_{sw:0}(M, m)$ is similar, except that Algorithm C3 is repeated forwardly. More specifically, starting with $m = V$ at time 0, Algorithm C3 with $\underline{s} = \underline{u}_V$, α_{l_1} and σ_{j_1} produces $Q_0(K_1, m \mid K_0)$ for $m \in \mathcal{N}_V$. Using these values as the starting values, Algorithm C3 can be employed again with α_{l_2} and σ_{j_2}. As shown in Exhibit 4(b), the procedure can be repeated until K_{i_0-1} is reached. For the final cycle, Algorithm C3 is applied for the time interval $[K_{i_0-1}\,\Delta t, M\Delta t]$ and the desired results $Q_{sw:0}(M, m)$ can be obtained. Algorithm C6 in Appendix C describes this algorithm in a pseudo-code form.

Assuming that both $\alpha(t)$ and $\sigma(t)$ are step functions as in Equation (31), let $\pi_{sw:c}(0 \mid M)$ be the price of a European call option associated with the discount bond of maturity at time $T = K_I \Delta t$, having the maturity of the option at time $\tau_M = M\Delta t$ with strike price of K_S. We are now in a position to establish a new algorithm to compute $\pi_{SW:c}(0 \mid M)$ via the Ehrenfest approach using Algorithms C5 and C6. More specifically, both the needed bond prices $D_{SW}(M, m \mid K_l)$ for $m \in \mathcal{N}_V$ and the necessary Arrow-Debreu prices $Q_{SW:0}(M, m)$ for $m \in \mathcal{N}_V$ can be produced via Algorithms C5 and C6 respectively. One then finds that

$$\pi_{SW:c}(0 \mid M) = \sum_{m \in \mathcal{N}_V} h_c(D_{SW}(M, m \mid K_l)) Q_{SW:0}(M, m) \quad (32)$$

This final algorithm is summarized in a pseudo-code form as Algorithm C7 in Appendix C.

Using Algorithm C5 and the modified trinomial tree approach given in Appendix A, Exhibits 5(a) and (b) exhibit the zero-coupon discount bond prices as a function of K_I for the following two different sets of stepwise reversion and volatility functions:

$$\alpha(t) = \begin{cases} 0.2, & 0 \le t < 2 \\ 0.3, & 2 \le t < 4 \\ 0.1, & 4 \le t \le 5 \end{cases} \quad \sigma(t) = \begin{cases} 0.02, & 0 \le t < 1 \\ 0.03, & 1 \le t < 4 \\ 0.01, & 4 \le t \le 5 \end{cases} \quad (33)$$

and

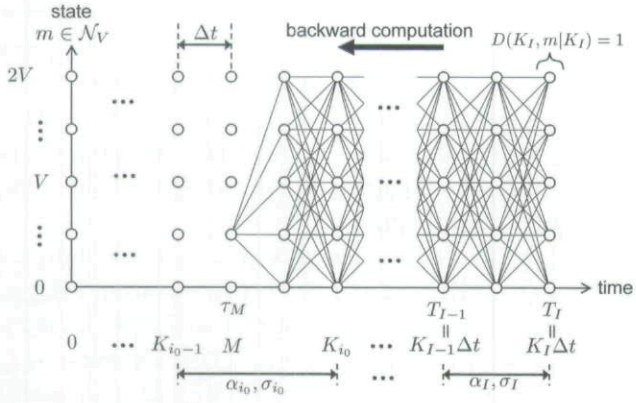
$$\alpha(t) = \begin{cases} 0.1, & 0 \le t < 2 \\ 0.3, & 2 \le t < 4 \\ 0.2, & 4 \le t \le 5 \end{cases} \quad \sigma(t) = \begin{cases} 0.01, & 0 \le t < 1 \\ 0.03, & 1 \le t < 4 \\ 0.02, & 4 \le t \le 5 \end{cases} \quad (34)$$

For both cases, we set $\phi = 0.05$, $R(0) = 0.05$ and $T = 5$. In each exhibit, "MHW Tree" indicates that the calculations are based on the modified trinomial tree approach while "Ehrenfest Approximation" indicates the proposed new approach. Since the analytical results are not available, the first difference $\Delta D_{SW}(K_I) \stackrel{\text{def.}}{=} | D_{SW}(0, 0 \mid K_I) - D_{SW}(0, 0 \mid K_I - 50) |$ and the second difference $\Delta^2 D_{SW}(K_I) \stackrel{\text{def.}}{=} | \Delta D_{SW}(K_I) - \Delta D_{SW}(K_I - 50) |$ are summarized in Exhibits 6(a) and (b). One sees that these

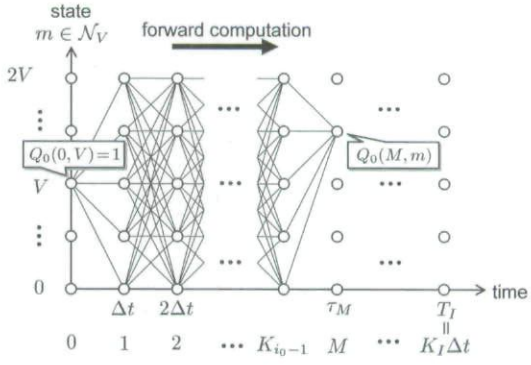
EXHIBIT 4

Backward and Forward Computation Schemes for Evaluating Prices

The left exhibit illustrates the backward computation for evaluating bond prices at each state, while the right exhibit illustrates the forward computation for evaluating Arrow-Debreu prices.



(a) Backward Computation via Algorithm C5

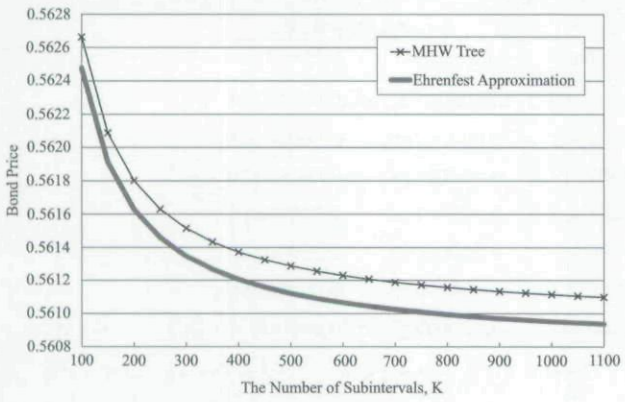


(b) Forward Computation via Algorithm C6

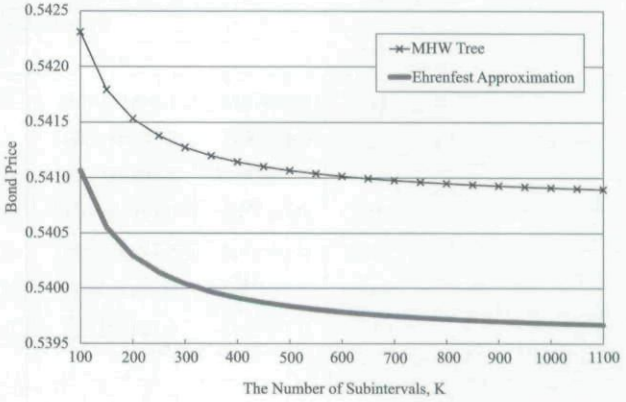
EXHIBIT 5

Convergence Behavior of Discount Bond Prices with Stepwise Functions

These two exhibits show computed discount bond prices as a function of the number K of subintervals in the time axis when different stepwise reversion and volatility functions are applied. "Ehrenfest Approximation" in each exhibit indicates the prices by the proposed Ehrenfest approximation, while "MHW Tree" indicates the modified trinomial tree approach.



(a) $\phi = 0.05$, $\alpha = [0.2, 0.3, 0.1]$, $\sigma = [0.02, 0.03, 0.01]$



(b) $\phi = 0.05$, $\alpha = [0.1, 0.3, 0.2]$, $\sigma = [0.01, 0.03, 0.02]$

differences for both the Ehrenfest approach and the modified trinomial tree approach decrease as K_I increases. Moreover, the Ehrenfest approach achieves similar first and second differences to the modified trinomial tree approach. The absolute relative differences between the two approaches are contained within 0.231% for Exhibits 5(a) and (b), demonstrating the robust accuracy of the two approaches.

Exhibits 7(a) and (b) show the corresponding European call option prices on the zero-coupon discount bond, with $K_S = 0.7$ and $\tau_M = 4$. The first two differences $\Delta\pi_{SW:c}(M) \stackrel{\text{def.}}{=} |\pi_{SW:c}(0|M) - \pi_{SW:c}(0|M-40)|$ and $\Delta\pi^2_{SW:c}(M) \stackrel{\text{def.}}{=} |\Delta\pi_{SW:c}(M) - \Delta\pi_{SW:c}(M-40)|$ are summarized in Exhibits 8(a) and (b), where $M = K_I \cdot \tau_M / T$. Again, the first and second differences for both the Ehrenfest approach and the modified trinomial tree approach decrease as K_I increases and the results of the two approaches are close to each other. The absolute relative differences between the two approaches are contained within 0.399% for Exhibits 7(a) and (b), assuring the trustworthiness of the two approaches.

Although the performances of the Ehrenfest approach and the modified trinomial tree approach seem to be more or less comparable, the former approach is guaranteed to work for any stepwise reversion functions while the latter approach may not, as discussed earlier. In order to illustrate this point, the following stepwise reversion and volatility functions are considered:

$$\alpha(t) = \begin{cases} 0.40, & 0 \le t < 2 \\ 0.20, & 2 \le t < 4 \\ 0.05, & 4 \le t \le 5 \end{cases} \quad \sigma(t) = \begin{cases} 0.040, & 0 \le t < 1 \\ 0.020, & 1 \le t < 4 \\ 0.005, & 4 \le t \le 5 \end{cases} \tag{35}$$

Let m_k^i be the positive integer such that $m_k^i \Delta x$ represents the highest value at time $k\Delta t$ on the modified trinomial tree in the i-th time interval, where Δx is the magnitude of each jump on the tree. With $\Delta t = 0.05$ for $i = 1, 2, 3, 4$, the range of m_k^i, denoted by $\mathrm{RG}(i)$, should satisfy (see Appendix A for details)

$$\begin{cases} \mathrm{RG}(1) = \mathrm{RG}(2) = [10, 40] & \text{for } 0 \le t < 2 \\ \mathrm{RG}(3) = [19, 81] & \text{for } 2 \le t < 4 \\ \mathrm{RG}(4) = [74, 326] & \text{for } 4 \le t \le 5 \end{cases} \tag{36}$$

Clearly the intersection of $\mathrm{RG}(i)$, $i = 1, 2, 3, 4$ in Equation (36) is empty so that the modified trinomial tree approach is not applicable. The Ehrenfest approach can deal with this case without any difficulty as demonstrated in Exhibits 9(a), (b) and Exhibit 10.

A computer with CPU Genuine Intel 1.66 GHz and 512 MB RAM was employed for all the computations presented in this article, using MATLAB as a programming language. The required CPU times were bounded by 7 seconds.

EXHIBIT 6

First and Second Differences in Discount Bound Price Computation

This exhibit shows the first difference $\Delta D_{SW}(K_I) = |D_{SW}(0, 0|K_I) - D_{SW}(0, 0|K_I - 50)|$ and the second difference $\Delta^2 D_{SW}(K_I) = |\Delta D_{SW}(K_I) - \Delta D_{SW}(K_I - 50)|$ of the computed prices of a discount bond when the underlying O-U process has stepwise reversion and volatility functions.

(a) $\alpha = [0.2, 0.3, 0.1]$, $\sigma = [0.02, 0.03, 0.01]$

K_I	Δt	First Difference		Second Difference	
		Ehrenfest	MHW Tree	Ehrenfest	MHW Tree
800	0.006250	0.00001408	0.00001430	0.00000201	0.00000204
850	0.005882	0.00001243	0.00001262	0.00000166	0.00000168
900	0.005556	0.00001105	0.00001122	0.00000138	0.00000140
950	0.005263	0.00000988	0.00001003	0.00000116	0.00000118
1000	0.005000	0.00000889	0.00000903	0.00000099	0.00000100
1050	0.004762	0.00000805	0.00000817	0.00000085	0.00000086
1100	0.004545	0.00000732	0.00000743	0.00000073	0.00000074

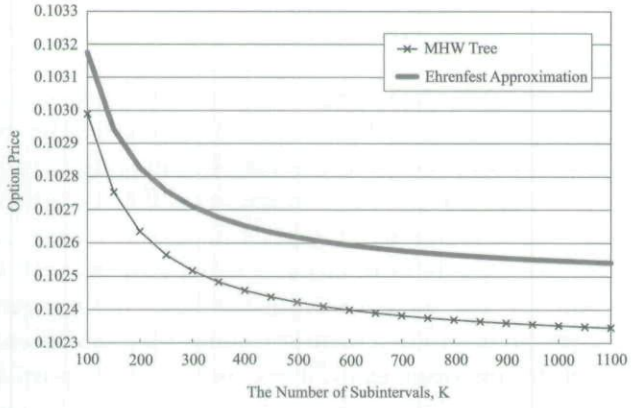
(b) $\alpha = [0.1, 0.3, 0.2]$, $\sigma = [0.01, 0.03, 0.02]$

K_I	Δt	First Difference		Second Difference	
		Ehrenfest	MHW Tree	Ehrenfest	MHW Tree
800	0.006250	0.00001276	0.00001296	0.00000182	0.00000185
850	0.005882	0.00001126	0.00001144	0.00000150	0.00000152
900	0.005556	0.00001001	0.00001016	0.00000125	0.00000127
950	0.005263	0.00000896	0.00000909	0.00000105	0.00000107
1000	0.005000	0.00000806	0.00000818	0.00000090	0.00000091
1050	0.004762	0.00000729	0.00000740	0.00000077	0.00000078
1100	0.004545	0.00000663	0.00000673	0.00000066	0.00000067

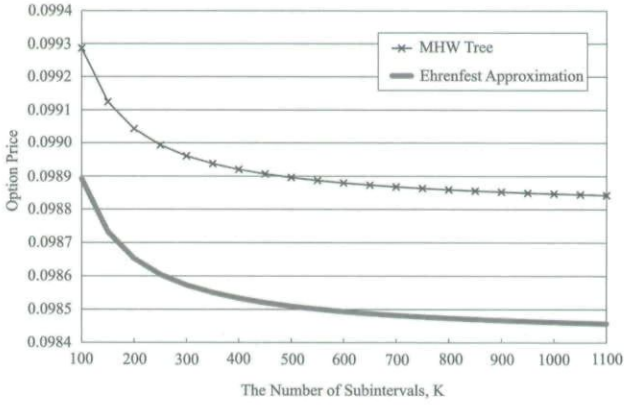
EXHIBIT 7

Convergence Behavior of European Call Option Prices with Stepwise Functions

These two exhibits show computed European call option prices on a discount bond as a function of the number K of subintervals in the time axis when different stepwise reversion and volatility functions are applied. "Ehrenfest Approximation" in each exhibit indicates the prices by the proposed Ehrenfest approximation, while "MHW Tree" indicates the modified trinomial tree approach.



(a) $\phi = 0.05$, $\alpha = [0.2, 0.3, 0.1]$, $\sigma = [0.02, 0.03, 0.01]$



(b) $\phi = 0.05$, $\alpha = [0.1, 0.3, 0.2]$, $\sigma = [0.01, 0.03, 0.02]$

EXHIBIT 8

First and Second Differences in European Call Option Price Computation

This exhibit shows the first difference $\Delta\pi_{SW:c}(K_I) = |\pi_{SW:c}(0|K_I) - \pi_{SW:c}(0|K_I - 40)|$ and the second difference $\Delta^2\pi_{SW:c}(K_I) = |\Delta\pi_{SW:c}(K_I) - \Delta\pi_{SW:c}(K_I - 40)|$ of the computed prices of a European call option when the underlying O-U process has stepwise reversion and volatility functions.

(a) $\alpha = [0.2, 0.3, 0.1]$, $\sigma = [0.02, 0.03, 0.01]$

K_I	Δt	First Difference		Second Difference	
		Ehrenfest	MHW Tree	Ehrenfest	MHW Tree
800	0.006250	0.00000580	0.00000588	0.00000083	0.00000084
850	0.005882	0.00000512	0.00000519	0.00000068	0.00000069
900	0.005556	0.00000455	0.00000461	0.00000057	0.00000058
950	0.005263	0.00000407	0.00000413	0.00000048	0.00000049
1000	0.005000	0.00000366	0.00000371	0.00000041	0.00000041
1050	0.004762	0.00000331	0.00000336	0.00000035	0.00000035
1100	0.004545	0.00000301	0.00000305	0.00000030	0.00000031

(b) $\alpha = [0.1, 0.3, 0.2]$, $\sigma = [0.01, 0.03, 0.02]$

K_I	Δt	First Difference		Second Difference	
		Ehrenfest	MHW Tree	Ehrenfest	MHW Tree
800	0.006250	0.00000398	0.00000407	0.00000057	0.00000058
850	0.005882	0.00000351	0.00000359	0.00000047	0.00000048
900	0.005556	0.00000312	0.00000319	0.00000039	0.00000040
950	0.005263	0.00000279	0.00000285	0.00000033	0.00000034
1000	0.005000	0.00000251	0.00000257	0.00000028	0.00000029
1050	0.004762	0.00000227	0.00000232	0.00000024	0.00000024
1100	0.004545	0.00000207	0.00000211	0.00000021	0.00000021

CONCLUDING REMARKS

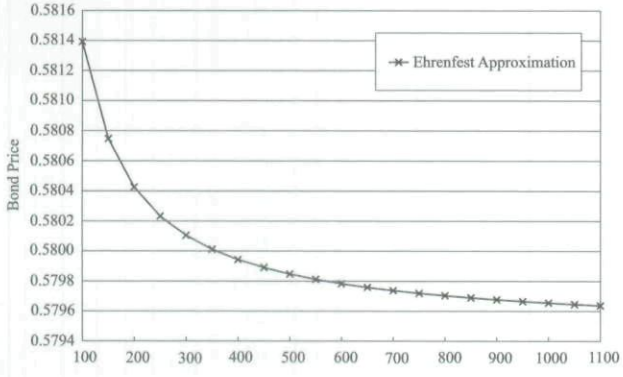
In this article, a novel approach is proposed for computing the prices of zero-coupon discount bonds and associated European options for the Hull-White model with stepwise reversion and volatility functions. The market fitting function is set to be constant, which does not affect our proposed approach in any essential way. We first focus on the Vasicek model, which is a special case of the Hull-White model with constant reversion and volatility functions. The underlying O-U process $\hat{X}_{OU}(t)$ is expressed as a sum of another O-U process $X_{OU}(t)$ and a shift function $\theta(t)$. Based on SGJ [2006], it is shown that a sequence of Ehrenfest processes with appropriate shifting and scaling converges in law to $X_{OU}(t)$ as the state space size goes to infinity. Using the uniformization procedure of Keilson [1979], a numerical algorithm is then developed to compute the time dependent transition probability matrix of the Ehrenfest process, which in turn enables one to approximate the conditional transition probability density of $\hat{X}_{OU}(t)$ systematically. Consequently, the prices of zero-coupon discount bonds and associated European options for the Vasicek model can be readily computed.

For the Hull-White model with stepwise reversion and volatility functions, computational algorithms are developed by patching a sequence of $\hat{X}^i_{OU}(t)$'s, each defined in a time interval in which both the reversion and the volatility functions are constant. For comparison purposes, a modified trinomial tree approach is also developed to cope with stepwise volatility and reversion functions based on the Hull-White trinomial tree approach. Numerical experiments show that the two approaches are comparable in their performances. However, while the Ehrenfest approach can deal with any stepwise reversion and volatility functions systematically and therefore can be automated easily, it may not be always possible to apply the modified trinomial tree approach. Because of this, the modified trinomial tree approach is limited in its

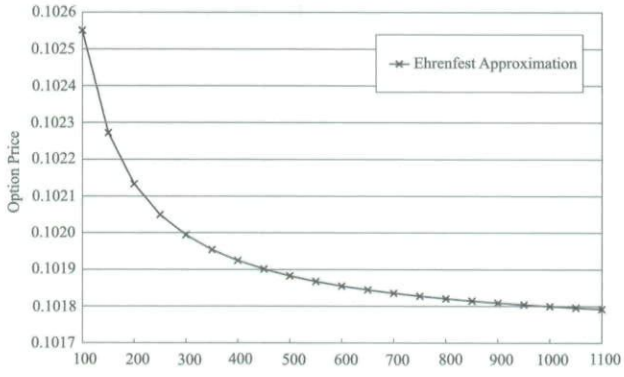
EXHIBIT 9

Convergence Behavior for $\phi = 0.05$, $\alpha = [0.4, 0.2, 0.05]$, $\sigma = [0.04, 0.02, 0.005]$

These two exhibits (a) and (b) show the computed prices of a discount bond and a European call, respectively, when the underlying O-U process has stepwise reversion and volatility functions for which the modified trinomial tree approach collapses.



(a) Discount Bond Prices



(b) European Call Option Prices

capacity for automated software development. Consequently, the Ehrenfest approach is more attractive than the modified trinomial tree approach for practical use.

As far as the computational complexity is concerned, $O(\bar{k}V)$ time complexity is required for computing the transition probabilities and the Arrow-Debreu prices for the whole state space and time axis, while $O(V^2I)$ is required for the backward computation of the bond and option prices, where $\bar{k} = \max\{k : \exp\{-\alpha Vt\}(\alpha Vt)^k/\ k! < \varepsilon,\ k > \alpha V\tau\}$ as in Algorithm C1. For a given set of parameter values, numerical examples presented in this article can be computed within 7 seconds of CPU time, using MATLAB as a programming language and a typical personal computer.

The Ehrenfest approach relies on dynamic analysis of the underlying Markov chain in continuous time. Accordingly, one can deal with a variety of boundaries with little alteration as well as evaluation of withdrawal option values. This implies that the Ehrenfest approach opens a new path to develop algorithmic procedures for computing the prices of various barrier options and American options. Such a study is in progress and will be reported elsewhere.

EXHIBIT 10

First and Second Differences for $\alpha = [0.4, 0.2, 0.05]$, $\sigma = [0.04, 0.02, 0.005]$

This exhibit shows the first differences, $\Delta D_{SW}(K_I) = |D_{SW}(0, 0 | K_I) - D_{SW}(0, 0 | K_I - 50)|$ and $\Delta\pi_{SW:c}(K_I) = |\pi_{SW:c}(0 | K_I) - \pi_{SW:c}(0 | K_I - 40)|$, and the second differences, $\Delta^2 D_{SW}(K_I) = |\Delta D_{SW}(K_I) - \Delta D_{SW}(K_I - 50)|$ and $\Delta^2\pi_{SW:c}(K_I) = |\Delta\pi_{SW:c}(K_I) - \Delta\pi_{SW:c}(K_I - 40)|$, of the computed prices of a discount bond and a European call option, respectively when the underlying O-U process has stepwise reversion and volatility functions for which the modified trinomial tree approach collapses.

K_I	Δt	Discount Bond		European Call	
		First Difference	Second Difference	First Difference	Second Difference
800	0.006250	0.00001604	0.00000229	0.00000694	0.00000099
850	0.005882	0.00001415	0.00000189	0.00000612	0.00000082
900	0.005556	0.00001258	0.00000157	0.00000544	0.00000068
950	0.005263	0.00001126	0.00000132	0.00000487	0.00000057
1000	0.005000	0.00001013	0.00000113	0.00000438	0.00000049
1050	0.004762	0.00000916	0.00000097	0.00000396	0.00000042
1100	0.004545	0.00000833	0.00000083	0.00000360	0.00000036

APPENDIX A

HULL-WHITE TRINOMIAL TREE APPROACH

The Original Hull-White Trinomial Tree Approach

When $\alpha(t)$ and $\sigma(t)$ in Equation (1) are constant, Hull and White [1993, 1994] have constructed a trinomial tree to represent the short rate stochastic process $\hat{X}_{OU}(t)$ specified in Equation (4) by discretizing both the state space and the time axis with the length of each time step Δt. The transition probabilities from a state to the three adjacent states are constructed in a risk-neutral manner so that the discretized tree process converges in law to $\hat{X}_{OU}(t)$ as $\Delta t \to 0$. The construction of the trinomial tree involves two separate stages as described below.

First Stage. The first stage is to build a tree representing $X_{OU}(t)$ specified in Equation (11) starting with $X_{OU}(0) = 0$. The magnitude of each jump on the tree, if any, is set to be $\Delta x = \sigma\sqrt{3\Delta t}$ for consideration of error minimization (see Hull [2000]). At time $t = k\Delta t$ and state $x = m\Delta x$, the corresponding node in the tree is denoted by (k, m), where k is a positive integer and m is an integer. A state transition in the tree takes one of the three types as shown in Exhibit A1. The corresponding upper, middle, and lower transition probabilities at node (k, m) are denoted by $p_{m:u}(\Delta t)$, $p_{m:m}(\Delta t)$, and $p_{m:d}(\Delta t)$ respectively. The probabilities are chosen so as to match the first two moments of the trinomial tree process with those of the change of $X_{OU}(t)$ in the second order of Δt. More specifically, for type A, the following three equations should be satisfied.

$$\begin{cases} p_{m:u}(\Delta t)\Delta x - p_{m:d}(\Delta t)\Delta x & = -\alpha m \Delta x \Delta t \\ p_{m:u}(\Delta t)\Delta x^2 + p_{m:d}(\Delta t)\Delta x^2 & = \sigma^2 \Delta t + \alpha^2 m^2 \Delta x^2 \Delta t^2 \\ p_{m:u}(\Delta t) + p_{m:m}(\Delta t) + p_{m:d}(\Delta t) & = 1 \end{cases} \tag{37}$$

Substituting $\Delta x = \sigma\sqrt{3\Delta t}$ into Equation (37), solving the three equations yields the transition probabilities given by

$$\begin{cases} p_{m:u}(\Delta t) & = \dfrac{1}{6} + \dfrac{\alpha^2 m^2 \Delta t^2 - \alpha m \Delta t}{2} \\ p_{m:m}(\Delta t) & = \dfrac{2}{3} - \alpha^2 m^2 \Delta t^2 \\ p_{m:d}(\Delta t) & = \dfrac{1}{6} + \dfrac{\alpha^2 m^2 \Delta t^2 + \alpha m \Delta t}{2} \end{cases} \tag{38}$$

with $-\frac{\sqrt{6}}{3\alpha\Delta t} < m < \frac{\sqrt{6}}{3\alpha\Delta t}$ to make the three probabilities positive. Similarly, for type B, the three transition probabilities are found as

$$\begin{cases} p_{m:u}(\Delta t) & = \dfrac{7}{6} + \dfrac{\alpha^2 m^2 \Delta t^2 - 3\alpha m \Delta t}{2} \\ p_{m:m}(\Delta t) & = -\dfrac{1}{3} - \alpha^2 m^2 \Delta t^2 + 2\alpha m \Delta t \\ p_{m:d}(\Delta t) & = \dfrac{1}{6} + \dfrac{\alpha^2 m^2 \Delta t^2 - \alpha m \Delta t}{2} \end{cases} \tag{39}$$

with $\frac{3-\sqrt{6}}{3\alpha\Delta t} < m < \frac{3+\sqrt{6}}{3\alpha\Delta t}$. The transition probabilities for type C are symmetric to those for type B given by

$$\begin{cases} p_{m:u}(\Delta t) & = \dfrac{1}{6} + \dfrac{\alpha^2 m^2 \Delta t^2 - \alpha \mid m \mid \Delta t}{2} \\ p_{m:m}(\Delta t) & = -\dfrac{1}{3} - \alpha^2 m^2 \Delta t^2 + 2\alpha \mid m \mid \Delta t \\ p_{m:d}(\Delta t) & = \dfrac{7}{6} + \dfrac{\alpha^2 m^2 \Delta t^2 - 3\alpha \mid m \mid \Delta t}{2} \end{cases} \tag{40}$$

with $-\frac{3+\sqrt{6}}{3\alpha\Delta t} < m < -\frac{3-\sqrt{6}}{3\alpha\Delta t}$.

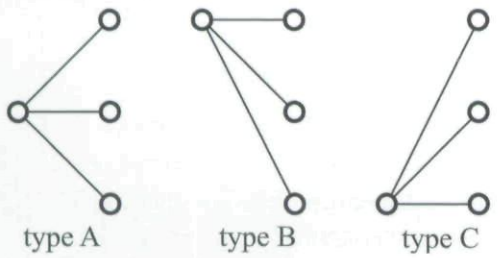
Let m_k be the maximum positive integer of m at time $k\Delta t$. To keep the three transition probabilities positive for all of the three types, m_k should satisfy the following inequality:

$$\left\lceil \frac{3-\sqrt{6}}{3\alpha\Delta t} \right\rceil \le m_k \le \left\lfloor \frac{\sqrt{6}}{3\alpha\Delta t} \right\rfloor \tag{41}$$

Let $\mathcal{N}_k \overset{\text{def}}{=} \{m : -m_k \le m \le m_k\}$. The values of x considered at time $k\Delta t$ consist of $m\Delta x$, $m \in \mathcal{N}_k$. As long as Equation (41) is satisfied at time $(k + 1)\Delta t$, state transitions at time $k\Delta t$ are of type A. If Equation (41) is violated at time$(k + 1)\Delta t$, a Type B transition takes place at node (k, m_k) and a Type C transition at $(k, -m_k)$ with all other transitions of Type A. More specifically, let $k_{max} = \max\{k : m_k \text{ satisfies (41)}\}$. Then, for $0 \le k < k_{max}$, the state transition probabilities are given by Equation (38) with $m_k = k$. For $k \ge k_{max}$, m_k remains to be k_{max} with the state

EXHIBIT A1

Types of Tree Branching



transition probabilities given by Equation (39) at (k, m_k), Equation (40) at $(k, -m_k)$ and Equation (38) at (k, m) with $m \neq m_k$, $m \neq -m_k$.

Second Stage. In the second stage, the trinomial tree constructed in Stage 1 for $X_{OU}(t)$ is shifted to represent $\hat{X}_{OU}(t)$, using the shift function $\theta(t)$ as given in Equation (12). The computational algorithms for evaluating the prices of zero-coupon discount bonds, Arrow-Debreu securities and associated European options are in parallel with those of the Ehrenfest approach in Equations (14), (17) and (19) respectively, where the transition probabilities obtained from the Ehrenfest process are replaced by those obtained in Stage 1. We note that Remark 1 is also applicable to the Hull-White trinomial tree approach and an arbitrary market fitting function $\phi(t)$ can be incorporated.

The Modified Trinomial Tree Approach

As for the Ehrenfest approach, the Hull-White trinomial tree approach can also be modified so as to cope with $R(t)$ in Equation (1) having stepwise reversion and volatility functions. The idea is again to use the patching procedure described in the first section. However, one important difference can be found in that the size of the state space for the Hull-White trinomial tree approach depends on the reversion factor α to satisfy Equation (41), while that for the Ehrenfest approach is independent of all the parameters characterizing $R(t)$.

For implementing the patching procedure, it is necessary to keep the state space size the same throughout different time intervals since the state probabilities at the end of one time interval become the initial state probabilities for the next time interval. Because of this, the modified trinomial tree approach has to satisfy Equation (41) simultaneously throughout I different time intervals, where I is given in Equation (28). Let $\mathrm{RG}(i)$ be defined by $\mathrm{RG}(i) = \left[\left\lceil \frac{3-\sqrt{6}}{3\alpha_i \Delta t} \right\rceil, \left\lfloor \frac{\sqrt{6}}{3\alpha_i \Delta t} \right\rfloor\right]$. It is then necessary to have $\bigcap_{i=1}^{I} \mathrm{RG}(i) \neq \phi$, for the modified trinomial tree approach to be applicable for patching. This condition, of course, is not guaranteed to hold all the time, thereby limiting the potential usefulness of the modified trinomial tree approach significantly.

APPENDIX B

UNIFORMIZATION PROCEDURE OF KEILSON [1979]

In what follows, the uniformization procedure of Keilson [1979] is first summarized, followed by the description of its application to the Ehrenfest process $N_{2V}(t)$.

For a continuous time Markov chain $N(t)$ defined on $\mathcal{N} = \{0, 1, 2, \ldots, N\}$ governed by $\underline{\underline{\nu}} = \{\nu_{mn}\}$ with $\nu_m \overset{\text{def.}}{=} \sum_{n \in \mathcal{N}} \nu_{mn} < \infty$, the transition probability matrix $\underline{\underline{P}}(t)$ can be given in terms of the infinitesimal generator $\underline{\underline{Q}}$ as

$$\underline{\underline{P}}(t) = e^{\underline{\underline{Q}}t}; \underline{\underline{Q}} = -\underline{\underline{\nu}}_D + \underline{\underline{\nu}}; \underline{\underline{\nu}}_D = diag\{\nu_m\} \tag{42}$$

For $\nu \geq \max_m \nu_m$, let $\underline{\underline{a}}_\nu$ be a stochastic matrix defined by

$$\underline{\underline{a}}_\nu \overset{\text{def}}{=} \underline{\underline{I}} - \frac{1}{\nu}\underline{\underline{\nu}}_D + \frac{1}{\nu}\underline{\underline{\nu}} = \underline{\underline{I}} + \frac{1}{\nu}\underline{\underline{Q}} \tag{43}$$

By substituting Equation (43) into Equation (42), one then finds that

$$\underline{\underline{P}}(t) = e^{-\nu t[\underline{\underline{I}} - \underline{\underline{a}}_\nu]} = \sum_{k=0}^{\infty} e^{-\nu t} \frac{(\nu t)^k}{k!} \underline{\underline{a}}_\nu^k \tag{44}$$

where $\underline{\underline{a}}_\nu^0 = \underline{\underline{I}}$. It should be noted that $\underline{\underline{a}}_\nu$ is a stochastic matrix. Accordingly, Equation (44) involves only nonnegative numbers, providing excellent numerical stability.

Let $\underline{\underline{P}}_{2V}(t) = [p_{2V;mn}(t)]$, $m, n \in \mathcal{N}_V$ be the transition probability matrix of $N_{2V}(t)$ and let $\underline{p}^{\mathrm{T}}_{2V;m}(t) = [p_{2V;m0}(t), p_{2V;m1}(t), \ldots, p_{2V;m,2V}(t)]$ be the corresponding state probability vector starting from state m. For $N_{2V}(t)$ it should be noted that the state space is $\mathcal{N}_V = \{0, 1, \ldots, 2V\}$ governed by the transition rates λ_n and μ_n, $n \in \mathcal{N}_V$ as given in Equation (8) with $\overline{\nu} = \frac{\alpha}{2}$ and $S = 2V$. By setting $\nu = \alpha V$ Equation (44) then becomes

$$\underline{\underline{P}}_{2V}(t) = \sum_{k=0}^{\infty} e^{-V\alpha t} \frac{(V\alpha t)^k}{k!} \underline{\underline{a}}_V^k$$

where

$$\underline{\underline{a}}_V = \frac{1}{\alpha V} \begin{array}{c} \begin{matrix} 0 & 1 & 2 & \cdots & 2V-1 & 2V \end{matrix} \\ \begin{pmatrix} 0 & \lambda_0 & 0 & \cdots & 0 & 0 \\ \mu_1 & 0 & \lambda_1 & \cdots & 0 & 0 \\ 0 & \mu_2 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & \lambda_{2V-1} \\ 0 & 0 & 0 & \cdots & \mu_{2V} & 0 \end{pmatrix} \end{array} \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ 2V-1 \\ 2V \end{matrix}$$

The time dependent state probability vector $\underline{p}^{\mathrm{T}}_{2V;m}(t)$ can be evaluated accordingly as

$$\underline{p}^{\top}_{2V;m}(t) = \underline{u}^{\top}_{m}\underline{\underline{P}}_{2V}(t) = \sum_{k=0}^{\infty} \underbrace{e^{-\alpha V t}\frac{(\alpha V t)^k}{k!}}_{A} \underbrace{(\underline{u}^{\top}_{m}\underline{\underline{a}}^{k}_{V})}_{B} \qquad (45)$$

where $\underline{u}^{\top}_{m}$ is the unit vector corresponding to state m. For each k in Equation (45), the underbraced parts A and B may be calculated separately. The matrix-vector multiplication denoted as B in Equation (45) can be reduced to vector-based operations. For any $(2V + 1)$-dimensional real vector $\underline{x} \stackrel{\text{def.}}{=} [x_0, x_1, x_2, \ldots, x_{2V-2}, x_{2V-1}, x_{2V}]^{\top}$, let $\underline{x}^0$ and $\underline{x}^1 \in \mathbb{R}^{2V}$ be defined by $\underline{x}^0 \stackrel{\text{def.}}{=} [x_0, x_1, x_2, \ldots, x_{2V-2}, x_{2V-1}]^{\top}$ and $\underline{x}^1 \stackrel{\text{def.}}{=} [x_1, x_2, x_3, \ldots, x_{2V-1}, x_{2V}]^{\top}$. For the transition rates described in the form of $\underline{\lambda} \stackrel{\text{def.}}{=} [\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{2V-2}, \lambda_{2V-1}]^{\top}$ and $\underline{\mu} \stackrel{\text{def.}}{=} [\mu_1, \mu_2, \mu_3, \ldots, \mu_{2V-1}, \mu_{2V}]^{\top}$, one has $\underline{x}^{\top}\underline{\underline{a}}_{V} = \frac{1}{\alpha V}[0, \underline{x}^{0\top} \otimes \underline{\lambda}^{\top}] + \frac{1}{\alpha V}[\underline{x}^{1\top} \otimes \underline{\mu}^{\top}, 0]$, where $\otimes$ denotes the componentwise multiplications. For the computation of A in Equation (45), by taking logarithm as

$$b(\alpha V, k, t) = \log\frac{e^{-\alpha V t}(\alpha V t)^k}{k!} = -\alpha V t + k\log(\alpha V t) - \sum_{m=1}^{k}\log m \qquad (46)$$

the following recurrence equation may be exploited: $b(\alpha V, k, t) = \log(\alpha V t) - \log k + b(\alpha V, k-1, t)$. The above computational procedure is summarized as Algorithm C1 in Appendix C.

Once the state probability vector $\underline{p}_{2V;m}(t)$ of $N_{2V}(t)$ is computed from Equation (45), that of $X_{\mathrm{OU}}(t)$ can be obtained through a sequence of state conversions. This in turn enables one to evaluate the prices of zero coupon discount bonds and associated European options defined on $X_{\mathrm{OU}}(t)$. The state conversions among $N_{2V}(t)$ and $X_{V}(t)$: are summarized in Exhibit B1.

APPENDIX C

SUMMARY OF BASIC COMPUTATIONAL ALGORITHMS

Algorithm C1 *(Transition Probability Matrix of $N_{2V}(t)$)*

Input:

- ▷ V : *parameter for the size of state space of $N_{2V}(t)$*
- ▷ τ : *future time as the argument of the transition probability matrix*
- ▷ α : *reversion factor*
- ▷ ε : *parameter for stopping criteria for the series expansion of Equation* (46)

Output: $\underline{\underline{P}}_{2V}(\tau)$: *transition probability matrix*

Procedure:

1) *Set* $\bar{k} = \max\{k: e^{b(\alpha V,k,\tau)} < \varepsilon, k > \alpha V\tau\}$, $k_0 = \min\{k: e^{b(\alpha V,k,\tau)} > \varepsilon, k, < \alpha V\tau\}$ *and* $m = 0$.
2) LOOP1: *Set* $\underline{p}_{2V;m}(\tau) \leftarrow \underline{0}$, $\underline{x} \leftarrow \underline{u}_m$ *and* $k \leftarrow 0$.
3) LOOP2: $\underline{x}^{\top} \leftarrow \underline{x}^{\top}\underline{\underline{a}}_{V}$.
4) *If* $k < k_0$, *set* $k \leftarrow k + 1$ *and go to* LOOP2.
5) LOOP3: $\underline{p}_{2V;m}(\tau) \leftarrow \underline{p}_{2V;m}(\tau) + e^{b(\alpha V,k,\tau)}\underline{x}$.
6) *If* $k < \bar{k}$, *set* $\underline{x}^{\top} \leftarrow \underline{x}^{\top}\underline{\underline{a}}_{V}$ $k \leftarrow k + 1$, *and go to* LOOP3.
7) *If* $m < 2V$, $m \leftarrow m + 1$, *and go to* LOOP1. *Otherwise, stop.*

Based on the above algorithm, the transition probability matrix of $N_{2V}(t)$ can be evaluated, which we denote by $\underline{\underline{P}}_{2V}(\tau) \leftarrow$ ALGOC1 $[V, \tau, \alpha]$ for notational convenience. From Equation (12), this in turn enables one to approximate the transition probabilities of $\hat{X}_{\mathrm{OU}}(t)$.

EXHIBIT B1

State Conversions

This exhibit summarizes state conversion relationships between two Ehrenfest processes, $N_{2V}(t)$ and $X_{V}(t)$, where $\eta_V(x) = \left\lceil \frac{\sqrt{\alpha V}}{\sigma} x \right\rceil$.

Process	State Conversion		State Space
	$x \in \mathbb{R} \rightarrow m \in \mathcal{N}_V$	$m \in \mathcal{N}_V \rightarrow x \in \mathbb{R}$	
$N_{2V}(t)$	$m = \eta_V(x) + V$	m	$\mathcal{N}_V = \{0, 1, \ldots, 2V\}$
$X_{V}(t)$	$\frac{\sigma}{\sqrt{\alpha V}}\eta_V(x)$	$x_V(m) = \frac{\sigma}{\sqrt{\alpha V}}\, m - \sigma\sqrt{\frac{V}{\alpha}}$	$\{-\sigma\sqrt{\frac{V}{\alpha}}, \ldots, \sigma\sqrt{\frac{V}{\alpha}}\}$

Algorithm C2 (*Discount Bond Prices of* $[D(M, m \mid K)]_{m \in \mathcal{N}_V}$)

Input:

- ▷ V : *size of the state space of* $N_{2V}(t)$
- ▷ T : *maturity time*
- ▷ τ_M: *time at which the discount bond prices are desired*
- ▷ Δt : *discretized time interval satisfying* $T = K\Delta t$ *and* $\tau_M = M\Delta t$ *for some positive integers* K *and* M
- ▷ α : *reversion factor*
- ▷ σ : *volatility factor*
- ▷ $\theta(t)$: *shift function*
- ▷ $\underline{d} = [d_0, \cdots, d_{2V}]$: *terminal value vector*

Output: $[D(M, m \mid K)]_{m \in \mathcal{N}_V}$: *discount bond prices*

Procedure:

1) *Set* $D(K, m \mid K) = d_m$ *for all* $m \in \mathcal{N}_V$, $k = K - 1$, *and obtain* $\underline{\underline{P}}_{2V}(\Delta t) \leftarrow$ ALGOC1 $(V, \Delta t, \alpha)$.
2) LOOP: *For all* $m \in \mathcal{N}_V$, *compute* $r(k, m) = x_V(m) + \theta(k\Delta t)$ *with* $x_V(m)$ *as in Equation* (10), *and generate* $D(k, m \mid K)$ *using* (14).
3) *If* $k > M$, *set* $k \leftarrow k - 1$ *and go to* LOOP. *Otherwise, stop.*

For notational convenience, we write $[D(M, m \mid K)]_{m \in \mathcal{N}_V} \leftarrow$ ALGOC2 $(V, \tau_M, T, \Delta t, \alpha, \sigma, \theta(t), \underline{d})$.

Algorithm C3 (*Arrow-Debreu Prices of* $[Q_0(M, m \mid k_0)]_{m \in \mathcal{N}_V}$)

Input:

- ▷ V : *size of the state space of* $N_{2V}(t)$
- ▷ τ_0 : *maturity time of the Arrow-Debreu security whose present values are known*
- ▷ τ_M: *maturity time of the Arrow-Debreu security whose present values are desired*
- ▷ Δt : *discretized time interval satisfying* $\tau_0 = k_0\Delta t$ *and* $\tau_m = M\Delta t$ *for some positive integers* k_0 *and* M
- ▷ α : *reversion factor*
- ▷ σ : *volatility factor*
- ▷ $\theta(t)$: *shift function*
- ▷ $\underline{s} = [s_0, \cdots, s_{2V}]$: *starting value vector*

Output: $[Q(M, m \mid k_0)]_{m \in \mathcal{N}_V}$: *Arrow-Debreu prices*

Procedure:

1) *Set* $Q_0(k_0, m \mid k_0) = s_m$ *for all* $m \in \mathcal{N}_V$, $k = k_0 + 1$, *and obtain* $\underline{\underline{P}}_{2V}(\Delta t) \leftarrow$ ALGOC1 $(V, \Delta t, \alpha)$.
2) LOOP: *For all* $m \in \mathcal{N}_V$, *compute* $r(k, m) = x_V(m) + \theta(k\Delta t)$ *with* $x_V(m)$ *as in Equation* (10), *and generate* $Q_0(k, m \mid k_0)$ *using Equation* (17).
3) *If* $k < M$, *set* $k \leftarrow k + 1$ *and go to* LOOP. *Otherwise, stop.*

As before, we write $[Q_0(M, m \mid k_0)]_{m \in \mathcal{N}_V} \leftarrow$ ALGOC3 $(V, \tau_0, \tau_M, \Delta t, \alpha, \sigma, \theta(t), \underline{s})$ for notational convenience.

Algorithm C4 (*European Call Option Price of* $\pi_c(0 \mid M)$)

Input:

- ▷ V : *size of the state space of* $N_{2V}(t)$
- ▷ T : *maturity time of the discount bond*
- ▷ τ_M: *maturity time of the option*
- ▷ Δt : *discretized time interval satisfying* $T = K\Delta t$ *and* $\tau_M = M\Delta t$ *for some positive integers* K *and* M
- ▷ K_S: *strike price of the option*
- ▷ α : *reversion factor*
- ▷ σ : *volatility factor*
- ▷ $\theta(t)$: *shift function*

Output: $\pi_c(0 \mid M)$: *price of European call option on discount bond*

Procedure:

1) *Set* $\underline{d} \leftarrow \underline{1}$, $\underline{s} \leftarrow \underline{u}_V$ *and* $\tau_0 = 0$.
2) *Obtain* $[D(M, m \mid K)]_{m \in \mathcal{N}_V} \leftarrow$ ALGOC2$(V, \tau_M, T, \Delta t, \alpha, \sigma, \theta(t), \underline{d})$.
3) *Compute* $[h_c(D(M, m \mid K))]_{m \in \mathcal{N}_V}$ *by using Equation* (18).
4) *Obtain* $[Q_0(M, m)]_{m \in \mathcal{N}_V} \leftarrow$ ALGOC3$(V, \tau_0, \tau_M, \Delta t, \alpha, \sigma, \theta(t), \underline{s})$.
5) *Compute* $\pi_c(0 \mid M)$ *by using Equation* (19) *and stop.*

Algorithm C5 (*Discount Bond Prices of* $[D_{SW}(M, m \mid K_I)]_{m \in \mathcal{N}_V}$)

Input:

- ▷ V : *size of the state space of* $N_{2V}(t)$
- ▷ T : *maturity time of the discount bond*
- ▷ τ_M: *time at which the discount bound prices are desired*
- ▷ $[\alpha_l, T_l^\alpha]_{l \in \{1, \ldots, L\}}$: *stepwise reversion function where* $\alpha(t) = \alpha_l$ *for* $t \in [T_{l-1}^\alpha, T_l^\alpha], 1 \le l \le L$, *with* $T_L^\alpha = T$
- ▷ $[\sigma_j, T_j^\sigma]_{j \in \{1, \ldots, J\}}$: *stepwise volatility function where* $\sigma(t) = \sigma_j$ *for* $t \in [T_{j-1}^\sigma, T_j^\sigma], 1 \le j \le J$, *with* $T_J^\sigma = T$
- ▷ $[\theta_l(t)]_{l \in \{1, \ldots, L\}}$: *shift functions with* $t \in [0, T]$

Output: $[D_{SW}(M, m \mid K_I)]_{m \in \mathcal{N}_V}$: *discount bond prices*

Procedure:

0) *Perform* $I \leftarrow$ JOIN$[L, J]$, *yielding* $[T_i]_{i \in \{1, \ldots, I\}}$ *and set* $\underline{d} \leftarrow \underline{1}$.
1) *Choose* $\Delta t > 0$ *so that Equation* (29) *is satisfied, yielding* $[K_i]_{i \in \{1, \ldots, I\}}$ *and* M.
2) *If* $i_0 = I$ *with* i_0 *as in Equation* (30), *obtain*

$$[D_{SW}(M, m \mid K_I)]_{m \in \mathcal{N}_V} \leftarrow \text{ALGOC2}\,(V, \tau_M, T, \Delta t, \alpha_{l_I}, \sigma_{j_I}, \theta_{l_I}(t), \underline{d})$$

and stop.

3) *Set* $i \leftarrow I$.
4) LOOP: *Set* $\underline{d} \leftarrow$ ALGOC2$(V, T_{i-1}, T_i, \Delta t, \alpha_{l_i}, \sigma_{j_i}, \theta_{l_i}(t), \underline{d})$
5) *If* $i > i_0 + 1$, *set* $i \leftarrow i - 1$ *and go to* LOOP.
6) *Obtain* $[D_{SW}(M, m \mid K_I)]_{m \in \mathcal{N}_V} \leftarrow$ ALGOC2 $(V, \tau_M, T_{i_0}, \Delta t, \alpha_{l_{i_0}}, \sigma_{j_{i_0}}, \theta_{l_{i_0}}(t), \underline{d})$ *and stop.*

Algorithm C6 (*Arrow-Debreu Prices of* $[Q_{\mathrm{SW:0}}(M, m)]_{m\in\mathcal{N}_V}$)

Input:

- ▷ V : *size of the state space of* $N_{2V}(t)$
- ▷ T : *maturity time of the discount bond*
- ▷ τ_M: *maturity time of the Arrow-Debreu security whose present values are desired*
- ▷ $[\alpha_l, T^{\alpha}_l]_{l\in\{1,\ldots,L\}}$: *stepwise reversion function where* $\alpha(t) = \alpha_l$ *for* $t \in [T^{\alpha}_{l-1}, T^{\alpha}_l], 1 \le l \le L$, *with* $T^{\alpha}_L = T$
- ▷ $[\sigma_j, T^{\sigma}_j]_{j\in\{1,\ldots,J\}}$: *stepwise volatility function where* $\sigma(t) = \sigma_j$ *for* $t \in [T^{\sigma}_{j-1}, T^{\sigma}_j], 1 \le j \le J$, *with* $T^{\sigma}_J = T$
- ▷ $[\theta_l(t)]_{l\in\{1,\ldots,L\}}$: *shift functions with* $t \in [0,T]$

Output: $[Q_{\mathrm{SW:0}}(M, m)]_{m\in\mathcal{N}_V}$: *Arrow-Debreu Prices*

Procedure:

0) *Perform* $I \leftarrow \mathrm{JOIN}[L, J]$, *yielding* $[T_i]_{i\in\{1,\ldots,I\}}$ *and set* $\underline{s} \leftarrow \underline{u}_V$.
1) *Choose* Δt *so that Equation* (29) *is satisfied, yielding* $[K_i]_{i\in\{1,\ldots,I\}}$ *and* M.
2) *If* $i_0 = 1$ *with* i_0 *as in Equation* (30), *obtain*

$$[Q_{\mathrm{SW:0}}(M, m)]_{m\in\mathcal{N}_V} \leftarrow \mathrm{ALGOC3}(V, 0, \tau_M, \Delta t, \alpha_{l_1}, \sigma_{j_1}, \theta_{l_1}(t), \underline{s})$$

and stop.
3) *Set* $i \leftarrow 1$.
4) LOOP : *Set* $\underline{s} \leftarrow \mathrm{ALGOC3}(V, T_{i-1}, T_i, \Delta t, \alpha_{l_i}, \sigma_{j_i}, \theta_{l_i}(t), \underline{s})$.
5) *If* $i < i_0 - 1$, *set* $i \leftarrow i + 1$ *and go to* LOOP.
6) *Obtain* $[Q_{\mathrm{SW:0}}(M, m)]_{m\in\mathcal{N}_V} \leftarrow \mathrm{ALGOC3}(V, T_{i_0-1}, \tau_M, \Delta t, \alpha_{l_{i_0}}, \sigma_{j_{i_0}}, \theta_{l_{i_0}}(t), \underline{s})$ *and stop.*

Algorithm C7 (*European Call Option Price of* $\pi_{\mathrm{SW:c}}(0 \mid M)$)

Input:

- ▷ V : *size of the state space of* $N_{2V}(t)$
- ▷ T : *maturity time of the discount bond*
- ▷ τ_M: *maturity time of the option*
- ▷ $[\alpha_l, T^{\alpha}_l]_{l\in\{1,\ldots,L\}}$: *stepwise reversion function where* $\alpha(t) = \alpha_l$ *for* $t \in [T^{\alpha}_{l-1}, T^{\alpha}_l], 1 \le l \le L$, *with* $T^{\alpha}_L = T$
- ▷ $[\sigma_j, T^{\sigma}_j]_{j\in\{1,\ldots,J\}}$: *stepwise volatility function where* $\sigma(t) = \sigma_j$ *for* $t \in [T^{\sigma}_{j-1}, T^{\sigma}_j], 1 \le j \le J$, *with* $T^{\sigma}_J = T$
- ▷ $[\theta_l(t)]_{l\in\{1,\ldots,L\}}$: *shift functions with* $t \in [0,T]$

Output: $\pi_{\mathrm{SW:c}}(0 \mid M)$: *European call option price*

Procedure:

1) *Obtain* $[D_{\mathrm{SW}}(M, m \mid K_l)]_{m\in\mathcal{N}_V}$ *via Algorithm C5.*
2) *Compute* $h_c(D_{\mathrm{SW}}(M, m \mid K_l)) = [D_{\mathrm{SW}}(M, m \mid K_l) - K_S]^+$ *for all* $m \in \mathcal{N}_V$.
3) *Find* $[Q_{\mathrm{SW:0}}(M, m)]_{m\in\mathcal{N}_V}$ *via Algorithm C6.*
4) *Compute the price based on Equation* (32) *and stop.*

ENDNOTE

This research is partly supported by Mizuho-Daiichi Financial Technologies, co., MEXT Grant-in-Aid for Scientific Research (C) 17510114, and MEXT Grant-in-Aid for Young Scientist (B) 17710125.

REFERENCES

Hull, J., and A. White. "Valuing Derivative Securities Using the Explicit Finite Differential Method." *Journal of Financial and Quantitative Analysis*, 25 (1990), pp. 87–100.

——. "One-factor Interest Rate Model and the Valuation of Interest Rate Derivative Securities." *Journal of Financial and Quantitative Analysis*, 28 (1993), pp. 235–254.

——. "Numerical Procedures for Implementing Term Structure Models I: Single-factor Models." *Journal of Derivatives*, 2 (1994), pp. 7–16.

Hull, J. *Options, Futures and Other Derivatives*. 4th ed., Prentice-Hall International, Inc., 2000.

Keilson, J. *Markov Chain Models: Rarity and Exponentiality*. Applied Mathematical Science Series, 28, New York: Springer, 1979.

Kijima, M. and I. Ngayama. "Efficient Numerical Procedures for the Hull-White Extended Vasicek Model." *Journal of Financial Engineering*, 3:3/4 (1994), pp. 275–292.

Pelsser, A. *Efficient Methods for Valuing Interest Rate Derivatives*. Spring-Verlag London Limited, 2000.

Sumita, U., J. Gotoh and H. Jin. "Numerical Exploration of Dynamic Behavior of the Ornstein-Uhlenbeck Process via Ehrenfest Process Approximation." *Journal of the Operations Research Society of Japan*, 49 (2006), pp. 256–278.

Vasicek, O. "An Equilibrium Characterization of the Term Structure." *Journal of Financial Economics*, 5 (1977), pp. 177–188.