Extracting and Applying Smooth Forward Curves From Average-Based Commodity Contracts with Seasonal Variation

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In this article, we propose a method of computing a smooth curve from observed forward prices with settlement over a period. We consider the electricity market, where such contracts are based on an average of the spot price for electricity over a specified period ranging from one day up to a year. From observed market data, we construct a smooth curve which is supposed to model the prices of forward contracts with settlement at a fixed time, if such would have been traded in the market. Since electricity prices are seasonally dependent, we propose to decompose the curve into a seasonal component and a residual or correction term. The correction term is defined as a polynomial spline function with a maximum smoothness property, so that the constructed curve replicates the observed market prices perfectly.

We apply our smoothing algorithm to market data from Nord Pool. The user supplied seasonality affects the shape of the forward curve in the long end. We investigate the term structure of volatility using historical data. From a modeling perspective, we suggest modeling the average-based contracts directly (swap-price modeling) in favour of fixed-delivery forwards (forward-curve modeling), since the former method seems less affected by the seasonality specification.

In many of the newer markets, the underlying "asset" of a financial contract is non-storable (like weather, bandwidth, or freight rates) or extremely costly to store (like electricity). The typical forward contracts in such markets are often settled against the average spot price (or some index like HDD/CDD in temperature) during a settlement period. If the delivery rate in the settlement period is constant, the contracts are called average-based forwards/futures. In this article, we propose an algorithm for constructing a smooth curve of fixed-delivery forward prices from average-based forward contracts, and we apply our method to analyzing forwards and futures data in the Nordic electricity exchange, Nord Pool.

Representing forward and futures prices by one continuous term structure curve is necessary when modeling the forward/futures price dynamics or when marking an OTC-product to the market. Cortazar and Schwartz [1994] adopted the Heath-Jarrow-Morton framework for commodity market modeling, where the term structure is represented by a continuous curve (see Bjerksund et al. [2000] for an application to the electricity market). Our method computes a smooth forward curve that can be used in such a modeling environment.

This article is closely related to the literature on fitting yield curves (see McCulloch [1971] for a seminal paper and Anderson and Deacon [1996] for a survey). The two main approaches in the fixed income literature are either to fit a parametric function to the entire yield curve by regression, or to fit all observed yields with a spline. We follow a mixture of the two approaches, assuming that the forward curve can be represented as the sum of a seasonal function and an adjustment function. In particular, we apply a "maximum smoothness" criterion in the specification of the adjustment term, first used in fixed income markets by Adams and Deventer [1994] (see also Forsgren [1998]; Lim and Xiao [2002]).

We cannot directly apply the methods developed by these authors for two reasons. First, our market data are prices on average-based contracts, not fixed point yields. Second, we must account for seasonally varying prices. There are many ways to do this, for example, based on spot price data which can be linked to forward prices. However, since there is no clear arbitrage-free connection between the spot prices and the forward curve in the electricity market, the choice of seasonal function will necessarily be, to some degree, ad hoc.[1]

We have implemented our algorithm and tested the methodology on data collected from the Nordic electricity exchange, Nord Pool. We provide several examples using different specifications of the seasonality function to show the flexibility of our approach. As an application, we perform a statistical study of historical forward curves, and analyze these for two different models of the forward price dynamics. In Benth and Koekebakker [2005], a Heath-Jarrow-Morton (HJM) approach for average-based forwards is considered (swap-price modeling) and, in light of their studies, we focus on the volatility term structure that can be extracted from the smoothed historical forward curves. Our investigations show that the HJM-approach suggested in Benth and Koekebakker [2005] seems to better capture the term structure of volatility than the traditional fixed-delivery forward approach suggested by Bjerksund et al. [2000]. Moreover, the specification of the seasonality function affects the volatility estimates significantly. The volatility in the long end of the curve seems to be upward biased using the latter approach compared to the former. The empirical results, however, are all preliminary, and a more thorough study must be conducted before any robust conclusions can be drawn.

The article is organized as follows: In the next section, we introduce the price dynamics of the different forward and futures contracts that we will analyze in this article. The following section contains the theory behind our smoothing algorithm, which is then applied to several examples in Section 4. The last section offers all conclusions.

AVERAGE-BASED CONTRACTS AND THE FORWARD CURVE

We work directly under the risk-neutral probability Q and assume that the market has a finite time horizon $T < \infty$. Consider a forward contract on a commodity where T^s and T^e denote the start and end of the settlement period, respectively, with $T^s < T^e \leq T$. Let $\tau \leq T^s < T^e$ and denote by $F(\tau, T^s, T^e)$ the price at time τ (today) for receiving a unit of the commodity at a continuous flow during the period $[T^s, T^e]$. If the price of the commodity at time u is $S(u)$, the payoff from the forward contract is

$$\int_{T^s}^{T^e} e^{-ru}\left\{S(u) - F(\tau, T^s, T^e)\right\} du$$

where r is the risk-free interest rate. Since it is (usually) costless to enter the forward contract, the arbitrage-free pricing relation (see for e.g., Benth and Koekebakker [2005]) yields that

$$F(\tau, T^s, T^e) = \int_{T^s}^{T^e} w(r;t)\mathbb{E}\left[S(t)|\mathcal{F}_\tau\right]dt$$

where

$$w(r;t) = \frac{e^{-rt}}{\frac{1}{r}(e^{-rT^s} - e^{-rT^e})}$$

and $\mathbb{E}$ is the risk-neutral expectation. Since it is known that the price of a forward at time τ with delivery at the fixed time $t \geq \tau$ is

$$f(\tau, t) = \mathbb{E}\left[S(t)|\mathcal{F}_\tau\right] \tag{1}$$

we find the following relationship between average-based and fixed-delivery forwards

$$F(\tau, T^s, T^e) = \int_{T^s}^{T^e} w(r;t) f(\tau, t)dt \tag{2}$$

In these derivations, we have assumed that the contracts are marked-to-market in the settlement period. If the contracts are settled at T^e only, w becomes $w(r;t) = 1/(T^e - T^s)$.

In practice, the contracts are not settled continuously over the delivery period, but at discrete times. Assuming settlement at N points in time $t_1 < t_2 \ldots < t_N$, with $T^s = t_1$ and $T^e = t_N$, the relationship becomes

$$F(0,T^s,T^e) = \sum_{k=1}^{N} w(r;t_k) f(0,t_k) \tag{3}$$

with

$$w(r;t_i) = \frac{e^{-r\tau_k}}{\sum_{j=1}^{N} e^{-rt_j}}$$

Note that these expressions suggest that our average-based forward contracts are equivalent to swaps. This is natural since the contracts are swapping a fixed price commodity against a floating price commodity in the settlement period. If the contracts are settled at the terminal time of delivery T^e only, w becomes simply $w(r;\, t) = 1/N$.

COMPUTING SMOOTH FORWARD CURVES

Note from Equation (3) that for $\tau = 0$, we have

$$F(0,T^s,T^e) = \sum_{k=1}^{N} w(r;t_k) f(0,t_k)$$

with discrete delivery. In this article, we are concerned with the construction of $f(0,t)$ based on the observations of $F(0, T^s, T^e)$ for different delivery periods. From now on, we write $f(t)$ for $f(0, t)$ and $F(T^s, T^e)$ for $F(0, T^s, T^e)$.

Assume that we observe m contracts on any given day. Furthermore, let t_0 be the start of the settlement period for the contract with the shortest time to delivery, and denote by t_n the end of the settlement period for the contract going longest into the future. In the following subsections, we will specify a certain functional form for f that makes the relationship in Equation (2) or Equation (3) hold for m contracts at a given point in time. We decompose the forward price into

$$f(t) = s(t) + \varepsilon(t), \quad t \in [t_0, t_n] \tag{4}$$

for two continuous functions $s(t)$ and $\varepsilon(t)$. We interpret $s(t)$ to be the seasonality of the forward curve, and $\varepsilon(t)$ to be an adjustment function that measures the forward curve's deviation from the seasonality.

Since there is no unique arbitrage-free relationship between the spot price dynamics and the forward curve for many commodity markets such as electricity, one may use the objective probability as the risk-neutral pricing measure. Thus, the forward price should simply be the average spot price, which is given by the seasonality function. However, it is natural to believe that the traders in the market are including a risk premium for the lack of a hedge of the average-based forward, which is reflected as a deviation from the seasonality as the forward curve. In this sense, the adjustment function can be understood as the market price of risk, and it is quite natural to believe that this is a function of time to delivery (see e.g., Benth et al. [2003]). In the short-end of the curve, the traders have much information about future price formation based, for instance, on weather conditions, hydro reservoir fillings, etc. The long-end of the curve may be several years ahead, and obviously the market's view on risk is less sensitive to time. This suggests a time-varying ε, but also that it should be flat at the long end. From now on, we therefore assume that

$$\varepsilon'(t_n) = 0$$

Note that the smoothness is calculated on the adjustment function, $\varepsilon(t)$, and not on the forward function $f(t)$. The reason for this is to better retain seasonal patterns.

In this article, we want to use a parametric form of $s(t)$, while the adjustment function is specified as a spline in order to have a smoothed forward curve perfectly matched to the observed average-based forward data. Loosely speaking, we fit the parameters of the seasonal function by least squares, and next use the spline specification of the adjustment function to ensure a perfect fit to the observed electricity forward prices of the market.

Maximum Smoothness Criteria

Adams and van Deventer [1994] applied the maximum smoothness criteria to construct yield curves. Let $C_0^2([t_0, t_n])$ be the set of real-valued functions on the interval $[t_0, t_n]$ which are twice continuously differentiable with zero derivative in t_n. Consider some subclass of functions C. With "smoothness" of a function expressed as the mean square value of its second derivative, we define the smoothest possible forward curve on an interval $[t_0, t_n]$ as the one that minimizes

$$\int_{t_0}^{t_n} \left[\varepsilon''(t)\right]^2 dt$$

over the set C, if such exists. We define $\varepsilon \in C$ to be this minimizing function, and interpret the smoothest forward curve to be the one for which ε solves the minimization problem above. The subclass C is chosen so that it is easy to take the swap price data into account, either by exact matching, or by a constraint on the bid-ask spread prices. The smoothness is calculated on the adjustment function, $\varepsilon(t)$, and not on the forward function $f(t)$, to better retain the seasonal patterns. In addition to be smoothest possible, we want the adjustment function to be twice continuously differentiable and horizontal at time t_n. It turns out that the class of polynomial spline functions of order four is appropriate for our purposes. We describe C in more detail below.

A Smooth Forward Curve Constrained by Closing Prices

Let

$$\Phi = \left\{ \left(T_1^s, T_1^e\right), \left(T_2^s, T_2^e\right), \ldots, \left(T_m^s, T_m^e\right) \right\}$$

be a list of start and end dates for the settlement period of m different average-based forward contracts. To be able to handle overlapping settlement periods, we construct a new list, $\{t_0, t_1, \ldots, t_n\}$, of dates where overlapping contracts are split into sub periods. This is illustrated in Exhibit 1.

As we can see from Exhibit 1, the new list is basically the elements in Φ sorted in ascending order with any duplicated dates removed. The bid and ask prices for the forward contract $i \in \{1, \ldots, m\}$ are denoted F_i^B and F_i^A respectively. For the time being, we concentrate on using the closing prices denoted by F_i^C.

We assume a zero interest rate, arguing that the interest rate effect is less than the effect from both the seasonality and adjustment function, and thus believe the approximation is valid. Similarly to Adam and van Deventer [1994], and Lim and Xiao [2002], we show in Appendix A that the smoothest adjustment function with the above properties is a polynomial spline of order four. This mean that we can write the adjustment function as

$$\varepsilon(t) = \begin{cases} a_1 t^4 + b_1 t^3 + c_1 t^2 + d_1 t + e_1 & t \in [t_0, t_1] \\ a_2 t^4 + b_2 t^3 + c_2 t^2 + d_2 t + e_2 & t \in [t_1, t_2] \\ \vdots & \\ a_n t^4 + b_n t^3 + c_n t^2 + d_n t + e_n & t \in [t_{n-1}, t_n] \end{cases}$$

To find the parameters

$$x^T = [a_1\ b_1\ c_1\ d_1\ e_1 \quad a_2\ b_2\ c_2\ d_2\ e_2 \ldots a_n\ b_n\ c_n\ d_n\ e_n]$$

to the adjustment function, we solve the following equality constrained convex quadratic programming problem

$$\min_x \int_{t_0}^{t_n} \left[\varepsilon''(t; \mathrm{x})\right]^2 dt \tag{5}$$

subject to the natural constraints in the connectivity and smoothness of derivatives at the knots, $j = 1, \ldots, n-1$,

EXHIBIT 1
Splitting Overlapping Average-Based Forward Contracts

$$(a_{j+1}-a_j)t_j^4+(b_{j+1}-b_j)t_j^3+(c_{j+1}-c_j)t_j^2 \\ +(d_{j+1}-d_j)t_j+e_{j+1}-e_j=0 \tag{6}$$

$$4(a_{j+1}-a_j)t_j^3+3(b_{j+1}-b_j)t_j^2 \\ +2(c_{j+1}-c_j)t_j+d_{j+1}-d_j=0 \tag{7}$$

$$12(a_{j+1}-a_j)t_j^2+6(b_{j+1}-b_j)t_j+2(c_{j+1}-c_j)=0 \tag{8}$$

and

$$\varepsilon'(t_n;x)=0 \tag{9}$$

$$F_i^C=\int_{T_i^s}^{T_i^e} w(r;t)(\varepsilon(t)+s(t))\,dt \tag{10}$$

for $i = 1, \ldots, m$.[2] This minimization problem has a total of $3n + m - 2$ constraints. By inserting the expression for $\varepsilon''(t)$ and integrating it, we can write the first part of the minimization problem as

$$\min_x x^T Hx$$

where

$$H=\begin{bmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{bmatrix},\quad h_j=\begin{bmatrix} \frac{144}{5}\Delta_j^5 & 18\Delta_j^4 & 8\Delta_j^3 & 0 & 0 \\ 18\Delta_j^4 & 12\Delta_j^3 & 6\Delta_j^2 & 0 & 0 \\ 8\Delta_j^3 & 6\Delta_j^2 & 4\Delta_j^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\Delta_j^l=t_{j+1}^l-t_j^l$$

The dimension of x is $5n \times 1$ and the dimensions of the symmetric H are $5n \times 5n$. All the constraints to Equation (5) are linear with respect to x. The constraints, Equations (6)–(10), can be formulated in matrix form as Ax = b, where A is a $(3n + m - 2) \times 5n$ matrix, and b is a $(3n + m - 2)$ 1 vector. We obtain an explicit solution by the Lagrange Multiplier Method. Let $\lambda^T = [\lambda_1, \lambda_2, \ldots, \lambda_{3n+m-2}]$ be the corresponding Lagrange multiplier vector to the constraints. We can now express Equation (5) as the following unconstrained minimization problem

$$\min_{x,\lambda} \quad x^T Hx+\lambda^T(Ax-b) \tag{11}$$

The solution $[x^*, \lambda^*]$ is thus obtained by solving the linear equation

$$\begin{bmatrix} 2H & A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix}=\begin{bmatrix} 0 \\ b \end{bmatrix} \tag{12}$$

The dimensions of the left matrix are $(8n + m - 2) \times (8n + m - 2)$. The solution vector and the rightmost vector have the dimensions of $(8n + m - 2) \times 1$. Numerically solving Equation (12) is standard, and can be done using various numerical techniques; we have chosen to use QR factorization. If n or m is large, one could improve the calculation speed by utilizing the sparseness of the matrix.

A Smooth Forward Curve Constrained by Bid-Ask Spreads

When the market is open for trading, we do not observe exact prices but, rather, a bid-ask spread. We will now extend the previous model to handle bid-ask prices. We replace the fixed closing price constraint in Equation (10) with

$$F_i^B \le \int_{T_i^s}^{T_i^e} w(r;t)(\varepsilon(t)+s(t))\,dt \le F_i^A \qquad i=1,\ldots,m$$

The smooth forward function is constrained between the market bid-ask prices. Unfortunately, all of the constraints are no longer binding by equality and it is, therefore, not possible to use the fast and easy Lagrange Multiplier Method.

There exist several methods to solve this problem numerically. See Judd [1998] for a short description of some of the most commonly used algorithms. In this article, we will use a method inspired by the "active set" approach. The strategy is to use the basic model in Equation (11) and change a pseudo close-price in the direction implied by the sign of the Lagrangian within the boundaries of the bid-ask spread. The algorithm is outlined as follows:

1. *Initialization:* Set the initial pseudo closing prices, F_i^C, closest to the seasonal function and solve Equation (11). One could, alternatively, use $F_i^C = (F_i^A + F_i^B)/2$ as initial values.
2. *Start optimization:* Let the close-price-Lagrangian, $\lambda_{3n-2}, \ldots, \lambda_{3n+m-2}$, with the largest absolute value, be called λ^C, and adjust the pseudo closing price according to

$$F_{new}^C = \begin{cases} F^A & \text{if } \lambda^C > 0 \text{ and } \lambda^A \geq 0 \\ F^C - \dfrac{\lambda^C(F^A - F^C)}{\lambda^A - \lambda^C} & \text{if } \lambda^C > 0 \text{ and } \lambda^A < 0 \\ F^C - \dfrac{\lambda^C(F^C - F^B)}{\lambda^C - \lambda^B} & \text{if } \lambda^C < 0 \text{ and } \lambda^B > 0 \\ F^B & \text{if } \lambda^C < 0 \text{ and } \lambda^B \leq 0 \end{cases}$$

   where λ^A denotes the contracts Lagrangian with an average price equal to the ask price F^A. Similarly λ^B denotes the contracts Lagrangian with the average price equal the bid price F^B.
3. *Stopping criteria:* We apply two different stopping criteria. The first one is to stop if the following is true for each F_i: (*a*) either the average price F_i^C is equal to F^B and λ_i^C is still negative or (*b*) the average price F_i^C is equal to F^A and λ_i^C is still positive. This means that it is not possible to improve the smoothness by changing F_i^C. The second criterion is to stop when the improvement of the smoothness is below some percentage γ.

$$1 - \frac{(x^T Hx)_k}{(x^T Hx)_{k-1}} < \gamma$$

   where k is the iteration number. The minimization ends when one of the following two criteria is fulfilled. If neither of the stopping criteria is satisfied, the algorithm returns to step 2.

The main advantage with this algorithm is the calculation speed. Convergence is usually obtained in m to $2m$ iterations. The reason for this rapid convergence is mainly due to the relative small bid-ask spread compared to the value of the adjustment function, that is, a small bid-ask spread usually implies that the bid or the ask constraint is binding and, thereby, reduces the number of constraints with inequalities.

SOME APPLICATIONS

In this section, we will apply our algorithm to real world data. All our examples use data collected from the Nord Pool electricity exchange. A detailed description of how the exchange operates and which contracts are traded can be found in Benth and Koekebakker [2005] or at www.nordpool.no.

Computing a Smooth Forward Curve from Nord Pool Market Data

In this example, we study the effect of the choice of seasonal function on the smoothed forward curve. The input data is from May 4, 2005, and is reported in Exhibit 2.[3] The six weekly contracts ("ENOWxxxx") are futures contracts and the rest of the contracts are forward contracts.

The seasonality is most easily seen from the quarterly contracts in both 2006 and 2007. The first quarter has the highest price and Q3 the lowest. To visualize the effect of the seasonal function on a smooth forward curve, we try three different specifications:

1. Zero seasonality, $s(t) = 0$,
2. A simple trigonometric function,

$$s(t) = 145.732 + 29.735 \times \cos\left((t + 6.691)\frac{2\pi}{365}\right)$$

   and
3. A spot forecast from a bottom-up model.

The first alternative has no seasonality. In markets with average-based contracts and no seasonality, this is a natural candidate (one may think of the oil market as having no seasonality). In the case of electricity it is not, but we include the specification for comparison. The second specification is a spot-based estimate of the seasonality in the Nord Pool market, taken from Lucia and Schwartz [2002], where the parameter values are those estimated for their one-factor arithmetic model. The third example is a forecast supplied by Agder Energi AS on

EXHIBIT 2
Market Data from Nord Pool, May 4, 2005

Ticker	Startdate	Enddate	Bid	Ask
ENOW19-05	2005-05-09	2005-05-15	252.5	253.00
ENOW20-05	2005-05-16	2005-05-22	248.75	250.00
ENOW21-05	2005-05-23	2005-05-29	253.25	257.00
ENOW22-05	2005-05-30	2005-06-05	252.00	255.00
ENOW23-05	2005-06-06	2005-06-12	252.00	255.00
ENOW24-05	2005-06-13	2005-06-19	252.00	255.00
ENOMJUN-05	2005-06-01	2005-06-30	252.00	254.50
ENOMJUL-05	2005-07-01	2005-07-31	236.00	236.50
ENOMAUG-05	2005-08-01	2005-08-31	256.00	258.00
ENOMSEP-05	2005-09-01	2005-09-30	260.00	263.00
ENOMOCT-05	2005-10-01	2005-10-31	263.00	269.00
ENOMNOV-05	2005-11-01	2005-11-30	275.00	277.00
FWV2-05	2005-10-01	2005-12-31	276.00	276.50
ENOQ1-06	2006-01-01	2006-03-31	280.50	283.50
ENOQ2-06	2006-04-01	2006-06-30	235.00	240.00
ENOQ3-06	2006-07-01	2006-09-30	230.00	233.00
ENOQ4-06	2006-10-01	2006-12-31	251.00	259.00
ENOQ1-07	2007-01-01	2007-03-31	266.50	272.00
ENOQ2-07	2007-04-01	2007-06-30	220.50	228.50
ENOQ3-07	2007-07-01	2007-09-30	218.50	226.50
ENOQ4-07	2007-10-01	2007-12-31	250.00	250.50
ENOYR-06	2006-01-01	2006-12-31	249.00	249.50
ENOYR-07	2007-01-01	2007-12-31	240.50	241.00
ENOYR-08	2008-01-01	2008-12-31	240.00	244.00

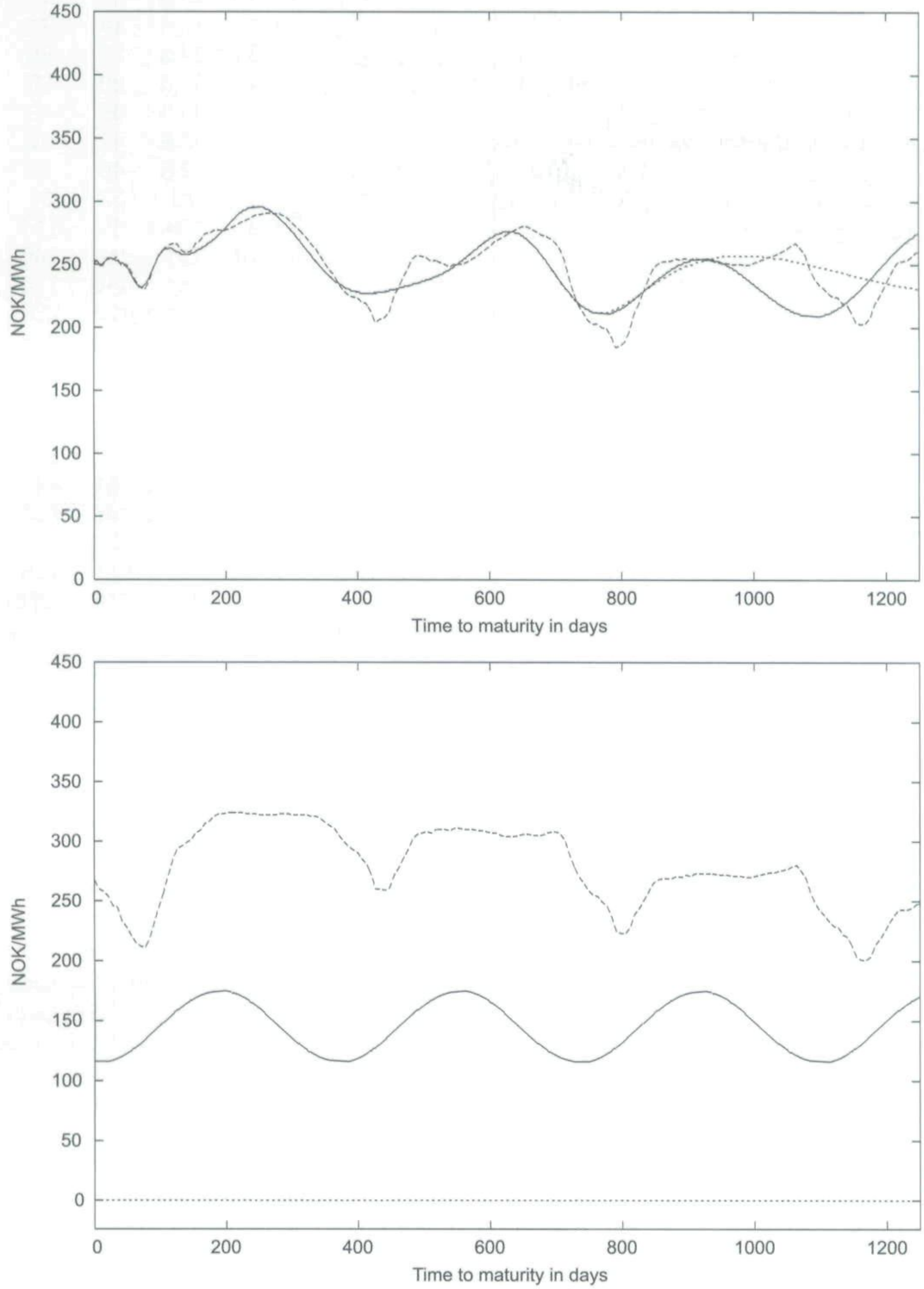
May 4, 2005 (see the lower panel of Exhibit 3), based on their internal bottom-up model.

The smoothed forward curves are constructed from $m = 21$ contracts and are represented by a spline consisting of $n = 32$ polynomials. The algorithm converged after about 28 iterations (depending on the seasonal function we used). The seasonal functions are plotted in the lower panel of Exhibit 3, and the corresponding smoothed forward curves are plotted in the upper panel. Note that the three forward curves are all plausible, in the sense that all three curves can reproduce average based forward contracts within the bid-ask prices reported in Exhibit 2.

We have deliberately chosen very different seasonal functions. The trigonometric function is much smoother, and with a regular wave pattern, compared to the bottom-up forecast. They also differ in level, with the zero alternative as an extreme. The effect of the three different specifications on the shape of the forward curve varies with maturity. All three seasonal functions produce more or less identical forward curves for maturities less than a year. The reason is simple. Since the delivery periods for short term contracts are short, the marked-to-market constraint on the forward curve completely dominates the deterministic seasonality part of the forward curve pricing function. For maturities between one and two years to maturity (roughly between 400 and 800 days to maturity), the bottom-up prognosis differs somewhat from the other two. The four quarterly contracts force the curve with flat seasonality to fluctuate over the yearly cycle, making it indistinguishable from the curve with trigonometric seasonality. The market price covering the rightmost end of the curve is from a yearly contract alone. The curve flattens without seasonality. Overall, this example shows that the choice of seasonal function has little effect on the smooth forward curve in the short end. With longer maturities, the forward curve can take on various shapes depending on the user-supplied seasonality function. In the long end, with price information from only one yearly contract, the seasonal function becomes imperative in shaping the forward curve.

EXHIBIT 3

The market data in this figure is from Nord Pool, May 4, 2005 (see Exhibit 2). The lower panel shows the three seasonal functions underlying the curves; $s(t) = 0$ (dotted line), $s(t) = 145.732 + 29.735 \times \cos\left((t + 6.691)\frac{2\pi}{236}\right)$ (solid line), and a forecast provided by Agder Energi AS, Norway (dashed line). The upper panel shows the corresponding smooth forward curves.



Analyzing Volatility Using Historical Data

Even in highly liquid futures markets, one will often need to estimate prices for other maturity dates than those observed in the market. In this subsection, an investigation of volatility based on historical data is conducted. It is shown that the way the panel data set is constructed affects the estimated volatility dynamics. Note that the analysis is for illustrative purposes only; a full scale empirical analysis is beyond the scope of this article.

Constructing historical prices. We use the first two seasonal functions given above, zero seasonality ($s(t) = 0$), and yearly variation estimated in Lucia and Schwartz [2002], ($s(t) = 145.732 + 29.735 \times \cos((t + 6.691)\frac{2\pi}{365})$). On a given date τ_n, $n = 1, \ldots, N$, we apply our algorithm to compute the forward curve with maximum smoothness, using all reported average-based forward contracts on Nord Pool constrained by their closing bid-ask spreads. The sample period starts January 2, 2001, and ends April 29, 2005, a total of $N = 1076$ trading days. Hence, we extract the total of 1076 smooth fixed-delivery forward curves.

We next extract panel data sets in two ways for each choice of seasonal function; a panel of fixed-delivery forwards and a panel of average-based forwards. From each day's computed forward curve, we pick a cross-section of forward prices, either fixed delivery or average based. The cross-sections of fixed-delivery forward prices are derived for a set of delivery times u_m, $m = 1, \ldots, M$, while for the average-based forwards, we use the relationship in Equation (2) to compute cross-sections of M average-based forward contracts, having delivery over the periods $\{T^s_m, T^e_m\}$, $m = 1, \ldots, M$. In our empirical study, we suppose $M = 22$.

The maturities of the fixed-delivery forwards and the delivery periods are given in Exhibit 4. The columns T^s_m and T^e_m are the start and end of the delivery periods for non-overlapping average-based forward contracts. The first contract starts the delivery period in seven days. It has a seven-day delivery period, thus a weekly contract. The first seven contracts are weekly contracts, the next 10 contracts are monthly contracts (30 days of delivery), then four quarterly contracts (90 days of delivery), and finally a yearly contract (360 days of delivery). This structure of delivery periods mimics roughly the actual traded contracts in this market, with delivery periods increasing with time to delivery. The column u_m provides the maturities of the fixed-delivery forwards. The maturities are chosen as the mid-points of the average-based forwards. Hence, the term structure spans a little less than three years (1076 days) in the data set of average-based contracts, and a little less than 2.5 years (896 days) of fixed-delivery forwards. Since we consider two different contract types and two choices of seasonal functions, we have four different data sets to investigate.

EXHIBIT 4

Maturities of Fixed-Delivery Forwards and Delivery Periods for Average-based Forwards (in Days)

m	u_m	T^s_m	T^e_m
1	10.5	7	14
2	17.5	14	21
3	24.5	21	28
4	31.5	28	35
5	38.5	35	42
6	45.5	42	49
7	52.5	49	56
8	72	56	86
9	101	86	116
10	131	116	146
11	161	146	176
12	191	176	206
13	221	206	236
14	251	236	266
15	281	266	296
16	311	296	326
17	341	326	356
18	401	356	446
19	491	446	536
20	581	536	626
21	671	626	716
22	896	716	1076

Arithmetic forward contract dynamics. Lucia and Schwartz [2002] investigated one- and two-factor models for the spot price of electricity, and fitted them to Nord Pool data. They studied both arithmetic and geometric models, and concluded that the arithmetic class of models had the best fit to the price observations. In what follows, we will stick to a forward based version of the arithmetic model. The main reason why we exclusively consider this class is the simple relationship between the dynamics of the fixed-delivery and average-based forwards, which becomes useful in empirical analysis of the volatility term structure.

Assume that there exists a continuum of fixed-delivery forwards $f(\tau, u)$, with $\tau \leq u \leq T$. Furthermore, let the forward price dynamics under the risk-neutral probability be given by

$$df(\tau,u)=\sigma(\tau,u)dW(\tau) \tag{13}$$

where $W(t)$ is a standard Brownian motion, and $\sigma(t,u)$ is a deterministic function of time τ, and maturity u of the forward contract. This is an arithmetic model for the forward curve evolution yielding normally distributed forward prices.

Alternatively, we can model the average-based contracts directly. Let the average-based forward contract for $0 \le \tau \le T^s < T^e \le T$ be given by

$$dF(\tau,T^s,T^e)=\Sigma(\tau,T^s,T^e)\,dW(\tau) \tag{14}$$

It is possible to show that (see Theorem 4.3 in Benth and Koekebakker [2005])

$$\Sigma(\tau,T^s,T^e)=\int_{T^s}^{T^e} w(r;t)\,\sigma(\tau;t)dt \tag{15}$$

The forward price models in Equations (13)–(14) describe the stochastic evolution under an equivalent martingale measure, and not under the real world measure where observations are made. Although there may be risk premia in the market that cause forward/futures prices to exhibit non-zero drift terms, the diffusion terms are equal under both measures. Hence, the volatility functions Σ and σ can be estimated from real world data. As noted by Cortazar and Schwartz [1994], this is only strictly correct when observations are sampled continuously. In our analysis, we use daily observations as a proxy to a continuously sampled data set. Let $f(\tau_n, u_m)$ denote the forward price at date τ_n, with maturity at date u_m, where $\tau_n \le u_m$. Our discrete approximations of the models in Equation (13) and Equation (14) are

$$df(\tau_n,u_m)\approx f(\tau_n,u_m)-f(\tau_{n-1},u_m)=x^f_{n,m}$$

and

$$dF\left(\tau_n,T^s_m,T^e_m\right)\approx F\left(\tau_n,T^s_m,T^e_m\right)-F\left(\tau_{n-1},T^s_m,T^e_m\right)=x^F_{n,m}$$

where $n = 1,\dots,N$ and $m = 1,\dots,M$. We construct two different data sets

$$X^i_{(N\times M)}=\begin{bmatrix} x^i_{1,1} & x^i_{1,2} & \cdots & x^i_{1,M} \\ x^i_{2,1} & x^i_{2,2} & \cdots & x^i_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ x^i_{N,1} & x^i_{N,2} & \cdots & x^i_{N,M} \end{bmatrix}$$

where $i = f, F$.

Estimated term structure of volatility. We compute for each column in our data set

$$\hat{\sum}_m=\sqrt{\frac{1}{N-1}\sum_{n=1}^{N}\left(x^F_{n,m}-\bar{x}^F_m\right)^2}$$

for the average-based price data, and

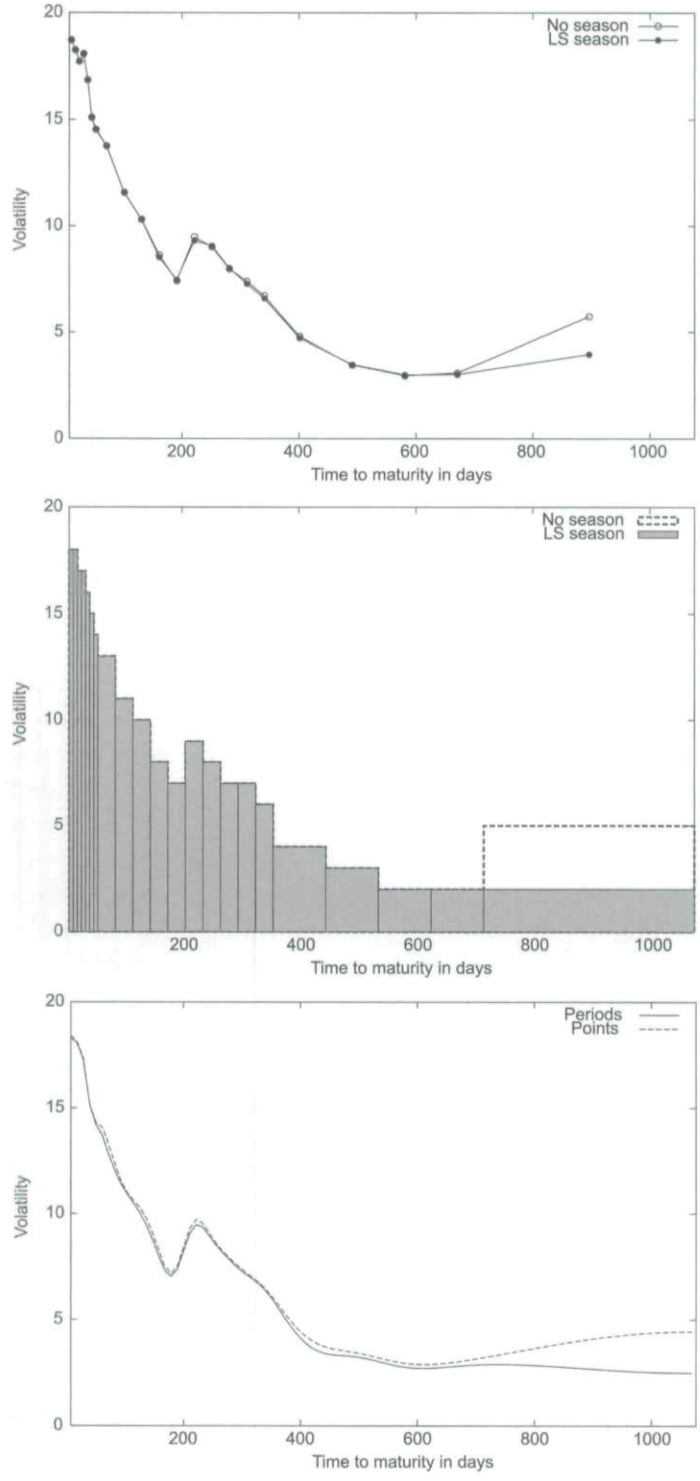
$$\hat{\sigma}_m=\sqrt{\frac{1}{N-1}\sum_{n=1}^{N}\left(x^f_{n,m}-\bar{x}^f_m\right)^2}$$

for the fixed maturity data, where $\bar{x}_m$ denotes the average daily price differences.

In Exhibit 5, we have plotted the estimated volatility curves based on the four different data sets. In the upper panel, the daily volatility based on fixed-delivery seasonal and non-seasonal prices is computed. Perhaps unsurprisingly, the volatility estimates are very similar in the short end of the term structure. Only in the last year do the estimates differ; the data set based on the zero seasonal function produces higher volatility compared to the data set based on the trigonometric seasonal function. Also, note that both volatility curves rise during the last year (albeit this is more pronounced for the data set constructed with zero seasonality). This is hard to justify theoretically. If volatility increases in time to maturity, the underlying stochastic process exhibits some sort of explosive behavior in the sense of becoming more and more volatile. The increase in volatility in the long end for the zero-seasonality data sets is more likely the result of misspecification. Ignoring seasonality results in an unrealistic forward curve in the long end (cf. Exhibit 3). Over time, such an unrealistic curve will necessarily fluctuate quite a lot to fit the seasonal market prices. Therefore, the

Exhibit 5

The upper panel shows the volatility estimates for the fixed-delivery forward price differences. The mid-panel shows the volatility estimates for the average-based contracts. The bottom panel shows a smoothed fixed-delivery volatility curve implied from average-based contracts compared to a smoothed fixed-delivery forward curve computed from fixed-delivery price differences – both using the trigonometric seasonality function.



volatility at the long end is prone to an upward bias for mis-specified volatility.

In the mid-panel, the volatility estimates for the seasonal and non-seasonal data sets of average-based contracts are plotted. We illustrate by using bar plots, where the width of each bar corresponds to the delivery period of that particular contract. Only for the last delivery period is it possible to tell the two estimates apart. The data set created without seasonality shows an increasing tendency, but this is not the case for the data with trigonometric seasonality. It seems that working with average-based contracts directly reduces the problem with seasonality induced volatility bias. This is quite natural, since our data set mimics the actual delivery periods for market traded contracts. In the long end, we only consider the volatility of a yearly average-based contract, not the volatility at a given point on the forward curve in the distant future. The former is naturally less sensitive to the seasonal specification.

We can make a more direct comparison of the $\hat{\Sigma}$ or $\hat{\sigma}$ using the relationship in Equation (15). From our estimate of $\hat{\Sigma}$, we can compute the *implied* volatility for the fixed-delivery forward curve during the delivery period. We will use the approximation $w(r;\ t) \approx \frac{1}{T^e - T^s}$, so that Equation (15) becomes $\Sigma(\tau, T^s, T^e) = \frac{1}{T^e - T^s}\int_{T^s}^{T^e} \sigma(\tau; t)\, dt$. In this model, the volatility of the average-based forwards is simply the average volatility of the fixed-delivery forwards during the delivery period. Assuming that the implied fixed-delivery forward volatility is smooth, we can use our proposed algorithm (with zero seasonality) to compute a smooth volatility curve. Simply replace average prices with average volatility in the derivation in the previous section. The result is a maximum smooth implied volatility function. In the lower plot of Exhibit 5 a volatility curve implied by $\hat{\Sigma}$ and a smoothed version of $\hat{\sigma}$ is plotted. Both curves are based on data sets with trigonometric seasonality. Between one and two years to maturity, the fixed forward approach leads to 5%–10% higher volatility for $\hat{\sigma}$ compared to the estimate implied by $\hat{\Sigma}$. Between two and three years to maturity, this number increases steadily to more than 50% of the volatility estimate implied by the average-based prices.

CONCLUDING REMARKS

In this article, we have proposed an algorithm to extract forward curves from average-based forward prices observed in the market. The method combines a seasonality function with the principle of maximum smoothness used in the interest-rate literature (see Adams and van Deventer [1994]). The seasonality function may be of a parametric form, or given by some bottom-up model (like in Fleten and Lemming [2003]). Motivated by the relation between spot and forward prices, and the market price of risk coming from arbitrage-free pricing principles, we include an adjustment function which measures the deviation between the seasonality function and the actual traded forward prices. Using the principle of maximum smoothness, this becomes a fourth order polynomial spline, ensuring a smooth forward curve, given the choice of seasonality function, which may, in fact, be non-smooth if desirable. The computation of the smooth curve is constrained on the bid-ask spread, to ensure a reasonable interpolation of prices. The algorithm is designed to handle average-based contracts, which may be overlapping in the delivery period. This feature is of importance in seasonal markets like electricity, gas, and weather, say.

We have applied our algorithm to study the smoothing of average-based forward contracts in the Nord Pool electricity market. This market is seasonal, and trades in average-based forward/futures contracts with partly overlapping delivery periods. The first empirical study confirmed that the shape of the smooth curve is dependent on the specification of the seasonality function, in particular for the contracts in the long end of the curve. For contracts with a short time left to maturity, the smoothed curve was much more stable with respect to different choices of seasonality function. We compared a trigonometric seasonality function estimated in Lucia and Schwartz [2002] with a zero seasonal function and that arising from a bottom-up model.

Maybe more interestingly, we applied our algorithm to extract and analyze the volatility term structure. Based on a simple arithmetic model for fixed-delivery and average-based forwards, we extracted the volatility term structure from data constructed based on the smooth forward curves over a long set of daily forward/futures market quotes. Apart from demonstrating the flexibility of the algorithm, this exercise showed that the estimated volatility term structure is dependent on the specification of the seasonality function. This observation was more pronounced for the fixed-delivery model than the average-based model. The study gave some statistical support to use average-based forward models like in Benth and Koekebakker [2005] rather than fixed-delivery models, since the former seems to give more intuitively correct volatility term structures when fitted to data.

Let $q(t) = \varepsilon''(t)$. Our constrained optimization problem can now be reformulated using Lagrange multipliers as

$$\min_{q,\lambda} L(q, \lambda)$$

where

$$\begin{aligned} L(q, \lambda) &= \int_{t_0}^{t_n} q^2(t)\,dt + \int_{t_0}^{t_n} \sum_{i=1}^{m} \lambda_i \mathbb{I}_{\{T_i^s < t < T_i^e\}} \left[\varepsilon(t) - F_i^C + s(t) \right] dt \\ &= \int_{t_0}^{t_n} q^2(t)\,dt + \int_{t_0}^{t_n} \sum_{i=1}^{m} \lambda_i \Big[\mathbb{I}_{\{t > T_i^e\}} \frac{1}{2}(\beta^2 - \alpha^2) q(t) \\ &\quad + \mathbb{I}_{\{t < T_i^s\}} (\alpha - \beta) \int_t^{t_n} q(v)\,dv \\ &\quad + \mathbb{I}_{\{T_i^s < t < T_i^e\}} \left[\frac{1}{2} t^2 q(t) - \frac{1}{2} \alpha^2 q(t) - \beta \int_t^{t_n} q(v)\,dv \right] \\ &\quad - (\alpha - \beta)\varepsilon(t_0) \Big] dt \\ &\quad + \int_{t_0}^{t_n} \sum_{i=1}^{m} \lambda_i \mathbb{I}_{\{T_i^s < t < T_i^e\}} \left[-F_i^C + s(t) \right] dt \end{aligned}$$

Suppose q^* is the solution of the optimization problem. Then

$$\frac{d}{d\varepsilon} L(q^* + \epsilon k) \big|_{\epsilon=0} = 0$$

for any continuos function k on the interval $[t_0, t_n]$ such that $k(t) = q(t) - q^*(t)$. Thus, the first order condition for a minimum becomes

$$\begin{aligned} &\int_{t_0}^{t_n} \Big[2q^*(t) + \sum_{i=1}^{m} \lambda_i \Big(\mathbb{I}_{\{t > T_i^e\}} \frac{1}{2}(\beta^2 - \alpha^2) \\ &\quad + \mathbb{I}_{\{T_i^s < t < T_i^e\}} \frac{1}{2}(t^2 - \alpha^2) \Big) \Big] k(t)\,dt \\ &\quad + \int_{t_0}^{t_n} \left[\sum_{i=1}^{m} \lambda_i \left(\mathbb{I}_{\{t < T_i^s\}} (\alpha - \beta) - \mathbb{I}_{\{T_i^s < t < T_i^e\}} \beta \right) \right] \\ &\quad \left[\int_t^{t_n} k(v)\,dv \right] dt = 0 \end{aligned} \tag{20}$$

Next, we make use of a variant of the lemma in Lim and Xiao [2002].

Lemma 1. *Suppose A and h are continuous functions and B is integrable. Then, for all $a \leq t \leq b$,*

$$\int_a^b A(t)h(t)\,dt + \int_a^b B(t) \int_t^b h(v)\,dv\,dt = 0$$

if and only if

APPENDIX A

FINDING THE SMOOTHEST FUNCTION

We will, in this section, find the smoothest possible adjustment function, $\varepsilon(t)$, that solves the following constrained minimization problem

$$\min_x \int_{t_0}^{t_n} \left[\varepsilon''(t; x) \right]^2 dt$$

subject to Equations (6)–(10). We use a similar approach as Adams and van Deventer [1994] and Lim and Xiao [2002] to find the adjustment function. To shorten our notation, we will use $\alpha = T_i^s$ and $\beta = T_i^e$. The first step of the proof is to use the Lagrange method to find an expression for the solution and then to show that the solution is a polynomial spline of order four.

Using partial integration, we can show that

$$\begin{aligned} \int_\alpha^\beta \varepsilon(t)\,dt &= \beta\,\varepsilon(\beta) - \alpha\,\varepsilon(\alpha) \\ &\quad + \frac{1}{2} \left[\alpha^2 \varepsilon'(\alpha) - \beta^2 \varepsilon'(\beta) + \int_\alpha^\beta t^2 \varepsilon''(t)\,dt \right] \end{aligned} \tag{16}$$

Since $\varepsilon(t) \in C^2$ and $\varepsilon'(t_n) = 0$, we can apply the fundamental theorem of calculus to obtain

$$\varepsilon(x) = \varepsilon(t_0) - \int_{t_0}^{x} \int_t^{t_n} \varepsilon''(v)\,dv\,dt \tag{17}$$

Equivalently,

$$\varepsilon'(x) = \varepsilon'(t_n) - \int_x^{t_n} \varepsilon''(t)\,dt = -\int_x^{t_n} \varepsilon''(t)\,dt \tag{18}$$

Combining Equations (16), (17), and (18), we can write the condition in Equation (10) for $i = 1,\ldots,m$ as

$$\begin{aligned} (\beta - \alpha)F_i^C - \int_\alpha^\beta s(t)\,dt &= (\alpha - \beta)\left[\int_{t_0}^{\alpha} \int_t^{t_n} \varepsilon''(v)\,dv\,dt - \varepsilon(t_0) \right] \\ &\quad - \beta \int_\alpha^\beta \int_t^{t_n} \varepsilon''(v)\,dv\,dt \\ &\quad + \frac{1}{2}(\beta^2 - \alpha^2) \int_\beta^{t_n} \varepsilon''(t)\,dt \\ &\quad - \frac{1}{2}\alpha^2 \int_\alpha^\beta \varepsilon''(t)\,dt + \frac{1}{2} \int_\alpha^\beta t^2 \varepsilon''(t)\,dt \end{aligned} \tag{19}$$

$$A(t) = -\int_a^t B(v)\,dv$$

By using the above lemma together with the first order condition in Equation (20), we get

$$2q^*(t) + \sum_{i=1}^{m} \lambda_i \left[\mathbb{I}_{\{t > T_i^e\}} \frac{1}{2}(\beta^2 - \alpha^2) + \mathbb{I}_{\{T_i^s < t < T_i^e\}} \frac{1}{2}(t^2 - \alpha^2) \right]$$
$$= -\int_{t_0}^{t} \sum_{i=1}^{m} \lambda_i \left[\mathbb{I}_{\{v < T_i^s\}}(\alpha - \beta) - \mathbb{I}_{\{T_i^s < v < T_i^e\}} \beta \right] dv$$

for all $t \in [t_0, t_n]$. After re-arranging, we get

$$q^*(t) = -\frac{1}{4}\sum_{i=1}^{m} \lambda_i t^2 + \frac{1}{2}\int_{t_0}^{t} \sum_{i=1}^{m} \lambda_i \beta\, dv + \frac{1}{4}\sum_{i=1}^{m} \lambda_i \alpha^2$$

for $t \in [\alpha, \beta]$, which proves that $q^*(t)$ is a quadratic polynomial. Using $q(t) = \varepsilon''(t)$, we see that the adjustment function can be written as the following fourth order polynomial spline function

$$\varepsilon(t) = \begin{cases} a_1 t^4 + b_1 t^3 + c_1 t^2 + d_1 t + e_1 & t \in [t_0, t_1] \\ a_2 t^4 + b_2 t^3 + c_2 t^2 + d_2 t + e_2 & t \in [t_1, t_2] \\ \vdots & \\ a_n t^4 + b_n t^3 + c_n t^2 + d_n t + e_n & t \in [t_{n-1}, t_n] \end{cases}$$

Thus, we have derived the adjustment function ε.

ENDNOTES

[1]Our methodology is related to Fleten and Lemming [2003], where they smooth a forward curve based on a bottom-up model called the MPS model see (Botnen et al. [1992]). The MPS model calculates weekly equilibrium prices and production quantities based on fundamental factors for demand and production, e.g., temperature, fuel costs, snow levels, capacities, etc. The MPS model may serve as a seasonal function in our context, however, we are not limited to such a specification. In fact, we may use any seasonality function, which gives much more flexibility for the user, since he may combine the MPS model with modifications as a seasonality function, or directly use any other seasonal function which he prefers. In this way, speculators without profound resource based models may conduct quantitative analyses using only financial market data.

[2]In the empirical calibration, we use the approximation $w(r, t) \approx \frac{1}{T_i^e - T_i^s}$ even though the contracts traded on Nord Pool are marked-to-market in the delivery period. This approximation is very good for reasonable levels of the interest rate (see Lucia and Schwartz [2002]).

[3]Quarterly and yearly contracts are traded in Euro. The NOK conversion is done via the spot NOK/EUR currency rate on the actual day.

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