Do Lead-Lag Effects Affect Derivative Pricing?

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In this article, we address the implementation of pricing models for derivatives whose prices depend on the covariance between asset returns. We show that lead-lag effects can have a strong impact on correlation-dependent derivatives. Simple adjustments to the annualized variance-covariance matrix of finite holding-period returns are derived that account for predictability. To illustrate the practical importance of this, we apply our results to three real-world problems: the valuation of index-linked stock option plans, the valuation of cross-currency derivatives, and the assessment of incentive fees charged by mutual funds.

At first glance, the predictability of an asset's return has no impact on prices of derivatives written on that asset. In the continuous-time, no-arbitrage pricing framework pioneered by Black and Scholes [1973] and Merton [1973], the drift rate of the asset price process under the physical measure does not appear in pricing formulae. Predictability,[1] however, is usually expressed through a particular drift specification.

Lo and Wang [1995] argue convincingly that predictability and drift specification do matter. For a fixed unconditional variance of finite holding-period asset returns, they show that changes in predictability alter the estimated value of the infinitesimal or instantaneous variance parameter σ^2. As σ^2 enters option pricing formulae, predictability has an impact on the option price. Lo and Wang demonstrate that this impact can be strong for realistic return autocorrelations, using the trending Ornstein-Uhlenbeck (O-U) process as a reference model. Their findings have important consequences for the implementation of option pricing models. Predictability must be taken into account when volatility is estimated from a time series of discretely sampled asset prices. In particular, the annualized sample variance of finite holding-period returns is no longer a consistent estimator of the instantaneous variance parameter if asset returns are predictable.

The main contribution of our article is an investigation into the impact of predictability on derivatives whose prices depend on the covariance between asset returns. There is a growing market for such correlation products (see Mahoney [1995] or Fengler and Schwendner [2004]) and we think that the issue of predictability is even more important for these derivatives than for options written on a single asset. There are two main reasons for this importance. First, implied covariance parameters are rarely available due to the lack of other covariance-dependent derivatives. Therefore, we have to rely on time series estimates, where the issue of predictability comes into play. Second, we show that there is a close link between the instantaneous covariance and the cross-autocorrelation of finite holding-period returns. Because lead-lag effects, i.e., non-zero cross-autocorrelations, appear frequently,[2] predictability affects many covariance-dependent derivatives.

The starting point of our analysis is the question of how to obtain the relevant input parameters for derivative pricing. As far as time series estimation is concerned, we have to know how the desired instantaneous parameters are related to the properties of finite holding-period returns. To answer this question, we present a model of the asset price dynamics which is flexible enough to explain important characteristics of asset returns, in particular, the commonly observed lead-lag patterns. To capture realistic lead-lag relations, we generalize the bivariate trending O-U process proposed by Lo and Wang [1995] in a way that allows for general feedback between the two univariate processes. Based on the model presented, we derive the relation between the instantaneous variance-covariance parameters and the variance-covariance matrix of finite holding-period returns. The understanding of this relation is useful for three reasons. First, we can investigate if and how instantaneous parameters and prices of derivatives are affected by different lead-lag patterns for otherwise fixed moments of finite holding-period returns. In particular, such an analysis helps us to identify and understand situations in which price effects are very large. Second, once we have established the link between finite holding-period and instantaneous variances and covariances, we can switch easily, and consistently, from one time scale to another without ignoring lead-lag effects. This switching is extremely useful if asset prices are observed on some frequency, e.g., daily, but for valuation purposes, another discretization is more appropriate. Third, we can analyze how the bias of the annualized sample variance-covariance matrix, as an estimator of the instantaneous parameters, alters with changing lead-lag patterns. The latter point raises the question of whether adjustments to the annualized sample variance-covariance matrix exist that use only autocorrelations and cross-autocorrelations and are, therefore, easy to implement. For a case that is important in practice, we explicitly derive such adjustments. We then apply these adjustments to three problems, namely, the valuation of index-linked stock option plans; the valuation of cross-currency derivatives; and the assessment of incentive fees charged by mutual funds. In the context of these problems, we provide numerical examples and case studies that stress the importance of lead-lag effects for derivative pricing. In the last section, we summarize the main results and give our conclusions.

DERIVATIVE PRICING AND VARIANCE-COVARIANCE PARAMETERS

If we assume competitive, frictionless, and arbitrage-free markets, in which securities trade in continuous time, we can easily derive a valuation equation for the price F of a derivative instrument whose pay-off depends solely on asset prices and time t. Let P and S be the prices of two traded assets whose log-price processes satisfy the following bivariate stochastic differential equation:

$$\begin{aligned} d\log(P(t)) &\equiv dp(t) = \mu_p(\cdot)dt + \sigma_p dW_p \\ d\log(S(t)) &\equiv ds(t) = \mu_s(\cdot)dt + \sigma_s dW_s \end{aligned} \tag{1}$$

The drift coefficients $\mu_p(\cdot)$ and $\mu_s(\cdot)$ can be functions of P, S, t, or the prices of other traded assets.[3] The diffusion coefficients $\sigma_p \geq 0$ and $\sigma_s \geq 0$ are constants, and W_p and W_s are two standard Wiener processes such that $dW_p\, dW_s = \rho_{ps} dt$. Let us further assume a known, constant, risk-free interest rate r.

Under our assumptions, we know from the Black-Scholes analysis that $F = F(P, S, t)$ is the solution to the partial differential equation

$$\begin{aligned} \frac{\partial F}{\partial t} + rP\frac{\partial F}{\partial P} + rS\frac{\partial F}{\partial S} + \frac{1}{2}\sigma_p^2\, P^2\frac{\partial^2 F}{\partial P^2} \\ + \frac{1}{2}\sigma_s^2 S^2\frac{\partial^2 F}{\partial S^2} + \sigma_p\sigma_s\rho_{ps} PS\frac{\partial^2 F}{\partial P \partial S} = rF \end{aligned} \tag{2}$$

subject to some appropriate terminal and boundary conditions. Of the coefficients in Equation (1), only the variances σ_p^2, σ_s^2 and the covariance $\sigma_p\sigma_s\rho_{ps}$ seem to affect the derivatives' prices, since the drift coefficients do not appear in the valuation equation. If we know σ_p^2, σ_s^2 and $\sigma_p\sigma_s\rho_{ps}$, we can solve for F, depending on the pay-off structure of the derivative instrument, either analytically or numerically. However, an important question is how to obtain the parameter values.

In principle, there are two estimation approaches. The first one is to imply the variance-covariance parameters from other derivative prices. This approach is not feasible if there are no liquid markets for other derivatives whose theoretical prices also depend on σ_p^2, σ_s^2, and

$\sigma_p\sigma_s\rho_{ps}$. In particular, implied estimates of the covariance will often not be available.

In the following analysis, we concentrate on the second approach, which is to estimate the parameters from the time series data of asset prices. Assume that we observe prices P and S at n equally spaced intervals of length τ over the period $[0,T]$. Hence, we observe $P(l\tau)$, $S(l\tau)$, with $l = 0, \ldots, n$ and $T = n\tau$. Let us further define continuously compounded returns as $\Delta p_k \equiv \log(P(k\tau)/P((k-1)\tau))$ and $\Delta s_k \equiv \log(S(k\tau)/S((k-1)\tau))$, $k = 1, \ldots, n$. A natural, easily calculable estimator of σ_p^2, σ_s^2 and $\sigma_p\sigma_s\rho_{ps}$ is the sample variance-covariance matrix of returns, divided by τ.

$$\begin{aligned}
\hat{\sigma}_p^2 &\equiv \frac{1}{n\tau}\sum_{k=1}^{n}\left(\Delta p_k - \frac{1}{n}\sum_{j=1}^{n}\Delta p_j\right)^2 \\
\hat{\sigma}_s^2 &\equiv \frac{1}{n\tau}\sum_{k=1}^{n}\left(\Delta s_k - \frac{1}{n}\sum_{j=1}^{n}\Delta s_j\right)^2 \\
\widehat{\sigma_s\sigma_p\rho_{ps}} &\equiv \frac{1}{n\tau}\sum_{k=1}^{n}\left(\Delta p_k - \frac{1}{n}\sum_{j=1}^{n}\Delta p_j\right)\left(\Delta s_k - \frac{1}{n}\sum_{j=1}^{n}\Delta s_j\right)
\end{aligned} \tag{3}$$

The estimator defined by Equations (3) is the maximum likelihood estimator if the vectors $(\Delta p_k, \Delta s_k)$, $k = 1, \ldots, n$, are IID normal random variables. The latter condition holds by construction when $\tau \to 0$. It also holds for an arbitrary interval length τ if the drift coefficients $\mu_p(\cdot)$ and $\mu_s(\cdot)$ in Equation (1) are constant over time, i.e., the log-price process follows a bivariate Brownian motion with drift.

However, the bivariate Brownian motion model is not compatible with both lead-lag effects and autocorrelated returns. If we observe lead-lag patterns between finite holding-period returns, it is unclear how the estimates obtained from Equations (3) relate to the desired parameters. In order to analyze this relation, and to derive simple modifications to estimator (3), we have to specify a more flexible reference model of the price dynamics.

A TWO-FACTOR LINEAR DIFFUSION MODEL

A reference model of price dynamics should be sophisticated enough to generate realistic lead-lag patterns, while being simple enough to be analytically tractable. The following variant of Equation (1), with explicit drift specification, is a natural candidate:

$$\begin{aligned}
dp(t) &= (-\alpha_p p(t) - \alpha_s s(t))\,dt + \sigma_p\,dW_p \\
ds(t) &= (-\beta_p p(t) - \beta_s s(t))\,dt + \sigma_s\,dW_s
\end{aligned} \tag{4}$$

where α_p, α_s, β_p, and β_s are constant parameters.

Model (4) should be understood as the detrended version of the more general trending bivariate O-U log-price process.[4] In fact, (4) is a bivariate extension of the univariate trending O-U log-price process analyzed by Lo and Wang [1995]. It encompasses familiar models as special cases. For example, with $\alpha_p = \alpha_s = \beta_p = \beta_s = 0$, we obtain the standard model for plain vanilla option pricing, a (correlated) bivariate Brownian motion with drift. Two (correlated) trended O-U processes result if $\alpha_s = \beta_p = 0$, $\alpha_p > 0$, and $\beta_s > 0$. In general, we allow for feedback between $p(t)$ and $s(t)$ through the drift specification, i.e., both α_s and β_p can be non-zero.[5] As the special cases show, model (4) generates both difference-stationary and trend-stationary processes, depending on the parameter values.

The stochastic differential Equations (4) are not well defined for all values of the parameters α_p, α_s, β_p, and β_s. For example, when $\alpha_s = \beta_p = \beta_s = 0$, we require that α_p is not negative. Otherwise, the $p(t)$ process would be explosive in the sense that the impact of past innovations on current prices increases with time, compared, for e.g., to an O-U process, where the impact decreases, or a Brownian motion, where the impact is constant.

In order to characterize the range of sensible parameter values, we express the processes $p(t)$ and $s(t)$ as linear combinations of two simpler processes, $x(t)$ and $y(t)$:

$$\begin{aligned}
p(t) &\equiv x(t) + a\cdot y(t) \\
s(t) &\equiv b\cdot x(t) + y(t)
\end{aligned} \tag{5}$$

where a and b are scalars, and $x(t)$ and $y(t)$ satisfy the following stochastic differential equations:

$$\begin{aligned}
dx(t) &= -\gamma_x x(t)\,dt + \sigma_x\,dW_x \\
dy(t) &= -\gamma_y y(t)\,dt + \sigma_y\,dW_y
\end{aligned} \tag{6}$$

with $\gamma_x \geq 0$, $\gamma_y \geq 0$, $\sigma_x \geq 0$, $\sigma_y \geq 0$, and $dW_x dW_y = \rho_{xy}dt$. Thus, the $x(t)$ and $y(t)$ processes are either O-U processes for positive γ-parameters or Brownian motions for zero γ-parameters. From representation (5) of $p(t)$ and $s(t)$, we can directly exclude explosive processes by demanding $\gamma_x \geq 0$ and $\gamma_y \geq 0$.[6]

In the analysis that follows, we can either work with the parameter set $\{\alpha_p, \alpha_s, \beta_p, \beta_s, \sigma_p, \sigma_s, \rho_{ps}\}$ or the parameter set $\{a, b, \gamma_x, \gamma_y, \sigma_x, \sigma_y, \rho_{xy}\}$.[7] The advantage of the former set is the more intuitive parameter interpretation. However, the latter parametrization leads to more parsimonious expressions for the moments of the processes $p(t)$ and $s(t)$.

Whatever parametrization is used, our reference model (4) is consistent with very different cross-autocorrelation patterns. Let us illustrate three insightful cases.

- The choice of $\alpha_s = \beta_p = 0$, $\alpha_p > 0$, and $\beta_s > 0$ implies that $p(t)$ and $s(t)$ are (correlated) O-U processes. Since both α_s and β_p equal zero, there is no feedback in the drift. Given a positive instantaneous correlation, both first order return cross-autocorrelations, $\rho^{\Delta}_{p_{-1}s} \equiv Cor(\Delta p_{-1}, \Delta s)$ and $\rho^{\Delta}_{ps_{-1}} \equiv Cor(\Delta p, \Delta s_{-1})$,[8] are negative due the negative autocorrelations of the O-U processes.
- The parameter choice $\alpha_p = \beta_p = 0$, $\alpha_s < 0$ and $\beta_s > 0$ describes a situation where $s(t)$ is an O-U process but $p(t)$ is a non-stationary process. $p(t)$ has a stochastic drift rate $\alpha_s s(t)$. This feature leads to a positive correlation between Δs_{-1} and Δp.
- Another important case, $\alpha_p > 0$, $\alpha_s < 0$ and $\beta_p = \beta_s = 0$, can be interpreted as a situation with two non-stationary processes and partial feedback: $s(t)$ is a Brownian motion, but $p(t)$ adapts to $s(t)$ because of an error-correction term $-\alpha_p(p(t) - \frac{-\alpha_s}{\alpha_p} s(t))$. The two processes are co-integrated. Obviously, $\rho^{\Delta}_{p_{-1}s} = 0$ and the error-correction term induces a positive lead of the $s(t)$ process.

DRIFT SPECIFICATION, VARIANCE-COVARIANCE PARAMETERS, AND SIMPLE ADJUSTMENTS

We are now prepared to analyze the relation between the instantaneous variance-covariance parameters σ_p^2, σ_s^2, $\sigma_p\sigma_s\rho_{ps}$, the estimates obtained from Equations (3), and the variance-covariance matrix of finite holding-period returns. If more and more data points become available,[9] estimator (3) converges to the unconditional variances $Var[\Delta p]$, $Var[\Delta s]$, and the covariance $Cov[\Delta p, \Delta s]$, divided by τ. Thus, if we establish the link between the unconditional moments of finite holding-period returns and the instantaneous variance-covariance parameters under the price dynamics of model (4), we can study the impact of different lead-lag patterns not only on the instantaneous variance-covariance parameters, but also on the asymptotic behavior of estimator (3).

Finite Holding-Period Returns and Instantaneous Variance-Covariance Parameters

Equations (5) lead to

$$\begin{pmatrix} Var[\Delta p] \\ Var[\Delta s] \\ Cov[\Delta p, \Delta s] \end{pmatrix} = \begin{pmatrix} 1 & a^2 & 2a \\ b^2 & 1 & 2b \\ b & a & (1+ab) \end{pmatrix} \begin{pmatrix} Var[\Delta x] \\ Var[\Delta y] \\ Cov[\Delta x, \Delta y] \end{pmatrix} \quad (7)$$

where $Var[\Delta x]$ and $Var[\Delta y]$ are the unconditional variances of changes of x and y, respectively, over an interval of length τ, and $Cov[\Delta x, \Delta y]$ is the corresponding covariance. In the following analysis, $A = A(a, b)$ denotes the 3×3 matrix on the right-hand side of Equation (7). Under the O-U processes (6), the unconditional variances and covariance amount to:[10]

$$\begin{aligned}
Var[\Delta x] &= \frac{1 - e^{-\gamma_x \tau}}{\gamma_x \tau} \sigma_x^2 \tau \equiv g^x(\gamma_x, \tau)\sigma_x^2 \tau \\
Var[\Delta y] &= \frac{1 - e^{-\gamma_y \tau}}{\gamma_y \tau} \sigma_y^2 \tau \equiv g^y(\gamma_y, \tau)\sigma_y^2 \tau \\
Cov[\Delta x, \Delta y] &= \frac{2 - e^{-\gamma_x \tau} - e^{-\gamma_y \tau}}{(\gamma_x + \gamma_y)\tau} \sigma_x \sigma_y \rho_{xy} \tau \\
&\equiv g^{xy}(\gamma_x, \gamma_y, \tau)\sigma_x \sigma_y \rho_{xy} \tau
\end{aligned} \quad (8)$$

By substituting the expressions on the right hand side of Equations (8) for $Var[\Delta x]$, $Var[\Delta y]$, and $Cov[\Delta x, \Delta y]$ in Equation (7), by using the relation $(\sigma_x^2, \sigma_y^2, \rho_{xy}\sigma_x\sigma_y)^T = A^{-1}(\sigma_p^2, \sigma_s^2, \rho_{ps}\sigma_p\sigma_s)^T$, and by dividing both sides by τ, we obtain:

$$\begin{pmatrix} \dfrac{Var[\Delta p]}{\tau} \\ \dfrac{Var[\Delta s]}{\tau} \\ \dfrac{Cov[\Delta p, \Delta s]}{\tau} \end{pmatrix} = A \begin{pmatrix} g^x & 0 & 0 \\ 0 & g^y & 0 \\ 0 & 0 & g^{xy} \end{pmatrix} A^{-1} \begin{pmatrix} \sigma_p^2 \\ \sigma_s^2 \\ \rho_{ps}\sigma_p\sigma_s \end{pmatrix} \quad (9)$$

Equation (9) provides the required relation between the annualized variance-covariance matrix of finite holding-period returns and the instantaneous variance-covariance parameters. For fixed variances and covariance of finite holding-period returns, Equation (9) shows how different lead-lag patterns change the instantaneous variances and covariance through g^x, g^y, g^{xy}, and A. Therefore, we can study how derivatives' prices are affected by predictability.

Moreover, Equation (9) helps us to switch from one time scale to another, while taking into account lead-lag effects. Suppose that you want to value a long-term (10-year) American option whose payoff depends on the two asset prices. Further assume that you have daily return data that deliver estimates of the unconditional daily return variances and covariance. For valuation, however, you want to use a binomial model with weekly time steps. To implement the model, you could use your daily moments and calculate the instantaneous ones via Equation (9). Then you could change the direction, start with the instantaneous moments, and obtain the weekly ones via Equation (9). Of course, the knowledge of the parameters a, b, γ_x, and γ_y that influence g^x, g^y, g^{xy}, and A is needed to do this. However, as we show below, there exists a simple way to obtain the required inputs for important special cases.

Finally, Equation (9) shows how the asymptotic properties of estimator (3) are affected by predictability. As we see from Equation (9), the components of estimator (3) are, in general, linear combinations of all three instantaneous parameters. Thus, there will be a bias, even asymptotically.[11] This bias disappears only when $g^x = g^y = g^{xy} = 1$, i.e., when the length τ of the sampling interval tends to zero, or when $\gamma_x = \gamma_y = 0$ and model (4) is reduced to a bivariate Brownian motion with drift. We note that A, g^x, g^y, and g^{xy} depend on a, b, γ_x, γ_y, and τ only, and are independent of the instantaneous variance-covariance parameters σ_p, σ_s, and ρ_{ps} themselves.

Simple Correction Formulae

In the analysis to follow, we derive simple adjustments for autocorrelations and lead-lag patterns to the finite holding-period annualized variance-covariance matrix as an estimator of the instantaneous parameters. These adjustments provide an easily accessible alternative to the more demanding estimation of the full model (4).[12] Moreover, as the adjustments depend on return cross-correlations and autocorrelations only, we can gain some insight into the effect of predictability on a particular derivative pricing problem simply by looking at sample moments.

We first determine the first order autocovariances $Cov(\Delta p, \Delta p_{-1})$ and $Cov(\Delta s, \Delta s_{-1})$, as well as the cross-autocovariances $Cov(\Delta p, \Delta s_{-1})$ and $Cov(\Delta s, \Delta p_{-1})$.

From model (5), it follows:

$$\begin{pmatrix} Cov[\Delta p, \Delta p_{-1}] \\ Cov[\Delta s, \Delta s_{-1}] \\ Cov[\Delta p, \Delta s_{-1}] \\ Cov[\Delta s, \Delta p_{-1}] \end{pmatrix} = \begin{pmatrix} 1 & a^2 & a & a \\ b^2 & 1 & b & b \\ b & a & 1 & ab \\ b & a & ab & 1 \end{pmatrix} \begin{pmatrix} Cov[\Delta x, \Delta x_{-1}] \\ Cov[\Delta y, \Delta y_{-1}] \\ Cov[\Delta x, \Delta y_{-1}] \\ Cov[\Delta y, \Delta x_{-1}] \end{pmatrix} \tag{10}$$

where $Cov[\Delta x, \Delta x_{-1}]$, $Cov[\Delta y, \Delta y_{-1}]$, $Cov[\Delta x, \Delta y_{-1}]$, $Cov[\Delta y, \Delta x_{-1}]$ are the first order autocovariances and cross-autocovariances of Δx and Δy.

Furthermore, under the O-U processes x and y in Equations (6), we can derive the autocovariances and cross-autocovariances on the right-hand side of Equation (10):

$$\begin{aligned} Cov[\Delta x, \Delta x_{-1}] &= -\frac{[1-e^{-\gamma_x \tau}]^2}{2\gamma_x}\sigma_x^2 \\ Cov[\Delta y, \Delta y_{-1}] &= -\frac{[1-e^{-\gamma_y \tau}]^2}{2\gamma_y}\sigma_y^2 \\ Cov[\Delta x, \Delta y_{-1}] &= -\frac{[1-e^{-\gamma_x \tau}]^2}{\gamma_x+\gamma_y}\sigma_x\sigma_y\rho_{xy} \\ Cov[\Delta y, \Delta x_{-1}] &= -\frac{[1-e^{-\gamma_y \tau}]^2}{\gamma_x+\gamma_y}\sigma_x\sigma_y\rho_{xy} \end{aligned} \tag{11}$$

In general, the four moments on the right hand side of Equation (10) depend non-linearly on the parameter set $\{a, b, \gamma_x, \gamma_y, \sigma_x, \sigma_y, \rho_{xy}\}$ and, thus, also on the parameter set $\{\alpha_p, \alpha_s, \beta_p, \beta_s, \sigma_p, \sigma_s, \rho_{ps}\}$. However, for special cases, we are able to determine the parameters a, b, γ_x, and γ_y explicitly from these moments. Since a, b, γ_x, and γ_y determine g^x, g^y, g^{xy}, and A, they suffice to construct simple correction formulae for the usual variance-covariance estimate in terms of "observable" variables.

We can derive simple correction formulae for three special cases. The first one refers to feedback parameters α_s and β_p equal to zero in model (4). This parameter restriction leads to two correlated O-U processes without feedback in the drift. Simple correction formulae also exist for a symmetric, co-integrated system where both feedback parameters are non-zero.[13] The third case, which we study in more detail, states that one process influences the drift rate of the other, but not vice versa, i.e., we have partial feedback. We consider this third case as most relevant in practice. For example, one could think of the asset that is influenced in the drift rate as a less liquid one compared to the other.

Formally, this third case corresponds to parameter values $\alpha_p > 0$, $\alpha_s < 0$, and $\beta_p = \beta_s = 0$. Thus, s follows a Brownian motion, while the drift of the p process has an error-correction term $-\alpha_p(p(t) - \frac{-\alpha_s}{\alpha_p} s(t))$. In terms of model (5), these restrictions transfer to $b = 0$, $\gamma_y = 0$, $\gamma_x = \alpha_p$, and $a = -\alpha_s/\alpha_p$, which shows directly that p and s are co-integrated. In this case, Equation (9) reduces to:

$$\begin{pmatrix} \sigma_p^2 \\ \sigma_s^2 \\ \rho_{ps}\sigma_p\sigma_s \end{pmatrix} = \begin{pmatrix} \frac{1}{g^x} & a^2(1-\frac{1}{g^x}) & 0 \\ 0 & 1 & 0 \\ 0 & a(1-\frac{1}{g^x}) & \frac{1}{g^x} \end{pmatrix} \begin{pmatrix} \frac{Var[\Delta p]}{\tau} \\ \frac{Var[\Delta s]}{\tau} \\ \frac{Cov[\Delta p, \Delta s]}{\tau} \end{pmatrix} \tag{12}$$

By stating the standardized cross-autocovariance

$$\rho^{\Delta}_{xy_{-1}} \equiv \frac{Cov[\Delta x, \Delta y_{-1}]}{Cov[\Delta x, \Delta y]} = \frac{Cov[\Delta p, \Delta s_{-1}]}{Cov[\Delta p, \Delta s] - aVar[\Delta s]} \tag{13}$$

in terms of moments of asset returns, we can derive the following expressions for g^x and a:[14]

$$g^x = \frac{\rho^{\Delta}_{xy_{-1}}}{ln(\rho^{\Delta}_{xy_{-1}} + 1)} \tag{14}$$

$$a = \frac{Cov[\Delta p, \Delta p_{-1}]}{Cov[\Delta p, \Delta s_{-1}]} + \sqrt{\frac{Cov[\Delta p, \Delta p_{-1}]^2}{Cov[\Delta p, \Delta s_{-1}]^2} - 2\frac{Cov[\Delta p, \Delta p_{-1}]Cov[\Delta p, \Delta s]}{Cov[\Delta p, \Delta s_{-1}]Var[\Delta s]} + \frac{Var[\Delta p]}{Var[\Delta s]}} \tag{15}$$

Equation (12) shows how to adjust the usual variance-covariance estimator for autocorrelation and lead-lag patterns. Through (14) and (15) we are able to express this adjustment in terms of moments of finite holding-period returns only. First, we do not need any adjustment for the estimate of the variance σ_s^2. This is not surprising since s follows a Brownian motion. Second, to analyze the adjustment for the estimate of σ_p^2, we rearrange the correction formula to be:

$$\begin{aligned}\sigma_p^2 &= \frac{ln(\rho^{\Delta}_{xy_{-1}} + 1)}{\rho^{\Delta}_{xy_{-1}}}\frac{Var[\Delta p]}{\tau} \\ &\quad + a^2\left(1 - \frac{ln(\rho^{\Delta}_{xy_{-1}} + 1)}{\rho^{\Delta}_{xy_{-1}}}\right)\frac{Var[\Delta s]}{\tau} \\ &= \frac{Var[\Delta p]}{\tau} + D\left(\frac{ln(\rho^{\Delta}_{xy_{-1}} + 1)}{\rho^{\Delta}_{xy_{-1}}} - 1\right)\end{aligned} \tag{16}$$

where $D \equiv \frac{Var[\Delta p]}{\tau} - a^2\frac{Var[\Delta s]}{\tau}$ is the difference between the annualized variances of Δp and $a\Delta s$. Since $\frac{ln(\rho^{\Delta}_{xy_{-1}}+1)}{\rho^{\Delta}_{xy_{-1}}} - 1 \leq 0$, Equation (16) shows that the usual estimate for σ_p^2 understates the true value when $D > 0$, but overstates it when $D < 0$. The more negative $\rho^{\Delta}_{xy_{-1}}$ becomes, the more severe the estimation error. A strong lead of s, and similar values of the variance of Δs and the covariance between Δs and Δp, indicate such situations.

We can derive a similar correction formula for the covariance term. Rearranging Equation (12) leads to:

$$\begin{aligned}\rho_{ps}\sigma_p\sigma_s &= a\left(1 - \frac{ln(\rho^{\Delta}_{xy_{-1}} + 1)}{\rho^{\Delta}_{xy_{-1}}}\right)\frac{Var[\Delta s]}{\tau} \\ &\quad + \frac{ln(\rho^{\Delta}_{xy_{-1}} + 1)}{\rho^{\Delta}_{xy_{-1}}}\frac{Cov[\Delta p, \Delta s]}{\tau} \\ &= \frac{Cov[\Delta p, \Delta s]}{\tau} + \bar{D}\left(\frac{ln(\rho^{\Delta}_{xy_{-1}} + 1)}{\rho^{\Delta}_{xy_{-1}}} - 1\right)\end{aligned} \tag{17}$$

where $\bar{D} \equiv \frac{Cov[\Delta p, \Delta s]}{\tau} - a\frac{Var[\Delta s]}{\tau}$. Again, the above equation shows that the usual estimate for $\rho_{ps}\sigma_p\sigma_s$ understates the true value when $\bar{D} > 0$ and overstates it when $\bar{D} < 0$. As before, the estimation error increases when the standardized cross-autocovariance $\rho^{\Delta}_{xy_{-1}}$ becomes more negative.

In contrast to the case with no feedback, an estimation error can occur even though asset returns are not autocorrelated. To see this, note that two parameter sets can lead to an autocorrelation ρ_p^Δ of zero. The first is $\gamma_x = 0$. This means that both processes follow a Brownian motion and therefore estimator (3) is adequate.

More interesting is the second parameter set, where ρ_p^Δ becomes zero also for $\gamma_x > 0$. Then, $\sigma_x = -2a\sigma_y\rho_{xy}$ or, equivalently, $\sigma_p = a\sigma_s$. In this case, the usual estimate for σ_p^2 is adequate, since D in Equation (16) equals zero.

However, the usual covariance estimate can seriously differ, even asymptotically, from the true parameter value. Exhibit 1 shows the asymptotic limit of the estimator for the correlation coefficient ρ_{ps} as a function of $\rho_{xy_{-1}}^\Delta$. The true parameters are $\sigma_p^2 = \sigma_s^2 = 0.3$, $a = 1$, and $\rho_{ps} = -1$, -0.5, 0, or 0.5. Thus, $\sigma_p = a\sigma_s$ and the autocorrelation ρ_p^Δ is equal to zero, although α_p (and therefore γ_x) is not. Since we fixed $a = 1$, we can state the adjustment for the estimate of $\sigma_p\sigma_s\rho_{ps}$ completely in terms of $\rho_{xy_{-1}}^\Delta$.

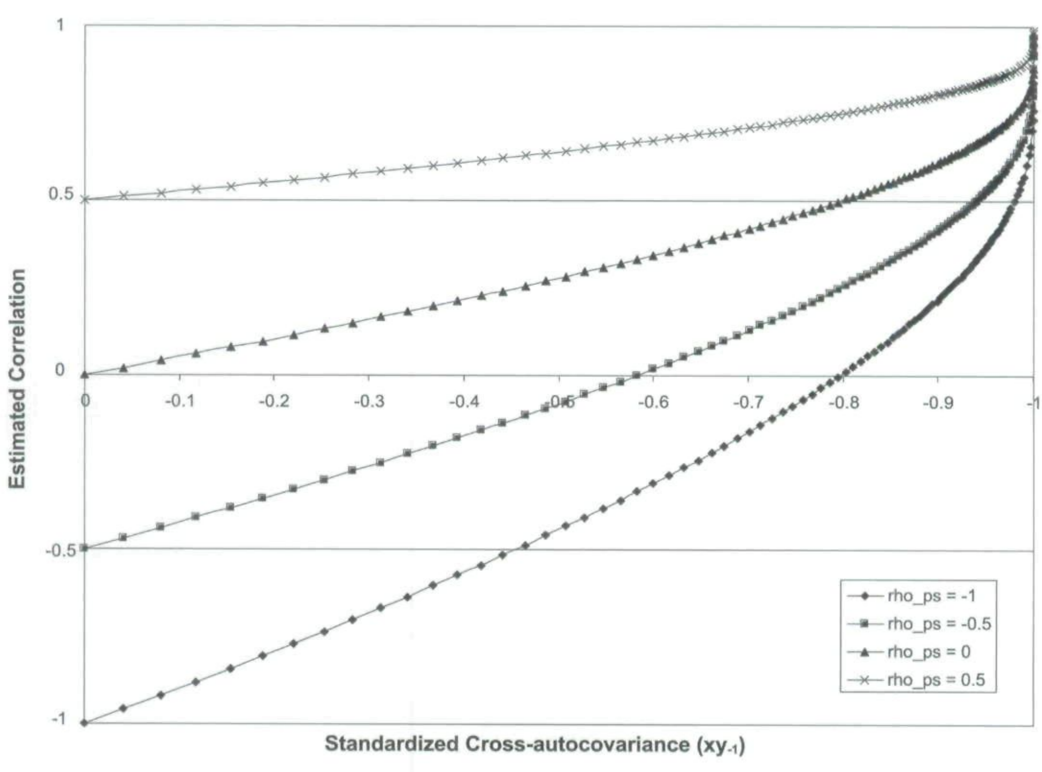
Due to the lead-lag patterns, we obtain an asymptotic bias when estimating the instantaneous correlation via estimator (3). The stronger the lead-lag relation, the more the estimated correlation coefficient overstates the true value. When $\rho_{xy_{-1}}^\Delta$ converges towards -1, which implies that the error-correction term leads to a very fast adaptation of p towards s, the estimated correlation coefficient tends towards 1, regardless of what the true value is. The intuition behind this result is that estimator (3) does not distinguish between price changes due to the error-correction term and those which are due to innovations, according to the Brownian motion. Therefore, we erroneously interpret the error correction as a very high instantaneous correlation, although the true correlation can take any value.

The analysis in this section demonstrates that changes in return autocorrelations and lead-lag patterns can strongly affect the instantaneous variance-covariance parameters. Next, we present different applications in order to highlight the impact on derivatives whose prices

EXHIBIT 1

Estimated Correlation ρ_{ps}

This exhibit shows the asymptotic limit of the usual estimator for the correlation coeffient ρ_{ps} under partial feedback as a function of $\rho_{xy_{-1}}^\Delta$. The chosen parameter values are $\sigma_p^2 = \sigma_s^2 = 0.3$, $a = 1$, and ρ_{ps} is either -1, -0.5, 0, or 0.5.



depend on the covariance between the returns on two assets.

APPLICATIONS

Derivatives depending on at least two traded assets arise in various contexts. Mahoney [1995] provides several common forms of such derivatives. In this section, we analyze three forms in more detail: index-linked stock option plans, cross-currency derivatives, and incentive fees of mutual funds. We show how the results of the previous section can be used to assess the importance of predictability for a certain valuation problem and how to come up with appropriate values of the instantaneous variance-covariance parameters.

Index-linked Stock Option Plans

Usually, a stock option gives an employee the right to buy a certain number of company shares at a so-called grant price within a certain time period. Historically, it was common in the U.S. to set the grant price equal to the market price of the stock on the date the employee received the options ($P(0)$). The main reason for the widespread use of such at-the-money call options lay in U.S. accounting rules that qualified such stock option plans as "fixed plans." For fixed plans, only the intrinsic value, which is zero for an at-the-money option, had to be expensed when the option was granted.[15]

However, since accounting rules have recently moved towards "fair value accounting," [16] it is likely that incentive issues will play a more important role for the design of stock option plans in the future. One indication is the situation in Germany, where accounting rules did not favor "fixed plans" in the past. Actually, we observe many index-linked plans in Germany. Index-linked plans require that the employee benefits from the plan only if the company's share price outperforms a certain index, i.e., there is less danger for windfall profits due to a general upward trend in the market.[17]

A usual way to determine the grant price in the case of an index-linked option plan is to multiply the share price $P(T)$ at the exercise date by the ratio of one plus the percentage index performance and one plus the percentage performance of the own stock. Thus, when exercising the option in T, the grant price[18] is

$$P(T)\frac{1+\frac{S(T)-S(0)}{S(0)}}{1+\frac{P(T)-P(0)}{P(0)}}=S(T)\frac{P(0)}{S(0)} \qquad (18)$$

where $S(0)$ and $S(T)$ denote the index levels at times 0 and T, respectively. For ease of exposition, assume that the current index level equals the current stock price. In this case, the exercise value for one share is

$$\max(0,\ P(T)-S(T)) \qquad (19)$$

which is exactly the pay-off from the option to exchange one risky asset for another.

Numerical examples. First, we provide the results of a comparative static analysis, which shows the quantitative effect of different lead-lag patterns on the value of a "stylized" index-linked employee's stock option. In order to value such an exchange option, we employ the option pricing formula derived by Margrabe [1978]. Of course, the Margrabe formula does not incorporate many features of specific stock option plans.[19] However, as our focus is on the analysis of the impact of predictability on derivatives whose prices depend on the covariance between two asset returns, it is a natural first step to gain some insight from a stylized plan with closed-form solutions. To highlight the impact of cross-autocorrelations, as opposed to autocorrelations, we chose the case of partial feedback and a zero return autocorrelation of both processes.

The three panels of Exhibit 2 show prices of exchange options for times to maturity of one, five, and 10 years, respectively. We fixed both the annualized finite holding-period return variances $Var[\Delta p^*]/\tau$ and $Var[\Delta s^*]/\tau$ at the same value of 0.1456. This value was chosen to ease comparison with Table I of Lo and Wang [1995]. The annualized covariance was fixed at $Cov[\Delta p^*, \Delta s^*]/\tau=0.0728$, which implies a correlation of $\rho^{\Delta}_{ps}=0.5$. Holding these moments fixed, we vary the cross-autocorrelation of finite holding-period returns $\rho^{\Delta}_{ps_{-1}}$ within the feasible parameter range of zero to 0.5. From Equations (12), (13), (14), and (15), we then obtain the instantaneous parameters σ^2_p, σ^2_s, and $\rho_{ps}\sigma_p\sigma_s$, which serve as inputs to the pricing model by Margrabe [1978]. Option prices are provided for a current share price of $P(0)=40$ and current index levels $S(0)$ of 40, 45, or 50.

We see that for all maturities and all current index levels, the option price rises with the cross-autocorrelation. As we have partial feedback with zero return

EXHIBIT 2

Option Prices for Different Cross-Autocorrelations

This exhibit shows option prices according to the model by Margrabe [1978] as functions of the finite holding-period return cross-autocorrelation $\rho^{\Delta}_{ps_{-1}}$. The current share price is always $P(0) = 40$, the current index level $S(0)$ is either 40, 45, or 50, and the time to maturity is either one year, five years, or 10 years. Variances and covariance of finite holding-period returns are fixed at $Var[\Delta p^*]/\tau = 0.1456$, $Var[\Delta s^*]/\tau = 0.1456$, and $Cov[\Delta p^*, \Delta s^*]/\tau = 0.0728$.

$S(0)$	$\rho^{\Delta}_{ps_{-1}}$						
	0	0.05	0.1	0.2	0.3	0.4	0.45
	Time to Maturity: One year						
40	6.052	6.210	6.388	6.828	7.456	8.532	9.590
45	4.230	4.391	4.571	5.019	5.663	6.774	7.873
50	2.922	3.071	3.241	3.667	4.288	5.380	6.477
	Time to Maturity: Five years						
40	13.214	13.542	13.910	14.811	16.078	18.200	20.200
45	11.665	12.010	12.396	13.345	14.680	16.913	19.031
50	10.338	10.693	11.092	12.071	13.453	15.775	17.982
	Time to Maturity: Ten years						
40	18.148	18.571	19.042	20.185	21.763	24.314	26.619
45	16.864	17.310	17.808	19.015	20.683	23.382	25.821
50	15.723	16.188	16.707	17.966	19.709	22.534	25.090
ρ_{ps}	0.5	0.473	0.442	0.361	0.236	-0.006	-0.279

autocorrelations and equal variances, $a = \sqrt{Var[\Delta p^*]}/\sqrt{Var[\Delta s^*]} = 1$. In this case, we know from the earlier sub-section on simple correction formulae that both σ_p^2 and σ_s^2 will be unaffected by changes of $\rho^{\Delta}_{ps_{-1}}$. Thus, the lead of Δs^* affects option prices solely through the instantaneous correlation ρ_{ps}. The last row of the exhibit shows the values of ρ_{ps} that correspond to different cross-autocorrelations. Though the correlation of simultaneous finite holding-period returns is as high as 0.5, the instantaneous correlation can become negative when the cross-autocorrelation exceeds values of about 0.4. When we look from the perspective of parameter estimation, we see again that the correlation of finite holding-period returns is an estimator of the instantaneous correlation that might not even give us the correct sign. In this context, we note that the properties of ρ^{Δ}_{ps} as an estimator of ρ_{ps} are independent of the length of the return interval τ, provided that the annualized variances $Var[\Delta p^*]/\tau$, $Var[\Delta s^*]/\tau$, and the covariance $Cov[\Delta p^*, \Delta s^*]/\tau$, are fixed.

A time to maturity of one year allows us to compare the results with those reported by Lo and Wang ([1995], Table 1, p. 98), which looked at the impact of return autocorrelation on the price of vanilla call options. For at-the-money options ($S(0) = 40$), a cross-autocorrelation of 0.45 increases the price of an exchange option by more than 58%, compared with the case of zero cross-autocorrelation. Even for $\rho^{\Delta}_{ps} = 0.2$, we still have a price increase of 12%. Lo and Wang report a price increase of 50% when the autocorrelation is –0.45, and an increase of 11% for an autocorrelation of –0.2. Therefore, the lead-lag effects are even stronger than the already strong autocorrelation effects reported by Lo and Wang. When we look at current index levels of 45 or 50, i.e., where the stock has to increase more than the index to receive a payment from the option plan, the relative impact of cross-autocorrelations on option prices is even more pronounced than in the base case of at-the-money options.

The results for the more realistic maturities of five and 10 years are provided in the next two panels of Exhibit 2. In absolute terms, the longer the time to expiration, the more strongly the option prices are affected by cross-autocorrelations. This result is intuitive, since the option's "vega" increases with the expiration date. When option prices are measured in percentage points of the price that prevails under zero cross-autocorrelation, the price impact of an increasing $\rho^{\Delta}_{ps_{-1}}$ even declines slightly with time to maturity.

The SAP long term incentive plan 2000. As a further illustration of our results, we look at the stock option plan launched by the German software firm SAP in 2000. According to this plan, a total amount of up to 6,250,000 call options on SAP shares[20] can be distributed among certain employees. These options are American-style options with a time to expiration of 10 years, but there are freeze-out periods of either two, three, or four years, before an option can be exercised. One option entitles the holder to buy one share for a grant price, which is equal to the then prevailing share price minus the out-performance of the SAP share over the GSTI Software Index. The exact terms lead to formula (18) for the grant price and we can, therefore, interpret the stock option as an option to exchange one risky asset (the index) for another (the SAP share), with a normalized index such that the index level, when converted into Euro, equals the share price at time 0. Shares needed when options are exercised can either be own shares or new shares. However, we assume that no new shares are issued, which means that we need not consider dilution effects. The stock option plan was accepted in a general meeting of the shareholders on January 18, 2000.

Imagine that, on December 31, 1999, one wanted to value an option according to the SAP suggestions in order to assist the shareholders with their decision on the plan and for accounting purposes.[21] Furthermore, imagine that one used daily log-returns of the SAP stock and the GSTI Software Index over the preceding 250-day period to estimate the instantaneous variances and the correlation. The first column of Exhibit 3 shows estimation results according to estimator (3). As expected, the SAP share has a higher volatility than the index ($\sigma_p > \sigma_s$), and we find a significant positive correlation coefficient. Given the standard errors of the parameter estimates, estimation errors seem to be a minor point for a sample size of 250.

Based on these parameter estimates, option prices are provided for freeze-out periods of two, three, and four years, using the XETRA closing price of €161.67 of one SAP share on December 31, 1999. For both the SAP share and the GSTI software index, a constant dividend yield of 1% per year is assumed. Due to the American feature of the options, option prices are obtained numerically. As Exhibit 3 shows, longer freeze-out periods lower option values only slightly.

The results in the first column of Exhibit 3 should be interpreted with caution, as the simple parameter estimates ignore both autocorrelations and lead-lag effects of asset returns. Exhibit 4 presents the estimated return autocorrelations and cross-autocorrelations for the stock and the index. The horizontal lines provide the boundaries of a 95% confidence interval. We find no significant autocorrelations, but a significant positive lead of the index, i.e., we observe a correlation of 19% between stock returns and index returns lagged by one day. All other cross-autocorrelations are insignificant. An explanation for the lead of the index could be that information about the software sector is primarily generated in the U.S. Since the index is dominated by U.S. firms, this information shows up in the index immediately, while producing only a delayed effect on firms in other regions.

Autocorrelations and cross-autocorrelations suggest a situation of partial feedback, which we analyzed in the sub-section on simple correlation formulae. This finding allows us to apply the corresponding correction formulae (12). We use only significant moments for this correction, i.e., all non-significant moments are set equal to zero. The second column of Exhibit 3 shows the corrected variance-covariance parameter estimates and option prices.[22] Since we have no autocorrelation and no lead of the SAP stock, the variances σ_p^2 and σ_s^2 remain unchanged compared to the results of estimator (3). However, the instantaneous correlation ρ_{ps} decreases substantially from 0.315 to 0.19. This bias correction is much larger than the average sampling error, which is quantified by the standard error of 0.054. Moreover, the bias does not even disappear asymptotically with an increasing number of daily observations. Due to the lower instantaneous correlation, option prices increase by about 6.7% from almost €92 to almost €98.

In summary, the results of the comparative static analysis and the SAP example show that substantial errors might occur when we ignore lead-lag effects in the assessment of a stock option plan. In the example of the SAP plan, such errors are easily detected and corrected by means of sample moments.

EXHIBIT 3

Parameter Estimates and Theoretical Values of SAP Stock Options

This exhibit shows parameter estimates resulting from estimator (3), corrected estimates obtained from Equations (12), and the corresponding theoretical prices (in €) of one SAP stock option with 10 years to maturity on December 31, 1999, when the stock price was €161.67. Option prices are given for freeze-out periods of 2, 3, and 4 years.

	Simple Parameter Estimates: Equations (3) (standard errors)	Corrected Parameter Estimates: Equations (12) (standard errors)
σ_p	0.503 (0.034)	0.503 (0.034)
σ_s	0.407 (0.020)	0.407 (0.020)
ρ_{ps}	0.315 (0.054)	0.190 (0.074)
Option Price 2 years	91.80	97.94
Option Price 3 years	91.78	97.92
Option Price 4 years	91.71	97.84

Cross-Currency Derivatives

Another class of contracts whose prices depend on the correlation between two assets are cross-currency derivatives, or quantos. The payoff of such derivatives is defined in terms of an asset price measured in one currency, but the payoff is (fully or partly) made in a second currency. For example, Benninga, Björk, and Wiener [2002] consider the valuation of an option on a stock denominated in U.K. pounds with a strike price in U.S. dollars. Other cross-currency derivatives, which are especially interesting with respect to covariance estimation, are quanto forwards on stock indexes. Denote by $S(0)$ the current index level, e.g., the level of the Brazilian BOVESPA index, by $S(T)$ the index level at maturity of the forward, and by F the forward price measured in Brazilian real. If the payoff is made in U.S. dollars, the payoff of the quanto forward equals

$$(S(T) - F)\,dollars \tag{20}$$

Further, denote by $P(t)$ the time t Brasilian real per U.S. dollar exchange rate and assume P and S satisfy Equations (1). If the forward contract is fully dividend protected, no-arbitrage valuation delivers the following forward price:

$$F = S(0)exp\{(r_f + \rho_{ps}\sigma_p\sigma_s)T\} \tag{21}$$

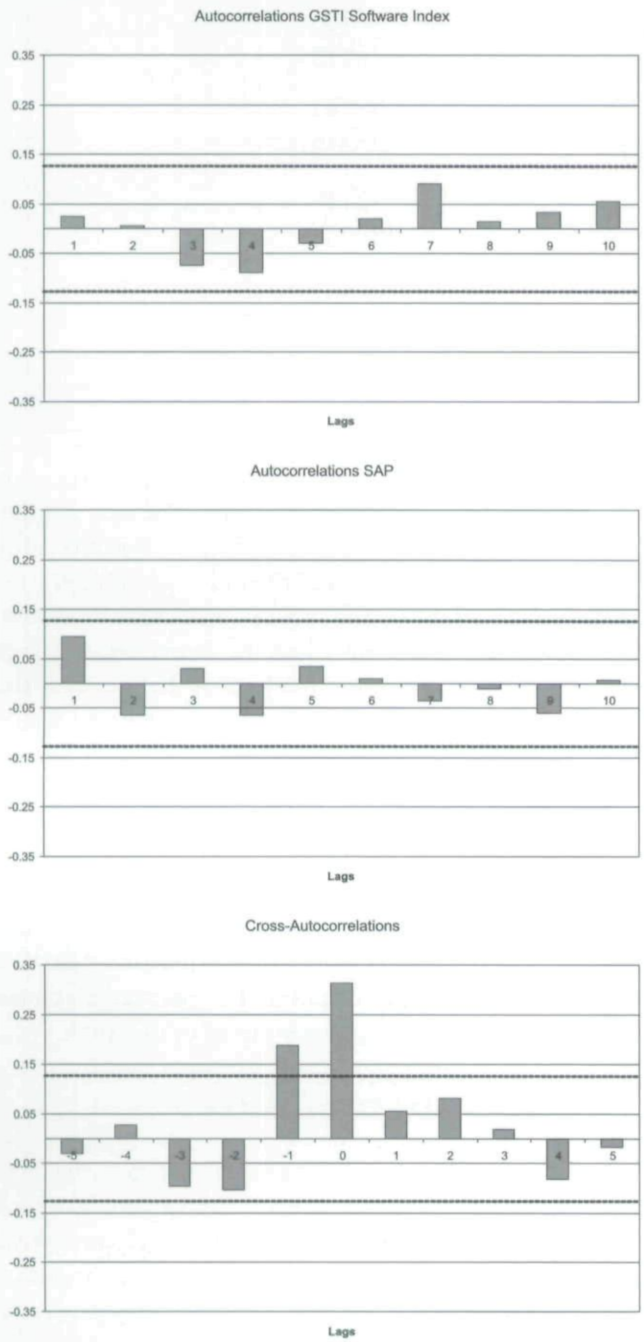
where r_f denotes the risk-free interest rate in Brazil, σ_s is the volatility of the log-index process, σ_p is the volatility of the log-value process of one U.S. dollar in Brazilian real, and ρ_{ps} denotes the correlation between the two processes. According to Equation (21), the forward price depends on the current index level, the risk-free interest rate in Brazil, the instantaneous covariance between index and exchange rate, and the time to maturity. Since the covariance is the only input that is not observable, covariance estimation is crucially important for the valuation of the contract.

Quantos on the BOVESPA index. Assume that we are currently at the end of the year 2006 and want to value quantos on the major Brazilian stock index BOVESPA,

Exhibit 4

Daily Return Autocorrelations and Cross-Autocorrelations of SAP Stocks and the GSTI Software Index

This exhibits shows estimated autocorrelations and cross-autocorrelations based on 250 daily observations from the period preceding December 31, 1999. Autocorrelations are provided for up to 10 lags and cross-autocorrelations for up to five lags. Negative lags for the cross-atutocorrelations refer to lagged returns of the GSTI Software Index and positive lags to lagged returns of SAP stocks. The horizontal lines are the boundaries of a 95% confidence interval.



with payments made in U.S. dollars. For this purpose, we use daily data on the index and the Brazilian real–U.S. dollar exchange rate over the year 2006. The resulting simple estimates, according to estimator (3), are provided in the first column of Exhibit 5. Note that the contemporaneous correlation between relative changes in the index and the exchange rate is strongly negative, indicating that positive (negative) news on the Brazilian stock market is usually accompanied by an appreciation (depreciation) of the real. This strong negative correlation translates into a covariance of –1.35%, i.e., the discount rate that determines the forward price according to Equation (21) should be 135 basis points below the risk-free interest rate.

However, the above analysis ignores lead-lag effects. An inspection of the cross-autocorrelations shows that the index leads the exchange rate by one day, with a cross-autocorrelation coefficient of –0.347. One possible explanation is that the development of the stock market provides important information on the economic situation of a country which, in turn, has an impact on the exchange rate. Note that other autocorrelations and cross-autocorrelations are not significant. Therefore, we again observe a situation of partial feedback and can use Equations (12) to correct our parameter estimates. Column two of Exhibit 5 provides the corrected values. The instantaneous correlation between the index and exchange rate has been reduced substantially in absolute terms from –0.413 to –0.264. This change translates into a change of the covariance from –1.35% to –0.86%. Therefore, ignoring lead-lag effects would result in an underestimation of the discount rate by 49 basis points, which is clearly economically significant.

Incentive Fees of Mutual Funds

Incentive fees, which make management compensation a function of performance, are often used to compensate investment managers. For example, managers of hedge funds frequently receive about 20% of the excess return above a benchmark. However, hedge funds usually employ either a value of zero or a cash rate of return as the benchmark. Therefore, correlation between fund and benchmark does not play an important role.

Things are different for mutual funds. Under U.S. regulation, incentive fees must take the form of a so called "fulcrum fee." The fee must be centered around an index, i.e., it is an index-linked fee. In practice, these incentive fees always have an upper and lower limit on size, which

EXHIBIT 5

Parameter Estimates and Quanto Forward Price Adjustments

This exhibit shows parameter estimates resulting from estimator (3), the corrected estimates obtained from Equations (12), and the corresponding discount rates from Equation (21). The data consists of daily returns on the BOVESPA Index and the relative changes of the value of U.S. $1 in Brazilian real over the year 2006.

	Simple Parameter Estimates: Equations (3) (standard errors)	Corrected Parameter Estimates: Equations (12) (standard errors)
σ_p	0.136	0.136
	(0.013)	(0.013)
σ_s	0.241	0.241
	(0.013)	(0.013)
ρ_{ps}	-0.413	-0.264
	(0.060)	(0.066)
$\rho_{ps}\rho_p\sigma_s$	-1.35%	-0.86%
Discount Rate	r_f - 135 BP	r_f - 86 BP

introduces option features.[23] Therefore, variance-covariance parameters influence the value of fees. In other countries, where regulation is different, index-linked fees can take the form of standard exchange options.

DWS Type-O Funds. As an example of mutual funds with incentive fees, we use the so called type-O family of DWS, an investment company that belongs to Deutsche Bank Group. Type-O funds are no load funds, but charge substantial incentive fees that are linked to different benchmark indexes. Consider, for example, the type-O fund that invests in European stocks. Essentially, the fund's yearly incentive fee equals up to 25% of the excess performance over the MSCI Europe index within a year.[24] Therefore, one fourth of the price of a one-year European option to exchange the benchmark index for the assets of the fund is a good approximation for the present value of the incentive fee. Since the fund retains dividends and the benchmark index is a total return index, we do not need dividend yields for the valuation of the exchange option. The only parameters that we have to specify are the volatilities and the correlation between fund returns and index returns.

Exhibit 6 provides the necessary input parameters for the valuation of the exchange option and the present value of the incentive fee. The parameter estimates are based on daily index returns and relative changes in the net asset value of the fund over the year 2006. The first column shows the estimation results according to Equations (3). As expected, the volatilities of the index and the fund are very similar and the instantaneous correlation is positive. However, there also exists a high positive cross-autocorrelation of 0.39 that refers to a lead of the index. If this cross-autocorrelation is considered, we obtain the results in the second column of Exhibit 6, which are based on the correction formulae (12). Note that neither return series contains significant autocorrelations. A comparison between the first and second columns of Exhibit 6 shows that the correction for the lead of the index return reduces the instantaneous correlation substantially which, as a result, leads to a higher value of the incentive fee. The difference equals 56 basis points per year, a substantial effect, especially for large investments in the fund.

SUMMARY AND CONCLUSION

Lo and Wang [1995] have shown that the predictability of an asset's return has an impact on derivative prices since, for a fixed variance of finite-holding period returns, predictability changes the instantaneous variance. We extend Lo and Wang's analysis by considering

EXHIBIT 6

Parameter Estimates and Incentive Fees

This exhibit shows parameter estimates resulting from estimator (3), the corrected estimates obtained from Equations (12), and the present value of the yearly incentive fee charged by the DWS European Stocks Type-O Fund (in percent of the net asset value). The data consists of daily returns of the fund and the MSCI Europe index over the year 2006.

	Simple Parameter Estimates: Equations (3) (standard errors)	Corrected Parameter Estimates: Equations (12) (standard errors)
σ_p	0.153 (0.012)	0.153 (0.012)
σ_s	0.152 (0.010)	0.152 (0.010)
ρ_{ps}	0.485 (0.067)	0.036 (0.249)
Incentive Fee (% of Asset Value)	1.54%	2.11%

derivatives that also depend on the covariance of two assets' returns and by looking at lead-lag effects as a frequently observed form of return predictability.

As a starting point of our analysis, we propose a bivariate linear diffusion model which generates a variety of lead-lag patterns. Based on our two-factor diffusion model, we establish the link between the annualized variance-covariance matrix of finite holding-period returns and the instantaneous variance-covariance matrix. The former will, in general, be a biased estimate of the latter. In particular, the correlation between finite holding-period returns might not even provide the right sign of the instantaneous correlation. To allow for a lead-lag structure of returns, we derive adjustments to the annualized variance-covariance matrix of finite holding-period returns. These adjustments only depend on return variances, covariances, autocovariances, and cross-autocovariances and are, therefore, easy to implement.

Our results imply that lead-lag effects can have a strong impact on correlation-dependent derivatives. To illustrate this point, we discussed some examples of index-linked stock option plans, cross-currency derivatives, and incentive fees of mutual funds. We find that, in all cases, our simple correction formulae were applicable. Moreover, the price effects have been substantial. For the SAP stock option plan, we see that the correction formulae lead to a value that is increased by about 6.7%. The relevant discount rate that determines the forward price of a quanto forward on the Brazilian BOVESPA index increases by 49 basis points, and the yearly performance fee of the DWS type-O European equity fund was judged to be 56 basis points higher using the corrected parameter values.

There are still other applications of our results. Derivatives prices have frequently been used to extract implied volatilities in order to forecast future volatilities. This idea has been extended to correlation forecasts by Siegel [1997]; Campa and Chang [1998]; Bodurtha and Shen [1999]; and Walter and Lopez [2000]. However, our analysis shows why and how the implied (instantaneous) parameters should be adjusted in the presence of lead-lag effects to predict variances and covariances over finite holding periods. In conclusion, we believe that a consideration of lead-lag patterns is relevant for many different problems related to correlation-dependent derivatives.

Appendix 1

In order to see how the restrictions $\gamma_x \geq 0$ and $\gamma_y \geq 0$ translate into restrictions on α_p, α_s, β_p, and β_s, we express γ_x and γ_y as functions of α_p, α_s, β_p, and β_s. Using matrix notation, we equate the drift rates of $p(t)$ and $s(t)$ given by the two parameterizations (4) and (5) and obtain

$$\begin{pmatrix} -\alpha_p & -\alpha_s \\ -\beta_p & -\beta_s \end{pmatrix}\begin{pmatrix} p(t) \\ s(t) \end{pmatrix} = \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}\begin{pmatrix} -\gamma_x & 0 \\ 0 & -\gamma_y \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \tag{22}$$

Substitution of the expressions on the right hand side of system (5) for $p(t)$ and $s(t)$ on the left hand side of Equation (22) leads to

$$\begin{pmatrix} -\alpha_p & -\alpha_s \\ -\beta_p & -\beta_s \end{pmatrix}\begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix} = \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}\begin{pmatrix} -\gamma_x & 0 \\ 0 & -\gamma_y \end{pmatrix} \tag{23}$$

Equation (23) is a non-linear system of equations we have to solve for a, b, γ_x, and γ_y, which can be equivalently written as follows:

$$-\alpha_p - b\alpha_s = -\gamma_x \tag{24}$$

$$\beta_p + b(\beta_s - \alpha_p) - b^2\alpha_s = 0 \tag{25}$$

$$-\beta_s - a\beta_p = -\gamma_y \tag{26}$$

$$\alpha_s + a(\alpha_p - \beta_s) - a^2\beta_p = 0 \tag{27}$$

From Equations (24) to (27) we obtain four solutions $(b_1, \gamma_{x1}, a_1, \gamma_{y_1})$, $(b_1, \gamma_{x_1}, a_2, \gamma_{y_2})$, $(b_2, \gamma_{x_2}, a_1, \gamma_{y_1})$, and $(b_2, \gamma_{x2}, a_2, \gamma_{y2})$, with $b_{1,2}$, $\gamma_{x1,2}$, $a_{1,2}$, $\gamma_{y1,2}$ given below for the case of $\alpha_s \neq 0$, $\beta_p \neq 0$ and $(\alpha_p - \beta_s)^2 + 4\beta_p\alpha_s \geq 0$:

$$b_{1,2} = \frac{\beta_s - \alpha_p}{2\alpha_s} \pm \sqrt{\frac{(\beta_s - \alpha_p)^2}{4\alpha_s^2} + \frac{\beta_p}{\alpha_s}} \tag{28}$$

$$-\gamma_{x_{1,2}} = \frac{-(\alpha_p + \beta_s)}{2} \pm \sqrt{\frac{(\beta_s - \alpha_p)^2}{4} + \alpha_s\beta_p} \tag{29}$$

$$a_{1,2} = \frac{\alpha_p - \beta_s}{2\beta_p} \pm \sqrt{\frac{(\alpha_p - \beta_s)^2}{4\beta_p^2} + \frac{\alpha_s}{\beta_p}} \tag{30}$$

$$-\gamma_{y_{1,2}} = \frac{-(\alpha_p + \beta_s)}{2} \pm \sqrt{\frac{(\alpha_p - \beta_s)^2}{4} + \alpha_s\beta_p} \tag{31}$$

Not all four solutions are appropriate in the sense of being useful models. First, consider the case $\alpha_s\beta_p > 0$. In this case, we obtain

$$\begin{aligned} 4a_1b_1\alpha_s\beta_p = &\left[(\beta_s - \alpha_p) + \sqrt{(\beta_s - \alpha_p)^2 + 4\alpha_s\beta_p}\right] \\ &\left[(\alpha_p - \beta_s) + \sqrt{(\beta_s - \alpha_p)^2 + 4\alpha_s\beta_p}\right] \\ &= (\beta_s - \alpha_p)(\alpha_p - \beta_s) + (\beta_s - \alpha_p)^2 + 4\alpha_s\beta_p \\ &= 4\alpha_s\beta_p \end{aligned}$$

which implies $a_1b_1 = 1$. Similar calculations show that $a_2b_2 = 1$. These cases are degenerate ones where $p(t)$ and $s(t)$ are identical up to a scale factor. Thus, only the solutions $(b_1, \gamma_{x_1}, a_2, \gamma_{y_2})$ and $(b_2, \gamma_{x_2}, a_1, \gamma_{y_1})$ correspond to a non-degenerate case. For these solutions, it holds that $\min[2\gamma_{x_1}, 2\gamma_{y_2}] = \min[2\gamma_{x_2}, 2\gamma_{y_1}] = (\alpha_p + \beta_s) - \sqrt{(\alpha_p - \beta_s)^2 + 4\alpha_s\beta_p}$.

Second, when $\alpha_s\beta_p < 0$, we find that $a_1b_2 = a_2b_1 = 1$. Consequently, we need only to consider the solutions $(\beta_1, \gamma_{x_1}, \alpha_1, \gamma_{y_1})$ and $(\beta_2, \gamma_{x_2}, \alpha_2, \gamma_{y_2})$. Of the parameters α_s and β_p one is positive and one negative, resulting in opposite signs in front of the square root terms in Equations (29) and (31). This implies that $\min[2\gamma_{x_1}, 2\gamma_{y_1}] = \min[2\gamma_{x_2}, 2\gamma_{y_2}] = (\alpha_p + \beta_s) - \sqrt{(\alpha_p - \beta_s)^2 + 4\alpha_s\beta_p}$.

Therefore, we have shown that for both γ_x and γ_y to be non-negative, we require

$$(\alpha_p + \beta_s) - \sqrt{(\alpha_p - \beta_s)^2 + 4\alpha_s\beta_p} \geq 0 \tag{32}$$

Note that the same condition (32) must hold when either one or both parameters α_s or β_p are zero.

Finally, we equate the diffusion coefficients of $p(t)$ and $s(t)$ given by the two parameterizations (4) and (5), which leads to

$$\begin{pmatrix} \sigma_p^2 \\ \sigma_s^2 \\ \sigma_p\sigma_s\rho_{ps} \end{pmatrix} = \begin{pmatrix} 1 & a^2 & 2a \\ b^2 & 1 & 2b \\ b & a & (1+ab) \end{pmatrix}\begin{pmatrix} \sigma_x^2 \\ \sigma_y^2 \\ \sigma_x\sigma_y\rho_{xy} \end{pmatrix} \tag{33}$$

Together with Equations (28) to (31), Equation (33) completely describes the relation between the two parameter sets $\{\alpha_p, \alpha_s, \beta_p, \beta_s, \sigma_p, \sigma_s, \rho_{ps}\}$ and $\{a, b, \gamma_x, \gamma_y, \sigma_x, \sigma_y, \rho_{xy}\}$.

APPENDIX 2

To express the adjustment function (12) in terms of moments of finite holding-period returns, we only need expressions for a and g^x. First, from the moments in Equations (8) and (11), we obtain:

$$\rho^{\Delta}_{xy_{-1}} \equiv \frac{Cov[\Delta x, \Delta y_{-1}]}{Cov[\Delta x, \Delta y]} = -(1 - e^{-\gamma_x \tau}) \tag{34}$$

and

$$\rho^{\Delta}_{x} \equiv \frac{Cov[\Delta x, \Delta x_{-1}]}{Var[\Delta x]} = -\frac{(1 - e^{-\gamma_x \tau})}{2} \tag{35}$$

Therefore,

$$\gamma_x = -\frac{ln(\rho^{\Delta}_{xy_{-1}} + 1)}{\tau} \tag{36}$$

and

$$g^x = \frac{\rho^{\Delta}_{xy_{-1}}}{ln(\rho^{\Delta}_{xy_{-1}} + 1)} \tag{37}$$

which yields to

$$\begin{pmatrix} \sigma_p^2 \\ \sigma_s^2 \\ \rho_{ps}\sigma_p\sigma_s \end{pmatrix} = \begin{pmatrix} \frac{ln(\rho^{\Delta}_{xy_{-1}}+1)}{\rho^{\Delta}_{xy_{-1}}} & a^2\left(1 - \frac{ln(\rho^{\Delta}_{xy_{-1}}+1)}{\rho^{\Delta}_{xy_{-1}}}\right) & 0 \\ 0 & 1 & 0 \\ 0 & a\left(1 - \frac{ln(\rho^{\Delta}_{xy_{-1}}+1)}{\rho^{\Delta}_{xy_{-1}}}\right) & \frac{ln(\rho^{\Delta}_{xy_{-1}}+1)}{\rho^{\Delta}_{xy_{-1}}} \end{pmatrix} \begin{pmatrix} \frac{Var[\Delta p]}{\tau} \\ \frac{Var[\Delta s]}{\tau} \\ \frac{Cov[\Delta p, \Delta s]}{\tau} \end{pmatrix} \tag{38}$$

Second, by stating the first order standardized cross-autocovariance

$$\rho^{\Delta}_{xy_{-1}} = \frac{Cov[\Delta p, \Delta s_{-1}]}{Cov[\Delta p, \Delta s] - aVar[\Delta s]} \tag{39}$$

as well as the first order autocorrelation

$$\rho^{\Delta}_{x} = \frac{Cov[\Delta p, \Delta p_{-1}] - aCov[\Delta p, \Delta s_{-1}]}{Var[\Delta p] + a^2 Var[\Delta s] - 2aCov[\Delta p, \Delta s]} \tag{40}$$

in terms of moments of asset returns by inverting the systems (7) and (10), we obtain the following quadratic equation in a from Equations (34) and (35):

$$\frac{Cov[\Delta p, \Delta s_{-1}]}{Cov[\Delta p, \Delta s] - aVar[\Delta s]} = 2\frac{Cov[\Delta p, \Delta p_{-1}] - aCov[\Delta p, \Delta s_{-1}]}{Var[\Delta p] + a^2 Var[\Delta s] - 2aCov[\Delta p, \Delta s]} \tag{41}$$

Within our model, Equation (41) has the unique[25] solution

$$a = \frac{Cov[\Delta p, \Delta p_{-1}]}{Cov[\Delta p, \Delta s_{-1}]} + \sqrt{\frac{Cov[\Delta p, \Delta p_{-1}]^2}{Cov[\Delta p, \Delta s_{-1}]^2} - 2\frac{Cov[\Delta p, \Delta p_{-1}]Cov[\Delta p, \Delta s]}{Cov[\Delta p, \Delta s_{-1}]Var[\Delta s]} + \frac{Var[\Delta p]}{Var[\Delta s]}} \tag{42}$$

ENDNOTES

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[1]There are various possible reasons for predictability. In any case, predictability can be perfectly consistent with market efficiency (see for e.g., LeRoy [1989]). This issue was also discussed in the context of co-integration (see for e.g., Dwyer and Wallace [1992]).

[2]Lead-lag patterns of financial asset returns have been documented in many studies. Cross-autocorrelations between different stocks or stock portfolios are reported by Lo and MacKinlay [1990]; Conrad, Kaul, and Nimalendran [1991] and Badrinath, Kale, and Noe [1995]; studies which find lead-lag patterns for international stock markets include Eun and Shim [1989]; Arshanapalli and Doukas [1993]; Richards [1995]; Soydemir [2000]; and Masih and Masih [2001]. Lead-lag relations between stock indices and stock index futures are investigated by Kawaller, Koch, and Koch [1987]; Stoll and Whaley [1990]; and Chan [1992]. Kwan [1996] documents that stock returns lead yield changes of bonds issued by the same firm. Baillie and Bollerslev [1989] find a co-integration vector in a

set of exchange rates, which also implies some form of lead-lag relation.

[3]Restrictions are imposed only by the regularity conditions which ensure the existence of a solution to the stochastic differential Equation (1).

[4]For our purposes, it suffices to look at the detrended process, as a deterministic time trend has no impact on return variances, covariances, autocorrelations, and cross-autocorrelations.

[5]Model (4) is more general than the bivariate trending O-U process of Lo and Wang [1995], in which only one of the parameters can be non-zero.

[6]Appendix 1 shows how the restrictions, $\gamma_x \geq 0$ and $\gamma_y \geq 0$, on the system of Equations (6) translate into restrictions on α_p, α_s, β_p, and β_p.

[7]The formal relation between the two parameter sets is elaborated in Appendix 1.

[8]Note that, from now on, we will skip the index k when we deal with unconditional moments, as the unconditional return distribution is identical for all $k \in (1, \ldots, n)$.

[9]Formally, we lengthen the data period $[0, T]$, but fix the length τ of the sampling interval. Hence, the number of observations n increases.

[10]A derivation can be obtained from the authors upon request.

[11]This is a crucial difference to the estimation risk investigated by Fengler and Schwendner [2004].

[12]Model (4) can be estimated by different methods. As the transition density functions are available in closed form, maximum likelihood estimation is feasible (see Lo [1988] for a discussion). Alternatively, GMM estimation can be applied based on conditional and unconditional moments (see for e.g., Hansen and Scheinkmann [1995] and Singleton [2001]).

[13]The correction formulae for the first two cases, together with a derivation, can be obtained from the authors upon request.

[14]See Appendix 2 for a derivation.

[15]See APB Opinion No. 25.

[16]See SFAS No. 123: Share-based Payment, and IFRS 2: Share-based Payment.

[17]For a discussion of the advantages of index-linked plans with respect to incentive effects, see Johnson and Tian [2000].

[18]In some cases, this grant price is adjusted with some prespecified factor.

[19]Typically, stock options are American-style options and the employee is required to hold the option during the so-called "freeze-out" period before exercising. In addition, dividends must be considered, as well as a possible dilution from exercising the options. Another aspect that we do not discuss here is the question of whether the stock option should be worth less than the market value for the employee, given that he might be unable to hold a well-diversified portfolio or hedge the option dynamically.

[20]Originally, the options were written on preferred stock, but in 2001, SAP converted all of its preferred stock to common stock.

[21]According to the International Financial Reporting Standard 2, the "fair value" of a stock option plan should either be determined from market prices or "using a valuation technique to estimate what the price of those equity instruments would have been on the measurement date in an arm's length transaction between knowledgeable, willing parties. The valuation technique shall be consistent with generally accepted methodologies for pricing financial instruments … ."

[22]As our correction formulae build on a closed form method of moments estimator, we obtain estimates of all free model parameters and can easily calculate standard errors.

[23]See Elton, Gruber, and Blake [2003] for a description of fulcrum fees and an analysis of their effects on the behavior of mutual fund managers.

[24]The actual calculation of the fee is a bit more complicated, since it is done on a daily basis to cope with daily inflows and outflows of capital.

[25]The solution with the negative root is inconsistent with $\gamma_x > 0$.

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