

Calibration Risk for Exotic Options

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Equity derivative pricing models are calibrated to market data of plain vanilla options by minimization of an error functional. From the economic viewpoint, there are several possibilities to measure the error between the market and the model. These different specifications of the error give rise to different sets of calibrated model parameters and the resulting prices of exotic options vary significantly.

We provide evidence for this calibration risk in a time series of DAX implied volatility surfaces from April 2003 to March 2004. We analyze factors influencing these price differences for exotic options in the Heston and in the Bates models and recommend an error functional. Moreover, we determine the model risk of these two stochastic volatility models for the time series and compare it to calibration risk.

Recently, there has been considerable interest, both from a practical and a theoretical point of view, in the risks involved in option pricing. Schoutens et al., [2004] have analyzed model risk in an empirical study and Cont [2004] has put this risk into a theoretical framework. Another source of risk is hidden in the calibration of models to market data. This calibration risk is also fundamental for the banking industry because it significantly influences the prices of exotic options. Moreover, calibration risk exists even if an appropriate model has been chosen and model risk does not exist anymore.

Calibration risk arises from the different possibilities to measure the error between the observations on the market and the corresponding quantities in the model world. A natural approach to specify this error is to consider the absolute price (AP) differences. (See e.g., Schoutens et al., [2004]). But the importance of absolute price differences depends on the magnitude of these prices. Hence, another useful way for measuring the error is relative price (RP) differences. (See e.g., Mikhailov et al., [2003]). As models are often judged by their capability to reproduce implied volatility surfaces, other measures can be defined in terms of implied volatilities. There are again the two possibilities of absolute implied volatilities (AI) and relative implied volatilities (RI). We consider these four ways to measure the difference between model and market data and explore the implications for the pricing of exotic options.

To this end, we focus on the stochastic volatility model of Heston. In order to analyze the influence of the goodness of fit on calibration risk, we consider in addition the Bates model, which is an extension of the Heston model with similar qualitative features. These two models are calibrated to the prices of plain vanilla options on the DAX. We use a time series of implied volatility surfaces from April 2003 to March 2004. As exotic options we consider down and out puts, up and out calls, and cliquet options for one, two, or three years to maturity.

In this framework we determine the size of calibration risk and analyze factors influencing it.

Besides calibration risk, there is also model risk, which represents wrong prices because a wrong parametric model has been chosen. We consider the model risk between the Heston and the Bates models and analyze the relation between the two forms of risk in pricing exotic options.

The next section introduces the models and describes their risk neutral dynamics that we use for option pricing. Moreover, this section contains information about the data used for the calibration. The following section describes the calibration method and defines the error functionals analyzed in this work. The goodness of fit is shown by representative surfaces and statistics on the errors. We then present the exotic options that we consider for calibration risk and price these products by simulation. In our next section we analyze the model risk for the two stochastic volatility models under the four error functionals. In the last section, we summarize the results and draw our conclusions.

MODELS AND DATA

In this section, we describe briefly the Heston model and the Bates model for which we are going to analyze calibration risk. Moreover, we provide some descriptive statistics of the implied volatility surfaces that we use as input data for the calibration.

Heston Model

We consider the popular stochastic volatility model of Heston [1993]:

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{V_t} dW_t^1 \quad (1)$$

where the volatility process is modelled by a square-root process:

$$dV_t = \xi(\eta - V_t)dt + \theta\sqrt{V_t}dW_t^2 \quad (2)$$

and W^1 and W^2 are Wiener processes with correlation ρ .

The variance process (V_t) remains positive if its volatility θ is small enough with respect to the product of the mean reversion speed ξ and the average variance level η :

$$\xi\eta > \frac{\theta^2}{2} \quad (3)$$

The dynamics of the price process are analyzed under a martingale measure under which the characteristic function of (S_t) is given by:

$$\begin{aligned} \phi_t^H(z) = \exp \left\{ \frac{-(z^2 + \mathbf{i}z)V_0}{\gamma(z) \coth \frac{\gamma(z)t}{2} + \xi - \mathbf{i}\rho\theta z} \right\} \\ \times \frac{\exp \left\{ \frac{\xi\eta(\xi - \mathbf{i}\rho\theta z)}{\theta^2} + \mathbf{i}ztr + \mathbf{i}z \log(S_0) \right\}}{\left(\cosh \frac{\gamma(z)t}{2} + \frac{\xi - \mathbf{i}\rho\theta z}{\gamma(z)} \sinh \frac{\gamma(z)t}{2} \right)^{\frac{2\xi\eta}{\theta^2}}} \end{aligned} \quad (4)$$

where $\gamma(z) \stackrel{\text{def}}{=} \sqrt{\theta^2(z^2 + \mathbf{i}z) + (\xi - \mathbf{i}\rho\theta z)^2}$, see e.g., Cont et al. [2004].

Bates Model

Bates [1996] extended the Heston model by considering jumps in the stock price process:

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sqrt{V_t} dW_t^1 + dZ_t \\ dV_t &= \xi(\eta - V_t)dt + \theta\sqrt{V_t}dW_t^2 \end{aligned} \quad (5)$$

where Z is a compound Poisson process with intensity λ and jumps k that have a lognormal distribution:

$$\log(1+k) \sim N \left(\log(1+\bar{k}) - \frac{\delta^2}{2}, \delta^2 \right) \quad (6)$$

We analyze the dynamics of this model under a martingale measure under which the characteristic function of (S_t) is given by:

$$\begin{aligned} \phi_t^B(z) &= \exp \{ t\lambda (e^{-\delta^2 z^2/2 + \mathbf{i}\{\log(1+\bar{k}) - \frac{1}{2}\delta^2\}z} - 1) \} \\ &\times \exp \left\{ \frac{-(z^2 + \mathbf{i}z)V_0}{\gamma(z) \coth \frac{\gamma(z)t}{2} + \xi - \mathbf{i}\rho\theta z} \right\} \\ &\times \frac{\exp \left\{ \frac{\xi\eta(\xi - \mathbf{i}\rho\theta z)}{\theta^2} + \mathbf{i}z t(r - \bar{k}) + \mathbf{i}z \log(S_0) \right\}}{\left(\cosh \frac{\gamma(z)t}{2} + \frac{\xi - \mathbf{i}\rho\theta z}{\gamma(z)} \sinh \frac{\gamma(z)t}{2} \right)^{\frac{2\xi\eta}{\theta^2}}} \end{aligned} \quad (7)$$

where $\gamma(z) \stackrel{\text{def}}{=} \sqrt{\theta^2(z^2 + \mathbf{i}z) + (\xi - \mathbf{i}\rho\theta z)^2}$. (See e.g., Cont et al. [2004].)

The Bates model has eight parameters while the Heston model has only five parameters. Because of these three additional parameters, the Bates model can better fit an observed surface but parameter stability is more difficult to achieve.

Data

Our data consists of EUREX-settlement implied volatilities on the DAX. Hence, these volatilities are given by inversion of the Black-Scholes formula, i.e., if you plug these volatilities into the Black-Scholes formula together with the other corresponding parameters then you get the settlement prices of European options on the DAX. In this context, we approximate the risk-free interest rates by the EURIBOR. On each trading day we use the yields corresponding to the maturities of the implied volatility surface. As the DAX is a performance index, it is adjusted to dividend payments. Thus, we do not consider dividend payments explicitly.

We analyze the time period from April 2003 to March 2004. Since March 2003 the EUREX trades plain vanilla options with maturities up to five years. Until March 2004, it has not changed its range of products. Hence, the data is homogeneous in the sense that the implied volatility surfaces are derived from similar products.

From this time period, we analyze the surfaces from all the Wednesdays when trading has taken place. We restrict ourselves to these days because of the computationally

intense Monte Carlo simulations for the pricing. Thus, we consider 51 implied volatility surfaces. We exclude observations that are deep out of the money because of illiquidity of these products. More precisely, we consider only options with moneyness $K/S_0 \in [0.75, 1.35]$ for small times to maturity $T \leq 1$. As we analyze exotic options that expire in one, two, or three years we exclude also plain vanillas with time to maturity less than three months.

Some information about the resulting implied volatility surfaces is summarized in Exhibit 1. The surfaces contain on average 140 prices and nine times to maturity with a mean moneyness range of 65%.

The values of the underlying in the sample period are shown in Exhibit 2. This exhibit also shows the (interpolated) at the money implied volatilities for one year to maturity. The market value of the DAX went up in this period and accordingly the implied volatilities went down as Exhibit 2 shows.

CALIBRATION

In this section, we specify the calibration routine and describe the four error functionals. The calibration results illustrate how well the plain vanilla prices can be replicated by the Heston and the Bates models.

Calibration Method

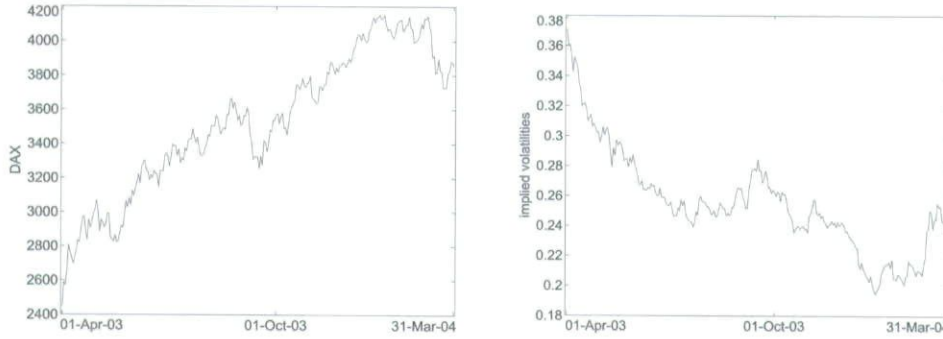
Carr and Madan [1999] found a representation of the price of a European call option by one integral for a whole class of option pricing models. Their method, that

EXHIBIT 1

	mean number of maturities	mean number of observations	mean money- ness range
short maturities ($0.25 \leq T < 1.0$)	3.06	64	0.553
long maturities ($1.0 \leq T$)	5.98	76	0.699
total	9.04	140	0.649

Note: Description of the implied volatility surfaces.

EXHIBIT 2



Note: DAX and ATM implied volatility with 1 year to maturity on the trading days from 01 April 2003 to 31 March 2004.

is applicable to the Heston and the Bates model, is based on the characteristic function of the log stock price under the risk neutral measure.

Carr and Madan showed that the price $C(K, T)$ of a European call option with strike K and maturity T is given by

$$C(K, T) = \frac{\exp\{-\alpha \ln(K)\}}{\pi} \int_0^{+\infty} \exp\{-i\nu \ln(K)\} \psi_T(\nu) d\nu \quad (8)$$

for a (suitable) damping factor $\alpha > 0$. The function ψ_T is given by

$$\psi_T(\nu) = \frac{\exp(-rT) \phi_T\{\nu - (\alpha + 1)i\}}{\alpha^2 + \alpha - \nu^2 + i(2\alpha + 1)\nu} \quad (9)$$

where ϕ_T is the characteristic function of $\log(S_T)$; see the prior section on Models and Data.

For the difference between market and model, we consider the following four objective functions based on the root weighted square error:

$$\begin{aligned} AP &\stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^n w_i (P_i^{\text{mod}} - P_i^{\text{mar}})^2} \\ RP &\stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^n w_i \left(\frac{P_i^{\text{mod}} - P_i^{\text{mar}}}{P_i^{\text{mar}}} \right)^2} \\ AI &\stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^n w_i (IV_i^{\text{mod}} - IV_i^{\text{mar}})^2} \\ RI &\stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^n w_i \left(\frac{IV_i^{\text{mod}} - IV_i^{\text{mar}}}{IV_i^{\text{mar}}} \right)^2} \end{aligned} \quad (10)$$

where *mod* refers to a model quantity and *mar* to a quantity observed on the market, P to a price, and IV to an implied volatility. The index i runs over all n_t observations of the surface on day t . The weights w_i are non negative with $\sum_i w_i = 1$. Hence, the objective functions can be interpreted as mean average errors. While the error functionals AP and RP measure the differences between option prices, the other two error measures focus on Black-Scholes implied volatilities because these quantities are normally used in reality for price quotations. The model implied volatilities IV^{mod} are computed from the prices of European options in the Heston and in the Bates models by numerical inversion of the Black-Scholes formula.

We choose the weights in such a way that on each day all maturities have the same influence on the objective function. In order to make different surfaces comparable, each maturity gets the weight $1/n_{\text{mat}}$ where n_{mat} denotes the number of maturities in this surface. Moreover, we assign the same weight to all points of the same maturity. This leads to the weights

$$w_i \stackrel{\text{def}}{=} \frac{1}{n_{\text{mat}} n_{\text{str}}^i} \quad (11)$$

where n_{str}^i denotes the number of strikes with the same maturity as observation i . This weighting leads asymptotically to a uniform density on each maturity.

Given these weights, the average time to maturity of an implied volatility surface can be measured by a modified duration:

$$\sum_{i=1}^n \frac{\tau_i w_i}{\sum_{i=1}^n w_i} \quad (12)$$

where τ_i is the time to maturity of the option i . The mean duration of the 51 surfaces is 2.02 and the minimal (maximal) is 1.70 (2.30). Thus, the point of balance for the maturities lies around 2 for our time series of surfaces. As we analyze exotic options with one, two or three years to maturity, this point of balance confirms a correct weighting for our purposes.

We consider only out-of-the-money prices. Thus, we use call prices for strikes higher than the spot and put prices for strikes below the spot. This approach ensures that we compare only prices of similar magnitude. It has no impact on the errors based on implied volatilities. Because of put-call parity, the use of OTM options has no impact on the absolute price error (AP). But the relative prices are weighted in such a way that the observations around the spot receive less weight. Hence, only the relative price error (RP) is influenced by this choice of prices.

In order to estimate the model parameters, we apply a stochastic global optimization routine and minimize the objective functions with respect to the model parameters. In addition to some natural constraints on the range of the parameters, we have taken into account inequality (3) that ensures the positivity of the variance process.

Calibration Results

We consider 51 implied volatility surfaces between April 2003 and March 2004. Each of these is calibrated with respect to the four error functions described in our prior sub-section. These calibrations are done for the Heston and the Bates models.

The resulting errors of these 408 calibrations have been summarized in Exhibit 3 for the Heston model and in Exhibit 4 for the Bates model. Descriptive statistics on the calibrated parameters are given in Exhibit 5 for the Heston model and in Exhibit 6 for the Bates model. Exhibit 7 shows the fit of the implied volatility surface in the Heston model on a day that is representative for the AI error.

Exhibit 3 reports in each line the means of the four errors when the objective function given in the left column is minimized. In the Heston model, we get a mean absolute price error of 7.3 and a mean relative price error of 9.7% when we calibrate with respect to AP. Using the RP error functional we get the opposite result with a mean absolute price error of 11 and a mean relative price error of 6.1%. The errors based on implied volatilities are smaller for the RP objective function than for

the AP objective function. The results for the AI and RI objective functionals differ only slightly: the mean absolute implied volatility error is about 0.68% and the mean relative implied volatility error is about 2.5%. Moreover, the price errors for these objective functions lie between the price errors of the other two objective functions. The calibration with respect to RI gives the best overall fit because it has the smallest RI error and the second best errors for the rest. The meaning of these numbers is illustrated by Exhibit 7, which shows an implied volatility fit that is representative for an AI error of 0.68%. In order to make the AP errors comparable for different days (with different values of the spot) we have computed the mean of AP/DAX as 0.21%, 0.34%, 0.27%, 0.25% for the four error functionals.

The calibrated parameters which are described by Exhibit 5 form two groups because the parameters for the RP, AI, and RI calibration are quite similar. The initial variance V_0 and the average variance level η are both about 0.07 for all objective functionals. For the AP calibration we get a reversion speed $\xi = 0.9$, a volatility of variance of $\theta = 0.34$, and a correlation $\rho = -0.82$. The other calibrations lead to similar parameters with a reversion speed $\xi = 1.3$, a volatility of volatility of $\theta = 0.44$, and a correlation $\rho = -0.75$. As the correlations are sig-

EXHIBIT 3

objective fct.	AP	RP	AI	RI
AP	7.3 (2.2)	0.097 (0.048)	0.0081 (0.0025)	0.031 (0.012)
RP	11. (4.6)	0.061 (0.025)	0.0074 (0.0022)	0.029 (0.010)
AI	9.4 (3.2)	0.073 (0.032)	0.0068 (0.0020)	0.026 (0.009)
RI	8.8 (2.9)	0.070 (0.030)	0.0070 (0.0021)	0.025 (0.009)

*Note: Mean calibration errors in the Heston model for 51 days.
(AP = absolute price differences, RP = relative price differences, AI = absolute implied volatility differences, RI = relative implied volatility differences)*

EXHIBIT 4

objective fct.	AP	RP	AI	RI
AP	7.0 (2.2)	0.130 (0.100)	0.0076 (0.0023)	0.028 (0.009)
RP	12 (4.8)	0.051 (0.019)	0.0067 (0.0020)	0.026 (0.009)
AI	8.9 (3.2)	0.064 (0.026)	0.0060 (0.0018)	0.023 (0.008)
RI	8.7 (3.2)	0.062 (0.028)	0.0062 (0.0020)	0.022 (0.008)

Note: Mean calibration errors in the Bates model for 51 days. (AP = absolute price differences, RP = relative price differences, AI = absolute implied volatility differences, RI = relative implied volatility differences)

nificantly greater than -1 , the calibrated Heston models have really two stochastic factors.

The Bates model exhibits similar qualitative results as the Heston model: the AP and the RP calibrations differ clearly while the AI and the RI calibrations lead to similar results. The Bates model can be regarded as an extension of the Heston model. The additional three parameters for the jumps in the spot process lead to better calibration results for all error functionals: the AP error is reduced (on average) by 4%, the RP error by 16%, the AI and the RI error both by 12%.

The calibrated parameters of the Bates model are given in Exhibit 6. As in the Heston model, they form two groups with the AP calibration on the one hand and the RP, AI, and RI calibrations on the other hand. The parameters ξ , η , θ , and V_0 are similar to the calibrations for the Heston model. Only the correlation ρ rises to a level of -0.93 for all objective functions. Hence, this criterion for distinguishing between the two groups disappears. It is replaced by the expected number of jumps per year: for the AP calibration we expect (on average) a jump every three years while we expect a jump every two years for the other calibrations. It is interesting that all calibrations lead to a mean jump up of about $+8\%$ for the returns. The expected jumps upwards correspond to the market going up as shown in Exhibit 2.

EXHIBIT 5

	ξ	η	θ	ρ	V_0
AP	0.87 (0.48)	0.07 (0.02)	0.34 (0.08)	-0.82 (0.08)	0.07 (0.02)
RP	1.38 (0.35)	0.07 (0.02)	0.44 (0.06)	-0.74 (0.03)	0.08 (0.02)
AI	1.32 (0.40)	0.07 (0.02)	0.43 (0.06)	-0.77 (0.04)	0.08 (0.02)
RI	1.20 (0.35)	0.07 (0.02)	0.41 (0.06)	-0.75 (0.05)	0.08 (0.02)

Note: Mean parameters (std. dev.) in the Heston model for 51 days.

Schoutens et al. [2004] found that the Heston and the Bates option models can both be calibrated well to the EuroStoxx50. In summarizing the results of this section, we can say that DAX implied volatility surfaces can be replicated by these models for different error functionals as well as the EuroStoxx50 data in Schoutens et al. [2004]. As in that work, we find that the Bates model gives only slightly better fits for the AP calibration. In addition we have shown that it leads to a considerable improvement in the fit for the other objective functions.

EXOTIC OPTIONS

In this section, we analyze the price differences of exotic options for calibrations with respect to different error measures. We consider barrier and cliquet options. The prices of these products are calculated by Monte Carlo simulations using Euler discretizations.

Simulation

We price all exotic options by Monte Carlo simulations. To this end, we use for each derivate one million

EXHIBIT 6

	ξ	η	θ	ρ	V_0	λ	$\bar{k}$	δ
AP	0.92 (0.50)	0.07 (0.02)	0.33 (0.08)	-0.94 (0.07)	0.07 (0.02)	0.33 (0.21)	0.07 (0.03)	0.08 (0.06)
RP	1.56 (0.47)	0.07 (0.02)	0.45 (0.07)	-0.89 (0.07)	0.08 (0.02)	0.54 (0.23)	0.05 (0.03)	0.08 (0.06)
AI	1.43 (0.44)	0.07 (0.02)	0.43 (0.06)	-0.95 (0.06)	0.07 (0.02)	0.50 (0.22)	0.06 (0.03)	0.09 (0.04)
RI	1.36 (0.44)	0.07 (0.02)	0.41 (0.07)	-0.93 (0.09)	0.07 (0.02)	0.52 (0.26)	0.05 (0.04)	0.08 (0.08)

Note: Mean parameters (std. dev.) in the Bates model for 51 days.

paths generated by Euler discretization. (See e.g., Glasserman [2004]). As we take into account the positivity constraint (3) the square root process for the variance can be simulated by truncation at zero

$$V_{t_{i+1}} = \left(V_{t_i} + \xi(\eta - V_{t_i})\Delta t + \theta\sqrt{V_{t_i}}\sqrt{\Delta t}Z_i \right)^+$$

where Δt is the time step and Z_i are independent standard normal variables. This simple scheme leads to an acceptable small bias when the positivity constraint is fulfilled. For each exotic option we consider three maturities: 1 year, 2 years, and 3 years. We analyze three exotic options: up and out calls, down and out puts, and cliquet options. These products are described in the following sections where remaining parameters are also specified.

The payoffs of barrier options depend only on whether the underlying price process exceeded the barrier in some time interval. Hence, the value of barrier options depends on the minimum or maximum of the underlying price process. We approximate such continuous extrema by discrete extrema using one observation for each trading day. We use 252 time steps to simulate a process for a year assuming 252 trading days a year.

The calibration results are presented in the following sections together with a discussion of the options. The accuracy of the Monte Carlo results is given by the relative standard error in Exhibit 8. This exhibit confirms that the estimators have sufficiently small variance after one million paths compared to the price differences that we observe in Exhibits 10, 13, and 16.

Barrier Options

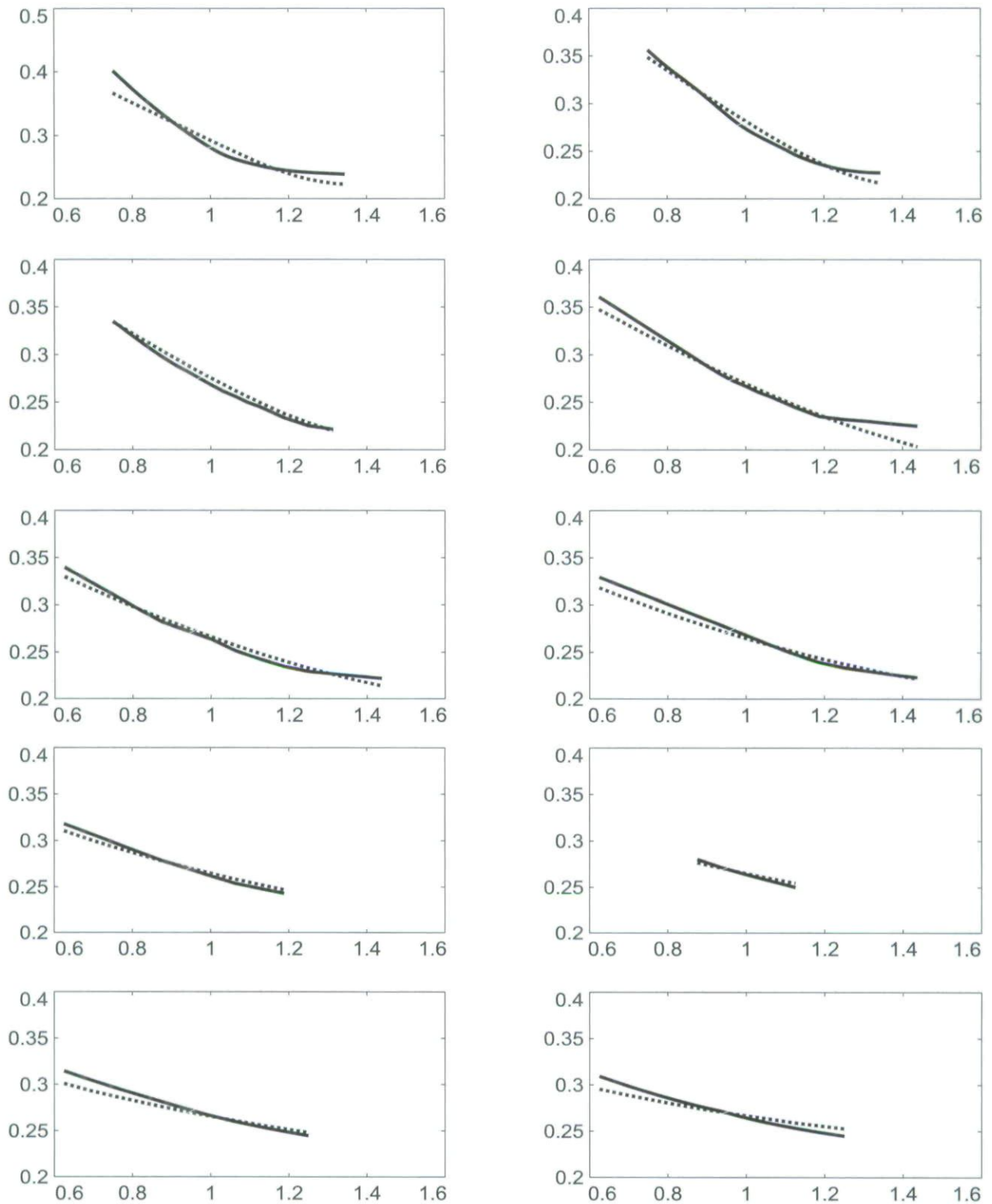
We consider two types of barrier options: up and out calls and down and out puts. These options are quite popular on the market. Down and out puts are, for example, sold together with zero-strike calls as "bonus certificates." These structured products are actively traded in Germany and also in many other markets.

Up and out call options The prices of up and out calls with strike K , barrier B , and maturity T on an underlying (S_t) are given by

$$\exp(-rT) E[1_{\{M_T < B\}}(S_T - K)^+] \quad (13)$$

where

EXHIBIT 7



Note: Implied volatilities in the market and in the Heston model for the maturities 0.26, 0.52, 0.78, 1.04, 1.56, 2.08, 2.60, 3.12, 3.64, 4.70 (left to right, top to bottom) for AI parameters on 25/06/2003. Solid: model, dotted: market. X-axis: moneyness.

$$M_T \stackrel{\text{def}}{=} \max_{0 \leq t \leq T} S_t \quad (14)$$

We choose as strike K and barrier B

$$K = (1 - 0.1T)S_0 \quad (15)$$

$$B = (1 + 0.2T)S_0 \quad (16)$$

where T denotes time to maturity. Up and out calls with such strikes and barriers are widely traded on the German market.

Up and out call options have the payoff profile of European call options if the underlying has not exceeded the barrier. Otherwise their payoff is zero. Thus, up and out calls are path dependent exotic options.

We want to analyze the difference between the prices of the exotic options when the underlying model has been calibrated with respect to different error measures. To this end, we have calibrated the Heston and the Bates models to implied volatility or price data on each day with respect to the four error functionals introduced in the prior sub-section on Calibration Method. Hence, we have four time series of calibrated models parameters that are described in the sub-section on Calibration Results. By Monte Carlo simulations we calculate on each day the prices of up and out calls for the four sets of model parameters.

In this way, we get four time series of up and out call prices corresponding to the four error measures for the Heston model and four corresponding time series of prices for the Bates model. We are interested in how the prices of exotic options differ when the four different error functionals are used. If the prices of up and out calls that are computed from the AP parameters are denoted by P_{AP}^{UOC} and the corresponding prices from the RP parameters by P_{RP}^{UOC} then we measure the difference between these prices by the ratio $P_{RP}^{UOC}/P_{AP}^{UOC}$. The other five price differences are measured by corresponding price ratios. Hence, we observe on each day six price ratios that describe the differences of the prices of exotic options resulting from different error measures.

The six time series of price ratios are summarized in Exhibit 9 for up and out calls with three years to maturity in the Heston model. In the boxplots the central line gives the median and the box contains 50% of the observations. Hence, the AP prices lie on average about 6% over the other prices and the AP prices are on 75% of the 51 days at least 4% higher than the other prices. The RP prices are about 2% below the AI or RI prices, which are very similar to each other.

We analyze the influence of time to maturity on these price ratios by also considering one year and two years to maturity (and by adjusting the barrier and the strike appropriately). The medians of the price ratios are presented in Exhibit 10 for all three times to maturity. This exhibit

EXHIBIT 8

	Heston			Bates		
	$T = 1$	$T = 2$	$T = 3$	$T = 1$	$T = 2$	$T = 3$
up and out calls	0.002	0.001	0.001	0.002	0.001	0.001
down and out puts	0.002	0.001	0.001	0.002	0.001	0.001
cliquet options	0.001	0.001	0.001	0.001	0.001	0.001

Note: Maximal ratio of standard error and price in Monte Carlo simulations. (Maximum over all time points and all objective functions.) Up and out calls with strike $1 - 0.1T$ and barrier $1 + 0.2T$ relative to initial spot, down and out puts with strike $1 + 0.1T$ and barrier $1 - 0.2T$ relative to initial spot, cliquets with reset times $t_i = Ti/3$ ($i = 0, \dots, 3$) and local caps of 8% and local floors of -8%. (T denotes time to maturity).

shows that the price differences become smaller for shorter times to maturity for the AP prices. The other price ratios remain almost constant. For one year to maturity, the price differences are about 2%–3% and the AP prices are lower than the other prices. For two years to maturity, the AP prices are again higher than the other prices.

In order to analyze the influence of the goodness of fit on the price ratios, we consider also the Bates model. The boxplots of the price ratios in this model are given in Exhibit 11 for three years to maturity. Compared to the Heston boxplots, the boxes are longer in the Bates model. Thus there is more variation between the prices for different error functionals. Moreover, the median ratios between the AP prices and the other prices are bigger than in the Heston model—especially for AP/AI and AP/RI. The ratios between RP, AI, and RI are similar to those in the Heston model. The corresponding results for one year and two years to maturity are presented in Exhibit 10. Qualitatively the situation is similar to the Heston model: For shorter times to maturity the price differences decrease—especially for AP prices.

Thus, the price ratios in the Heston and in the Bates model are similar among the RP, AI, and RI prices while the AP price differences are bigger in the Bates model. Moreover, the variation of the price ratios is higher in the Bates model.

Down and out p put options The prices of the down and out puts with strike K , barrier B , and maturity T on an underlying (S_t) are given by

$$\exp(-rT) E[1_{\{m_T > B\}}(K - S_T)^+] \quad (17)$$

where

$$m_T \stackrel{\text{def}}{=} \min_{0 \leq t \leq T} S_t \quad (18)$$

For our analysis, we use the strike K and the barrier B

$$K = (1 + 0.1T)S_0 \quad (19)$$

$$B = (1 - 0.2T)S_0 \quad (20)$$

where T denotes time to maturity. The strikes and barriers are set analogously to those for the up and out calls. Such down and out puts are often part of bonus certificates.¹

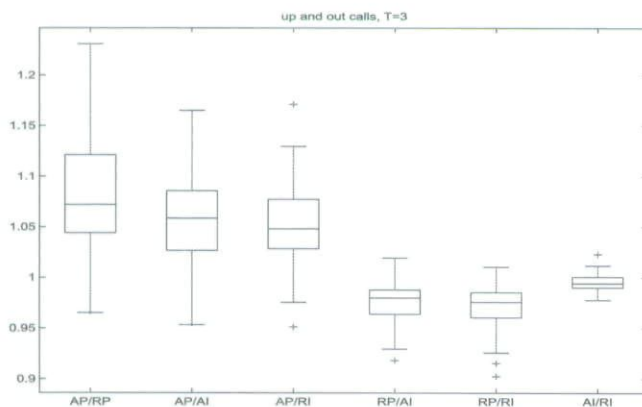
Down and out put options have the payoff profile of European put options if the underlying has been above the barrier during the lifetime of the option. Otherwise their payoff is zero.

As described above, we calculate on each day the prices of the down and out puts for the four parameter sets. The resulting six time series of price ratios are shown in the Exhibit 12 for three years to maturity in the Heston model. The AP prices are (on average) about 3.5% smaller than the other prices and 75% of the AP prices are at least 2% smaller than the other prices. The RP prices lie above the prices from the calibrations to implied volatilities. These AI and RI prices are quite similar so that we can identify again the two groups that we have already observed for the up and out calls.

Compared to the up and out calls, the price differences are smaller for the down and out puts. This can be seen also from Exhibit 13 that reports the median of the price ratios for one, two, and three years to maturity. This exhibit shows that the price ratios change for increasing time to maturity: For one year to maturity, the AP prices lie above the other prices but with increasing time to maturity, the AP prices become relatively smaller. The RP and AI prices remain on a similar level for all times to maturity and the RI prices tend to this level for longer times to maturity.

The situation in the Bates model that gives a better fit to the plain vanilla data is described in Exhibit 13 and Exhibit 14. The AP prices lie about 7% below the other prices. Thus this difference is bigger than in the Heston model. The other price ratios still lie on average on the same level but their variance has grown compared to the Heston model.

EXHIBIT 9



Note: Relative prices of the up and out calls in the Heston model for 3 years to maturity.

EXHIBIT 10

		AP/RP	AP/AI	AP/RI	RP/AI	RP/RI	AI/RI
Heston	$T = 1$	0.986	0.968	0.967	0.984	0.984	0.999
	$T = 2$	1.051	1.024	1.022	0.979	0.978	0.998
	$T = 3$	1.072	1.059	1.048	0.980	0.976	0.994
Bates	$T = 1$	0.988	0.985	1.002	1.002	1.006	1.012
	$T = 2$	1.070	1.083	1.104	0.970	0.986	1.018
	$T = 3$	1.106	1.123	1.129	0.972	0.975	1.013

Note: Median of price ratios of up and out calls.

The situation for the barrier options can be summarized as follows: the AP prices differ significantly from the other prices for both types of barrier options. While the AP prices are higher for up and out calls, they are lower for down and out puts relative to the other prices. In this sense the situation is symmetrical. The differences become bigger for longer times to maturity and

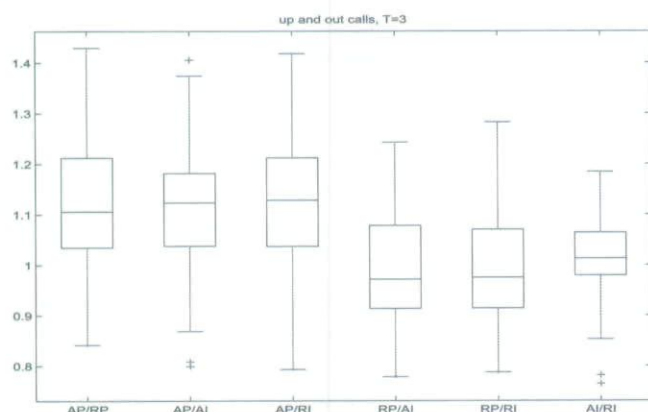
the better fit of the Bates model does not lead to smaller price differences.

Cliquet Options

We consider cliquet options with prices

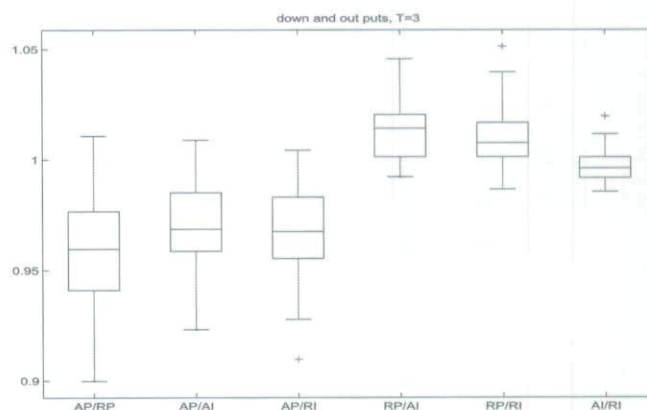
$$\exp(-rT)E[H] \quad (21)$$

EXHIBIT 11



Note: Relative prices of the up and out calls in the Bates model for 3 years to maturity.

EXHIBIT 12



Note: Relative prices of the down and out puts in the Heston model for 3 years to maturity.

EXHIBIT 13

		AP/RP	AP/AI	AP/RI	RP/AI	RP/RI	AI/RI
Heston	$T = 1$	1.025	1.031	1.005	1.007	0.980	0.977
	$T = 2$	0.983	0.994	0.984	1.011	0.997	0.986
	$T = 3$	0.960	0.969	0.968	1.014	1.008	0.996
Bates	$T = 1$	1.021	1.012	1.019	1.004	1.006	0.998
	$T = 2$	0.968	0.975	0.966	1.031	1.022	0.990
	$T = 3$	0.922	0.935	0.931	1.026	1.022	0.995

Note: Median of price ratios of down and out puts.

where the payoff H is given by

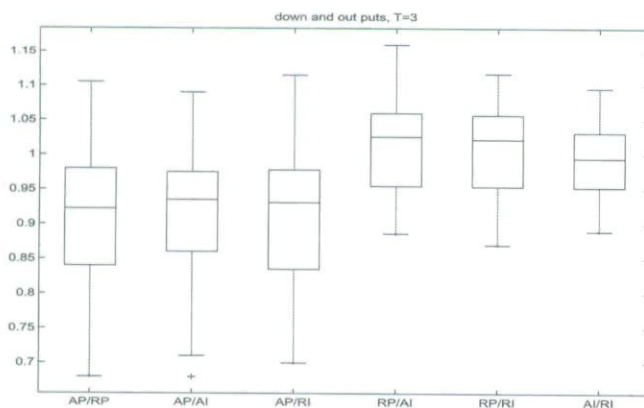
$$H^{\text{def}} = \min \left(c_g, \max \left[f_g, \sum_{i=1}^N \min \left\{ c_l^i, \max \left(f_l^i, \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right) \right\} \right] \right) \quad (22)$$

Here $c_g (f_g)$ is a global cap (floor) and $c_l^i (f_l^i)$ is a local cap (floor) for the period $[t_{i-1}, t_i]$.

We consider three periods with $t_i = \frac{T}{3}i (i = 0, \dots, 3)$ and the caps and floors are given by

$$\begin{aligned} c_g &= \infty \\ f_g &= 0 \\ c_l^i &= 0.08, i = 1, 2, 3 \\ f_l^i &= -0.08, i = 1, 2, 3 \end{aligned} \quad (23)$$

EXHIBIT 14

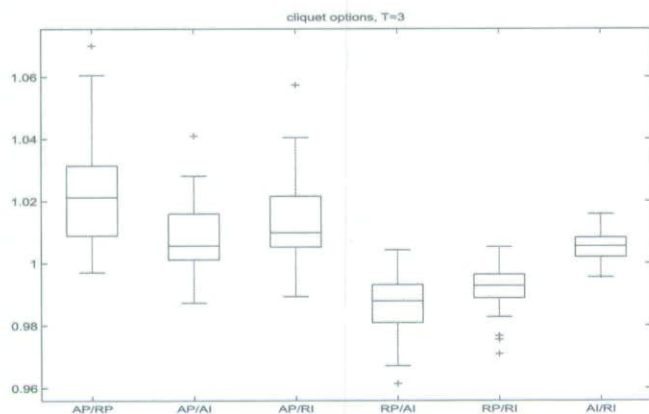


Note: Relative prices of the down and out puts in the Bates model for 3 years to maturity.

While barrier options are simple exotic options, cliquet options are more difficult because of their forward structure. Moreover, there exist many different types corresponding to the parameters. Hence, our specification cannot give a representative picture of all the traded cliquets. But these caps and floors are typical because the option holder cannot lose money and the returns are bounded above only by the local return bounds.

Cliquet options pay out basically the sum of the returns $R_i^{\text{def}} = (S_{t_i} - S_{t_{i-1}})/S_{t_{i-1}}$. In order to reduce risk, local and global floors f are introduced for the returns R . In the same way the returns are bounded from above by local and global caps c .

EXHIBIT 15



Note: Relative prices of the cliquet options in the Heston model for 3 years to maturity.

The distributions of the six time series of price ratios for cliquet options are described in Exhibit 15 for three years to maturity in the Heston model. The ratios are closer to 1 than in the case of the barrier options. The AP prices lie above the other prices but the difference is significant only for the AP and RP prices. The differences between the other prices is also small. Thus, we cannot recognize directly from this exhibit the two groups that we identified for the barrier options.

Exhibit 16 that reports the median price ratios for one, two, and three years to maturity gives some insight into this situation: the AP prices are about 2% smaller than the other prices for one year to maturity. With increasing time to maturity, the AP prices grow relatively and are about 1.5% higher than the other prices for three years to maturity. As Exhibit 16 confirms, the other price ratios remain relatively constant for different times to maturity. Thus there are again the two groups that we have identified for the barrier options: The changing relative AP prices on the one hand and the constant other price ratios on the other hand.

The relative prices of the cliquet options in the Bates model are presented in Exhibit 17 for three years to maturity. Here we see that the AP prices are about 7% smaller than the other prices. The RP prices lie about 2% under the AI prices that are 3% higher than the RI prices. The RP and RI prices are similar. Thus, there are quite big differences for the cliquet options in the Bates model. Moreover, the variance is larger relative to the Heston model. Exhibit 16 describes the situation of

different times to maturity and shows that the AP prices grow relatively with increasing time to maturity while the other price ratios remain rather constant for different times to maturity.

Comparing the results for the two types of barrier options and the cliquet options we see in all cases two groups, the AP prices and the other prices. The AP prices differ a lot from the other prices and in addition this difference changes for different times to maturity. Moreover, the variance of the price ratios with AP prices is bigger in general than for the other price ratios. The other group of RP, AI, and RI prices shows similar prices and small variances. The Bates model that gives better fits to implied volatility surfaces has higher price differences (with higher variances).

MODEL RISK

In the last section, we have described price differences that result from the calibration with respect to the four error functionals. In this section, we consider model risk and its relation to calibration risk and compare our results with the findings of Schoutens et al. [2004]. Model risk is generally understood as the risk of "wrong" prices because an inappropriate parametric model has been chosen for modelling the price process of the underlying.

In order to analyze this model risk for the two stochastic volatility models, we consider the ratios of the prices of the exotic options in the Bates model and the corresponding prices in the Heston model. The distribution of these ratios for up and out calls with three years to maturity is described by Exhibit 18. The prices in the Bates model lie below the prices in the Heston model for all four error functionals: The difference varies between 2% for the AP prices and 6% for the RI prices. Thus model risk is not independent of the calibration method, i.e., calibration risk. The results for smaller times to maturity are given in Exhibit 19. The exhibit suggests that model risk does not change significantly for different times to maturity.

The model risk of down and out puts is shown in Exhibit 20 for three years to maturity. The prices in the Bates model lie below the prices in the Heston model for all error functionals. Compared to the up and out calls, the model risk is bigger for the down and out puts: it varies between 9% for AP prices and 14% for RI prices. But again we observe the highest difference for RI prices

EXHIBIT 16

		AP/RP	AP/AI	AP/RI	RP/AI	RP/RI	AI/RI
Heston	$T = 1$	0.983	0.976	0.989	0.993	1.006	1.013
	$T = 2$	1.002	0.991	1.000	0.989	0.998	1.010
	$T = 3$	1.022	1.008	1.014	0.987	0.992	1.005
Bates	$T = 1$	0.917	0.899	0.917	0.987	1.005	1.024
	$T = 2$	0.931	0.903	0.923	0.980	0.999	1.029
	$T = 3$	0.946	0.912	0.933	0.976	0.995	1.029

Note: Median of price ratios of cliquet options.

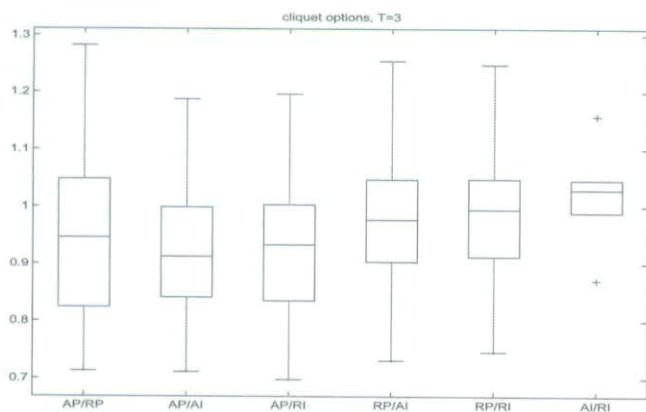
and the smallest for AP prices. Moreover, the variance is bigger than for the up and out calls. Exhibit 19 that gives the results for smaller times to maturity can be interpreted in such a way that the model risk becomes smaller for shorter times to maturity.

Finally, we consider the model risk of cliquet options in Exhibit 21. For these options the Bates prices lie above

the corresponding Heston prices for all calibration methods. The smallest price difference that appears for the AP prices is about 8% while the biggest difference of 16% is for RI prices. Exhibit 19 shows again smaller price differences for shorter times to maturity.

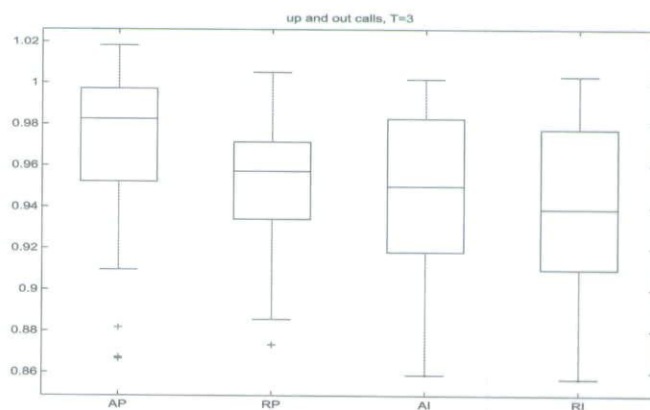
The model risk between the Heston and the Bates model can be described for barrier and cliquet options as

EXHIBIT 17



Note: Relative prices of the cliquet options in the Bates model for 3 years to maturity.

EXHIBIT 18



Note: Ratio of Bates to Heston prices for up and out calls with 3 years to maturity on 51 days.

EXHIBIT 19

		AP	RP	AI	RI
up and out calls	$T = 1$	0.973	0.953	0.944	0.941
	$T = 2$	0.980	0.954	0.953	0.940
	$T = 3$	0.983	0.957	0.950	0.939
down and out puts	$T = 1$	0.933	0.892	0.877	0.878
	$T = 2$	0.918	0.883	0.872	0.860
	$T = 3$	0.916	0.881	0.873	0.860
cliquets	$T = 1$	1.057	1.100	1.109	1.119
	$T = 2$	1.076	1.128	1.130	1.144
	$T = 3$	1.086	1.138	1.140	1.162

Note: Median of ratios of Bates to Heston prices.

follows: model risk measured by the price ratios in the two models increases for longer times to maturity. Moreover, it is ordered with respect to the calibration method. The calibration with respect to implied volatilities leads to bigger price differences than the calibration with respect to prices. The model risk is smallest for the AP calibration and bigger for the RP calibration. It is even bigger for the AI calibration and the price differences are the biggest for the RI calibrations. This emphasizes the importance of the implied volatility surfaces and their calibration. Moreover, model risk differs across option types. The more exotic cliquet options have a higher model risk than the barrier options.

Schoutens et al. [2004] consider up and out calls (with strike equal to spot) and cliquet options with three years to maturity. For a barrier 50% above the spot, they find a model risk for the up and out calls of about 14%. For the cliquet options, they do not find a significant model risk. These results do not correspond completely to our AP results. There may be several reasons for these different results: while

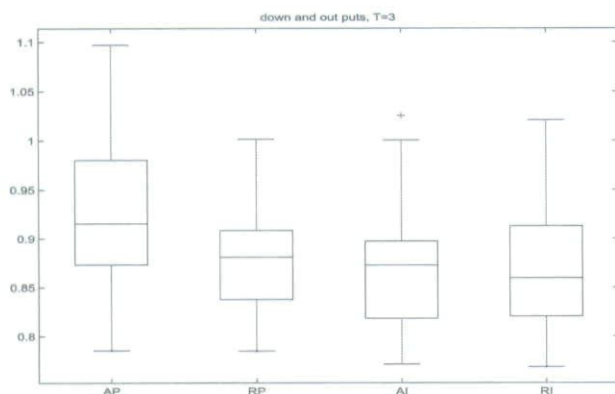
we look at a time series of 51 implied volatility surfaces, they focus only on one day. Moreover, they have analyzed EuroStoxx50 prices and we use DAX data.

CONCLUSION

We have looked at the stochastic volatility model of Heston and analyzed different calibration methods and their impact on the pricing of exotic options. Our analysis has been carried out for a time series of DAX implied volatility surfaces from April 2003 to March 2004.

We have shown that different ways to measure the error between the model and the market in the calibration routine lead to significant price differences for exotic options. We have considered the four error measures that are defined by the root mean squared error of absolute or relative differences between prices or implied volatilities in the market and in the model. Among these measures we have identified two groups. Calibrations with respect to relative prices, absolute implied volatilities, or relative

EXHIBIT 20

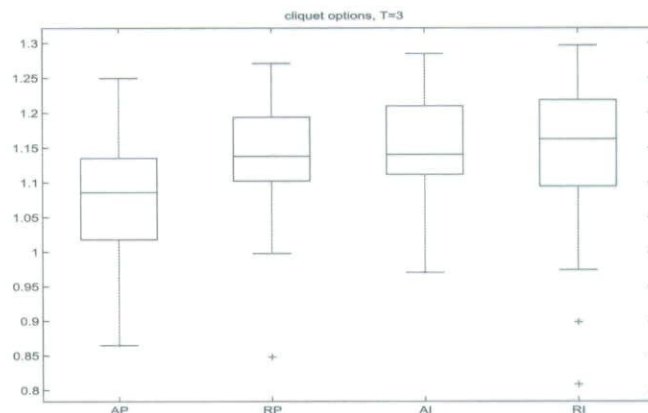


Note: Ratio of Bates to Heston prices for down and out puts with 3 years to maturity on 51 days.

implied volatilities lead to similar prices of exotic options. Calibrations with respect to absolute prices imply exotics prices that are quite different from the prices of the first group. As implied volatilities are of comparable size for different strikes and different times to maturity, calibrations with respect to absolute or relative implied volatility differences lead to similar prices of exotic options. As these prices are also rather similar to those computed from calibrations with respect to relative price differences, we conclude that all three of these error measures have a similar weighting of the errors. The price differences between the described two groups of error measures increase for longer times to maturity. Moreover, the differences do not decrease in the Bates model although it is an extension of the Heston model with similar qualitative features and a better fit to plain vanilla data. The price differences of exotic options differ also across option types and are bigger for barrier options than for cliques.

The observed price differences of exotic options between the two groups of error measures can be explained in the following way: as we use out of the money options for the calibration, measuring the error by absolute prices puts most weight at the money. The characteristic feature of implied volatility surfaces around the money is the skew because the smile can be identified only from deep out-of-the-money calls. Hence, errors in terms of absolute prices focus on the skew. In the Heston model, the skew is mainly controlled by the correlation between the Brownian motions of the stock and the variance process. Our calibrated model parameters confirm

EXHIBIT 21



Note: Ratio of Bates to Heston prices for clique options with 3 years to maturity on 51 days.

this because the calibrations with respect to absolute prices lead to higher (absolute) correlations between these two processes. At the same time such calibrations imply smaller volatilities of variance which control the smile. These two effects make the distribution of the stock more right skewed so that up and out calls (down and out puts) are more (less) expensive in models that are calibrated to absolute price differences.

Moreover, we have looked at the model risk between the Heston and the Bates model. Model risk and calibration risk are not independent because model risk is lowest for calibrations with respect to absolute prices and highest for calibrations with respect to relative implied volatilities. As this holds for all considered options, model risk seems to be ordered with respect to the error measure used in the calibration. In the analyzed time period the DAX experienced a strong upward trend. This is also reflected in the positive expected jumps in the Bates model. These jumps can explain why the barrier (clique) options are more (less) expensive in the Heston model than in the Bates model. Because of the jumps, the considered down and out puts often finish out of the money. The up and out calls are knocked out more often because of the upward jumps. The higher prices of the clique options in the Bates model may be due to the higher returns that result from the upward jumps. If the market has a downward trend that is also reflected in negative expected jumps in the Bates model, then model risk should have the same size in general but could have a different sign, e.g., for barrier options.

As model risk is bigger than calibration risk, calibrations should be carried out with respect to absolute prices if the choice of an appropriate model is unclear. But if a model has already been chosen we suggest measuring the error between the model and the market in terms of (relative) implied volatilities, because this error measure best reflects the characteristics of the model that are essential for pricing exotic options.

ENDNOTE

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¹A bonus certificate is a structured product incorporating a zero-strike call and a down and out put option. The payoff structure at expiry can be described as follows: if the value of the underlying at expiry is above the strike then the investor receives the underlying. If the price of the underlying is below the barrier then the investor bears the full loss of the underlying because she gets only the underlying. Otherwise the investor gets the underlying and in addition a bonus if the barrier has not been exceeded before expiration. Because of its payoff profile, this product is constructed mainly for sideways markets.

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