

A Tree Model for Pricing Convertible Bonds with Equity, Interest Rate, and Default Risk

DONALD R. CHAMBERS AND QIN LU

DONALD R. CHAMBERS is the KPMG Walter E. Hanson professor of finance in the Department of Economics and Business at Lafayette College in Easton, PA. chambers@lafayette.edu

QIN LU is an assistant professor in the Mathematics Department at Lafayette College in Easton, PA. luq@lafayette.edu

This article presents a binomial tree model for pricing convertible bonds. Our model is a two-factor model (interest rates and equity prices) in which the potential for default is modeled in the manner of Jarrow and Turnbull [1995]. Interest rates are modeled using the Ho-Lee [1986] lognormal model. Equity prices are modeled using the Cox-Ross-Rubinstein (CRR) model.

Our model differs from Das and Sundaram [2006] through differences in the specification of the correlation between interest rates and stock prices. Our model also differs from the model of Hung and Wang [2002] who, among other differences, assume no correlation. We model correlation analogously to the approach of Hull [2003, p. 474].

We demonstrate the simplicity of our model and compare the pricing of our model with the pricing of Hung and Wang using the numerical examples from their article. We find moderate pricing differences between the models and provide sensitivity analysis of the effect of correlation between the interest rate and equity factors.

The pricing of convertible bonds is a complex and important task. There are three primary methods: PDE-based solutions, simulations and tree models. Generally, in pricing convertible bonds, it is impossible to solve the PDEs analytically because most convertible bonds are callable, or have coupons and/or other complicating features. Another method for pricing convertibles is the simulation

method. A random generator is used to get numerous paths for the random variables. The price of a convertible bond is estimated by discounting back and averaging all the paths. Few articles use the simulation method: "due to the fact that the optimal early exercise strategy is a free boundary problem, the literature on convertible bonds has only considered finite differences and lattice (tree) methods." See Lvov, Yigitbasioglu, and Bachir [2004].

Convertible bonds are usually priced in practice using tree models. In their simplest form, such models are single factor (the underlying stock price). For example, see Tsiveriotis and Fernandes [1998] and Hull [2003, p. 653]. At each node, the convertible bond's value is separated into "equity" and "debt" components. The risk-free rate is used to discount the equity component and a risky bond rate is used to discount the debt component.

There are a few two-factor models available. Ho and Pfeffer's [1996] model is a two-factor tree model where the stock price and the interest rate are the two factors. They also considered the correlation between the stock price and the interest rate. Their model "shows that the correlation of stock risk and interest rate risk may affect convertible bond prices significantly." Their model assumes that the credit risk of the bond is captured by a constant option-adjusted spread added to the treasury interest rate tree at each node point. Ho and Pfeffer's model did not consider a recovery rate on defaulted bonds.

The first section of this article introduces our model's two factors, both without and with correlation, but without the probability of default.

The second section of the article details the modeling of default risk and risky corporate bond yields. This section combines the two-factor model of the previous section with default risk to create the full model.

The third section provides a detailed numerical example of our model. The final section provides conclusions.

THE INTEREST RATE AND EQUITY FACTORS WITHOUT DEFAULT

This section describes our model in the absence of default. The first subsection introduces the interest rate factor model; the second subsection introduces the equity factor model; the third subsection combines the two factors into a single model; and the final subsection allows for non-zero correlation between the factors.

The Interest Rate Factor

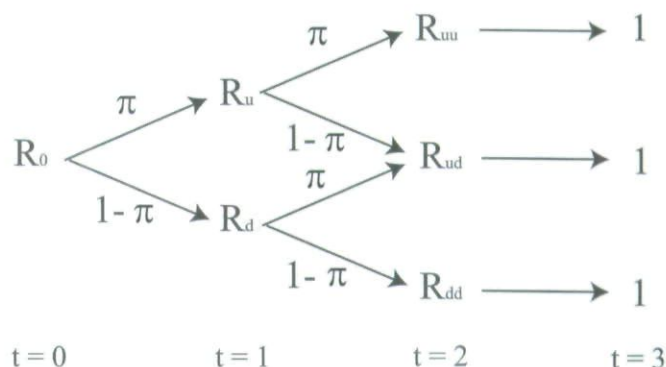
Like Hung and Wang, and Das and Sundaram, we use the Ho-Lee [1986] lognormal model (also known as the Black-Derman-Toy [1990] model with a constant variance). Details can be found in Baz and Chacko [2004, p. 162]. For simplicity, here we only build a three-period tree and we assume that each period Δt is one year. It can be easily generalized to an n -period tree and each period Δt can be any small positive number.

In order to build a three-period tree, we need the input of Treasury zero coupon rates with maturities at $t = 1$, $t = 2$, $t = 3$, respectively R_0 , R_1 , and R_2 (which are annual interest rates). We assume constant volatility σ_r for the Log of risk-free interest rate. Exhibit 1 describes the interest rate factor of our tree.

This tree model transforms the Treasury term structure into a tree of random variables, namely single period risk-free interest rates. In Exhibit 1, R_0 is the interest rate between $t = 0$ and $t = 1$. R_u and R_d are the two possible states of the interest rates between $t = 1$ and $t = 2$. R_{uu} , R_{ud} , and R_{dd} are the three possible states of the interest rates between $t = 2$ and $t = 3$. The probability for each up node is π and the probability for each down node is $1 - \pi$. In our article, we will follow the common convention that $\pi = 1/2$. In the tree in Exhibit 1, the number "1" stands for the bond's

EXHIBIT 1

Three-Period Risk-Free Interest Rate Tree



face value of \$1 which is received at the bond's three period maturity.

To make the tree model have the same volatility as σ_r (our assumed log short term interest rate volatility), the quotient of the up node over the down node must be equal to $e^{2\sigma_r\sqrt{\Delta t}}$, namely $R_u = R_d e^{2\sigma_r\sqrt{\Delta t}}$, $R_{uu} = R_{ud} e^{2\sigma_r\sqrt{\Delta t}}$, $R_{ud} = R_{dd} e^{2\sigma_r\sqrt{\Delta t}}$. Hence, once we obtain the values of the lowest nodes R_d and R_{dd} , the entire tree is determined. The method to find R_d and R_{dd} is called backward induction of the tree. Details can be found in Hull [2003]. Using backward induction on Exhibit 1, we can discount back our face value of \$1 to the initial node and compare this price with the market price of the three-year Treasury zero coupon bond implied by the Treasury term structure. The lower nodes are set so that bond prices found through backward induction of all maturities are equal to observed market prices.

Intuitively, at each point in time the dispersion of the nodes of the tree is set in order to match the volatility input by the user, while the level of the interest rate nodes is set so that the computed prices of all of the bonds will be consistent with the observed term structure of interest rates.

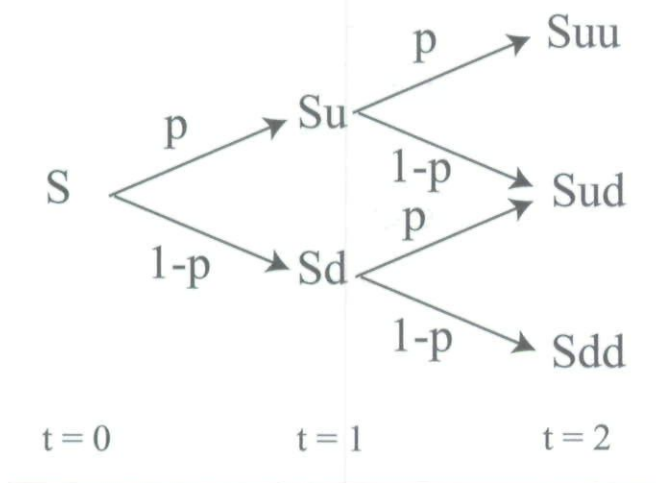
The Equity Factor

Like Hung and Wang, and Das and Sundaram, we adopt the Cox-Ross-Rubinstein (CRR) model [1985] for the equity price tree. Assume that the stock price, S , can go up or down during each period as shown in Exhibit 2.

In the CRR model, it is necessary to specify the movements and probabilities to be consistent with a lack of arbitrage opportunities.

EXHIBIT 2

Three-Period CRR Stock Tree



$$\begin{aligned} S_u &= S * u \\ S_d &= S * d \end{aligned}$$

where $u = e^{\sigma \sqrt{\Delta t}}$ and $d = \frac{1}{u}$

and

$$p = \frac{e^{r_f \Delta t} - d}{u - d}$$

For a detailed discussion of the particular selection of these parameters by CRR and their relationship to the use of risk neutrality, please see Nawalkha and Chambers [1995].

The Two-Factor Model without Correlation

Exhibit 3 combines the two factors into a single tree and recombines the nodes where possible. Binomial trees with a single factor which combine have a number of nodes after n time periods equal to $n + 1$. For a two-factor model, the number of nodes after n time periods is $(n + 1)^2$. However, in the final time period the number of nodes is reduced because the interest rate is no longer necessary for valuation purposes.

For two-factor models there are four pathways or “children” to each node, corresponding to the two possible equity movements times the two possible interest rate movements. The probability for each pathway, denoted

p_1 through p_4 , is given in Exhibit 4 under the assumptions that $\pi = 0.5$ and that the equity and interest rate factors are uncorrelated.

The Two-Factor Model with Correlation

Hung and Wang’s two-factor model did not permit non-zero correlation between the interest rate factor and the equity factor. Other two-factor models allow for correlation (for example, see Ho and Pfeffer [1996] and Hull [2003, p. 474]) but do not model credit risk (default). Our model allows both correlation and credit risk (discussed in the next section).

We assume that the stock price and the logarithm of the interest rate have a constant correlation coefficient of ρ .

We derived the values for the p_1, p_2, p_3, p_4 as shown in Exhibit 5. The derivation is in Appendix A. Das and Sundaram also allow for correlation between the stock price and the interest rates. However, Das and Sundaram model the correlation differently and therefore obtain a different set of probabilities. Our particular modeling choices facilitate analysis of the conditions in order to ensure that the probabilities are bounded by 0 and 1 as detailed in the following section.

THE MODELING OF DEFAULT (CREDIT) RISK

The previous section constructed a two-factor binomial tree with correlation. In this section, we add default risk and create a model including risky interest rates. The resulting model will be demonstrated in the next section using two simple numerical examples.

The method of including default risk is from Jarrow and Turnbull [1995]. Jarrow and Turnbull modified a risk-free interest rate tree by adding onto each node a default branch with a specified default probability during the upcoming period. Their model also specified a recovery value in the event of default.

The intuition of the Jarrow and Turnbull approach is that the term structure of interest rates for risky bonds (i.e., zero-coupon yields on non-convertible defaultable bonds of equal credit risk to the convertible bond being priced) can be used to infer probabilities of default. In other words, the default probability for each time period is set equal to that value which allows the tree to correctly price all zero-coupon non-convertible defaultable bonds with the given maturities (and given recovery rate).

EXHIBIT 3

Combining Interest Rate Ho-Lee Tree with Stock CRR Tree

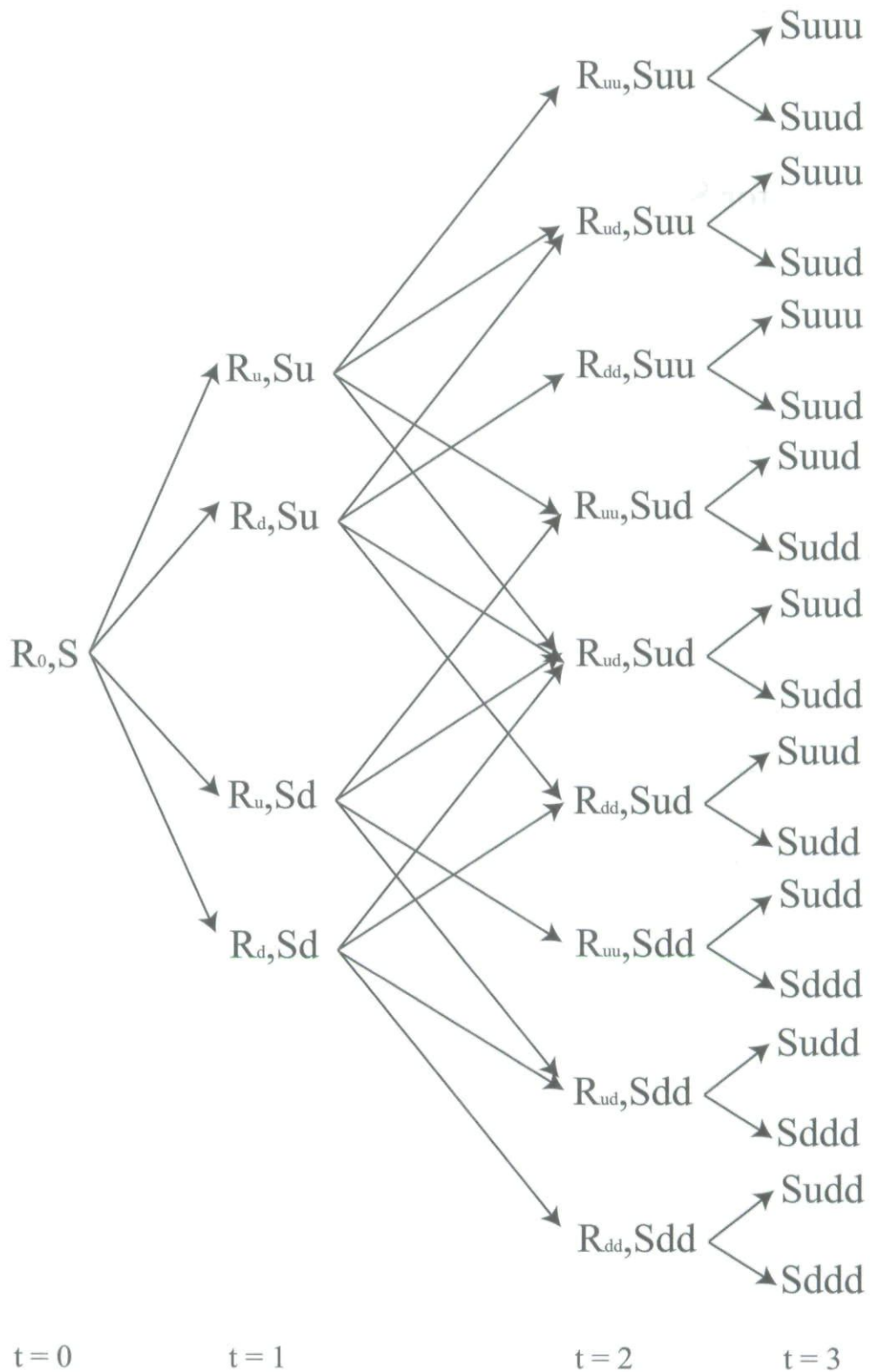


EXHIBIT 4

The Probability Measure for Two-Factor Tree Without Correlation

R\S	Su	Sd	Marginal for R
R_u	$p_1 = p/2$	$p_3 = (1-p)/2$	$\frac{1}{2} = \pi$
R_d	$p_2 = p/2$	$p_4 = (1-p)/2$	$\frac{1}{2} = 1 - \pi$
Marginal for S	p	$1-p$	1

The next five subsections detail the components of the final model. The last subsection discusses the final model in the context of a convertible bond.

Inserting the Default Probability and Recovery Rate into a Tree

Following generally the approach of Jarrow and Turnbull, Exhibit 6 illustrates the modeling of default with a two period single factor tree (for simplicity, we temporarily ignore the stock price factor). For each node there is an "extra" path for a defaultable bond's pricing in order to capture the economic effects of default. In Exhibit 6, the default path between $t = 0$ and $t = 1$ is illustrated vertically from the first node with probability λ_1 and with a recovery rate (on the \$1 face value of the bond) of δ . The default is shown as a vertical path only to simplify the exposition of the tree. This is especially useful when two factors are illustrated. However, the default is assumed to occur over the same time interval as the interest rate paths and to generate a recovery that is received at the end of the period. For example, in the case of the vertical default branch at $t = 0$, the default occurs between time $t = 0$ and $t = 1$ and the recovery is made at $t = 1$,

the same point in time at which either the interest rate R_u or R_d is observed. In Jarrow and Turnbull, the recovery is assumed to occur another period later ($t = 2$).

Note that in Exhibit 6, the total probability of an upward movement in R remains as π . The probability that the first up movement in R will occur without default is $\pi(1 - \lambda_1)$ and the probability that the up movement in R will occur with default is $\pi\lambda_1$. Exhibit 6 does not distinguish using an extra path between whether the default occurs with an upward or downward movement in the factor (R).

The recovery rate, δ , is exogenous and constant. The probability of default λ_1 can differ across time periods, and is determined as that probability that prevents arbitrage given the term structure of risky bond yields as indicated in the next subsection.

Determining the Default Probability in Each Period

The methods to find the first period (between $t = 0$ and $t = 1$) default probability, λ_1 , and all subsequent default probabilities share the same intuition. In the case of λ_1 , we first find the one-period risky bond price (i.e., defaultable

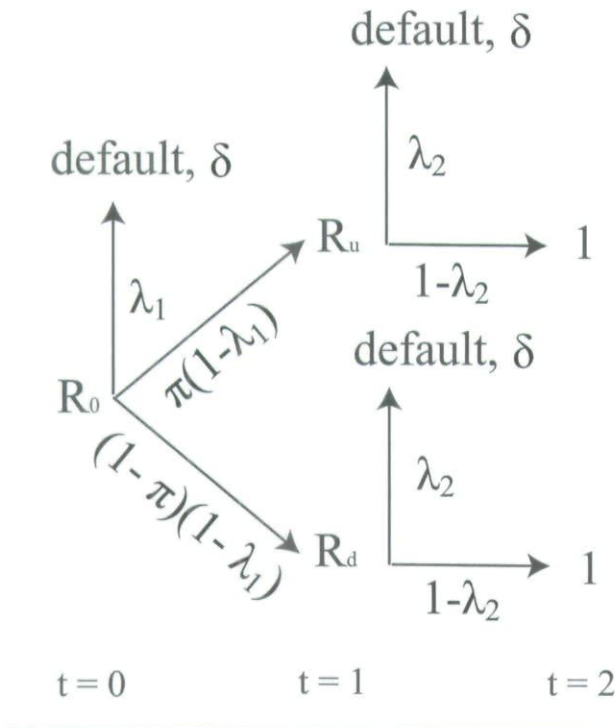
EXHIBIT 5

The Probability Measure for Two Factor Tree With Correlation

R\S	Su	Sd	Marginal for R
R_u	$p_1 = \frac{1}{2}(p + \sqrt{p(1-p)\rho})$	$p_3 = \frac{1}{2}(1-p - \sqrt{p(1-p)\rho})$	$\frac{1}{2} = \pi$
R_d	$p_2 = \frac{1}{2}(p - \sqrt{p(1-p)\rho})$	$p_4 = \frac{1}{2}(1-p + \sqrt{p(1-p)\rho})$	$\frac{1}{2} = 1 - \pi$
Marginal for S	p	$1-p$	1

EXHIBIT 6

Two-Period Interest Tree with Default Added



non-convertible zero coupon with \$1 face value) by backward induction using the one-period interest rate tree with default. We denote the one-year risky interest rate in period t as R_t^* . Therefore the one period risky bond price is $e^{-R_0^*}$ and:

$$e^{-R_0^*} = [1(1 - \lambda_1) + \delta \lambda_1] e^{-R_0}$$

The observed risky interest rate, the observed riskless interest rate and the assumed recovery rate are inserted into the above equation and solved for λ_1 .

$$\lambda_1 = (1 - e^{R_0 - R_0^*}) / (1 - \delta)$$

The intuition of the determination of λ_1 can be seen in the above equation by setting the recovery rate (δ) equal to zero—in which case the default probability is determined by the spread between the risky (R^*) and riskless (R) bond yields.

Given λ_1 , the two period risky bond price ($e^{-2R_1^*}$) and the riskless interest rate tree, we can solve λ_2 in the following equation since it is the only unknown.

$$e^{-2R_1^*} = \{[1(1 - \lambda_2) + \delta \lambda_2] e^{-R_u} \pi(1 - \lambda_1) + [1(1 - \lambda_2) + \delta \lambda_2] e^{-R_d} (1 - \pi)(1 - \lambda_1) + \delta \lambda_1\} e^{-R_0}$$

Similarly, given a full set of zero-coupon bond yields with identical risk to the risk of the convertible bond being priced, we can solve for the default probabilities in subsequent time periods and can build a risky interest rate tree for the model.

In practice, the zero-coupon yields would typically be inferred from a sample of similar credit risk (i.e., the same rating) coupon bonds. In addition to adjusting for coupons using a methodology to strip the coupons and form a zero-coupon curve, the task may involve interpolating between maturities and adjusting for the difference between the exact credit risk of the convertible bond and the average credit risk of the rating group. It should also be noted that the sample of coupon bonds may be heterogeneous with regard to issues such as recovery rates.

In the event that a reasonably sized sample of similarly rated risky bonds is not available, a reasonable approach might be: 1) estimate the zero-coupon curve for a sample of risky bonds with an appropriate sample size, and 2) shift the estimated curve upward or downward so that it intersects with the observed or estimated yield of a straight bond with the same credit risk as the convertible bond being priced.

Credit default swap (CDS) rates also provide a potential source of estimation of our default rates, λ_1 . Duffie [1999] demonstrates the relationship between CDS rates and parameters reflecting default rates and recovery rates. Duffie uses hazard rates and an expected loss parameter that can be transformed into the default probabilities and recovery rate used as inputs in our tree model. Duffie shows that, given the terms of the CDS and the riskless term structure, the CDS spread is a function of the probability of default and the recovery rate. Therefore, given an assumed recovery rate, CDS swap rates can be used to infer a default probability term structure. Duffie also notes the ability to use risky bond yields to estimate the same parameters. Therefore, CDS rates can be used to estimate default rates for our tree model when CDS swap rates are judged as having more available data, more applicable data, or better data (e.g., higher liquidity).

For expositional simplicity our remaining exhibits do not illustrate the risky interest rates, but rather specify the default probabilities and recovery rate.

The Two-Factor Tree with Default and Recovery

Exhibit 7 illustrates the default probabilities and recoveries in a three-period two-factor tree. F denotes the face value of the risky bond.

Note that all nodes except the terminal nodes in the combining tree in Exhibit 7 have five children: two from the stock tree, two from the risk-free interest rate tree, and one from the default event (which we mark in the vertical direction for simplicity). Comparing our tree with Hung and Wang's tree, we have reduced each node's six children to five children and have recognized recombining nodes.

Specification of Non-Arbitrage Path Probabilities

The purpose of this subsection is to derive path probabilities in the presence of default that satisfy no-arbitrage conditions and allow the use of risk neutrality modeling. We adapt the Cox-Ross-Rubinstein (CRR) tree for default as illustrated in Exhibit 8.

In order to develop a risk neutral non-arbitrage tree with a probability of default, it is necessary to adjust the CRR probabilities. With default as a possibility, there are three possible stock prices: S_u , S_d , and 0. The stock price is zero whenever default occurs. We denote the adjusted probabilities as $\tilde{p}$.

Referring to Exhibit 8, there is a probability λ of default (illustrated with the vertical default path) and therefore a λ probability that the stock price will be 0. The remaining probability of no default $(1 - \lambda)$ is divided into two paths based on the stock price. Given that default does not occur, there is a probability $\tilde{p}$ of an upward stock movement and $(1 - \tilde{p})$ of a downward movement. Therefore the unconditional probabilities are that there is a $\tilde{p}(1 - \lambda)$ probability of S_u , a $(1 - \tilde{p})(1 - \lambda)$ probability of S_d and a λ probability that the stock price will be 0. Following the general approach of CRR and adjusting for the default probability, λ :

$$p = \frac{e^{r_f \Delta t} / (1 - \lambda) - d}{u - d}$$

We claim that if $p = \frac{e^{r_f \Delta t} / (1 - \lambda) - d}{u - d}$, then the set $\{\lambda, \tilde{p}(1 - \lambda), (1 - \tilde{p})(1 - \lambda)\}$ is the risk-neutral measure that

we can use to price the derivatives. (Note that $\tilde{p}$ differs from the p in CRR model in that it divides $e^{r_f \Delta t}$ by $(1 - \lambda)$ to incorporate default. We still assume that $u = e^{\sigma_S \sqrt{\Delta t}}$ and $d = \frac{1}{u}$, as in the CRR model.)

We will offer a Naïve proof of the need to adjust the CRR probability here and a formal proof in Appendix B. Since we can view the stock as a simple derivative of itself, a risk-neutral probability measure must price the stock correctly. By backward induction,

the stock price

$$\begin{aligned} &= e^{-r_f \Delta t} [0\lambda + S_u p (1 - \lambda) + S_d (1 - p)(1 - \lambda)] \\ &= e^{-r_f \Delta t} \left[S_u \frac{e^{r_f \Delta t} / (1 - \lambda) - d}{u - d} (1 - \lambda) \right. \\ &\quad \left. + S_d \frac{u - e^{r_f \Delta t} / (1 - \lambda)}{u - d} (1 - \lambda) \right] \\ &= e^{-r_f \Delta t} \left[S \frac{ue^{r_f \Delta t} - (1 - \lambda)}{u - d} + S \frac{(1 - \lambda) - de^{r_f \Delta t}}{u - d} \right] \\ &= e^{-r_f \Delta t} \left[S \frac{ue^{r_f \Delta t} - (1 - \lambda)}{u - d} + S \frac{(1 - \lambda) - de^{r_f \Delta t}}{u - d} \right] \\ &= S \end{aligned}$$

Specification of Non-Arbitrage Path Probabilities with Two Factors and Correlation

In order to use backward induction in solving our two-factor tree, we need to get the probabilities for the four non-default children as shown in Exhibit 9. When we assign the probabilities p_1, p_2, p_3, p_4 , we make the assumption that the correlation between the stock price and the logarithm of the interest rate is ρ and we use the adjusted (for default) probability $\tilde{p}$.

We utilize the same general approach for deriving the values for p_1, p_2, p_3, p_4 (as discussed in The Two-Factor Model with Correlation section and in Appendix A) except that we use $\tilde{p}$ rather than p (i.e., we use our default adjusted CRR probability rather than the original CRR probability as discussed in the previous subsection). The resulting values are shown in Exhibit 10.

$\tilde{p}$ in Exhibit 10 is defined as before, $p = \frac{e^{R \Delta t} / (1 - \lambda) - d}{u - d}$ where $u = e^{\sigma_S \sqrt{\Delta t}}$, $d = \frac{1}{u}$, and R is the interest rate.

R can change every period, hence $p = \frac{e^{R \Delta t} / (1 - \lambda) - e^{-\sigma_S \sqrt{\Delta t}}}{e^{\sigma_S \sqrt{\Delta t}} - e^{-\sigma_S \sqrt{\Delta t}}}$ is indeed a function of R , where R is the interest rate for the parent node.

EXHIBIT 7

Three-Period Two-Factor Tree with Default and Recovery

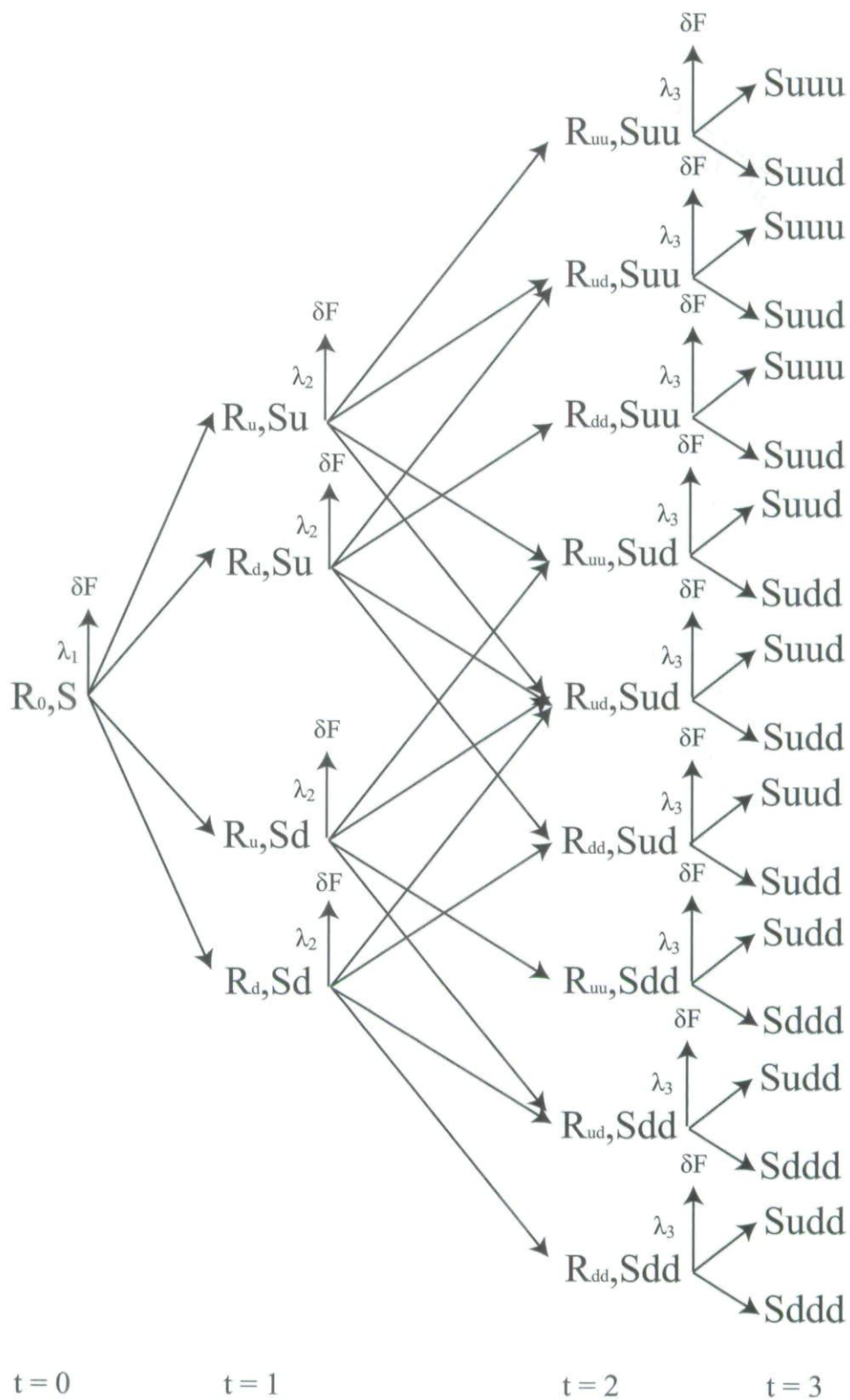
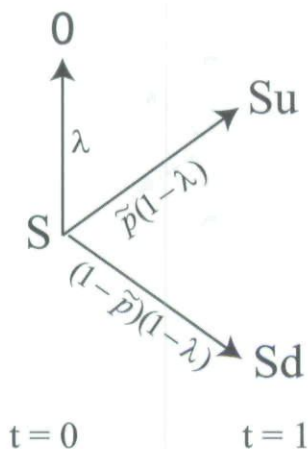


EXHIBIT 8

CRR Tree with Default Added



Note that ρ is never close to $+1(-1)$, hence p_1, p_2, p_3, p_4 will not be negative in practice. In fact, under the mild condition that $\rho^2/(1+\rho^2) \leq \tilde{p} \leq 1/(1+\rho^2)$, all four probabilities (p_1, p_2, p_3, p_4) will be between 0 and 1. (See Appendix C for the proof.) Thus our modeling of correlation following the approach of Hull [2003, p. 474] generates reasonable probabilities.

Now we need to confirm that the combined two-factor tree in Exhibit 7 prices the stock correctly, in other words, that the probability measure is still risk neutral.

The price of the stock

$$\begin{aligned}
 &= e^{-r_f \Delta t} [0\lambda_i + Sup_1(1-\lambda_i) + Sup_2(1-\lambda_i) \\
 &\quad + Sdp_3(1-\lambda_i) + Sdp_4(1-\lambda_i)] \\
 &= e^{-r_f \Delta t} \left[Su \frac{1}{2} (p + \sqrt{p(1-p)\rho})(1-\lambda_i) \right. \\
 &\quad + Su \frac{1}{2} (p - \sqrt{p(1-p)\rho})(1-\lambda_i) \\
 &\quad + Sd \frac{1}{2} (1-p - \sqrt{p(1-p)\rho})(1-\lambda_i) \\
 &\quad \left. + Sd \frac{1}{2} (1-p + \sqrt{p(1-p)\rho})(1-\lambda_i) \right] \\
 &= e^{-r_f \Delta t} [Sup(1-\lambda_i) + Sd(1-p)(1-\lambda_i)]
 \end{aligned}$$

By the same argument as in the previous subsection, the remaining right hand side of the above equation is equal to S —proving that the stock price derived through

backward induction and risk neutrality is equal to the current stock price used to create the tree. Hence, our recombining tree satisfies the no-arbitrage condition in order to assume risk-neutral pricing. In fact, our tree can be used to price derivatives other than convertible bonds.

Pricing the Convertible Bond with Backward Induction

Using the tree in Exhibit 7 and the probabilities derived in the previous subsection, we demonstrate backward induction in the case of a convertible bond by starting with the terminal nodes. We assume that the terminal value of the convertible bond is equal to δF when default happens at the bond's expiration. When there is no default, the value is equal to $\text{Max} [\text{Conversion value, Bond face } F]$ as shown in Exhibit 11. Note that Conversion value is defined as the product of the conversion ratio n and the stock price at the given node. We first calculate the weighted average by the probabilities $\lambda_n, (1-\lambda_n)\tilde{p}$, or $(1-\lambda_n)(1-\tilde{p})$ respectively, ($\tilde{p}$ is the probability previously defined) and then discount back by $e^{-R\Delta t}$, where R is the previous node's interest rate.

EXHIBIT 9

Probability Measure for Exhibit 7

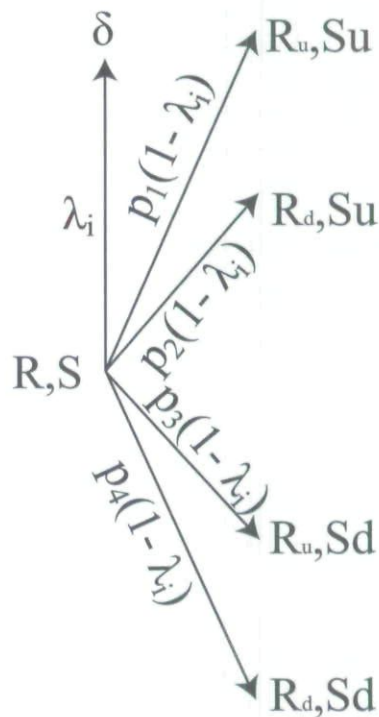


EXHIBIT 10

Probability Measure for Our Model

R\S	Su	Sd	Marginal for R
R_u	$p_1 = \frac{1}{2} (\tilde{p} + \sqrt{\tilde{p}(1-\tilde{p})}\rho)$	$P_3 = \frac{1}{2} (1 - \tilde{p} - \sqrt{\tilde{p}(1-\tilde{p})}\rho)$	$\frac{1}{2} = \pi$
R_d	$p_2 = \frac{1}{2} (\tilde{p} - \sqrt{\tilde{p}(1-\tilde{p})}\rho)$	$P_4 = \frac{1}{2} (1 - \tilde{p} + \sqrt{\tilde{p}(1-\tilde{p})}\rho)$	$\frac{1}{2} = 1 - \pi$
Marginal for S	$\tilde{p}$	$1 - \tilde{p}$	1

The resulting value is called a *rollback value* at this particular node. In general, we can get the rollback value for the convertible bond at each node by discounting the probability weighted average of all its children. The rollback value is not necessarily the convertible bond value at each node. The convertible bond value at each node can be stated as:

$$\text{Max} [\text{Min}(\text{rollback value}, \text{Call price}_t), \text{Conversion value}_t, \text{Put value}_t]$$

The above expression permits the modeling of a put feature and call feature to the convertible bond. Call price_t (or Put value_t) is the call price (put price) at time t. If it is not callable at t, then we can set Call price_t = +∞. If the bond is not puttable at t, then we can set Put value_t = 0.

At the initial node, it is obvious that Call price₀ = +∞ and Put value₀ = 0 (not callable and not puttable), hence

the optimal convertible bond value is $\text{Max} [\text{Min} (\text{rollback value}, \infty), \text{Conversion value}, 0] = \text{Max} [\text{rollback value}, \text{Conversion value}]$.

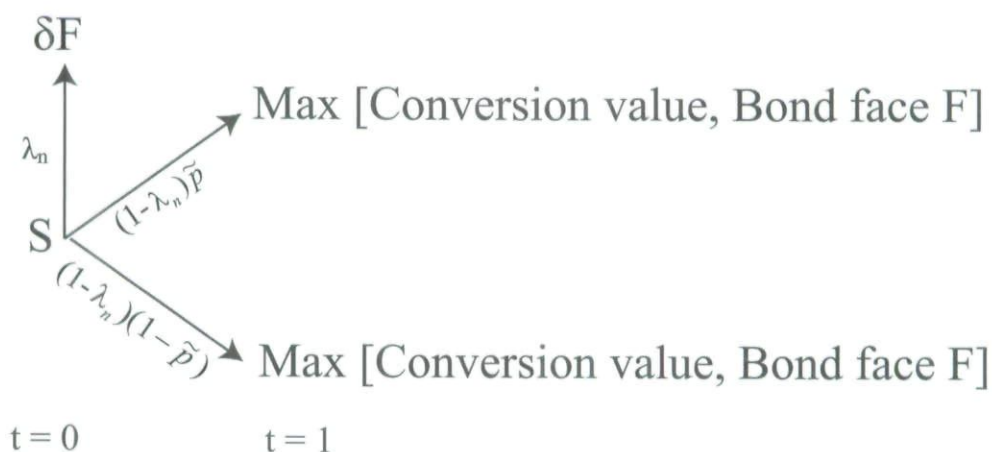
This completes the convertible bond pricing process of our model. We have made some extensions to traditional two-factor models including allowing correlation between the stock price and the riskless interest rate, recombining nodes and adjusting the CRR probabilities for default. The resulting tree has relatively few nodes and is arbitrage free.

A NUMERICAL EXAMPLE OF USING THE TREE TO PRICE A CONVERTIBLE BOND

This section details two numerical examples of our model using similar input data to Hung and Wang's examples: a hypothetical example and an actual example.

EXHIBIT 11

Convertible Bond Terminal Node



We try to provide extensive detail to assist readers in constructing and verifying a tree model. The first example is detailed in the first four subsections, and the second example, based on an actual Lucent bond, is detailed in the remaining subsections.

The Assumptions of the First Example

Example 1 is a three-year zero-coupon, no dividend convertible bond with a call feature.

Input:

$S_0 = 30$, $\sigma_s = 0.23$, $T = 3$, $\Delta t = 1$, $\text{Call} = 105$, $F = 100$, $\delta = 0.32$, $n = 3$, $R_0 = R_1 = R_2 = 0.10$, $R_0^* = R_1^* = R_2^* = 0.15$, $\sigma_r = 0.10$, $\rho = -0.1$ (all variables as previously defined).

This example from the Hung and Wang article is a hypothetical example in which all of the values are simply assumed. Note that the term structure is flat in the above example. Both Hung and Wang's article and our article only use this assumption for simplicity in the example and for comparing with other models that assume a flat term structure.

The Risk-Free Interest Rate Tree

Our risk-free interest rate tree is shown in Exhibit 12 and is almost the same as Hung and Wang's. We believe the difference is due to different rounding. It is easy to check the relation between the up node and the down node since $R_u = R_d e^{2\sigma_r \sqrt{\Delta t}}$.

The Default Probabilities (the Risky Interest Rate Tree)

Our default probabilities are shown in Exhibit 13. We should expect some differences from Hung and Wang

because our model has a slightly different tree since we assume that recovery occurs at the end of the default period rather than at the end of the subsequent period.

The Convertible Bond Price

Exhibit 14 compares our results with those of Hung and Wang.

In order to confirm the results of our model, we perform two tests. If we turn off the stock price process by setting $S_0 = 0$ and $\rho = 0$, the convertible bond price is 63.76 which matches today's risky bond term structure ($63.76 = 100 e^{-0.15 \cdot 3}$). If we also turn off the default process by setting $\lambda = 0$ for all time periods, the convertible bond price is 74.08 which matches today's Treasury yield curve ($74.08 = 100 e^{-0.10 \cdot 3}$).

Assumptions of the Second Example

Example 2 is Lucent's six-year zero-coupon, no dividend yield convertible bond with a call schedule—a real bond with market data as inputs.

Inputs:

$S_0 = 15.006$, $\sigma_s = 0.353836$, $T = 6$, $\Delta t = 1$, $\text{Call}_3 = 94.205$, $\text{Call}_4 = 96.098$, $\text{Call}_5 = 98.030$, $F = 100$, $\delta = 0.32$, $n = 5.07524$, $R_0 = 0.05969$, $R_1 = 0.06209$, $R_2 = 0.06373$, $R_3 = 0.06455$, $R_4 = 0.06504$, $R_5 = 0.06554$, $R_0^* = 0.0611$, $R_1^* = 0.0646$, $R_2^* = 0.0663$, $R_3^* = 0.0678$, $R_4^* = 0.0683$, $R_5^* = 0.06894$, $\sigma_r = 0.10$, $\rho = -0.1$.

The \$15.006 share price for Lucent is the previous closing price for the underlying shares on the day before the bond's issuance as reported by Hung and Wang. The historical volatility of the underlying stock was computed by Hung and Wang from daily stock prices from May 15, 1995, to May 15, 1997. The call prices are terms of the bond contract. The 32% recovery rate was assumed by

EXHIBIT 12

Risk-Free Interest Tree Output

Hung and Wang's	Ours
$R_0=0.10$, $R_u=0.1099$, $R_d=0.09$, $R_{uu}=0.1209$, $R_{ud}=0.099$, $R_{dd}=0.081$.	$R_0=0.10$, $R_u=0.1099$, $R_d=0.09$, $R_{uu}=0.1213$, $R_{ud}=0.0993$, $R_{dd}=0.0813$.

EXHIBIT 13

Default Probabilities Output

Hung and Wang's	Ours
$\lambda_1 = 0.0717, \lambda_2 = 0.0735, \lambda_3 = 0.0755$	$\lambda_1 = 0.0717214, \lambda_2 = 0.0774391,$ $\lambda_3 = 0.0843733$

EXHIBIT 14

The Convertible Bond Price

Hung and Wang's	Ours
Rollback value at initial node=86.15	Rollback value at initial node=93.15353
The convertible bond price= Max [rollback value, Conversion value]=90.	The convertible bond price= Max [rollback value, Conversion value]=93.15353.

EXHIBIT 15

The Risk-Free Interest Tree Output

Hung and Wang's	Ours
Not listed in paper	$R_0=0.05969, R_u=0.0709403, R_d=0.05808,$ $R_{uu}=0.08114, R_{ud}=0.0664319,$ $R_{dd}=0.0543898,$ $R_{uuu}=0.0893761, R_{uud}=0.0731752,$ $R_{udd}=0.0599109, R_{ddd}=0.049051,$ $R_{uuuu}=0.0985016,$ $R_{uuud}=0.0806464, R_{uudd}=0.0660279,$ $R_{uddd}=0.0540592, R_{dddd}=0.04426,$ $R_{uuuuu}=0.110334, R_{uuuud}=0.0903339,$ $R_{uuudd}=0.0739593,$ $R_{uuddd}=0.0605529, R_{udddd}=0.0495766,$ $R_{ddddd}=0.04059$

EXHIBIT 16

Default Probabilities Output

Hung and Wang's	Ours
$\lambda_1 = 0.0021, \lambda_2 = 0.0053, \lambda_3 = 0.0040,$ $\lambda_4 = 0.0078, \lambda_5 = 0.0049, \lambda_6 = 0.0061$	$\lambda_1 = 0.00207207, \lambda_2 = 0.00537085,$ $\lambda_3 = 0.00424289, \lambda_4 = 0.00817737,$ $\lambda_5 = 0.00558215, \lambda_6 = 0.00703698$

EXHIBIT 17

The Convertible Bond Price

Hung and Wang's	Ours
the convertible bond price= 90.4633 Implied volatility=0.322459 (Market 88.7060)	the convertible bond price= 90.83511 Implied volatility=0.315995 (Market 88.7060)

Hung and Wang and in practice would require analysis of historical recovery rates and professional judgment. The riskless interest rates were taken from Bloomberg on the date of issue by Hung and Wang, as were the risky rates (which were of the rating Aa2). The 10% volatility of the risk-free zero-coupon bond yield was assumed by Hung and Wang and in practice could be derived from historical data. The correlation of -0.10 was our assumed correlation of interest rates and stock prices and could be formed from historical analysis and professional judgment.

The Risk-Free Interest Rate Tree—Second Example

Exhibit 15 shows the risk-free interest rate generated by our model for the Lucent bond example (Hung and Wang did not publish their rates).

The Default Probabilities (The Risky Interest Rate Tree)—Second Example

Exhibit 16 shows the default probabilities from our model and those reported by Hung and Wang. As before,

we should expect some differences because our model has a slightly different assumption regarding the timing of recovery.

The Convertible Bond Price—The Second Example

Exhibit 17 shows the results of the prices of our model and those of Hung and Wang for the Lucent bond example.

If we keep all other inputs in Lucent's example and change the correlation to $\rho = 0.1$, then the convertible bond price = 91.09971—a change in our model of approximately 0.26.

Exhibit 18 summarizes the sensitivity of our model's convertible bond price with respect to the correlation ρ .

CONCLUSION

Our tree model for pricing convertible bonds is based on the most common two-factor models using CRR modeling of stock prices and Ho-Lee modeling of interest rates. We utilized the approach of Jarrow and Rudd to model default probability and we followed the general approach of Hull to model correlation.

EXHIBIT 18

Correlation Sensitivity

Correlation ρ	Convertible bond price in our model
-0.15	90.76877
-0.1	90.83511
-0.05	90.90137
0	90.96755
0.05	91.03367
0.10	91.09971
0.15	91.16569

The numerical examples indicate that our model produces a moderately different convertible bond price than that found by Hung and Wang. The numerical examples also indicate a moderate sensitivity of convertible bond prices to the assumed correlation between the factors. Correlation between the stock price and interest rate levels has been observed to be especially important in the pricing of the convertible bonds of financial institutions.

A primary path for future research is to allow the default rate to vary with one or more of the factors. In our model, default is modeled as a sudden event unrelated to the level of the stock price. While this is consistent with numerous recent bankruptcies including Worldcom, Enron, and Global Crossings, it is not consistent with bankruptcies that occur slowly and with a declining stock price. The default probability can be easily modeled within the tree to depend on the stock price level. However, it is a very complex problem to retain risk neutral pricing.

APPENDIX A

Derivation of Probabilities with Correlated Factors

In our proof, we will assume that $\Delta t = 1$ in all trees. In fact, Δt in the numerator and denominator can be cancelled. So, our proof can be easily generalized.

As detailed in the first section, we model interest rates using the methodology of BDT tree as in Baz and Chacko ([2004, p. 162]) with a constant variance, where $R_u = R_d e^{2\sigma_r}$ and σ_r^2 is the variance of the natural logarithm of the short term riskless interest rate. The interest rate tree is illustrated in Exhibit 19.

It follows that $\ln R_u - \ln R_d = 2\sigma_r$. Therefore,

$$\text{Var}(\ln R) = \left[\frac{1}{2} (\ln R_u - \ln R_d) \right]^2$$

As detailed in the first section, we model equity prices as a CRR stock tree as Hull [2003]. The stock price tree is illustrated in Exhibit 20.

With $S_u = S^*u$ and $S_d = S^*d$, therefore,

$$\text{Var}(S) = E(S^2) - (ES)^2$$

By definition, $E(S^2) = (S^2u^2)p + (S^2d^2)(1-p)$ and $ES = (Su)p + (Sd)(1-p)$

Therefore,

$$\begin{aligned} \text{Var}(S) &= S^2u^2p + S^2d^2(1-p) - (Su p + Sd(1-p))^2 \\ &= S^2u^2p(1-p) + S^2d^2(1-p)p - 2S^2udp(1-p) \\ &= p(1-p)S^2(u-d)^2 \\ &= [\sqrt{p(1-p)}S(u-d)]^2 \end{aligned}$$

As shown in the first section, we combine the two trees into one and assign probabilities p_1, p_2, p_3, p_4 as illustrated in Exhibit 21 under the assumption that the correlation between S and $\ln R$ is ρ .

By definition,

$$\begin{aligned} \text{Correlation between}(S, \ln R) &= \frac{\text{Cov}(S, \ln R)}{\sqrt{\text{Var}(\ln R)}\sqrt{\text{Var}(S)}} \\ &= \frac{E(S \ln R) - E(S)E(\ln R)}{\frac{1}{2}(\ln R_u - \ln R_d)\sqrt{p(1-p)}S(u-d)} \\ &= \frac{Su \ln R_u p_1 + Su \ln R_d p_2 + Sd \ln R_u p_3 + Sd \ln R_d p_4 - (pSu + (1-p)Sd)(\frac{1}{2} \ln R_u + \frac{1}{2} \ln R_d)}{\frac{1}{2}(\ln R_u - \ln R_d)\sqrt{p(1-p)}S(u-d)} \end{aligned}$$

EXHIBIT 19
BDT Interest Rate Tree

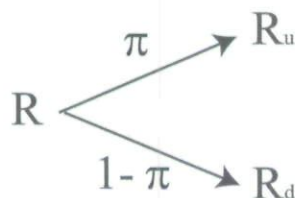


EXHIBIT 20
CRR Stock Tree

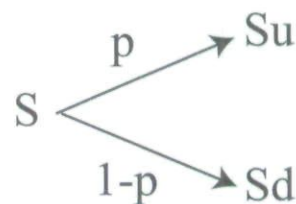
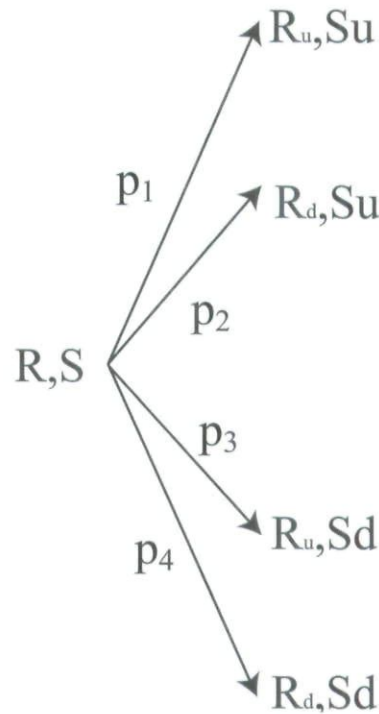


EXHIBIT 21

Two Factor Tree



We need to find expressions for p_1 , p_2 , p_3 , and p_4 such that the above equation equals to ρ .

$$\frac{SuLnR_u p_1 + SuLnR_d p_2 + SdLnR_u p_3 + SdLnR_d p_4 - (pSu + (1-p)Sd) \left(\frac{1}{2} LnR_u + \frac{1}{2} LnR_d \right)}{\frac{1}{2} (LnR_u - LnR_d) \sqrt{p(1-p)} S(u-d)} = \rho$$

It is equivalent to solving the following equation:

$$\begin{aligned} & SuLnR_u p_1 + SuLnR_d p_2 + SdLnR_u p_3 + SdLnR_d p_4 - (pSu + (1-p)Sd) \left(\frac{1}{2} LnR_u + \frac{1}{2} LnR_d \right) \\ &= \frac{1}{2} (LnR_u - LnR_d) \sqrt{p(1-p)} S(u-d) \rho \end{aligned}$$

It is equivalent to solving the following equation:

$$\begin{aligned} & SuLnR_u p_1 + SuLnR_d p_2 + SdLnR_u p_3 + SdLnR_d p_4 = (pSu + (1-p)Sd) \left(\frac{1}{2} LnR_u + \frac{1}{2} LnR_d \right) \\ &+ \frac{1}{2} (LnR_u - LnR_d) \sqrt{p(1-p)} S(u-d) \rho \end{aligned}$$

Expanding the right hand side of the equation, we have

$$\begin{aligned} & \left(\frac{1}{2} p + \frac{1}{2} \sqrt{p(1-p)} \rho \right) SuLnR_u + \left(\frac{1}{2} p - \frac{1}{2} \sqrt{p(1-p)} \rho \right) SuLnR_d + \left(\frac{1}{2} (1-p) - \frac{1}{2} \sqrt{p(1-p)} \rho \right) SdLnR_u \\ &+ \left(\frac{1}{2} (1-p) + \frac{1}{2} \sqrt{p(1-p)} \rho \right) SdLnR_d \end{aligned}$$

In order to match the left hand side, we have

$$p_1 = \frac{1}{2}(p + \sqrt{p(1-p)}\rho), p_2 = \frac{1}{2}(p - \sqrt{p(1-p)}\rho), p_3 = \frac{1}{2}(1-p - \sqrt{p(1-p)}\rho) \\ p_4 = \frac{1}{2}(1-p + \sqrt{p(1-p)}\rho)$$

So p_1, p_2, p_3, p_4 produce the correlation ρ . We also need to check that the marginal distributions match the original stock tree and interest tree respectively.

It is clear that from Exhibit 22 that it satisfies the condition.

APPENDIX B

Proof of risk neutral measure

We will show that the probability measure $\{\lambda, \tilde{p}(1-\lambda), (1-\tilde{p})(1-\lambda)\}$ is a risk-neutral measure we can use to price the derivatives.

Assume that we have the derivative tree as shown in Exhibit 23. The derivative has a payoff f_u when the stock price is up; it has a payoff f_d when the stock price is down; it has a payoff f_{default} when default happens.

By the probability measure $\{\lambda, \tilde{p}(1-\lambda), (1-\tilde{p})(1-\lambda)\}$, the price of our derivative today is:

$$\begin{aligned} f &= e^{-r_f \Delta t} [f_{\text{default}}\lambda + f_u p(1-\lambda) + f_d(1-p)(1-\lambda)] \\ &= e^{-r_f \Delta t} \left[f_{\text{default}}\lambda + f_u \frac{e^{r_f \Delta t}/(1-\lambda) - d}{u-d} (1-\lambda) + f_d \frac{u - e^{r_f \Delta t}/(1-\lambda)}{u-d} (1-\lambda) \right] \\ &= e^{-r_f \Delta t} \left[f_{\text{default}}\lambda + f_u \frac{e^{r_f \Delta t} - d(1-\lambda)}{u-d} + f_d \frac{u(1-\lambda) - e^{r_f \Delta t}}{u-d} \right] \\ &= e^{-r_f \Delta t} \left[f_{\text{default}}\lambda + \frac{f_u e^{r_f \Delta t} - f_u d(1-\lambda)}{u-d} + \frac{f_d u(1-\lambda) - f_d e^{r_f \Delta t}}{u-d} \right] \\ &= e^{-r_f \Delta t} \left[f_{\text{default}}\lambda + \frac{f_u e^{r_f \Delta t} - f_u d(1-\lambda) + f_d u(1-\lambda) - f_d e^{r_f \Delta t}}{u-d} \right] \\ &= e^{-r_f \Delta t} \left[f_{\text{default}}\lambda + \frac{(f_u - f_d)e^{r_f \Delta t} + (f_d u - f_u d)(1-\lambda)}{u-d} \right] \\ &= e^{-r_f \Delta t} \left[f_{\text{default}}\lambda + \frac{(f_u - f_d)e^{r_f \Delta t}}{u-d} + \frac{(f_d u - f_u d)(1-\lambda)}{u-d} \right] \\ &= e^{-r_f \Delta t} f_{\text{default}}\lambda + \frac{(f_u - f_d)}{u-d} + e^{-r_f \Delta t} \frac{(f_d u - f_u d)(1-\lambda)}{u-d} \quad (*) \end{aligned}$$

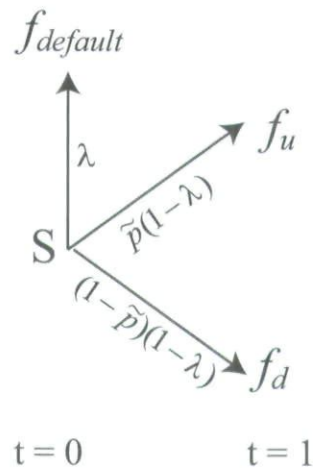
EXHIBIT 22

Probability Measure for Two-Factor Tree with Correlation

R \ S	Su	Sd	Marginal for R
R_u	$p_1 = \frac{1}{2}(p + \sqrt{p(1-p)}\rho)$	$p_3 = \frac{1}{2}(1-p - \sqrt{p(1-p)}\rho)$	$\frac{1}{2} = \pi$
R_d	$p_2 = \frac{1}{2}(p - \sqrt{p(1-p)}\rho)$	$p_4 = \frac{1}{2}(1-p + \sqrt{p(1-p)}\rho)$	$\frac{1}{2} = 1 - \pi$
Marginal for S	p	$1-p$	1

EXHIBIT 23

CRR Stock Tree with Default Added



Now we consider the following replicating portfolio:

- 1) Long a defaultable bond with value x one period later.
- 2) Short a riskless bond with value y one period later.
- 3) Buy Δ shares of stock

Exhibit 24 demonstrates the payoff of the replicating portfolio in different states (r_b stands for the risky interest rate). Recall that δ is the recovery rate. In order to make our portfolio have the same payoff as (replicate) the derivative, we have the following equations:

$$\begin{cases} Su\Delta - y + x = f_u & (1) \\ Sd\Delta - y + x = f_d & (2) \\ -y + x\delta = f_{default} & (3) \end{cases}$$

Equation (1) and (2) implies

$$\Delta = \frac{f_u - f_d}{Su - Sd} \quad (4)$$

and Equation (3) implies

$$y = x\delta - f_{default} \quad (5)$$

Now inserting Equation (5) in (1), we have

$$Su\Delta - x\delta + f_{default} + x = f_u \quad (6)$$

It further implies

$$-x\delta + x = f_u - f_{default} - Su\Delta \quad (7)$$

which is equivalent to

$$x = \frac{f_u - f_{default} - Su\Delta}{1 - \delta} \quad (8)$$

EXHIBIT 24

Portfolio Value in Different States

Today's value of the portfolio:	One period later value
$V = S \Delta - e^{-r_f \Delta t} y + e^{-r_b \Delta t} x$	Stock up: $S_u \Delta - y + x$ Stock down: $S_d \Delta - y + x$ Default happens: $0 \Delta - y + x \delta$

Now we consider today's value of our replicating portfolio:

$$\begin{aligned}
 & S\Delta - e^{-r_f \Delta t} y + e^{-r_b \Delta t} x \\
 &= S\Delta - e^{-r_f \Delta t} (x\delta - f_{\text{default}}) + e^{-r_b \Delta t} x \quad (\text{insert eq.(3)}) \\
 &= S\Delta - e^{-r_f \Delta t} x\delta + e^{-r_f \Delta t} f_{\text{default}} + e^{-r_b \Delta t} x \\
 &= S\Delta - e^{-r_f \Delta t} x\delta + e^{-r_b \Delta t} x + e^{-r_f \Delta t} f_{\text{default}} \\
 &= S\Delta + \frac{f_u - f_{\text{default}} - Su\Delta}{1 - \delta} (-e^{-r_f \Delta t} \delta + e^{-r_b \Delta t}) + e^{-r_f \Delta t} f_{\text{default}} \\
 &= S\Delta + \frac{f_u - f_{\text{default}} - Su\Delta}{1 - \delta} (-e^{-r_f \Delta t} \delta + e^{-r_b \Delta t}) + e^{-r_f \Delta t} f_{\text{default}}
 \end{aligned} \tag{9}$$

It does not look exactly the same as the equation (*). However, we know that r_f , r_b , λ , and δ are not independent.

Exhibit 25 illustrates a defaultable bond. The left hand side of Equation 10 is the expected value of the bond's cash flows discounted at the riskless rate. The right hand side discounts the face value of the bond at the risky rate.

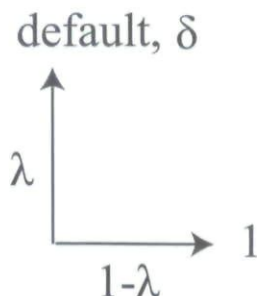
$$e^{-r_f \Delta t} (1(1 - \lambda) + \delta \lambda) = e^{-r_b \Delta t} 1 \tag{10}$$

Equation (10) is equivalent to:

$$e^{-r_f \Delta t} (1 - \lambda) + e^{-r_f \Delta t} \delta \lambda = e^{-r_b \Delta t} \tag{11}$$

EXHIBIT 25

One Period Bond with Default



Subtracting $e^{-r_f \Delta t} \delta$ on both side of Equation (11), we get

$$e^{-r_f \Delta t} (1 - \lambda) + e^{-r_f \Delta t} \delta \lambda - e^{-r_f \Delta t} \delta = -e^{-r_f \Delta t} \delta + e^{-r_f \Delta t} \delta \quad (12)$$

After simplifying the left hand side, we have

$$e^{-r_f \Delta t} (1 - \delta)(1 - \lambda) = -e^{-r_f \Delta t} \delta + e^{-r_f \Delta t} \delta \quad (13)$$

Now inserting (13) into (9), we have

$$\begin{aligned} S\Delta + e^{-r_f \Delta t} (f_u - f_{\text{default}} - Su\Delta)(1 - \lambda) + e^{-r_f \Delta t} f_{\text{default}} \\ = S\Delta + e^{-r_f \Delta t} (f_u - Su\Delta)(1 - \lambda) - e^{-r_f \Delta t} f_{\text{default}} (1 - \lambda) + e^{-r_f \Delta t} f_{\text{default}} \\ = S\Delta + e^{-r_f \Delta t} (f_u - Su\Delta)(1 - \lambda) + e^{-r_f \Delta t} f_{\text{default}} \lambda \end{aligned}$$

Now insert equation (4) in Δ ,

$$\begin{aligned} \text{We have } S \frac{f_u - f_d}{Su - Sd} + e^{-r_f \Delta t} \left(f_u - Su \frac{f_u - f_d}{Su - Sd} \right) (1 - \lambda) + e^{-r_f \Delta t} f_{\text{default}} \lambda \\ = \frac{f_u - f_d}{u - d} + e^{-r_f \Delta t} \left(f_u - \frac{uf_u - uf_d}{u - d} \right) (1 - \lambda) + e^{-r_f \Delta t} f_{\text{default}} \lambda \\ = \frac{f_u - f_d}{u - d} + e^{-r_f \Delta t} \left(\frac{f_u u - f_u d - uf_u + uf_d}{u - d} \right) (1 - \lambda) + e^{-r_f \Delta t} f_{\text{default}} \lambda \\ = \frac{f_u - f_d}{u - d} + e^{-r_f \Delta t} \left(\frac{uf_d - f_u d}{u - d} \right) (1 - \lambda) + e^{-r_f \Delta t} f_{\text{default}} \lambda \end{aligned}$$

After rearranging the terms, it is exactly the same as (*)

Since our replicate portfolio's payoff is the same as the derivative's payoff, our value of the replicate portfolio today is the price of the derivative. It matches with the price we obtain when we use the probability measures $\{\lambda, \tilde{p}(1 - \lambda), (1 - \tilde{p})(1 - \lambda)\}$. Hence this probability measure is the risk-neutral measure.

APPENDIX C

The proof of our mild condition

In order to make our probability p_1 (see Exhibit 10) satisfying $0 \leq p_1 \leq 1$, we only need

$$0 \leq \frac{1}{2} (p + \sqrt{p(1-p)\rho}) \leq 1 \quad (14)$$

It is equivalent to

$$0 \leq (p + \sqrt{p(1-p)\rho}) \leq 2$$

Note that $p + \sqrt{p(1-p)\rho} \leq 2$ is always true because $0 \leq \tilde{p} \leq 1$ and $\sqrt{p(1-p)\rho} \leq 1$.

We only need

$$0 \leq (p + \sqrt{p(1-p)\rho})$$

It is equivalent to show

$$-\sqrt{p(1-p)}\rho \leq p \quad (15)$$

Note that it is true for any $\rho \geq 0$.

For $\rho < 0$, Inequality (15) is equivalent to

$$p(1-p)\rho^2 \leq p^2 \quad (16)$$

It is easy to see the inequality (16) can be solved, namely

$$\frac{\rho^2}{1+\rho^2} \leq p \quad \text{for } \rho < 0 \quad (17)$$

In order to make our probability p_4 satisfying $0 \leq p_4 \leq 1$, we only need

$$0 \leq \frac{1}{2}(1-p+\sqrt{p(1-p)}\rho) \leq 1 \quad (18)$$

Assume $\tilde{q} = 1 - \tilde{p}$. Then inequality (18) is equivalent to

$$0 \leq \frac{1}{2}(q+\sqrt{q(1-q)}\rho) \leq 1 \quad (19)$$

Note that inequality (19) is equivalent inequality (14).

Hence, we can get solution

$$\frac{\rho^2}{1+\rho^2} \leq q \quad \text{for } \rho < 0 \quad (20)$$

Substituting $\tilde{q} = 1 - \tilde{p}$ in inequality (20), we have

$$p \leq \frac{1}{1+\rho^2} \quad \text{for } \rho < 0 \quad (21)$$

Putting (17) and (21) together, we have

$$\frac{\rho^2}{1+\rho^2} \leq p \leq \frac{1}{1+\rho^2} \quad \text{for } \rho < 0 \quad (22)$$

Note that from p_1 and p_4 , by replacing ρ with $-\rho$, we will get the p_2 and p_3 .

Hence, the condition for making p_2 and p_3 to be probability is:

$$\frac{\rho^2}{1+\rho^2} \leq p \leq \frac{1}{1+\rho^2} \quad \text{for } \rho \geq 0 \quad (23)$$

Putting (22) and (23) together, we need

$$\frac{\rho^2}{1+\rho^2} \leq p \leq \frac{1}{1+\rho^2}$$

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