Higher Order Greeks

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We examine whether any second order derivatives other than gamma and any third order derivatives are important in explaining changes in the prices of S&P 500 futures options over one week holding periods. We find that while gamma is normally most important, several other higher order derivatives have considerable incremental explanatory power. Particularly important in accounting for option price changes are the derivatives of delta with respect to volatility and time-to-expiration, and the derivatives of gamma with respect to the asset price, and volatility. The first three are more important for away-from-the-money options, while the fourth is most important for at-the-money options. For shorter-term options, consideration of higher order derivatives reduces the mean absolute unexplained price change by 60% for at-the-money options and by at least 75% for away-from-the-money options. We find that in spite of its theoretical problems and inability to explain the cross-sectional option price pattern (the smile), the Black-Scholes model's Greeks accurately describe the time series option price changes once higher order Greeks are incorporated. We further find that making delta-gamma-vega neutral portfolios of S&P 500 options neutral in terms of these four higher order Greeks leads to a substantial reduction in the risk of an unhedged price change.

Traditionally the riskiness of options (or assets or portfolios with option characteristics) has been described in terms of four first derivatives: delta, theta, vega, and rho and one second derivative: gamma. To our knowledge, every major derivatives text uses these five derivatives and these alone to analyze option price changes. The focus on these five "Greeks" raises the question whether any higher order derivatives besides gamma are important in explaining option price changes and when hedging. In other words, do any other higher order derivatives represent significant risks? Including cross-partials, such as the derivative of delta with respect to volatility, there are ten second derivatives for the Black-Scholes (BS) model but only gamma has received much attention to date. Is this focus on gamma to the exclusion of other second derivatives justified? Are any third order derivatives important? When hedging, is it important or worthwhile to make the portfolio neutral in terms of any of these higher moments?

In this article, we analyze option Greeks focusing particular attention on the higher order derivatives. Specifically, we measure the ability of Greeks measured at the beginning of a week to explain subsequent one week changes in S&P 500 futures option prices over one week holding periods from January 1, 1988 through March 31, 2003. Our findings include the following. One, the profession's focus on gamma appears justified in that it is normally the most important of the higher order derivatives in explaining option price changes over one week holding periods in this market. Two, nonetheless, other higher order derivatives have considerable incremental explanatory power. For instance, in our sample of at-the-money 4 to 8 week options, the average absolute unexplained price change is reduced 61% by incorporating other second and third order terms into the model and their incremental explanatory power is even greater for out-of-the-money options. Three, in our sample, the four most important higher order derivatives besides gamma are: $\partial^2O/\partial F\partial\sigma$, $\partial^2O/\partial F\partial T$, $\partial^3O/\partial F^3$, and $\partial^3O/\partial F^2\partial\sigma$ where O is the option price, F is the underlying asset price, σ is volatility, and T is time to expiration. Four, relative importance varies with the strike price (also time to expiration). Some higher order derivatives, such as $\partial^2O/\partial F\partial\sigma$, $\partial^2O/\partial F\partial T$, and $\partial^3O/\partial F^3$, are close to zero for at-the-money options and much larger away-from-the-money, while others, such as $\partial^3O/\partial F^2\partial\sigma$, peak close-to-the-money. Five, hedging these four higher order moments leads to a substantial reduction in risk. Across the eight portfolios that we consider, the mean absolute unhedged value change is reduced by an average of approximately 90% relative to that of the delta-gamma-vega neutral portfolio when the portfolios are re-balanced to make them neutral in terms of these four higher moments as well.

The results of this analysis allow us to examine how accurately the Black-Scholes (hereafter BS) model explains the time series pattern of option prices. The theoretical shortcomings of the BS model, in particular its assumption of log-normal returns and constant (or deterministic) volatility, are well known and its inability to explain the cross-sectional pattern of option prices (the smile) is well documented. We examine whether changes over time in option prices conform to those predicted by the BS model Greeks. For example, if a call's BS delta (calculated at the beginning of the week) is .4 and its gamma is .03 and the underlying asset price rises \$2.0, does the call price increase .4(2.0) +.5(.03)(4.0) = \$.86? We find that if higher order Greeks are not considered, one week changes in option prices do not match those predicted by the five traditional BS Greeks very well. However, once higher order Greeks are added to the predictive model, time series changes in option prices closely conform to those predicted by BS. For instance, the average absolute difference between the actual one week change in S&P 500 option prices and that predicted by the BS Greeks is only about 1% for at-the-money options when second and third order Greeks are included. Despite its theoretical shortcomings and inability to explain the cross-sectional pattern of option prices, we find that the BS model explains the time series price pattern quite well, though this conclusion is contingent on the appropriateness of our volatility measure.

The article is organized as follows. In the following section we discuss the issues to be examined in this article and describe our methodology. Our data and calculations are described in *Data and Measures*. In section *Explaining Option Price Changes* we apply these measures and tests to the five traditional Greeks. Second order derivatives other than gamma are analyzed in section *Second Order Greeks* and third order in *Third Order Greeks*. In section *Hedging with Higher Order Greeks*, we consider the implications of our results for hedging—specifically how making hedge portfolios neutral in terms of higher order moments affects hedge effectiveness. Section *Conclusion* concludes the article.

ISSUES AND BASIC PROCEDURE

As noted above, five derivatives have been used in the options literature to describe the risks associated with options and other derivatives: delta, theta, vega, and rho (which are the derivatives of the option price with respect to the underlying asset price (F), the time to expiration (T), the volatility of returns on the underlying asset (σ), and the interest rate (r) respectively) and gamma, the second derivative of the option price with respect to F.[1] Formulas for each according to the Black version of the BS model for options on futures (Black [1976]) are reproduced in the Appendix. Searches of the on-line options literature revealed occasional references to two other second derivatives, "vomma" (or "volga"), which is described as the second derivative of the option price with respect to σ, and "vanna," the derivative of vega with respect to F.[2] However, such references are rare.

The focus on gamma to the exclusion of other higher order derivatives is probably due to the fact that in the derivation of the BS formula (Black and Scholes [1973]), the fundamental differential equation is a relation between delta, theta, gamma, and the call price. However, it doesn't necessary follow that gamma is uniquely important in explaining option price changes over discrete periods of time. Since changes in the underlying asset price appear responsible for most large changes in option prices, intuitively one would expect derivatives of delta to be more important than those of vega, theta, or rho, but gamma is not the only derivative of delta. The derivatives of delta with respect to volatility and time might also be important. An issue addressed in this article is how important these and other higher order derivatives are in accounting for changes in option prices, specifically options on S&P 500 futures, and, relatedly, whether it is important to hedge portfolios in terms of these higher moments.

Our approach is to use a Taylor series expansion with the various derivatives to account for option price changes. If $Y = f(X_1, X_2, \dots X_N)$ where f is continuously differentiable and N is the number of determinants of Y, then according to a Taylor series expansion, the change in Y can be approximated by:

$$\delta Y = \sum_{n=1}^{N} f_n(\delta X_n) + \frac{1}{2}\sum_{n=1}^{N}\sum_{j=1}^{N} f_{nj}(\delta X_n)(\delta X_j) + \frac{1}{6}\sum_{n=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} f_{njk}(\delta X_n)(\delta X_j)(\delta X_k) + \dots + \varepsilon \tag{1}$$

where $f_n = \frac{\partial f}{\partial X_n}$, $f_{nj} = \frac{\partial^2 f}{\partial X_n \partial X_j}$, $f_{njk} = \frac{\partial^3 f}{\partial X_n \partial X_j \partial X_k}$ and δX_n is the change in X_n. Much of the analysis in this article revolves around Equation (1). For instance, based on the five traditional Greeks, the "predicted" change in option price O_i from time t to t + 1 according to Equation (1) is:

$$\delta\hat{O}(5)_{it} = \Delta_{it}(\delta F_t) + \theta_{it}(\delta T) + \nu_{it}(\delta\sigma_{it}) + \rho_{it}(\delta r_{it}) + .5\Gamma_{it}(\delta F_t)^2 \tag{2}$$

where F is the underlying futures price, T is time-to-expiration,[3] σ is volatility of the underlying futures price, r is the interest rate, and Δ_{it}, θ_{it}, ν_{it} ρ_{it} and Γ_{it} represent option i's delta, theta, vega,[4] rho, and gamma respectively measured at time t. δX_t denotes the change in variable X from t to t + 1. The product of a derivative k, the change in the variable(s) to which it relates, and the Taylor series coefficient (1 for first order derivatives, 1/2 for second order, and 1/6 for third order) will be termed the "k risk factor," or simply the "k factor." For example, $\Delta_{it}(\delta F_t)$, will be referred to as the delta risk factor, or delta factor, and $.5\Gamma_{it}(\delta F_t)^2$ as the gamma factor. Since unlike changes in the other three price determinants, the passage of time is totally predictable, the theta factor is not really a *risk* factor, but to simplify the exposition we will use the same terminology for all five. The "5" in parentheses in $\delta\hat{O}(5)_{it}$ identifies it as the change in the option price "predicted" by the five traditional BS Greeks. Note that we use the term "predicted" in an ex-post sense to mean the option price change explained by the BS Greeks given knowledge of how the underlying asset price, volatility, time-to-expiration, and interest rate changed over the holding period. The derivatives at the beginning of the holding period are multiplied by the actual change in their associated determinants over the holding period. Thus, $\delta\hat{O}(5)_{it}$ is a measure of the change in the option price accounted for by the five Greeks.

Adding the other second derivatives to Equation (2) yields:

$$\delta\hat{O}(2nd)_{it} = \Delta_{it}(\delta F_t) + \theta_{it}(\delta T) + \nu_{it}(\delta\sigma_{it}) + \rho_{it}(\delta r_{it}) + .5\sum_{j=1}^{4}\sum_{k=1}^{4}\left(\frac{\partial^2 O}{\partial X_j \partial X_k}\right)_{it}(\delta X_j)_{it}(\delta X_k)_{it} \tag{3}$$

where $X_1 = F$, $X_2 = T$, $X_3 = \sigma$, and $X_4 = r$. Note that gamma is now included in the summation term with the other second order factors. Therefore $\delta\hat{O}(2nd)_{it}$ represents the price change over the period from t to t + 1 for option i predicted or accounted for by the second order model. To avoid adding a multitude of new terms to the options lexicon, we use combinations of the traditional Greek terms to describe the added second derivatives. For instance $\partial^2 O/\partial F\partial\sigma$ will be referred to as the delta-vega (or vega-delta) derivative and $(\frac{\partial^2 O}{\partial F \partial \sigma})_t$ (δF_t) $(\delta\sigma_t)$ as the delta-vega factor. Adding third order derivatives yields:

$$\delta\hat{O}(3rd)_{it} = \Delta_{it}(\delta F_t) + \theta_{it}(\delta T) + \nu_{it}(\delta\sigma_{it}) + \rho_{it}(\delta r_{it}) + .5\sum_{j=1}^{4}\sum_{k=1}^{4}\left(\frac{\partial^2 O}{\partial X_j \partial X_k}\right)_{it}(\delta X_j)_{it}(\delta X_k)_{it} + \left(\frac{1}{6}\right)\sum_{j=1}^{4}\sum_{k=1}^{4}\sum_{l=1}^{4}\left(\frac{\partial^3 O}{\partial X_j \partial X_k \partial X_l}\right)_{it}(\delta X_j)_{it}(\delta X_k)_{it}(\delta X_l)_{it} \tag{4}$$

Note that since cross-partials enter Equations (3) and (4) more than once, their Taylor coefficient depends on the number of entries. For instance, since $\partial^2 O/\partial F\partial\sigma = \partial^2 O/\partial\sigma\partial F$, the delta-vega factor is $(.5 + .5)\ (\partial^2 O/\partial F\partial\sigma)_{it}\ (\delta F_t)(\delta\sigma_{it})$ while the gamma factor is $.5\Gamma_{it}(\delta F_t)^2$.

One issue we explore is how well the $\delta\hat{O}_{it}$ from Equations (2), (3), and (4) explain the actual change δO_{it}. The issue is whether the five traditional risk factors account for most changes in market value and whether adding higher order Greeks substantially increases the model's explanatory power. We use two measures of fit: The first is the mean absolute error, MAE = $(1/(T \times I)) \Sigma_{t=1}^{T} \Sigma_{i=1}^{I} |FE_{it}|$ where $FE_{it} = \delta O_{it} - \delta\hat{O}_{it}$, δO_{jt} is the actual price change and $\delta\hat{O}_{it}$ is the change predicted by Equations (2), (3), or (4). The second is the root mean squared error, RMSE = $[(1/(T \times I)) \Sigma_{t=1}^{T} \Sigma_{i=1}^{I} FE_{it}^{2}]^{.5}$. Obviously, RMSE places more weight on large errors than MAE. Since prices of in-the-money options change much more than far-out-of-the-money options, we also report the MAE (RMSE) as a percentage of the mean absolute (root mean squared) price change. If MAE and RMSE for Equations (3) and (4) are little changed from the MAE and RMSE for Equation (2), then the profession's concentration on the five traditional Greeks is clearly justified.

We are also interested in the relative importance of the various risk factors. For this, we measure the incremental increase in MAE or RMSE when that factor is dropped from the model. Specifically, we first calculate the MAE and RMSE from Equation (4) with all first, second, and third order terms included. We then re-estimate MAE and RMSE with one risk factor removed. The increase in MSE and/or RMSE when a factor is removed provides a measure of that factor's incremental explanatory power.

DATA AND MEASURES

Data and Market

We collect daily prices of S&P 500 futures options and S&P 500 futures from January 1, 1988 through March 31, 2003. This is one of the most active and probably the most studied options market.[5] For our purposes, options on S&P index futures have a number of advantages over options on the S&P (cash) index or options on individual stocks: 1) since futures don't pay dividends, the number of option price determinants is reduced by one and the BS equation simplified, 2) the options and futures markets close at the same time minimizing the non-synchronous trading problem, and 3) arbitrage between options and the futures contract is easier than between options and all 500 stocks. In this market the implied volatility smile is quite pronounced (Bollen and Whaley [2004]; Jackwerth [2000]; and Ederington and Guan [2002]). Since the BS model obviously fails to explain the cross-sectional option price pattern, how well it explains the time series pattern of option prices in this market is of particular interest.

To ameliorate the problem of non-synchronous prices, we use daily settlement prices for both options and futures. Settlement prices have the advantage of being measured at the same time for both the option and the underlying futures and are the prices used to mark positions to market. However, if an option is thinly traded, they may not represent actual closing trade prices. If there was no trade near the close for an option with a particular expiration and strike, a settlement price is set by an exchange committee based on trades close to the close of options with the same expiration and similar strikes. Hence, we restrict our sample to actively traded strikes. Our profit/loss and price change measures are in terms of one S&P 500 index unit, e.g., a change in the S&P 500 index from 1100 to 1101 means a \$1 price change. Since a futures or options contract is for 250 index units, the actual change in value of a single futures or options contract is 250 times the figures reported here.

The Holding Period

We measure option price changes, δO_{jt}, over one week holding periods. Measures of the relative importance of the various Greeks are clearly sensitive to this choice. Over very short holding periods only the first derivatives should matter. As the holding period lengthens, second and third derivatives should be more important. We wanted a period long enough so that the noise in option price changes due to bid-ask bounce and price discreteness would be minor but short enough so that changes in the option Greeks should also be minor. Over shorter, e.g., one-day, holding periods, we would expect delta to be relatively dominant. Over longer, e.g., monthly, holding periods, we would expect gamma, theta, and the higher order derivatives to be relatively more important.

Strike and Expiry Samples

Since the Greeks differ by both strike and expiry, we form separate subsamples based on both the time to expiration and strike. The Chicago Mercantile Exchange trades S&P index futures options maturing in March, June, September, and December (the calendar options) and in each of the two nearby months (the serial options) not covered by a calendar option. We construct datasets for two expiries. The first consists of options expiring in 4 to 8 weeks. That is we switch from the nearby option to the next contract on the nineteenth trading day before expiration. Our second dataset consists of options maturing in 13 to 26 weeks—specifically the second calendar option. We do not examine longer term options since trading in these is much lighter.

The Greeks also differ by strike price. As is well known, delta is a decreasing function of the strike price for calls and increasing function of the strike for puts, while vega, gamma, and theta are highest for at-the-money options.[6] As we shall see, other derivatives also differ by strike. Consequently we construct separate subsamples for different strikes. As shown by the trading volume statistics in Exhibit 1, trading is heavy in at-the-money (ATM) and out-of-the-money (OTM) options but quite light in deep in-the-money (ITM) options. Hence, we are able to construct subsamples for far out-of-the-money options but not for deep in-the-money options. Specifically we construct five different subsamples: 1) at-the-money strikes (ATM), 2) slightly out-of-the-money strikes (S-OTM), 3) moderately out-of-the-money strikes (M-OTM), 4) far out-of-the- money strikes (F-OTM),[7] and 5) (slightly) in-the-money strikes (ITM). The terms "slightly," "moderately," and "far" out-of-the-money are intended to represent relative positions only since obviously opinions may differ on how far is "far."

The ATM sub-samples consist of the calls and puts with strikes just above and below the underlying futures price, i.e., four options. To avoid overlap with the ITM and S-OTM sub-samples, we impose the restriction that the strikes be within 2% of the futures price, which eliminates a few options early in the data period when the S&P index was low. In the 4 to 8 week sample, the S-OTM

EXHIBIT 1A

Average Daily Trading Volumes for Options on S&P 500 Futures 4–8 weeks to Expiration Arranged by Strike Price (1988–2003)

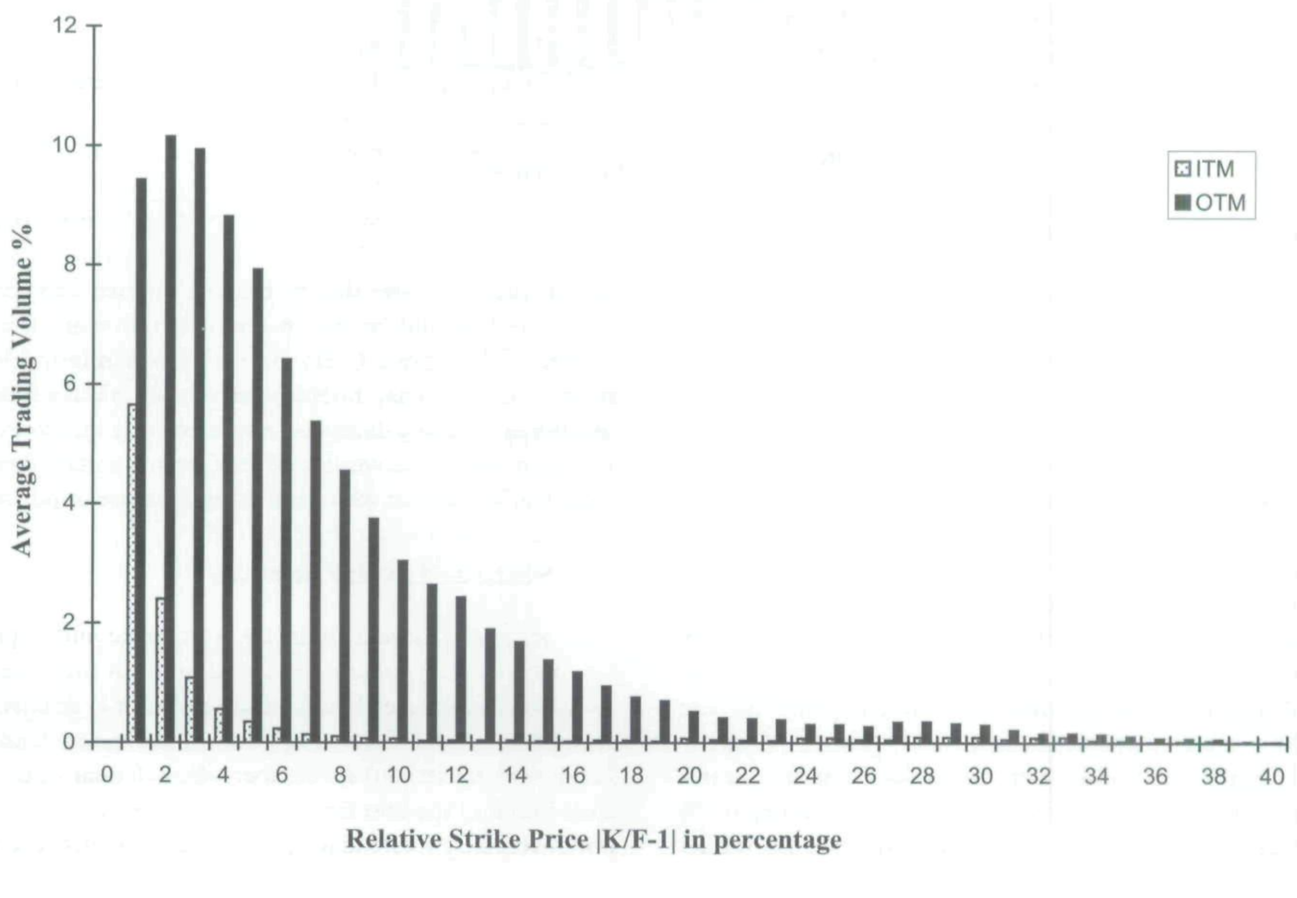
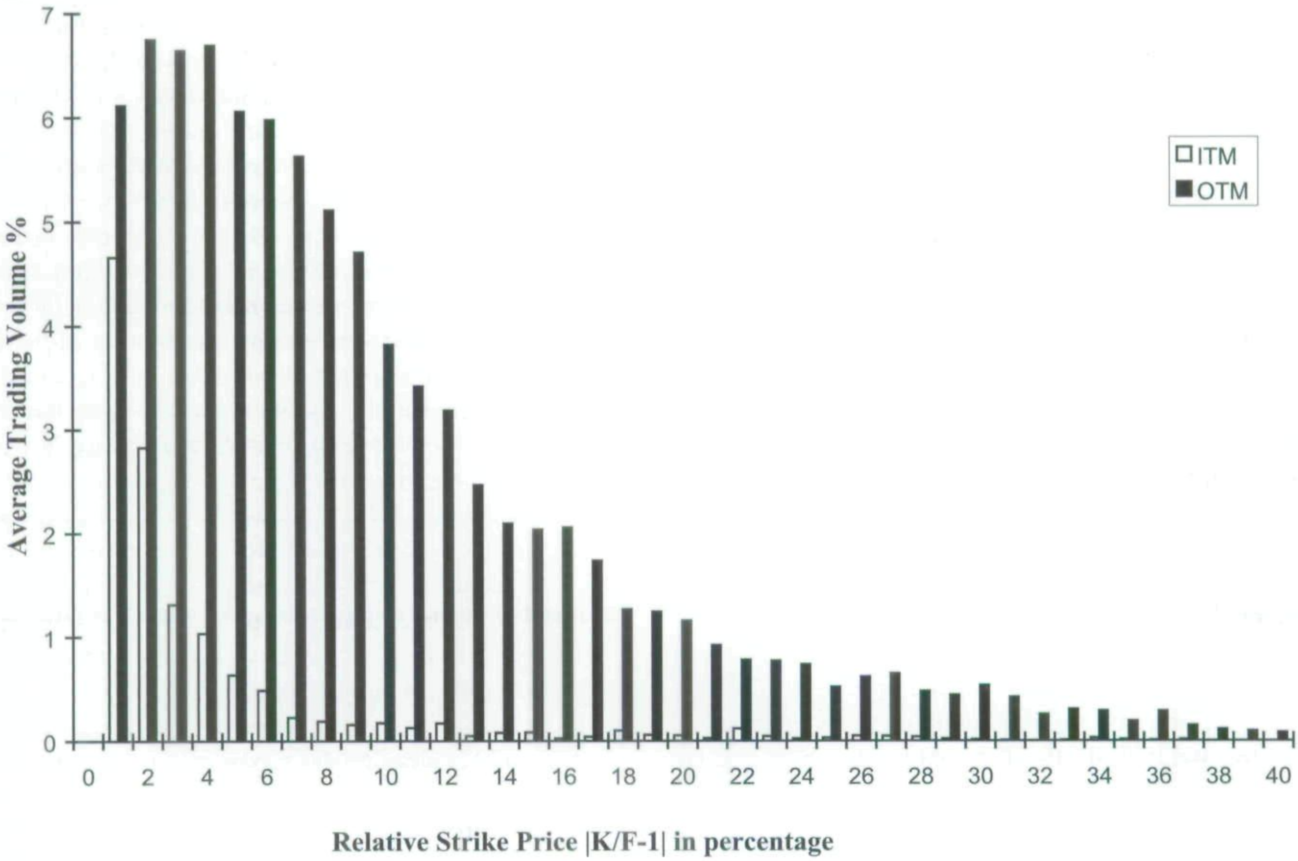


EXHIBIT 1B

Average Daily Trading Volumes for Options on S&P 500 Furtures 13–26 weeks to Expiration Arranged by Strike Price (1988–2003)



sub-sample consists of the calls (puts) which are closest to $1.04F_t$ ($.96F_t$) and within the range of $1.02F_t$ and $1.06F_t$ ($.94F_t$ and $.98F_t$ for puts) where F_t is the S&P 500 futures on day t. In the 13 to 16 week sample, we collect the options with strikes closest to $.93F_t$ and $1.07F_t$.[8] To provide an idea of how out-of-the-money the S-OTM strikes are, based on the average implied volatility of 19.1%, a price move from F_t to $F_{t+1} = 1.04F_t$ in the shorter term sample ($1.07F_t$ in the longer term sample) would represent a price move of about .62 (.60) standard deviations over the life of the option. For the ITM sub-sample, we collect observations for in-the-money strikes closest to $.96F_t$ and $1.04F_t$ in the shorter term sample and $.93F_t$ and $1.07F_t$ in the longer term sample. The shorter term M-OTM subsample consists of calls with strikes approximately 8% above the current futures price and puts 8% below. For the longer term sample, the M-OTM strikes are approximately 14% above or below the current futures. The F-OTM subsamples are constructed using strikes 12% from the futures in the shorter term sample and 21% in the longer term. Based on the mean ATM implied volatility over our data period and the mean times to expiration, it would take about a 1.85 (1.80) standard deviation movement in the underlying futures over the time-to-expiration for the F-OTM strikes to come into the money in the shorter (longer) term sample.

Greek and Risk Factor Calculations

All Greeks are calculated using Black's options on futures model. The deficiencies of the BS model (and by extension the Black version for options on futures) are well known. Specifically, it assumes log normal returns and constant (or deterministic) volatility. Also, S&P 500 futures options are American options, although the possibility of early exercise is fairly remote for our dataset since futures do not pay dividends and we include no deep-in-the-money options. By comparing the predicted option price changes from Equation (4) using Black model derivatives with actual price changes, we are able to test how well the Black model Greeks explain option price changes. Note that to the extent the Black model derivatives are inaccurate measures of the true sensitivities, our measures of the incremental explanatory power of individual risk factors are biased against finding that higher order derivatives are important.

Since there are four option price determinants in the Black model, there are four first derivatives, ten second derivatives, and twenty third derivatives. However, in our data set, all derivatives and risk factors involving the interest rate are inconsequential. Unless the time to expiration is long, the impact of interest rate changes on option prices is small; this is particularly true for options on futures since there is no time t cash flow on a futures purchase or sale. As documented below, the rho risk factor is inconsequential in our data set, and the higher order risk factors involving the interest rate are even smaller with no discernable impact on the predicted price change, $\delta\hat{O}_{it}$, for any option in any period. Due to its traditional standing, we include the rho factor in our estimated models, but to avoid cluttering the article with meaningless terms we exclude all higher order derivatives involving the interest rate. This reduces the set of second order risk factors to six and the set of third order factors to ten.

First, second, and third order derivatives are calculated for each option i on each day t using the formulae presented in the Appendix and these are multiplied by the change in the relevant determinant or determinants over the subsequent one week period to obtain the risk factor. As noted above, for F_t we utilize the future's settlement price on the S&P 500 contract on day t. $T_{i,t}$ is measured as the time in years until option i expires assuming 252 trading days in a year. Note that for all i and t, $\delta T_{it} = \delta T = -5/252 = -.01984$ years . For r_{it}, we use the yield to maturity (specifically the bond equivalent yield) on day t on the U.S. Treasury bill whose maturity date is closest to that of option i.

The Volatility Measure

Measuring the change in option prices due to changes in volatility, σ, requires special care. While the BS model assumes that either volatility is constant or that any changes are deterministic, changes in the market's volatility expectation clearly occur as new information is received, and implied volatilities provide a measure of these expectations. However, if implied volatilities for option i at times t and t + 1 are calculated from option i's prices on those dates and this change in the implied volatility ($\delta\sigma_{it}$) is then used in the equations to explain the price change from t to t + 1, the vega factor will tend to proxy for any price determinants other than F, T, and r and its importance will be exaggerated. For instance, suppose that there is no real change in the equilibrium option price from t to t + 1 but that the observed option price rises slightly because a trade at the bid price is observed at time t and a trade at the ask at time t + 1. This bid-ask bounce would result in a higher calculated implied volatility at time t + 1. Consequently, the increase in the option price would be attributed to the vega factor and not to the error term $\varepsilon_{i,t}$.

What is needed is a measure of the market's expectation of likely volatility over the life of option i which is not derived from option i's price. Theoretically, implied volatilities calculated from any option with the same expiry should suffice. However, it is well known that implied volatilities differ across strikes in a smile or smirk pattern and that this pattern shifts considerably over time (e.g., Bollen and Whaley [2004]). Consequently, we use strikes close to i's to calculate a proxy for i's volatility using the following procedure. Using Black's [1976] model for options on futures, day t closing prices and T-bill rates, we calculate implied volatilities of the calls and puts at strikes just above and below i's strike. We also calculate the implied volatility of the opposite type option (puts for calls and vice-versa) at the same strike. We then average these implied volatilities to obtain $\sigma_{i,t}$.[9] This procedure avoids impounding influences specific to option i, such as bid-ask bounce, into i's volatility measure. However, if factors not considered by BS, such as Bollen-Whaley [2004] type hedging pressures, impact prices at both i and all strikes in the vicinity of i, they will be impounded. Consequently, it is possible that some of the option price changes we identify as due to volatility changes could reflect option price determinants not considered by BS which impact option prices at a range of strikes.

Descriptive Statistics

Means and standard deviations of the resulting risk factor estimates for the five traditional Greeks over five day holding periods are reported in Exhibit 2. Since signs of the delta, vega, and rho risk factors differ depending on whether the change in F, σ, or r is positive or negative, we report means of absolute values. The statistics in Exhibit 2 provide an indication of a risk factor's impact on $\delta\hat{O}_{it}$ if 1) the BS derivatives accurately measure an option price's sensitivity to that determinant and 2) the factors are independent. They provide therefore a preliminary measure of each factor's *potential* importance.

As already noted, the rho factor is minuscule and inconsequential by both measures. As one might expect, by both measures, the delta risk factor dominates in the ATM and ITM subsamples. Its dominance is lower in the F-OTM subsample where deltas are lower, but even there its mean and standard deviation exceed the other risk factors. In comparing these statistics across risk factors, it should be kept in mind that they are for one week holding periods. As discussed above, if returns are independent, δF should be roughly proportional to $(\delta T)^{.5}$ while $(\delta F)^2$ should be roughly proportional to δT. Consequently, as the holding period is lengthened, both the mean and standard deviation of the theta and gamma factors should rise relative to those for delta.[10]

As is well-known and shown in the formulae in the Appendix, theta and gamma are decreasing functions of the time to expiration while vega is an increasing function. This is reflected in their relative magnitudes in the shorter-term and longer-term samples.[11]

EXPLAINING OPTION PRICE CHANGES

The first question we explore is how well the five traditional Greeks explain option price changes and how much (if any) is explained by higher order derivatives. In

EXHIBIT 2

Risk Factor Statistics: The Five Traditional Greeks

Means and standard deviations are reported for risk factors corresponding to the five traditional Greeks , i.e., the product of the Greek, the subsequent change in the option price determinant over a one-week period, and the appropriate Taylor series cofficient (.5 for Gamma and 1.0 for the others). Since the signs of the Delta, Vega, and Rho risk factors are differ depending on whether the price, volatility or interest rate increases or decreases, the reported means for all factors are their absolute values. Standard deviations are calculated from the signed data. ATM, S-OTM, M-OTM, F-OTM, and ITM designate at-the-money, slightly, moderately, and far out-of-the-money, and in-the-money respectively. The data period is Jaunary 1, 1988 through March 31, 2003.

Risk Factor	ATM options		S-OTM options		M-OTM Options		F-OTM Options		ITM Options	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
Panel A - 4 to 8 Week Options										
Delta	6.6344	10.5463	3.3821	5.9038	1.9794	3.6677	1.1384	2.2132	9.9933	15.3383
Theta	1.5260	1.1562	1.2182	1.0529	.9531	.8980	.7349	.7295	1.3364	1.0812
Vega	.9170	1.4283	.7470	1.2355	.5969	1.0199	.4526	.7573	.8265	1.3294
Rho	.0018	.0036	.0008	.0018	.0004	.0012	.0003	.0007	.0035	.0064
Gamma	1.2253	2.6085	.9270	2.1034	.6615	1.5902	.3992	1.0034	1.0499	2.3120
Panel B - 13 to 26 week Options										
Delta	5.5548	8.9981	2.8739	5.1650	1.7454	3.3081	1.1199	2.0700	8.1124	13.0801
Theta	.7585	.6004	.6145	.5297	.4987	.4645	.4365	.4029	.6163	.5352
Vega	.8781	1.4277	.7361	1.2831	.6526	1.2005	.5895	.9824	.7836	1.3542
Rho	.0081	.0157	.0036	.0080	.0020	.0046	.0012	.0025	.0151	.0260
Gamma	.5502	1.2619	.4431	1.1767	.3431	.9459	.2345	.6383	.4825	1.2682

Exhibit 3, we present measures of the unexplained price change or forecast error, $FE_{it} = \delta O_{it} - \delta\hat{O}_{it}$, where δO_{jt} is the actual price change on option j over the holding period starting at time t and $\delta\hat{O}_{it}$ is the change predicted by Equations (2), (3), or (4) in the shorter term sample. Analogous figures for the longer term sample are in Exhibit 4. Mean absolute forecast errors are reported in Panel A and root mean squared forecast errors in Panel B. These are expressed as percentages of the mean absolute price change and the root mean squared price change in Panels C and D respectively. Within each panel, results for the Taylor series expansion based on the five traditional Greeks (Equation (2)) are shown on the first line; results for Equation (3) which adds the other second order risk factors are shown on the second line; and results for Equation (4) which adds third order risk factors on the third line.

The following results stand out in Exhibits 3 and 4. One, the five traditional Greeks explain most, but not all option price changes over one week holding periods. For instance, as shown in Panel C of Exhibit 3, the percentage of the mean absolute option price change accounted for by the five traditional Greeks ranges from 97.43% for ATM options to 80.17% for F-OTM options in the shorter term sample.

Two, although less important than the five traditional Greeks, the second and third order Greeks have considerable incremental explanatory power. In every sub-sample, introducing second and third order risk factors lowers both the MAE and RMSE. In most cases, the reduction is substantial. For instance, averaged over the five sub-samples, in the shorter term sample, the mean absolute forecast error for the second order model is about half that for the five traditional Greeks, while the mean absolute forecast error for the third order model is about half that for the second order. For the five traditional Greeks, MAE in the shorter term sample ranges from \$.1753 for one index unit in the ATM sub-sample to \$.2924 in the M-OTM sub-sample; for the third-order model, MAE ranges from \$.0492 in the F-OTM subsample to \$.0684 in the ATM sub-sample. To put the latter figures in perspective, in this market the minimum tick size is \$.05 per index unit; but for premiums of \$10 and up, most prices are in increments of \$.10, so MAEs for the third order model are approaching the minimum possible.

The success of Equation (4) in explaining option price changes over time has important implications. As noted above, the BS assumptions of log-normal returns and constant volatility are clearly violated for most assets, and the cross-sectional option price pattern (e.g., the smile) is incompatible with the BS model. Nonetheless, the time-series pattern of option price changes matches that predicted by BS model Greeks quite closely. As shown in Panel C, the average percentage of the weekly price change explained by the Taylor series expansion of the Black model, once second and third derivatives are considered and volatility changes are introduced, is about 99% for ATM options and higher for ITM. In the shorter term sample, the percentage explained is at least 95% for all three OTM subsamples. In all shorter term subsamples, MAE is less than \$.07 per index unit. Only in the case of F-OTM options in the longer term sample, where the average MAE is about \$.12 and the mean absolute price change explained is 89%, are the predictions of the Black model Greeks not so satisfactory.

Three, the incremental explanatory power of the second and third order factors is greater in the shorter term sample than in the longer term sample. In the shorter term sample, introducing the higher order Greeks reduces the average MAE 76%; in the longer term sample only 43%. Also, once second and third order factors are introduced, MAE and RMSE are approximately the same in the two samples.

Four, the incremental explanatory power of the second order factors is greater for ITM and OTM than for ATM options. When only the five traditional Greeks are considered, MAE is considerably lower for short-term ATM options than for other strikes. The hypothesis that there is no difference in the MAE for ATM options and the other subsamples is rejected at the .05 level. If observing results based on only the five traditional Greeks, a casual observer might be tempted to hypothesize that higher explanatory power for ATM options is due to greater liquidity (and therefore less noise) for ATM options. However, it is clear from Exhibits 3 and 4 that the difference is because second and third order factors are more important for non-ATM options. In the shorter-term sample, adding second order factors to the Taylor series lowers the MAE for the ATM subsample, by \$.0491, or about one tick. In the ITM and OTM subsamples the reduction is at least \$.10. Once second and third order factors are added to the Taylor series, the differences in MAE and RMSE by strike are no longer significant. In the longer term sample, adding the second order factors lowers MAE only \$.0108 for ATM options but at least \$.04 for non-ATM options. As we shall see below, this is because the more important second derivatives are close to their nadir for ATM options.

EXHIBIT 3

Explanatory Power of Traditional and Higher Order Greeks—Shorter term sample

Measures of the ability of the Greeks to account for changes in option prices in the shorter term (4 to 8 week) sample are presented. Statistics on $FE_{it} = \delta O_{it} - \delta\hat{O}_{it}$ where δO_{jt} is the change in option j's price over a one week holding period and $\delta\hat{O}_{it}$ is the change predicted by j's Greeks at time t and the change in the option price determinant(s) in a Taylor series expansion are presented. $MAE = (1/(T \times I)) \Sigma^{T}_{t=1} \Sigma^{I}_{i=1} |FE_{it}|$ and $RMSE = [(1/(T \times I)) \Sigma^{T}_{t=1} \Sigma^{I}_{i=1} FE^{2}_{it}]^{.5}$ are presented in Panels A & B. The percentage mean absolute forecast error in Panel C is calculated as the ratio of MAE to the mean absolute price change and the percentage root mean squared forecast error in Panel D as the ratio of RMSE to the root mean squared price change. ATM, S-OTM, M-OTM, F-OTM, and ITM designate at-the-money, slightly, moderately, and far out-of-the-money, and in-the-money respectively. The data period is January 1, 1988 through March 31, 2003.

	ATM Options	S-OTM Options	M-OTM Options	F-OTM Options	ITM Options
Panel A - Mean absolute forecast error, MAE					
Five Traditional Greeks	\$.1753	\$.2512	\$.2924	\$.2347	\$.2725
Ten Factor Second Order Model	\$.1262	\$.1350	\$.1318	\$.0945	\$.1544
Twenty Factor Third Order Model	\$.0684	\$.0556	\$.0561	\$.0492	\$.0662
Panel B - Root mean squared forecast error, RMSE					
Five Traditional Greeks	\$.4350	\$.6469	\$.7646	\$.5545	\$.7052
Ten Factor Second Order Model	\$.3599	\$.4257	\$.4392	\$.3369	\$.5034
Twenty Factor Third Order Model	\$.2404	\$.2022	\$.1805	\$.1331	\$.2959
Panel C - Percentage mean absolute forecast error					
Five Traditional Greeks	2.566%	7.304%	14.547%	19.826%	2.650%
Ten Factor Second Order Model	1.847%	3.927%	6.559%	7.984%	1.501%
Twenty Factor Third Order Model	1.002%	1.618%	2.790%	4.155%	0.643%
Panel D - Percentage root mean squared forecast error					
Five Traditional Greeks	3.978%	10.358%	18.681%	21.395%	4.485%
Ten Factor Second Order Model	3.290%	6.816%	10.731%	12.998%	3.202%
Twenty Factor Third Order Model	2.198%	3.237%	4.410%	5.136%	1.882%

SECOND ORDER GREEKS

Description

Since it is apparent from the results in Exhibits 3 and 4 that the higher order Greeks add explanatory power, we next ask which second and third order factors are important and when. We also seek to explain the statistical regularities pointed out in the previous section—specifically why are second order factors more important for non-ATM options and why are second and third order

EXHIBIT 4

Explanatory Power of Traditional and Higher Order Greeks—Longer term sample

Measures of the ability of the Greeks to account for changes in option prices in the longer term (13 to 26 week) sample are presented. Statistics on $FE_{it} = \delta O_{it} - \delta \hat{O}_{it}$ where δO_{jt} is the change in option j's price over a one week holding period and $\delta \hat{O}_{it}$ is the change predicted by j's Greeks at time t and the change in the option price determinant(s) in a Taylor series expansion are presented. Results for the measures: $MAE = (1/(T \times I)) \Sigma_{t=1}^{T} \Sigma_{i=1}^{I} |FE_{it}|$ and $RMSE = [(1/(T \times I)) \Sigma_{t=1}^{T} \Sigma_{i=1}^{I} FE_{it}^2]^{.5}$ are presented in Panels A & B. The percentage mean absolute forecast error in Panel C is calculated as the ratio of MAE to the mean absolute price change and the percentage root mean squared error in Panel D as the ratio of RMSE to the root mean squared price change.

	ATM Options	S-OTM Options	M-OTM Options	F-OTM Options	ITM Options
Panel A - mean absolute forecast error , MAE					
Five Traditional Greeks	\$.0749	\$.1141	\$.1911	\$.2160	\$.1301
Ten Factor Second Order Model	\$.0641	\$.0726	\$.1228	\$.1538	\$.0819
Twenty Factor Third Order Model	\$.0550	\$.0530	\$.0991	\$.1390	\$.0712
Panel B - Root mean squared forecast error, RMSE					
Five Traditional Greeks	\$.1492	\$.3032	\$.6035	\$.4591	\$.3396
Ten Factor Second Order Model	\$.1287	\$.1954	\$.3840	\$.2768	\$.2566
Twenty Factor Third Order Model	\$.1005	\$.1268	\$.3139	\$.2367	\$.2230
Panel C - Percentage mean absolute forecast error					
Five Traditional Greeks	1.323%	3.866%	10.644%	17.710%	1.578%
Ten Factor Second Order Model	1.133%	2.458%	6.841%	12.612%	0.993%
Twenty Factor Third Order Model	0.971%	1.794%	5.520%	11.395%	0.863%
Panel D - Percentage root mean squared forecast error					
Five Traditional Greeks	1.623%	5.589%	16.942%	18.633%	2.557%
Ten Factor Second Order Model	1.400%	3.601%	10.780%	11.235%	1.932%
Twenty Factor Third Order Model	1.091%	2.336%	8.813%	9.608%	1.679%

factors more important in the shorter-term sample. Second order factors are considered in this section and third order in section *Third Order Greeks*.

Excluding inconsequential derivatives involving the interest rate, the Black model has six second order derivatives: $\partial^2O/\partial F^2$ (gamma), $\partial^2O/\partial\sigma^2$, $\partial^2O/\partial T^2$, $\partial^2O/\partial F\partial\sigma$, $\partial^2O/\partial F\partial T$, and $\partial^2O/\partial\sigma\partial T$. Since Gamma is well-known, we concentrate on the other five. Since delta is the most important of the first order derivatives, one intuitively expects that its derivatives could be important as well. To avoid confusing the options lexicon with many new terms, we use combinations of the existing terms to refer to the second order derivatives and risk factors. For instance, we will refer to $\partial^2 O/\partial F \partial \sigma$ as the delta-vega derivative and to the term $(\partial^2 O/\partial F \partial \sigma)$ $[\delta F\ \delta\sigma]$ as the "delta-vega" risk factor.[12] Black model formulae for the five second order derivatives, excepting gamma which is listed in Panel A, are in Panel B in the Appendix. The formulae are identical for calls and puts.

As is clear from the formulae in the Appendix, three second order derivatives, $\partial^2 O/\partial \sigma^2$, $\partial^2 O/\partial F \partial \sigma$, and $\partial^2 O/\partial F \partial T$, are approximately zero for ATM options. Specifically, $\partial^2 O/\partial F \partial \sigma = 0$ when the relation between F and the strike X is such that $d_2 = 0$. Since the $r\Delta$ term is small, $\partial^2 O/\partial F \partial T = 0$ at approximately the same point. $\partial^2 O/\partial \sigma^2 = 0$ when $d_1 = 0$ and again when $d_2 = 0$. For the parameters in our sample, differences between these points tend to be small. This is illustrated in Exhibit 5 where the five higher order derivatives (for calls) are plotted as functions of the strike price setting the values of F, σ, and r equal to their approximate sample means ($F = 725$, $\sigma = .175$, and $r = 5\%$) and $t = 6$ weeks. In order to fit all five derivatives on one graph, they are standardized by dividing by their maximum absolute value so that the scaled derivatives in Exhibit 5 vary from -1 to $+1$. As argued, $\partial^2 O/\partial \sigma^2$, $\partial^2 O/\partial F \partial \sigma$, and $\partial^2 O/\partial F \partial T$ are close to zero for ATM options. Consequently, these three factors should be more important for ITM and OTM options than for ATM options. On the other hand, $\partial^2 O/\partial T^2$ is highest for close-to-the-money options.

The second-order factors are of the form $(c)(f_{jk})(\delta X_j)(\delta X_k)$ where f_{jk} is the second derivative, δX_j is the change in determinant j, and c is the Taylor series coefficient, which is 1.0 for the three cross partials and .5 for $\partial^2 O/\partial \sigma^2$ and $\partial^2 O/\partial T^2$. For instance, the delta-vega factor for option i on day t is calculated as $(\partial^2 O/\partial F \partial \sigma)_{it}$ (δF_t) $(\delta \sigma_{it})$ where $(\partial^2 O/\partial F \partial \sigma)_{it}$ is the partial derivative of the option i's price with respect to F and σ at time t, and (δF_t) and $(\delta \sigma_{it})$ are the changes in the S&P 500 index futures and implied volatility over the week.

Means (of absolute values) and standard deviations of the resulting risk factors are reported in Exhibit 6.[13] As explained above, these statistics provide measures of the potential direct impact (that is, ignoring correlations with other factors) of that factor on the option price. Since as shown in Exhibit 6, the vega-vega and theta-theta factors have relatively small means and standard deviations in all subsamples, we focus attention on the three cross-partials. Consistent with the fact that the cross-partials $\partial^2 O/\partial F \partial \sigma$, and $\partial^2 O/\partial F \partial T$ are close to zero for ATM options, means and standard deviations for the delta-vega and delta-theta factors are small in the ATM subsamples. Both of these factors are fairly sizable in the non-ATM subsamples in the shorter term sample, and the delta-vega factor appears potentially important in the non-ATM subsamples in the longer term sample. This explains our earlier observation

EXHIBIT 5

Black Model Second Derivatives as a Function of the Strike Price for a Call Option where F = 725, Volatility = .175, r = 5% and Time to Expiration = 6 Weeks. Derivatives are Standardized to Vary from –1 to +1

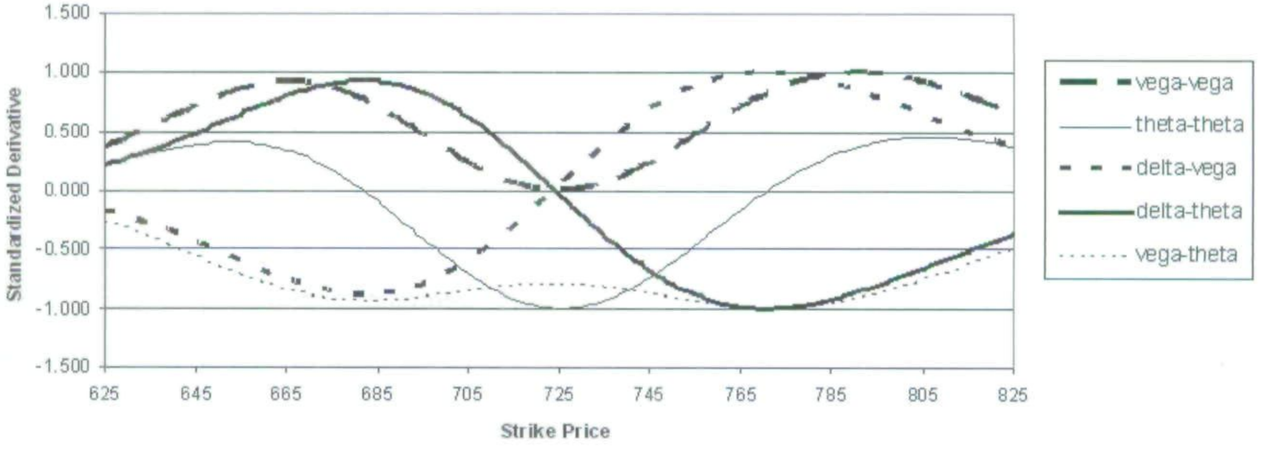


EXHIBIT 6
Descriptive Statistics: Second Order Risk Factors

Means (of absolute values) and standard deviations are reported for second order risk factors (excepting gamma) for a one-week holding period. ATM, S-OTM, M-OTM, F-OTM, and ITM designate at-the-money, slightly, moderately, far out-of-the-money, and in-the-money respectively. The data period is Jaunary 1, 1988 through March 31, 2003.

Risk Factor	Description	ATM		S-OTM		M-OTM		F-OTM		ITM	
		Mean	StdDev	Mean	StdDev	Mean	StdDev	Mean	StdDev	Mean	StdDev
Panel A - 4 to 8 Week Options											
Delta-vega	$\frac{\partial^2 O}{\partial F\,\partial\sigma}(\delta F)(\delta\sigma)$	.02164	.05416	.16238	.45128	.23246	.69864	.20789	.56516	.17243	.47535
Delta-theta	$\frac{\partial^2 O}{\partial F\,\partial T}(\delta F)(\delta T)$	.03096	.04982	.20952	.34301	.26180	.43201	.22908	.38068	.23971	.38470
Vega-theta	$\frac{\partial^2 O}{\partial\sigma\partial T}(\delta\sigma)(\delta T)$	.08034	.12893	.09273	.15236	.11527	.19346	.12247	.19679	.10110	.16256
Vega-vega	$.5\frac{\partial^2 O}{\partial\sigma\,\partial\sigma}(\delta\sigma)^2$	.00035	.00138	.01577	.04688	.04149	.13935	.05655	.16276	.01596	.05113
Theta- Theta	$.5\frac{\partial^2 O}{\partial T\,\partial T}(\delta T)^2$	.06958	.05917	.03525	.04187	.01912	.02631	.03088	.02681	.04069	.04433
Panel B - 13 to 26 week options											
Delta-vega	$\frac{\partial^2 O}{\partial F\,\partial\sigma}(\delta F)(\delta\sigma)$	.01160	.03237	.07751	.20467	.14287	.54557	.16302	.44634	.08020	.21076
Delta-theta	$\frac{\partial^2 O}{\partial F\,\partial T}(\delta F)(\delta T)$	.00977	.01865	.05402	.09317	.07243	.12769	.07176	.11656	.06701	.11262
Vega-theta	$\frac{\partial^2 O}{\partial\sigma\partial T}(\delta\sigma)(\delta T)$	.02345	.03821	.02735	.04452	.03708	.06435	.04744	.07225	.02878	.04670
Vega-vega	$.5\frac{\partial^2 O}{\partial\sigma\,\partial\sigma}(\delta\sigma)^2$	.00013	.00055	.00777	.01764	.02789	.12123	.04951	.13796	.00779	.01762
Theta- Theta	$.5\frac{\partial^2 O}{\partial T\,\partial T}(\delta T)^2$	.01203	.01043	.00627	.00725	.00292	.00434	.00526	.00397	.00685	.00762

in Exhibits 3 and 4 that the incremental explanatory power of the second order factors is lower in the ATM sub-sample than in the other subsamples. Clearly the reason is that the more important second order derivatives are approximately zero for ATM options. From Exhibit 6, it is also apparent why the second order terms are more important in the short-term sample (Exhibit 3) than in the longer term sample (Exhibit 4), since the higher order terms involving the time to expiration are relatively inconsequential in the longer-term sample.

Comparing the figures for gamma in Exhibit 2 with those for the other second order risk factors in Exhibit 6, it is clear that the gamma factor is the largest and most variable of the second order factors. Its mean and standard deviation exceed those of the other second order factors in every sub-sample. In summary, if restricted to one second derivative, gamma is the clear choice, but it appears that delta-vega and delta-theta may be important in explaining the price movements of moderately-to-far-out-of-the-money options.

Incremental Explanatory Power

While the statistics in Exhibit 6 are indicative of the *potential* importance of the various risk factors, such an interpretation assumes the Black derivatives are accurate.

Measures of the actual incremental explanatory power of the first and second order risk factors are reported in Exhibit 7. As explained above, in this Exhibit, we measure the increase in the mean absolute forecast error, IMAE, and the root mean squared forecast error, IRMSE, when each factor is dropped from the 20 factor model in Equation (4),which includes first, second, and third order factors. Below the IMAE and IRMSE of each factor, we report these figures as a percentage of the mean absolute option price change and the root mean squared option price change, respectively.

The more important results from Exhibit 7 include the following. One, among the first-order factors, the dominance of the delta factor and the insignificance of rho are reconfirmed. When the delta factor is dropped from the Taylor series expansion, the remaining terms have very little explanatory power. When rho is dropped, the explanatory power of the model is basically unaffected. Two, among the second order factors, the incremental explanatory power of gamma exceeds that of the other second order factors in all subsamples. As one would expect, gamma is more important in explaining large price changes than small price changes, that is, the percentage IRMSE for gamma exceeds its percentage IMAE. In contrast, theta's percentage IRMSE is smaller than its percentage IMAE, which is expected since the change in the time to expiration is the same for all observations.

EXHIBIT 7

Incremental Explanatory Power—First and Second Order Risk Factors

Measures of the incremental explanatory power of first and second order risk factors in explaining changes in S&P 500 futures options over one-week holding periods are presented. The incremental mean absolute forecast error (IMAE) and incremental root mean squared forecast error (IRMSE) are measured as the reduction in the mean absolute forecast error (MAE) and root mean squared forecast error (RMSE) when that risk factor is added to a Taylor series equation (Equation (4) in the text) which includes the other nineteen first, second and third order risk factors. These are expressed as percentages of the mean absolute price change (MAPC) and root mean squared price change (RMSPC) for S&P 500 options in parentheses below each term.

Risk Factor	Derivative with respect to	ATM Options		S-OTM Options		M-OTM Options		F-OTM Options		ITM Options	
		IMAE	IRMSE	IMAE	IRMSE	IMAE	IRMSE	IMAE	IRMSE	IMAE	IRMSE
Panel A - 4 to 8 Week Options											
Delta	F	6.5592 (96.01%)	10.2984 (94.16%)	3.3238 (96.65%)	5.6886 (91.08%)	1.9296 (96.02%)	3.4905 (85.28%)	1.1080 (93.59%)	2.1055 (81.24%)	9.9212 (96.46%)	15.0516 (95.73%)
Theta	T	1.4889 (21.79%)	1.7286 (15.81%)	1.1734 (34.12%)	1.4355 (22.98%)	.8961 (44.59%)	1.1415 (27.89%)	.6837 (57.85%)	.9026 (34.83%)	1.2808 (12.45%)	1.4559 (9.26%)
Vega	σ	.8359 (12.24%)	1.1611 (10.62%)	.6894 (20.05%)	1.0381 (16.62%)	.5464 (27.20%)	.8677 (21.20%)	.4061 (34.31%)	.6460 (24.92%)	.7604 (7.39%)	1.0496 (6.68%)
Rho	r	-.0000 (-0.00%)	-.0002 (-0.00%)	.0000 (0.00%)	-.0000 (-0.00%)	.0000 (0.00%)	.0000 (0.00%)	.0000 (0.00%)	.0000 (0.00%)	-.0000 (-0.00%)	-.0001 (-0.00%)
Gamma	F, F	1.1370 (16.64%)	2.5073 (22.93%)	.8704 (25.31%)	2.0647 (33.06%)	.6212 (30.91%)	1.5851 (38.72%)	.3662 (30.93%)	.9988 (38.54%)	.9892 (9.62%)	2.2162 (14.09%)
Delta-Vega	F, σ	.0068 (0.10%)	.0092 (0.08%)	.1111 (3.23%)	.2152 (3.44%)	.1805 (8.98%)	.4796 (11.72%)	.1703 (14.39%)	.4247 (16.39%)	.1134 (1.10%)	.1850 (1.18%)
Delta-Theta	F, T	.0082 (0.12%)	.0000 (0.00%)	.1652 (4.80%)	.2079 (3.33%)	.2106 (10.48%)	.2889 (7.06%)	.1708 (14.43%)	.2431 (9.38%)	.1819 (1.77%)	.1954 (1.24%)
Vega-Theta	σ, T	.0540 (0.79%)	.0536 (0.49%)	.0629 (1.83%)	.0573 (0.92%)	.0776 (3.86%)	.0793 (1.94%)	.0909 (7.68%)	.1172 (4.52%)	.0657 (0.64%)	.0483 (0.31%)

EXHIBIT 7 *(Continued)*

Risk Factor	Derivative with respect to	ATM Options		S-OTM Options		M-OTM Options		F-OTM Options		ITM Options	
		IMAE	IRMSE	IMAE	IRMSE	IMAE	IRMSE	IMAE	IRMSE	IMAE	IRMSE
Vega-Vega	σ, σ	-.0000 (-0.00%)	-.0001 (-0.01%)	.0063 (0.18%)	.0149 (0.23%)	.0268 (1.33%)	.0743 (1.82%)	.0372 (3.14%)	.0949 (3.66%)	.0064 (0.06%)	.0130 (0.08%)
Theta-Theta	T, T	.0449 (0.66%)	.0304 (0.28%)	.0175 (0.51%)	.0111 (0.18%)	.0064 (0.32%)	.0044 (0.11%)	.0119 (1.01%)	.0086 (0.33%)	.0169 (0.16%)	.0076 (0.05%)
Panel B - 13 to 26 Week options											
Delta	F	5.4940 (97.04%)	8.8838 (96.66%)	2.8278 (95.80%)	5.0414 (92.90%)	1.6997 (94.66%)	3.0810 (86.49%)	1.0924 (89.57%)	1.9357 (78.56%)	8.0562 (97.72%)	12.8761 (96.97%)
Theta	T	.7102 (12.55%)	.8801 (9.58%)	.5677 (19.23%)	.7017 (12.93%)	.4260 (23.72%)	.4436 (12.45%)	0.3369 (27.62%)	.4007 (16.26%)	.5522 (6.70%)	.6201 (4.67%)
Vega	σ	.8230 (14.54%)	1.3265 (14.43%)	.6814 (23.08%)	1.1550 (21.28%)	.5407 (30.12%)	.8515 (23.90%)	.4328 (35.49%)	.7568 (30.71%)	.7245 (8.79%)	1.1559 (8.70%)
Rho	r	.0010 (0.02%)	.0011 (0.02%)	.0007 (0.02%)	.0005 (0.01%)	.0003 (0.01%)	.0000 (0.00%)	.0001 (0.01%)	.0001 (0.00%)	.0003 (0.00%)	.0003 (0.00%)
Gamma	F, F	.5026 (8.88%)	1.2556 (13.66%)	.3995 (13.53%)	1.1105 (20.46%)	.2649 (14.75%)	.7408 (20.80%)	.1573 (12.90%)	.4952 (20.10%)	.4326 (5.25%)	1.1520 (8.68%)
Delta-Vega	F, σ	.0031 (0.05%)	.0068 (0.07%)	.0502 (1.70%)	.1146 (2.11%)	.1032 (5.75%)	.3194 (8.97%)	.1122 (9.20%)	.3011 (12.22%)	.0401 (0.49%)	.0358 (0.27%)
Delta-Theta	F, T	.0010 (0.02%)	.0001 (0.00%)	.0216 (0.73%)	.0283 (0.52%)	.0027 (0.15%)	-.0066 (-0.18)	-.0147 (-1.21%)	-.0295 (-1.20%)	.0429 (0.52%)	.0698 (0.53%)
Vega-Theta	σ, T	.0073 (0.13%)	.0089 (0.10%)	.0112 (0.38%)	.0111 (0.20%)	.0149 (0.83%)	.0225 (0.63%)	.0212 (1.74%)	.0284 (1.15%)	.0071 (0.09%)	.0039 (0.03%)
Vega-Vega	σ, σ	.0000 (0.00%)	.0000 (0.00%)	.0012 (0.04%)	.0003 (0.01%)	.0020 (0.11%)	.0158 (0.44%)	.0091 (0.74%)	.0323 (1.31%)	.0018 (0.02%)	.0015 (0.01%)
Theta-Theta	T, T	.0023 (0.04%)	.0025 (0.03%)	.0009 (0.03%)	.0009 (0.02%)	-.0002 (-0.01%)	-.0001 (-0.00)	-.0003 (-0.03%)	.0002 (0.01%)	.0005 (0.01%)	.0001 (0.00%)

Three, the delta-vega factor has considerable incremental explanatory ability in the M-OTM and F-OTM subsamples. When this factor is dropped from the 20 factor Equation, MAE and RMSE at least double. For instance, in the shorter term M-OTM subsample, MAE is \$.056 when this factor is included and \$.237 when it is not. In the F-OTM subsample, MAE is \$.049 with the delta-vega factor and \$0.219 without.

Four, the delta-theta factor has sizable incremental explanatory power in the M-OTM and FOTM subsamples in the shorter-term sample but not in the longer-term sample. For instance, in the M-OTM subsample in the shorter term sample, MAE is \$.056 with the delta-theta factor and \$.267 without. In the longer term sample, MAE is \$.099 with delta-theta and \$.102 without. We expected the delta-theta factor to be less important for longer term options since theta is smaller for longer term options. Also delta-theta's mean and standard deviation were small in the longer term sample in Exhibit 6. Nonetheless, the fact that delta-theta has virtually no incremental explanatory power in the longer term sample was a surprise. We shall return to this issue below.

Five, consistent with the results in Exhibit 6, the incremental explanatory power of the vega-vega and theta-theta factors is nil and that of the vega-theta factor is relatively small in most subsamples.

In summary, gamma is normally the most important of the second order derivatives, so in this sense the profession's concentration on this derivative is justified. However, several other second-order derivatives are important in non-ATM subsamples. Specifically, the derivatives of delta with respect to volatility and time-to-expiration (or equivalently, the derivatives of vega and theta with respect to the underlying asset price) are important in explaining price changes for moderately-to-far-out-of-the-money shorter-term option prices over one week holding periods, and delta-vega is important in explaining changes in longer term option prices. Again we would caution that these measures of relative importance are likely sensitive to the chosen holding period length—which is one week for our analyses. Since both delta-vega and delta-theta are close to zero for ATM options (and the other second derivatives are relatively inconsequential in general), the second order factors (excepting gamma) are more important for non-ATM than for ATM options.

THIRD ORDER GREEKS

Description

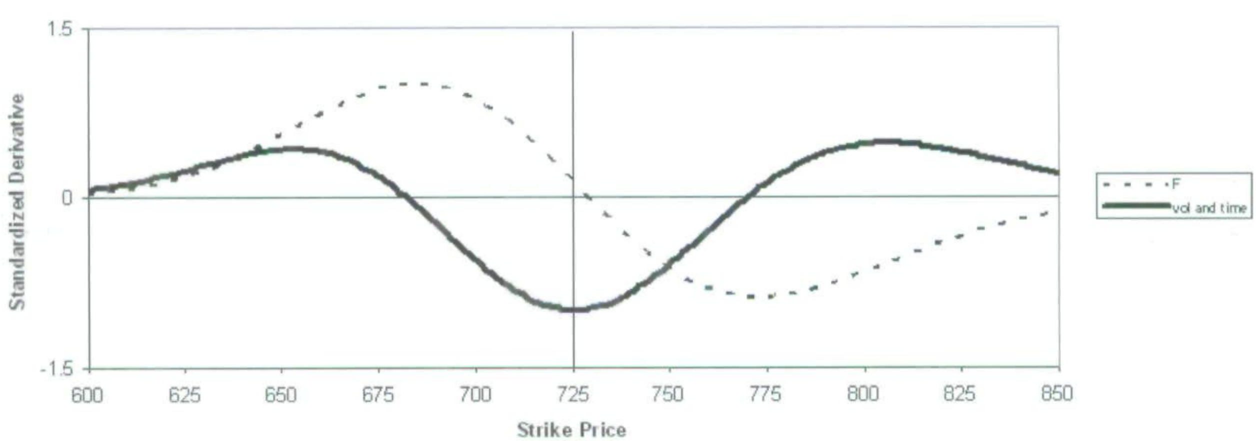
Attention is now turned to the third order Greeks. As shown in Exhibits 3 and 4, as a group, the third order factors have considerable incremental explanatory power. In the shorter-term sample, adding the third order factors to the Taylor series expansion cuts both the mean absolute forecast error and the root mean squared forecast error in half in all except the ATM sub-sample, where the reductions are still substantial. Their incremental explanatory power in the longer term sample is less dramatic but still substantial.

We now ask which of these third order factors are most important in accounting for option price changes over one week periods. Since gamma proved to be the most important of the second order derivatives, we anticipate that its derivatives with respect to F, σ, and T could be meaningful. Since the delta-vega and delta-theta factors also had explanatory power, their derivatives could also be important . Since the vega-vega, theta-theta, vega-theta factors proved relatively inconsequential, we anticipate that their derivatives will be inconsequential as well.

Formulae for the third order Black model derivatives (excluding those with respect to the interest rate) are presented in Panel C in the Appendix. The behavior of these derivatives across different strikes is illustrated in Exhibits 8 and 9. In Exhibit 8, we chart the derivatives of gamma with respect to the underlying asset price, volatility, and time-to-expiration as functions of the strike price for call options when F = 725, σ = .175, r = .05 (approximate means in our data set), and t = 6

EXHIBIT 8

Black Model Derivatives of Gamma with Respect to the Underlying Asset Price, F, Volatility, and Time to Expiration for a Call Option where F = 725, Vol. = .175, r = 5%, and T = 6 Weeks. The Derivatives are Standardized to Vary between –1 and +1



weeks. As in Exhibit 5, the derivatives are standardized so that their maximum positive value is 1.0 and their minimum negative value is –1.0. When standardized in this manner, $\partial^3O/\partial F^2\partial\sigma$ and $\partial^3O/\partial F^2\partial T$ are so close as to be indistinguishable in Exhibit 8. As shown in Exhibit 8, $\partial^3O/\partial F^3$ is close to zero for ATM options, while $\partial^3O/\partial F^2\partial\sigma$ and $\partial^3O/\partial F^2\partial T$ are near their absolute maximums for ATM options. At strikes about one standard deviation from F, which would generally fall between our S-OTM and M-OTM subsamples, $\partial^3O/\partial F^2\partial\sigma$ and $\partial^3O/\partial F^2\partial T$ are close to zero and $\partial^3O/\partial F^3$ nears its absolute maximum.

The derivatives $\partial^3O/\partial F\partial\sigma^2$, $\partial^3O/\partial F\partial T^2$, and $\partial^3O/\partial F\partial\sigma\partial T$ are graphed as a function of the strike price in Exhibit 9. All three are approximately zero for ATM options and are close to their absolute maximums at strikes about two-thirds of a standard deviation from F—roughly our S-OTM and ITM subsamples. As we shall see shortly, the third derivatives which do not involve F are inconsequential, so we have not graphed them as a function of the strike price.

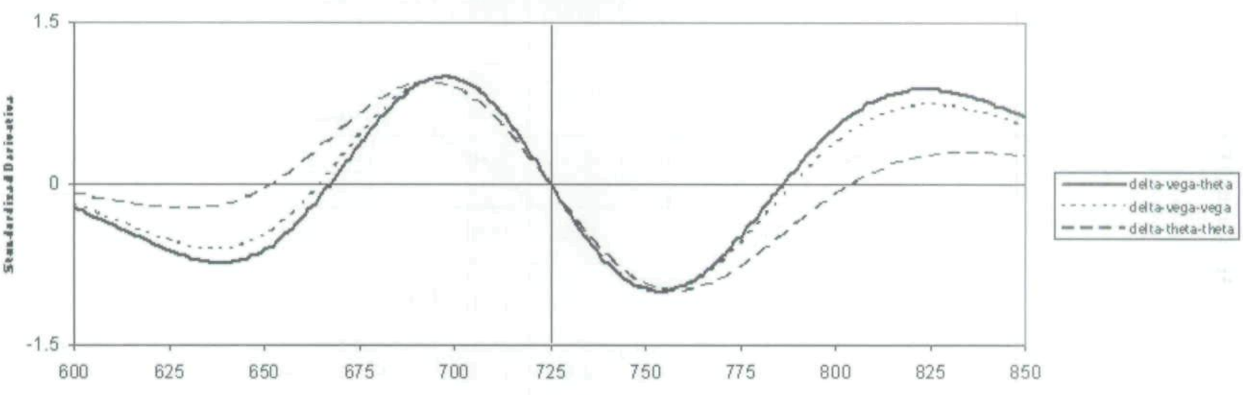
Means of absolute values and standard deviations of the ten third order factors are reported in Exhibit 10. As expected, the risk factors involving F have the largest means and standard deviations, and in general the more a factor depends on F, the larger its mean and standard deviation. To wit, with the exception of the ATM sub sample, where $\partial^3O/\partial F^3$ is close to zero, $(\partial^3O/\partial F^3)(\delta F)^3$ has the largest mean and standard deviation of the ten third order factors in all subsamples. The next largest means and standard deviations are for $(\partial^3O/\partial F^2\partial\sigma)$ $(\delta F)^2$ $(\delta\sigma)$ and $(\partial^3O/\partial F^2\partial T)$ $(\delta F)^2$ (δT), followed by $(\partial^3O/\partial F\partial\sigma^2)$ (δF) $(\delta\sigma)^2$, $(\partial^3O/\partial F\partial T^2)$ (δF) $(\delta T)^2$, and $(\partial^3O/\partial F\partial\sigma\partial T)$ (δF) $(\delta\sigma)$ (δT). As expected given our analysis above, $(\partial^3O/\partial F^2\partial\sigma)$ $(\delta F)^2$ $(\delta\sigma)$ and $(\partial^3O/\partial F^2\partial T)(\delta F)^2$ (δT) are the largest and most variable factors in the ATM subsample. Means and standard deviations of the four third order factors which do not involve F are minuscule in all subsamples.

Incremental Explanatory Power

Measures of the incremental explanatory power of the ten third order factors are reported in Exhibit 11. According to the results shown there, the incremental explanatory power of the third order factors is largely due to two third order factors: $(\partial^3O/\partial F^3)$ $(\delta F)^3$ and $(\partial^3O/\partial F^2\partial\sigma)$ $(\delta F)^2$ $(\delta\sigma)$. Only these two seem to account for meaningful changes in option price changes in our shorter-term sample and only $(\partial^3O/\partial F^3)$ $(\delta F)^3$ in the longer term sample. The relative insignificance of $(\partial^3O/\partial F^2\partial\sigma)(\delta F)^2$ $(\delta\sigma)$ in the longer term sample and $(\partial^3O/\partial F^2\partial T)(\delta F)^2$ (δT) in the shorter term sample is somewhat surprising, since according to the results in Exhibit 10, these two factors are sizable in several subsamples. Nonetheless, their incremental explanatory power is low.

EXHIBIT 9

Selected Black Model Third Derivatives for a Call as a Function of the Strike Price where F = 725, Volatility = .175, r = 5%, and T = 6 Weeks. The Derivatives are Standardizes to Vary between –1 and +1



HEDGING WITH HIGHER ORDER GREEKS

Next we explore how incorporating higher order Greeks into a hedge affects hedge effectiveness. For instance, given a portfolio which is delta, gamma, and vega neutral, is it worthwhile or important to make it neutral in terms of the higher moments as well? Setting up this comparison requires care since 1) the results are sensitive to the higher order Greeks of the original portfolio and 2) re-balancing to make a portfolio neutral in terms of one Greek usually changes the others.

Consider the first issue. Suppose we start with a delta-gamma-vega neutral portfolio (hereafter abbreviated as DGV) which is initially non-neutral in terms of one or more of the higher order Greeks and is then re-balanced to make it neutral in terms of the higher order Greeks as well. If we compare the hedging effectiveness of the second portfolio with the first, results will likely depend on how far from zero the higher order Greeks were in the original portfolio. It is therefore important to start with portfolios which have likely and realistic Greeks.

Consider the second issue. Suppose we start with a DGV portfolio and re-balance to make it neutral in terms of one of the higher moments. This re-balancing will change the other higher moments so could well lead to a portfolio which is less hedged against other moments. This problem is not limited to re-balancing in terms of

EXHIBIT 10
Descripptive Statistics for Third Order Risk Factors

Means (of absolute values) and standard deviations are reported for third order risk factors for a one-week holding period. ATM, S-OTM, M-OTM, F-OTM, and ITM designate at-the-money, slightly, moderately, and far out-of-the-money, and in-the-money respectively. The data period is Jaunary 1,1988 through March 31, 2003.

Risk Factor	Description	ATM		S-OTM		M-OTM		F-OTM		ITM	
		Mean	StdDev	Mean	StdDev	Mean	StdDev	Mean	StdDev	Mean	StdDev
Panel A - 4 to 8 Week Options											
Gamma-delta	$\frac{\partial^3 O}{\partial F^3}(\delta F)^3$	.03074	.12019	.11576	.43390	.14290	.57773	.10304	.41348	.12573	.46982
Gamma-vega	$\frac{\partial^3 O}{\partial F^2 \partial\sigma}(\delta F)^2(\delta\sigma)$	.08778	.38426	.04404	.21628	.03924	.26509	.05344	.27238	.05104	.24030
Gamma-theta	$\frac{\partial^3 O}{\partial F^2 \partial T}(\delta F)^2(\delta T)$	.10914	.2418	.05179	.13500	.02935	.08449	.03648	.09164	.06073	.15054
Delta-vega-vega	$\frac{\partial^3 O}{\partial F \partial\sigma^2}(\delta F)(\delta\sigma)^2$	.00227	.01221	.01523	.10563	.01344	.08492	.01889	.13112	.01648	.10985
Delta-theta-theta	$\frac{\partial^3 O}{\partial F \partial T^2}(\delta F)(\delta T)^2$	.00367	.00603	.02473	.04438	.02020	.03839	.01199	.02475	.02744	.04771
Delta-vega-theta	$\frac{\partial^3 O}{\partial F \partial\sigma \partial T}(\delta F)(\delta\sigma)(\delta T)$	.00195	.00501	.00825	.02282	.01524	.06536	.02910	.10110	.00909	.02411
Vega-vega-theta	$\frac{\partial^3 O}{\partial\sigma^2 \partial t}(\delta\sigma)^2(\delta t)$	.00004	.00013	.00077	.00241	.00348	.01852	.01018	.04289	.00080	.00240
Theta-theta-vega	$\frac{\partial^3 O}{\partial\sigma \partial T^2}(\delta\sigma)(\delta T)^2$	.00384	.00655	.00488	.00885	.00416	.00796	.00410	.00911	.00539	.00946
Vega-vega-vega	$\frac{\partial^3 O}{\partial\sigma^3}(\delta\sigma)^3$	.00005	.00041	.00197	.01683	.00316	.02223	.00378	.03118	.00203	.01951
Theta- theta-theta	$\frac{\partial^3 O}{\partial T^3}(\delta T)^3$	.00634	.00641	.00221	.00331	.00311	.00351	.00415	.00457	.00258	.00368

EXHIBIT 10 *(Continued)*

Panel B - 13 to 26 week options											
Risk Factor	Description	ATM		S-OTM		M-OTM		F-OTM		ITM	
		Mean	StdDev	Mean	StdDev	Mean	StdDev	Mean	StdDev	Mean	StdDev
Gamma-delta	$\frac{\partial^3 O}{\partial F^3}(\delta F)^3$	.01129	.05352	.03019	.13579	.04370	.21521	.03684	.15846	.03144	.13955
Gamma-vega	$\frac{\partial^3 O}{\partial F^2 \partial \sigma}(\delta F)^2(\delta \sigma)$	.02240	.10621	.01369	.07930	.01264	.09011	.02557	.14061	.01515	.08341
Gamma-theta	$\frac{\partial^3 O}{\partial F^2 \partial t}(\delta F)^2(\delta t)$	.01626	.04054	.00851	.02591	.00421	.01439	.00640	.01884	.00955	.02814
Delta-vega-vega	$\frac{\partial^3 O}{\partial F \partial \sigma^2}(\delta F)(\delta \sigma)^2$	.00053	.00330	.00448	.02209	.00652	.07021	.01055	.07275	.00465	.02247
Delta-theta-theta	$\frac{\partial^3 O}{\partial F \partial t^2}(\delta F)(\delta t)^2$	.00026	.00048	.00222	.00409	.00195	.00373	.00124	.00244	.00238	.00426
Delta-vega-theta	$\frac{\partial^3 O}{\partial F \partial \sigma \partial t}(\delta F)(\delta \sigma)(\delta t)$	.00032	.00092	.00143	.00412	.00240	.01005	.00652	.02544	.00150	.00424
Vega-vega-theta	$\frac{\partial^3 O}{\partial \sigma^2 \partial t}(\delta \sigma)^2(\delta t)$	.00001	.00003	.00014	.00037	.00053	.00230	.00239	.00977	.00014	.00039
Theta-theta-vega	$\frac{\partial^3 O}{\partial \sigma \partial t^2}(\delta \sigma)(\delta t)^2$	.00038	.00064	.00049	.00083	.00048	.00092	.00049	.00101	.00052	.00087
Vega-vega-vega	$\frac{\partial^3 O}{\partial \sigma^3}(\delta \sigma)^3$	.00001	.00010	.00057	.00258	.00211	.03229	.00243	.01421	.00058	.00254
Theta- theta-theta	$\frac{\partial^3 O}{\partial t^3}(\delta t)^3$	.00034	.00035	.00012	.00018	.00016	.00018	.00026	.00025	.00013	.00019

the higher moments. If we start with a non-DGV portfolio and re-balance to make it delta, gamma, and/or vega neutral, this re-balancing will normally change the higher moments as well and may move the portfolio either toward or away from neutrality in terms of these higher moments. For instance, consider a single ATM option. As we have seen, gamma-vega ($\partial^3 O/\partial F^2 \partial \sigma$) is sizable for ATM options but the other major higher order Greeks are negligible. We can delta hedge the ATM option using the underlying asset without changing any of the other Greeks. But to hedge against gamma or vega, we must introduce other options. If some of these options are not ATM, then risk in terms of the other major higher order Greeks, $\partial^2 O/\partial F \partial \sigma$, $\partial^2 O/\partial F \partial T$, and $\partial^3 O/\partial F^3$, is introduced where it did not exist before.

In light of these issues, we proceed as follows. To begin, we restrict our attention to delta, gamma, vega, and the four higher Greeks, $\partial^2 O/\partial F \partial \sigma$, $\partial^2 O/\partial F \partial T$, $\partial^3 O/\partial F^3$ and $\partial^3 O/\partial F^2 \partial \sigma$, which our analysis above has shown are most important. To construct the portfolios, we choose one option, such as an ATM call, as the numeraire—that is, the portfolio always contains a long position in one unit of this option. We then form delta-neutral and DVG portfolios which leave the other moments unchanged. For the delta-neutral portfolio, this simply involves shorting delta units of the underlying futures contract. Since, other than delta, all Greeks are zero for the futures contract, this makes the portfolio delta-neutral without changing any of the other Greeks. Making the delta-neutral portfolio gamma and delta neutral requires a minimum of two different options or strikes, but that approach will tend to change the higher order moments. For instance, if the numeraire is an ATM option, the delta-neutral portfolio will be approximately neutral in terms of three of the four higher order Greeks: delta-vega, delta-theta, and delta-gamma. Adding non-ATM options to the portfolio to make it vega and gamma neutral introduces risk in terms of these three Greeks. How much higher order Greek risk is introduced depends on which options are in the portfolio.

Our solution to this problem is to form DVG portfolios which are delta, gamma, and vega neutral but leave the four higher order moments unchanged. In other words, our delta-neutral and DVG portfolios have the same four higher order Greeks as our numeraire option. Conceptually, this allows us to compare hedging effectiveness as follows. Starting with a portfolio consisting of a single option, we first add futures to make it delta neutral without changing any of the other Greeks and measure how well the hedge works. Next we add options to make the portfolio gamma and vega neutral (and still delta neutral) without changing the four higher order Greeks and check its hedging effectiveness. Finally we re-balance the portfolio to make it neutral in terms of all seven Greeks, including the four higher order Greeks, and check hedging effectiveness.

The DGV and completely neutral portfolios are formed by solving seven linear equations of the form $G_p = \Sigma_i w_i G_i$, where G_p is one of the portfolio Greeks, such as delta,[14] gamma, etc., and G_i is the same Greek for option i. Since we consider seven different Greeks, there are seven such equations. The seven equations are then

EXHIBIT 11

Incremental Explanatory power—Third Order Risk Factors

Means of the incremental explanatory power of third order risk factors in explaining changes in S&P 500 futures options over one-week holding periods are presented. The incremental mean absolute forecast error (IMAE) and incremental root mean squared forecast error (IRMSE) are measured as the reduction in the mean absolute forecast error (MAE) and root mean squared forecast error (RMSE) when that risk factor is added to a Taylor series equation (Equation 4 in the text) which includes the other nineteen first, second and third order risk factors. These are expressed as percentages of the mean absolute price change (MAPC) and root mean squared price change (RMSPC) in the price of S&P 500 options in parentheses below each term.

Risk Factor	Derivative with respect to	ATM Options		S-OTM Options		M-OTM Options		F-OTM Options		ITM Options	
		IMAE	IRMSE	IMAE	IRMSE	IMAE	IRMSE	IMAE	IRMSE	IMAE	IRMSE
Panel A - 4 to 8 week options											
Gamma-delta	F, F, F	.0009 (0.01%)	-.0173 (-0.16%)	.0814 (2.37%)	.2036 (3.26%)	.1153 (5.74%)	.3764 (9.20%)	.0877 (7.41%)	.3152 (12.16%)	0.0899 (0.87%)	0.1840 (1.17%)
Gamma-vega	F, F, σ	.0682 (1.00%)	.2925 (2.67%)	.0261 (0.76%)	.1023 (1.64%)	.0243 (1.21%)	.1611 (3.94%)	.0338 (2.86%)	.1809 (6.98%)	.0308 (0.30%)	.1022 (0.65%)
Gamma-theta	F, F, T	.0361 (0.53%)	-.0064 (-0.06%)	.0182 (0.53%)	.0168 (0.27%)	.0059 (0.30%)	.0053 (0.13%)	.0105 (0.89%)	-.0026 (-0.10%)	.0225 (0.22%)	.0158 (0.10%)
Delta-vega-vega	F,σ, σ	.0007 (0.01%)	.0013 (0.01%)	.0076 (0.22%)	.0583 (0.93%)	.0064 (0.32%)	.0352 (0.86%)	.0132 (1.12%)	.0543 (2.09%)	.0072 (0.07%)	.0450 (0.29%)
Delta-theta-theta	F, T, T	.0001 (0.00%)	-.0001 (-0.00)	.0040 (0.12%)	.0070 (0.11%)	.0019 (0.09%)	.0033 (0.08%)	.0003 (0.02%)	-.0005 (-0.02%)	.0010 (0.01%)	.0043 (0.03%)
Delta-vega-theta	F, σ, T	-.0003 (-0.00%)	-.0009 (-0.01%)	-.0013 (-0.04%)	-.0069 (-0.11%)	.0083 (0.42%)	.0202 (0.49%)	.0155 (1.31%)	.0445 (1.72%)	-.0014 (-0.01%)	-.0054 (-0.03%)
Vega-vega-theta	σ,σ, T	-.0000 (-0.00%)	-.0000 (-0.00)	.0001 (0.00%)	.0004 (0.01%)	.0003 (0.02%)	-.0006 (-0.02%)	.0048 (0.41%)	.0073 (0.28%)	.0001 (0.00%)	.0003 (0.00%)
Theta-theta-vega	T, T, σ	.0009 (0.01%)	.0013 (0.01%)	.0007 (0.02%)	.0007 (0.01%)	.0001 (0.01%)	.0001 (0.00%)	-.0003 (-0.03%)	.0000 (0.00%)	.0006 (0.01%)	.0006 (0.00%)
Vega-vega-vega	σ, σ, σ	-.0000 (-0.00%)	.0000 (0.00%)	-.0003 (-0.01%)	-.0031 (-0.05%)	-.0004 (-0.02%)	-.0035 (-0.08%)	.0008 (0.07%)	.0045 (0.17%)	-.0004 (-0.00%)	-.0024 (-0.02%)
Theta- theta-theta	T, T, T	.0012 (0.02%)	.0016 (0.01%)	.0002 (0.01%)	.0003 (0.01%)	.0003 (0.01%)	.0004 (0.01%)	.0003 (0.03%)	.0006 (0.02%)	.0001 (0.00%)	.0002 (0.00%)

EXHIBIT 11 *(Continued)*

Panel B - 13 to 26 week options												
Risk Factor	Derivative with respect to	ATM Options		S-OTM Options		M-OTM Options		F-OTM Options		ITM Options	
		IMAE	IRMSE	IMAE	IRMSE	IMAE	IRMSE	IMAE	IRMSE	IMAE	IRMSE
Gamma-delta	F, F, F	.0019 (0.03%)	.0079 (0.09%)	.0183 (0.62%)	.0546 (1.01%)	.0343 (1.91%)	.1108 (3.11%)	.0311 (2.55%)	.0879 (3.57%)	.0077 (0.09%)	.0233 (0.18%)
Gamma-vega	F, F, σ	.0119 (0.21%)	.0575 (0.63%)	.0072 (0.24%)	.0309 (0.57%)	-.0022 (-0.12%)	.0237 (0.66%)	-.0021 (-0.17%)	-.0005 (-0.02%)	.0053 (0.06%)	.0144 (0.11%)
Gamma-theta	F, F, T	.0007 (0.01%)	-.0013 (-0.01%)	-.0004 (-0.01%)	-.0027 (-0.05%)	.0008 (0.04%)	.0028 (0.08%)	.0015 (0.12%)	-.0014 (-0.06%)	.0016 (0.02%)	.0020 (0.02%)
Delta-vega-vega	F, σ, σ	.0000 (0.00%)	.0000 (0.00%)	.0009 (0.03%)	.0025 (0.05%)	-.0003 (-0.02%)	-.0327 (-0.92%)	.0080 (0.65%)	.0264 (1.07%)	.0020 (0.02%)	.0030 (0.02%)
Delta-theta-theta	F, T, T	-.0000 (-0.00%)	-.0000 (-0.00%)	-.0002 (-0.01%)	-.0000 (-0.00%)	-.0007 (-0.04%)	-.0007 (-0.02%)	-.0004 (-0.03%)	-.0004 (-0.02%)	.0005 (0.01%)	.0004 (0.00%)
Delta-vega-theta	F, σ, T	-.0000 (-0.00%)	-.0000 (-0.00%)	.0000 (0.00%)	-.0001 (-0.00%)	-.0007 (-0.04%)	-.0024 (-0.06%)	-.0022 (-0.18%)	.0004 (0.02%)	-.0002 (-0.00%)	-.0003 (-0.00%)
Vega-vega-theta	σ, σ, T	-.0000 (-0.00%)	-.0000 (-0.00%)	-.0000 (-0.00%)	-.0000 (-0.00%)	.0002 (0.01%)	.0006 (0.02%)	.0011 (0.09%)	.0010 (0.04%)	.0000 (0.00%)	.0000 (0.00%)
Theta-theta-vega	T, T, σ	.0000 (0.00%)	.0000 (0.00%)	.0001 (0.00%)	.0001 (0.00%)	.0000 (0.00%)	.0002 (0.01%)	-.0002 (-0.01%)	-.0003 (-0.01%)	-.0000 (-0.00%)	-.0000 (-0.00%)
Vega-vega-vega	σ, σ, σ	-.0000 (-0.00%)	-.0000 (-0.00%)	.0001 (0.00%)	.0001 (0.00%)	.0010 (0.06%)	.0138 (0.39%)	-.0006 (-0.05%)	-.0012 (-0.05%)	-.0001 (-0.00%)	-.0001 (-0.00%)
Theta- theta-theta	T, T, T	.0000 (0.00%)	.0000 (0.00%)	.0000 (0.00%)	.0000 (0.00%)	-.0000 (-0.00%)	-.0000 (-0.00%)	-.0000 (-0.00%)	-.0000 (-0.00%)	.0000 (0.00%)	-.0000 (-0.00%)

solved for the w_i, which represent the number of each option type, e.g., ATM call, ITM put, etc. in the portfolio. For the numeraire option, $w_1 = 1$.

Since there are seven Greeks, each portfolio consists of six options besides the numeraire and a futures position. We choose the options to include in the portfolio according to the following rules. First, we require that there be at least two options with strikes below the strike of the numeraire option and two above.[15] Second, we choose the strikes closest to the strike of the numeraire option. Third, if more than one combination meets these two requirements, we use three calls and three puts. So when an ATM call (put) is the numeraire, the portfolio includes: ITM, S-OTM, and M-OTM calls (puts) and S-OTM, ATM, and ITM puts (calls). For an ITM call (put) numeraire, the portfolio includes: ATM and S-OTM calls (puts) and F-OTM, M-OTM, S-OTM and ATM puts (calls); and for an S-OTM call (put) numeraire: ATM, M-OTM, and F-OTM calls (puts) and S-OTM, ATM, and ITM puts (calls). The requirement that at least two options have lower strikes and two higher strikes rules out portfolios with M-OTM and F-OTM numeraires. As we have seen in Exhibits 7 and 11, three of the four higher order Greeks tend to be larger for M-OTM and F-OTM options than for closer to the money options, while delta, gamma, and vega tend to be lower. Hence if hedging the higher order Greeks proves important in the portfolios we examine, it should be for further OTM portfolios as well.

Since to measure the change in the price of the portfolio, prices of all seven options have to be observed on both the date the portfolio is formed and a week later, the data requirements are rather stringent and we lose a number of observations due to missing price observations. The problem is especially severe for F-OTM calls, which are traded less than a third of the time. Thus we were forced to drop portfolios with F-OTM call options—that is, portfolios with S-OTM call and ITM put numeraires. We also restrict the samples to observations when the hedge portfolios include no more than 30 options, i.e., $\Sigma_i |w_i| \leq 30$.[16]

We utilize two measures of hedging effectiveness. The first is the mean absolute unexpected or unhedged change in the value of the portfolio = $1/T\ \Sigma_{t=1}^{T} |\delta P_t - E(\delta P_t)|$ where δP_t is the change in the value of the portfolio over the week from t to t+1 and $E(\delta P_t)$ is the expected change based on the portfolio theta, that is $E(\delta P_t) = \Sigma_i w_i \Theta_{it}(5/252)$ where Θ_{it} is the theta of option i at time t. The second is the root mean squared unexpected change in the value of the portfolio = $[1/T\ \Sigma_{t=1}^{T}\ (\delta P - E(\delta P_t))^2]^{.5}$. Obviously the latter puts more weight on large unhedged changes in portfolio value. These are reported for 1) one unhedged unit of the numeraire option, 2) the delta neutral portfolio, 3) the DGV portfolio (which is delta, gamma, and vega neutral), and 4) the hedge portfolio which is neutral in terms of all seven Greeks including the four higher order Greeks. For ease of interpretation and comparison across portfolios, we also present results when the last three are divided by the first to measure the hedged risk as a percentage of the unhedged risk.

Results are reported in Exhibit 12 for our 4 to 8 week options and in Exhibit 13 for the 13 to 26 week options. As shown there, while delta, gamma, and vega hedges remove most risk from unhedged portfolios, making the portfolios neutral in terms of the four higher moments reduces the remaining risk considerably. For instance, for unhedged shorter-term ATM calls, the average absolute price change is \$4.344. When delta hedged, this falls to \$0.980, and when delta, gamma and vega hedged to \$0.390. But when hedged against all seven Greeks, the average unhedged absolute value change is only \$0.043. Across the four shorter-term option portfolios in Exhibit 12, making the portfolios neutral in terms of the four higher order Greeks reduces the mean absolute unhedged value change by between 89% and 91% as compared with the DVG hedge. Moreover, across the four portfolio types in Exhibit 12, the average unhedged value change ranges from \$0.043 for ATM calls to \$0.059 for ITM calls when all seven Greeks equal zero, while the minimum for DVG hedges is \$0.390. Considering that S&P 500 options are quoted in increments of \$0.05 and that the average portfolio consists of 3.92 options, it appears that the hedged value change is approaching the minimum possible when hedged against all seven Greeks.

In terms of the root mean squared unhedged price change, the results are even more dramatic. Again, hedging in terms of delta, gamma, and vega removes most of the risk. The root mean squared change in value for the shorter-term DVG portfolios in Exhibit 12 ranges from 8.62% of the unhedged root mean squared price change for ITM calls to 20.08% for S-OTM puts. When hedged against the four higher order moments as well, this figure ranges from 0.82% (ATM calls) to 2.12% (ATM puts). Making the DVG portfolios neutral in terms of the four higher Greeks reduces the root mean squared unhedged price change by a minimum of 87%.

While data restrictions prevented us from examining portfolios with M-OTM and F-OTM numeraires, we would anticipate that hedging the higher order Greeks would be even more important in these portfolios since, as seen in Exhibits 7 and 11, three of the higher order Greeks are higher for these more OTM options while delta, gamma, and vega are lower.

Results for longer term options in Exhibit 13 are roughly the same as for the shorter term. While the average absolute price change for the all-neutral portfolio is a little higher—ranging from \$0.065 for ATM calls to \$0.092 for ITM calls—making the four higher order Greeks zero still reduces the mean absolute price change by 88% to 89% relative to the DVG portfolio.

In summary, our hedging results largely duplicate our findings on the relative importance of the various Greeks above. Making a portfolio delta neutral eliminates most risk of unexpected value changes—reducing the mean absolute unexpected price change by an average of 76% across the eight portfolios we consider. Making the portfolio gamma and vega neutral as well normally further reduces the risk. On average across our eight portfolios, the mean unexpected absolute price change for the DVG portfolio is only 17.6% of the mean unexpected absolute price change for the unhedged portfolio. However, when the portfolios are made neutral in terms of the four higher order moments as well, this figure falls to only 1.8%, so it is clear that hedging in terms of the higher moments leads to a substantial further risk reduction. While we have examined only a subset of the possible portfolios, it appears that by hedging against the risk represented by these four higher moments, price risk can be virtually eliminated.

CONCLUSION

This article has explored whether any higher order derivatives besides gamma are important in explaining

EXHIBIT 12

Hedging Effectiveness for 4 to 8 Week Options

Measures of hedging effectiveness are reported for portfolios which are hedged in terms of 1) delta, 2) delta, gamma, and vega, and 3) delta, gamma, vega, and the four higher order Greeks: delta-vega, delta-theta, delta-gamma, and gamma-vega. The unexpected change in portfolio value is measured as the actual change in value over a one-week holding period minus the change predicated by the portfolio theta. All portfolios contain one long unit of the numeraire option and all unhedged portfolio Greeks equal the Greeks of the numeraire option. The unhedged and delta neutral portfolios contain the numeraire option and the numeraire option and futures respectively. The DVG and all-neutral portfolios contain seven options and futures.

	Numeraire Option			
	ATM Call	ATM Put	ITM Call	S-OTM Put
Mean Absolute Unexpected Change in Portfolio Value:				
Unhedged	\$4.3445	\$3.6017	\$7.7514	\$2.6697
Delta neutral	\$0.9801	\$0.9975	\$1.0881	\$1.0587
Delta, gamma, & vega neutral (DGV)	\$0.3900	\$0.4432	\$0.6332	\$0.6037
Neutral in terms of all seven Greeks	\$0.0427	\$0.0486	\$0.0593	\$0.0525
Root Mean Squared Unexpected Change in Portfolio Value:				
Unhedged	\$7.8321	\$6.0998	\$12.0881	\$5.4843
Delta neutral	\$2.1271	\$2.0506	\$2.1805	\$2.2590
Delta, gamma, & vega neutral (DGV)	\$0.6768	\$1.0138	\$1.0423	\$1.1011
Neutral in terms of all seven Greeks	\$0.0639	\$0.1295	\$0.1155	\$0.0912
Mean Absolute Unexpected Value Change as a Percentage of the Mean Unhedged Value Change:				
Delta neutral	22.559%	27.695%	14.038%	39.655%
Delta, gamma, & vega neutral (DGV)	8.976%	12.307%	8.169%	22.615%
Neutral in terms of all seven Greeks	0.983%	1.351%	0.765%	1.966%
Root Mean Squared Unexpected Value Change as a Percentage of the Unhedged Value Change:				
Delta neutral	27.159%	33.617%	18.038%	41.191%
Delta, gamma, & vega neutral (DGV)	8.641%	16.621%	8.622%	20.078%
Neutral in terms of all seven Greeks	0.816%	2.123%	0.956%	1.664%
Observations	782	1280	970	782

EXHIBIT 13

Hedging Effectiveness for 13 to 26 Week Options

Measures of hedging effectiveness are reported for portfolios which are hedged in terms of 1) delta, 2) delta, gamma, and vega, and 3) delta, gamma, vega, and the four higher order Greeks: delta-vega, delta-theta, delta-gamma, and gamma-vega. The unexpected change in portfolio value is measured as the actual change in value over a one-week holding period minus the change predicated by the portfolio theta. All portfolios contain one long unit of the numeraire option and all unhedged portfolio Greeks equal the Greek of the numeraire option. The unhedged and delta neutral portfolios contain the numeraire option and the numeraire option and futures, respectively. The DVG and all-neutral portfolios contain seven options and futures.

	Numeraire Option			
	ATM Call	ATM Put	ITM Call	S-OTM Put
Mean Absolute Unexpected Change in Portfolio Value:				
Unhedged	$4.0414	$2.6232	$6.8603	$2.5320
Delta neutral	$0.7356	$0.5653	$0.8277	$0.8312
Delta, gamma, & vega neutral (DGV)	$0.5966	$0.6119	$0.8249	$0.8135
Neutral in terms of all seven Greeks	$0.0645	$0.0733	$0.0923	$0.0822
Root Mean Squared Unexpected Change in Portfolio Value:				
Unhedged	$5.9022	$3.6094	$10.1584	$5.2536
Delta neutral	$1.2211	$0.8313	$1.8718	$1.9600
Delta, gamma, & vega neutral (DGV)	$0.9353	$0.8818	$1.1061	$1.2488
Neutral in terms of all seven Greeks	$0.0950	$0.1434	$0.1326	$0.1146
Mean Absolute Unexpected Value Change as a Percentage of the Mean Unhedged Value Change:				
Delta neutral	18.200%	21.550%	12.065%	32.829%
Delta, gamma, & vega neutral (DGV)	14.761%	23.326%	12.025%	32.130%
Neutral in terms of all seven Greeks	1.596%	2.796%	1.345%	3.247%
Root Mean Squared Unexpected Value Change as a Percentage of the Unhedged Value Change:				
Delta neutral	20.689%	23.032%	18.426%	37.308%
Delta, gamma, & vega neutral (DGV)	15.846%	24.430%	10.889%	23.770%
Neutral in terms of all seven Greeks	1.609%	3.974%	1.305%	2.180%
Observations	464	747	479	405

changes in S&P 500 index futures options over one week holding periods—that is, whether any other higher order derivatives represent significant risk factors. We find that while gamma is generally the most important of the higher order derivatives in explaining option price changes over a one-week holding period, several others have significant explanatory power—particularly for moderately to far out-of-the-money options. Adding second and third order factors to a Taylor series expansion leads to a considerable reduction in forecast errors in all subsamples. Particularly important over one-week holding periods are $\partial^2 O/\partial F\partial\sigma$, $\partial^2 O/\partial F\partial T$, $\partial^3 O/\partial F^3$, and $\partial^3 O/\partial F^2\partial\sigma$, where O is the option price, F is the underlying asset price, T is time to expiration and σ represents volatility. Since the first three derivatives are approximately zero for ATM options, they are more important in explaining changes in the prices of ITM and OTM options. On the other hand, $\partial^3 O/\partial F^2\partial\sigma$ tends to peak at the money so is more important in explaining price changes among ATM options. It bears repeating that our measures are over one week holding periods. If returns are independent, δF should be approximately proportional to the square root of the holding period while $(\delta F)^2$ and δT should be proportional to the length of the holding period. Consequently as the holding period is lengthened, factors involving $(\delta F)^2$ and δT should become more important relative to those involving δF.

A Taylor series expansion based on the five traditional Greeks explains most one week changes in option prices but leaves a considerable portion of the option price changes unexplained—particularly for moderately to far out-of-the-money options. However, when second and third order risk factors are added to the Taylor series, the average absolute unexplained price change is only one or two ticks and is roughly the same for ATM, ITM , and OTM options. Our evidence indicates that, despite its theoretical deficiencies and its inability to explain the cross-sectional pattern in option prices, the Black model's Greeks explain the time-series changes in option prices quite well.

Finally we find that while unexpected price changes on delta, gamma, and vega neutral portfolios are fairly small, making these portfolios neutral in terms of the four higher moments which we found were most important (delta-vega, delta-theta, gamma-vega, and gamma-delta) removes most of the remaining risk—roughly 90% in the portfolios we examine.

APPENDIX

Black Model Greeks

Panel A - Traditional Greeks. Formulae for option Greeks are presented according to Black's model for options on futures: $O = e^{-rT}$ [FN(d1) $-XN(d_2)$] for calls and $O = e^{-rT}$ $[XN(-d_2) - FN(-d_1)]$ for puts where: F is the underlying futures price, X the exercise price, O the price of the option, σ the volatility, T the time-to-expiration, and r the risk-free interest rate. $d_1 = [\ln(F/X) + .5\sigma^2 T] / \sigma\sqrt{T}$ and $d_2 = d_1 - \sigma\sqrt{T}$. N(.) represents the cumulative normal distribution, and n(.) the normal density.

Greek	Derivative	Calls	Puts
Delta (Δ)	$\partial O/\partial F$	$e^{-rT} N(d_1)$	$e^{-rT} [N(d_1)-1]$
Gamma (Γ)	$\partial^2 O/\partial F^2$	$\frac{e^{-rT}}{F\sigma\sqrt{T}} n(d_1)$	
Vega	$\partial O/\partial \sigma$	$Fe^{-rT}\sqrt{T}\, n(d_1)$	
Theta (θ)	$\partial O/\partial T$	$\frac{e^{-rT} F\sigma}{2\sqrt{T}} n(d_1) - rO$	
Rho (ρ)	$\partial O/\partial r$	-TO	

Panel B - Second Order Greeks. Formulae for second order derivatives (excepting Gamma which is in panel A) are presented according to Black's model for options on futures, which is: $O = e^{-rT}$ $[F\,N(d_1) - X\,N(d_2)]$ for calls and $O = e^{-rT}$ $[X\,N(-d_2) - F\,N(-d_1)]$ for puts where: F is the underlying futures price, X the exercise price, O the price of the option, σ the volatility, T the time-to-expiration, and r the risk-free interest rate. $d_1 = [\ln(F/X) + .5\,\sigma^2 T]/\sigma\sqrt{T}$ and $d_2 = d_1 - \sigma\sqrt{T}$. N(.) represents the cumulative normal distribution, and n(·) the normal density. θ, Δ, and Γ represent the Greeks defined in Panel A and $W = [e^{-rT} F\sigma n(d_1)]/(2\sqrt{T})$.

Derivative	Formula
$\frac{\partial^2 O}{\partial \sigma^2}$	$[\text{Vega}]\,(d_1)\left[\frac{d_2}{\sigma}\right]$
$\frac{\partial^2 O}{\partial T^2}$	$-r\,\theta - W\left(r + \frac{(1 - d_1 d_2)}{2T}\right)$
$\frac{\partial^2 O}{\partial F\,\partial \sigma}$	$-e^{-rT} n(d_1)\,[d_2/\sigma]$
$\frac{\partial^2 O}{\partial F\,\partial T}$	$-r\Delta - e^{-rT} n(d_1)\left[\frac{d_2}{2T}\right]$
$\frac{\partial^2 O}{\partial \sigma\,\partial T}$	$[\text{Vega}]\left[\frac{1 + d_1 d_2}{2T} - r\right]$

Panel C - Third Order Greeks. Formulae for third order derivatives are presented according to Black's model for options on futures which is: $O = e^{-rT}[F\,N(d_1) - X\,N(d_2)]$ for calls and $O = e^{-rT}[X\,N(-d_2) - F\,N(-d_1)]$ for puts where: F is the the underlying futures price, X the exercise price, O the price of the option, σ the volatility, T the time-to-expiration, and r the risk-free interest rate. $d_1 = [\ln(F/X) + .5\,\sigma^2 T]/\sigma\sqrt{T}$ and $d_2 = d_1 - \sigma\sqrt{T}$. $N(\cdot)$ represents the cumulative normal distribution, and $n(.)$ the normal density. $W = [e^{-rT} F\sigma\, n(d_1)]/(2\sqrt{T})$ and $Y = \partial W/\partial F = -e^{-rT}\, n(d_1)\,(d_2/2T)$.

Derivative	Formula
$\frac{\partial^3 O}{\partial F^3}$	$-\Gamma\left(\frac{1}{F}\right)\left[\frac{d_1}{\sigma\sqrt{T}} + 1\right]$
$\frac{\partial^3 O}{\partial F^2\,\partial\sigma}$	$\Gamma\left(\frac{d_1 d_2 - 1}{\sigma}\right)$
$\frac{\partial^3 O}{\partial F^2\,\partial T}$	$\Gamma\left(-r + \frac{d_1 d_2 - 1}{2T}\right)$
$\frac{\partial^3 O}{\partial F\,\partial\sigma^2}$	$\left(\frac{\partial^2 O}{\partial F\,\partial\sigma}\right)\left(\frac{1}{\sigma}\right)\left[d_1 d_2 - \frac{d_1}{d_2} - 1\right]$
$\frac{\partial^3 O}{\partial F\,\partial T^2}$	$-r\left(\frac{\partial^2 O}{\partial F\,\partial T}\right) + Y\left[-r + \frac{1}{2T}\left(d_1 d_2 - \frac{d_1}{d_2} - 2\right)\right]$
$\frac{\partial^3 O}{\partial T\,\partial\sigma^2}$	$\left(\frac{\partial^2 O}{\partial\sigma\partial T}\right)\left(\frac{d_1 d_2}{\sigma}\right) - \frac{Vega}{2T\sigma}(d_1^2 + d_2^2)$
$\frac{\partial^3 O}{\partial T^2\partial\sigma}$	$\left(\frac{\partial^2 O}{\partial\sigma\partial T}\right)\left(\frac{1 + d_1 d_2}{2T} - r\right) - \frac{Vega}{(2T)^2}[2 + (d_1 + d_2)^2]$
$\frac{\partial^3 O}{\partial F\,\partial\sigma\,\partial T}$	$\left(\frac{\partial^2 O}{\partial F\,\partial\sigma}\right)\left[-r + \frac{1}{2T}\left(d_1 d_2 - \frac{d_1}{d_2}\right)\right]$
$\frac{\partial^3 O}{\partial\sigma^3}$	$\left(\frac{\partial^2 O}{\partial\sigma^2}\right)\left(\frac{1}{\sigma}\right)\left[d_1 d_2 - \frac{d_1}{d_2} - \frac{d_2}{d_1} - 1\right]$
$\frac{\partial^3 O}{\partial T^3}$	$-r\left(\frac{\partial^2 O}{\partial T^2}\right) + W\left(r + \frac{1 - d_1 d_2}{2T}\right)^2 - \frac{W}{(2T)^2}[(d_1 + d_2)^2 - 2]$

ENDNOTES

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[1]For one of the earliest expositions of the five traditional derivatives, see Cox and Rubinstein [1985].

[2]See for instance the Derivatives Dictionary at http://www.margrabe.com/dictionary.html

[3]While the time to expiration differs from option to option and over time, δT_{it} is the same for all options at all times so has neither an i nor a t subscript.

[4]Since "vega" is not a Greek letter, we use the Greek symbol for nu, ν, for "vega" due to its similarity to the English letter v.

[5]Recent major studies of this market include Bakshi et al. [1997], Bollen and Whaley [2004], Jackwerth [2000], Ederington and Guan [2002], and Bates [2000].

[6]According to the Black model, gamma and vega are maximized at a strike slightly above the underlying asset price, specifically $Fe^{.5\sigma^2 t}$. For the S&P 500 futures options that we examine the exponential term is close to 1.0, so the point at which gamma and vega are maximized is very close to F.

[7]Trading is relatively light in F-OTM calls so this sample is smaller.

[8]The choices of points 4% from the mean for the shorter term sample and 7% for longer term are comparable in the following sense. If returns are independent, volatility over the time interval to expiration is proportional to the square root of the time-to-expiration. Based on a mean time to expiration of approximately 6 weeks in the shorter sample and 19.5 in the longer, that ratio is $(19.5/6)^{.5} = 1.8$ and $.04(1.8) = .072$. We round the latter off to .07. To avoid overlap with the other subsamples we require that the S-OTM observations be within 2% and 6% of the futures in the shorter-term sample and 3.5% and 10.5% in the longer term.

[9]For instance, to estimate the implied volatility of a call with a strike price of 800, we average together the implied volatilities of the calls with strikes of 795 and 805 and puts with strikes of 795, 800, and 905. Far from the money strikes may be in increments of 10 or more instead of 5 or may not all be actively traded. Consequently for the deep out-of-the-money sample we include strikes within 10 index points of j's instead of 5 as in the other three subsamples. Particularly in the early years in our sample, all five surrounding strikes are not always traded, in which case the average is based on the number available requiring at least two.

[10]How the relative importance of the vega factor varies with the length of the holding period will depend on the serial correlation in volatility changes.

[11]However, since implied volatilities for longer term options are more stable than those for shorter term options over our one week holding period, the means (of absolute values) and standard deviations of the vega factor are not consistently larger in the longer-term sample than in the shorter term, even though the vega derivative in the shorter-term sample is consistently smaller.

[12]Of course we could equally as well refer to it as the vega-delta derivative and factor.

[13]We do not present separate statistics for the derivatives, since these depend on the scale used to measure the determinants. Once multiplied by the change in the determinants, scale differences cancel out.

[14]The delta equation also includes the underlying futures contract as one of the i.

[15]Trial runs with less than two options on either side of the numeraire option resulted in large, unwieldy, and unrealistic portfolios of hundreds of contracts.

[16]If the sum of the absolute number of options exceeds 30 in either the DVG or all neutral portfolios, both portfolio observations are dropped from the sample.

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