

Fast Pricing of Cliquet Options with Global Floor

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We investigate the pricing of cliquet options with global floor when the underlying asset follows the Bachelier-Samuelson model. These options have a payoff structure which is a function of the sum of truncated periodic stock returns over the life span of the option.

Fourier integral formulas for the price and greeks are derived, and a fast and robust numerical integration scheme for the evaluation of these formulas is proposed. This algorithm seems much faster than quasi-Monte Carlo and finite difference techniques for a given level of computational accuracy.

This millennium started with a recession and rapidly falling stock markets. Investors who had relied on annual returns on investments exceeding 20% suddenly became aware of the risk inherent in owning shares and many turned their attention to safer investments like bonds and ordinary bank accounts. As an attempt to capitalize on this fear of losses, a variety of equity-linked products with capital guarantees were introduced on the market. Among the most successful is the so-called *cliquet option with global floor*, which is usually packaged with a bond and sold to retail investors under names like *equity-linked bond with capital guarantee* or *equity-index bond*.

Today, quasi-Monte Carlo and finite difference methods are the most common methods to compute the price and greeks of these

options. Since the payoff is rather complex, these methods are relatively time consuming. In this article, we propose a Fourier integral method which seems faster than existing methods for a given level of accuracy. Moreover, the method allows us to compute the greeks directly, avoiding the finite difference approximations of the partial derivatives often employed in the context of Monte Carlo or finite difference methods.

Smaller banks wanting to offer these structured products to their retail clients may lack the scale to support a separate exotic options desk. A fast computational method could allow these banks to hedge their cliquet options with global Floor together with their vanilla options, without suffering risky delays due to slow computations.

For simplicity, we use the standard Bachelier-Samuelson market model, but in separate notes we show how the method may be used in connection with more advanced market models.

This article is organized as follows: The second section introduces the type of options considered in this article and the third section Fixes notation and introduces the market model. The pricing formula is derived in the fourth section and the fifth section discusses some additional payoffs not considered in the second section that may be priced with this methodology. A numerical integration scheme is proposed in the sixth and seventh sections.

The Monte Carlo and PDE methods, used as benchmarks, are discussed briefly in the eighth section and pricing examples and results from benchmark tests are presented in the ninth section. Finally, the last section, containing conclusions and suggestions for future research, concludes the article.

CLIQUET OPTIONS WITH GLOBAL FLOOR

Let T be a future point in time, and divide the interval $[0, T]$ into N subintervals called *reset periods* of equal length $\Delta T = T_n - T_{n-1}$, where $\{T_n\}_{n=0}^N$, $T_0 = 0$, $T_N = T$, are called the *reset days*. The return of an asset with price process S_t over a reset period $[T_{n-1}, T_n]$ is then defined as

$$R_n = \frac{S_{T_n}}{S_{T_{n-1}}} - 1 \quad (1)$$

Truncated returns, $\bar{R}_n = \max(\min(R_n, C), F)$, are returns truncated at some floor and cap levels F and C , respectively, with $F < C$. Absence of floor and/or cap corresponds to $F = -1$ and $C = +\infty$.

A cliquet option with global floor has a payoff Y at time T of

$$Y = B \times \max \left(\sum_{n=1}^N \bar{R}_n, F_g \right) \quad (2)$$

where F_g is the *global floor* and B is a notional amount. This type of option is usually packaged with a zero-coupon bond with the same principal and maturity. If in addition $F_g = 0$, the resulting packaged product has a capital guarantee. In this article, we focus on the pricing of the cliquet option component of these packages and for simplicity take $B = 1$. More payoffs that may be priced with the methods in this article are presented in the fifth section.

MARKET MODEL

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete filtered probability measure generated by the Wiener process W_t . Furthermore, let $r > 0$ be the deterministic risk-free rate and for simplicity take P to be the risk-neutral probability measure. Under this measure, expectations are written $\mathbb{E}$.

The stock price S_t is assumed to follow the Bachelier-Samuelson dynamics

$$dS_t/S_t = rdt + \sigma dW_t \quad (3)$$

for $\sigma > 0$. This implies that under P the returns are independent and of the form

$$R_n \sim e^{a + bX_n} - 1 \quad (4)$$

where $X_n \sim N(0, 1)$ and

$$\begin{cases} a = (r - \frac{\sigma^2}{2})\Delta T \\ b = \sigma_n \sqrt{\Delta T} \end{cases} \quad (5)$$

Assuming that the current reset period is m (i.e., $T_{m-1} \leq t < T_m$), this changes to $R_m \sim (s/\bar{s})e^{a_m + b_m X_m} - 1$ where $s = S_t$ is the current share price, $\bar{s} = S_{T_{m-1}}$ is the share price at the last reset date, $X_m \sim N(0, 1)$, and

$$\begin{cases} a_m = (r - \frac{\sigma^2}{2})(T_m - t) \\ b_m = \sigma \sqrt{T_m - t} \end{cases} \quad (6)$$

Below, we write R_m^t instead of R_m to indicate that s and $\bar{s}$ are known at time t , and we define $\bar{R}_m^t = \max(\min(R_m^t, C), F)$.¹

PRICING FORMULAS

In this section, we derive integral formulas for the price V_t and greeks of a cliquet option with global floor.

Assuming that $t \in [T_{m-1}, T_m]$, we define the *performance up to date*

$$z = \sum_{n=1}^{m-1} \bar{R}_n \quad (7)$$

and the auxiliary variable $A = z + (N - m + 1)C - F_g$, which represents the maximal possible payoff given the performance up to date, less the capital guarantee F_g . The characteristic function of a random variable X is written $\varphi_X(\xi) = E[e^{i\xi X}]$.

The form of the price formula can be divided into the following three cases, two of which have trivial solutions, whereas the third requires a more thorough analysis:

- 1) $A \leq 0$: Performance has been so poor that the payoff will be F_g regardless of future share price development.
- 2) $z + (N - m + 1)F \geq F_g$: Performance has been so good that the payoff will be higher than F_g regardless of future share price development. This results in the analytical formulas for the price and the greeks given in Proposition 1.
- 3) $A > 0$: General case; the formula in Proposition 2 is valid. Case (2) is included in this case, but we prefer to treat it separately due to the existence of the analytical formulas for the price and the greeks in Proposition 1.

In Case (2) above, we have a portfolio of forward start performance options, a derivative that pays the holder $Y = \max(\frac{S_T}{S_{T_0}} - K, 0)$ at some time $T > T_0$. Hence, it is straightforward to compute formulas for the price V_t and we give it without proof in Proposition 1. Here $c(t, s, K, T, \sigma, r)$ denotes the formula for the price at time t of a European call option with strike K and maturity T in the Black-Scholes model (3) with parameters r and σ . A derivation may be found in Hull [1999].

Proposition 1. If $z + (N - m + 1)F \geq F_g$, the price V_t of a cliquet option with global floor is given by

$$V_t = e^{-r(T-t)} \left\{ z + (N - m + 1)F + (N - m) \left[c(0, 1, 1 + F, \Delta T, \sigma, r) - c(0, 1, 1 + C, \Delta T, \sigma, r) \right] + e^{r(T_m - t)} \left[c(0, s/\bar{s}, 1 + F, T_m - t, \sigma, r) - c(0, s/\bar{s}, 1 + C, T_m - t, \sigma, r) \right] \right\}$$

Proposition 1 simply tells us that the cliquet with global floor equals a portfolio of call options and bonds, all of which could be valued easily in our Bachelier-Samuelson framework. The greeks are found by taking partial derivatives of V_t in Proposition 1.

Before stating and proving the formulas for the price and greeks in Case (3), we will rewrite the payoff function (1). First, we introduce the random variables $\tilde{R}_n = C - \bar{R}_n$ and $\tilde{R}_m^t = C - \bar{R}_m^t$, which are non-negative. Then, we have (remember that $B = 1$, for simplicity)

$$Y = \max \left(\sum_{n=1}^N \tilde{R}_n, F_g \right) = F_g + \max \left(NC - F_g - \sum_{n=1}^N \tilde{R}_n, 0 \right) \quad (8)$$

This may be interpreted as a portfolio consisting of a bond paying the capital guarantee F_g , plus the payoff of a so-called *reversed cliquet option*. This instrument pays at most $NC - F_g$ if all the N returns are negative. If a return is positive, it is subtracted from the maximum amount, but the amount subtracted is capped at $C - F$. The derivative then pays the greater of zero or the final amount after subtracting all positive returns (subject to the cap $C - F$, of course).

Before proceeding to the main proposition of this article, let $\mathbb{I}$ be the distribution function of a $N(0, 1)$ random variable, $\phi = \mathbb{I}'$, and the constants a_m , a , b , and b_m given in equations (5) and (6).

Proposition 2. If $A > 0$, the price V_t of a cliquet option with global floor is given by

$$V_t = e^{-r(T-t)} \left\{ F_g + A^2 \int_{-\infty}^{\infty} \text{sinc}^2 \left(\frac{\xi A}{2} \right) \times \phi_{\tilde{R}_m}(\xi) \times (\phi_{\tilde{R}_n}(\xi))^{N-m} \frac{d\xi}{2\pi} \right\} \quad (9)$$

where

$$\phi_{\tilde{R}_m}(\xi) = e^{i\xi(C-F)} - i\xi \int_0^{C-F} \mathbb{I} \left(\frac{a_m - \log(1 + C - x)}{b_m} \right) e^{i\xi x} dx$$

and

$$\phi_{\tilde{R}_n}(\xi) = e^{-i\xi(C-F)} - i\xi \int_0^{C-F} \mathbb{I} \left(\frac{a - \log(1 + C - x)}{b} \right) e^{i\xi x} dx$$

The proof is found in the appendix.² It uses the independence of returns and Fourier transforms of the payoff (8) to

convert the $(N - m + 1)$ -dimensional integral of (13) into the set of one-dimensional integrals of Proposition 2. This may be faster to compute if $(N - m + 1)$ is large enough.³

Since differentiation is allowed inside the integral (9), expressions similar to (9) may be obtained for the greeks in Case (3).

EXTENSION TO OTHER PAYOFF FUNCTIONS

The methodology used to derive the price formula in Proposition 2 can be used to price other related derivatives.

In the absence of a local cap (i.e., when $C = \infty$), the formulas in Proposition 2 are not valid. However, by inserting a large virtual cap C , they can be used to obtain arbitrarily good approximations. An upper bound of the truncation error is given in Proposition 3 below. Here $\hat{R}_n = \max(R_n, F)$ is a return with a lower truncation only. In reality, limiting the downside without limiting the upside would result in a very expensive option.

Proposition 3. Let V_t^C and V_t^∞ be the price of floored cliquet options with local caps $C < \infty$ and $C = \infty$, respectively. Then, with $\Delta T = T_{n+1} - T_n$,

$$V_t^\infty - V_t^C \leq 2e^{-r(T-t)} \{ e^{r(T_m-t)} c(0, s/\bar{s}, 1+C, T_m-t, \sigma, r) + (N-m) e^{r\Delta T} c(0, 1, 1+C, \Delta T, \sigma, r) \}$$

Proof. Following the first steps of the derivation of Proposition 2, the truncation error can be written as

$$\begin{aligned} (V_t^\infty - V_t^C)/e^{-r(T-t)} &= E \left[\max \left(\sum_{n=1}^N \hat{R}_n, F_g \right) | \mathcal{F}_t \right] \\ &\quad - E \left[\max \left(\sum_{n=1}^N \bar{R}_n, F_g \right) | \mathcal{F}_t \right] \\ &= E \left[\max \left(\hat{R}_m + \sum_{n=m+1}^N \hat{R}_n, F_g - z \right) \right. \\ &\quad \left. - \max \left(\bar{R}_m + \sum_{n=m+1}^N \bar{R}_n, F_g - z \right) \right] \end{aligned}$$

If $x \geq y$, we have

$$\max(x, a) - \max(y, a) = \begin{cases} 0, & x \leq a, \\ x - a, & x \geq a \geq y, \\ x - y, & y \geq a \end{cases}$$

Using this with $X = \hat{R}_m + \sum_{n=m+1}^N \hat{R}_n$, $Y = \bar{R}_m + \sum_{n=m+1}^N \bar{R}_n$ and $a = F_g - z$ yields

$$\begin{aligned} (V_t^\infty - V_t^C)/e^{-r(T-t)} &= E[X - a; X \geq a \geq Y] + E[X - Y; Y \geq a] \\ &\leq E[X - Y; X \geq a \geq Y] + E[X - Y; Y \geq a] \\ &\leq 2E[X - Y] \\ &= 2E \left[\left(\hat{R}_m - \bar{R}_m \right) + \sum_{n=m+1}^N \hat{R}_n - \sum_{n=m+1}^N \bar{R}_n \right] \end{aligned}$$

where the inequality follows from the fact that $a \geq Y$ on $X \geq a \geq Y$ and the integrands are non-negative. Computing the expectation, and identifying the Black-Scholes call-option price formula, completes the proof.

It is also possible to derive a formula similar to that of Proposition 2 if a global cap C_g is added, in which case the holder of the derivative receives

$$Y = \min \left(\max \left(\sum_{n=1}^N \bar{R}_n, F_g \right), C_g \right)$$

at time T . To see this, note that

$$\begin{aligned} Y &= \min \left(\max \left(z + \bar{R}_m + \sum_{n=m+1}^N \bar{R}_n, F_g \right), C_g \right) \\ &= F_g + \Lambda_A \left(\bar{R}_m + \sum_{n=m+1}^N \bar{R}_n \right) - \Lambda_{A-C_g+F_g} \\ &\quad \left(\bar{R}_m + \sum_{n=m+1}^N \bar{R}_n \right) \end{aligned}$$

and proceed as in the proof of Proposition 2. Algebraic manipulations of this type allow us to price other cliquet-style derivatives that appear on the market, for example, the cliquet with global floor and coupon credit K and the reversed cliquet which pay the holder $Y = \max(\sum_{n=1}^N \bar{R}_n - K, F_g)$ and $Y = \max(C_g + \sum_{n=1}^N \bar{R}_n, F_g)$, respectively.

NUMERICAL COMPUTATION OF THE CHARACTERISTIC FUNCTIONS

To compute the pricing formula given in Proposition 2 we must compute the characteristic functions

$$E[e^{i\xi\hat{R}_n}] = e^{i\xi(C-F)} - i\xi \int_0^{C-F} \mathbb{I}\left(\frac{a_n - \log(1+C-x)}{b_n}\right) e^{i\xi x} dx \quad (10)$$

for each ξ . Due to the rapid oscillation of the integrand inside (10) for large ξ , this would be computationally very heavy if done directly by numerical integration. The monotonicity and high degree of smoothness of $\mathbb{I}\left(\frac{a_n - \log(1+C-x)}{b_n}\right)$ suggest that interpolation with complete cubic splines over the interval $[0, C-F]$ may be a good idea. Initially, this interval is divided into N_p equally long subintervals $[x_n, x_{n+1}]$, $n = 0, \dots, N_p$, and a cubic polynomial $p_3^{(n)}(x) = c_3^{(n)}x^3 + c_2^{(n)}x^2 + c_1^{(n)}x + c_0^{(n)}$ is assigned to each interval. The coefficients are then chosen such that they interpolate the function at the spline knots x_n , $n = 0, \dots, N_p + 1$ and have continuous first and second derivatives. In addition, we require that the derivative of the spline and the function to be interpolated coincide at the endpoints 0 and $C-F$. For more details about complete cubic spline construction see De Boor [1978, pp. 53–55].

To summarize, the cubic spline approximation $\hat{\mathbb{I}}$ of $\mathbb{I}$ can be written as

$$\hat{\mathbb{I}}\left(\frac{a - \log(1+C-x)}{b}\right) = \sum_{n=0}^{N_p-1} \chi[x_n, x_{n+1}](x) p_3^{(n)}(x)$$

with χ being the indicator function. Replacing $\mathbb{I}$ by $\hat{\mathbb{I}}$ in (10) and evaluating the integrals yields an approximation $\hat{\varphi}_{\hat{R}_n}(\xi)$ of $\varphi_{\hat{R}_n}(\xi)$ as

$$\begin{aligned} \hat{\varphi}_{\hat{R}_n}(\xi) = & \sum_{n=0}^{N_p-1} \left\{ c_3^{(n)} \left[\frac{e^{i\xi x}}{(i\xi)^4} ((i\xi x)^3 - 3(i\xi x)^2 + 6i\xi x - 6) \right] \right. \\ & + c_2^{(n)} \left[\frac{e^{i\xi x}}{(i\xi)^3} ((i\xi x)^2 - 2i\xi x + 2) \right]_{x_n}^{x_{n+1}} \\ & \left. + c_1^{(n)} \left[\frac{e^{i\xi x}}{(i\xi)^2} (i\xi x - 1) \right]_{x_n}^{x_{n+1}} + c_0^{(n)} \left[\frac{e^{i\xi x}}{i\xi} \right]_{x_n}^{x_{n+1}} \right\} \end{aligned}$$

Despite its horrible appearance, the formula is very fast to evaluate on a computer. To compute the distribution

function of a normal random variable at the spline knots, a fractional approximation proposed in Hull [1999], which promises five to six correct decimals with little computational effort, is used.

The next proposition states that φ converges to φ uniformly. We start by stating a lemma; the proof can be found in De Boor [1978, pp. 68–69].

Lemma 1. If $f(x) \in C^4$, $h = x_n - x_{n-1}$, and $p_3(x)$ is the cubic spline approximation of f on $[a, b]$, then

$$|f'(x) - p_3'(x)| \leq \frac{h^3}{24} \sup_{x \in [a, b]} \left| \frac{d^4 f}{dx^4} \right|$$

Proposition 4. Let $\hat{\varphi}_{\hat{R}_n}(\xi)$ be the approximation of $\hat{\varphi}_{\hat{R}_n}(\xi)$ and N_p be the number of spline intervals of length $h = (C-F)/N_p$. Then, $\hat{\varphi} \rightarrow \varphi$ uniformly in ξ when $h \rightarrow 0$. More specifically, we have

$$|\hat{\varphi}_{\hat{R}_n}(\xi) - \varphi_{\hat{R}_n}(\xi)| \leq \frac{h^3}{24} (C-F) \sup_{x \in [0, C-F]} \left| \frac{d^4}{dx^4} \mathbb{I}\left(\frac{a - \log(1+C-x)}{b}\right) \right|$$

Proof. Let $E(x) = \mathbb{I}\left(\frac{a - \log(1+C-x)}{b}\right) - \hat{\mathbb{I}}\left(\frac{a - \log(1+C-x)}{b}\right)$. Then, by lemma 1,

$$\begin{aligned} |\hat{\varphi}_{\hat{R}_n}(\xi) - \varphi_{\hat{R}_n}(\xi)| &= \left| -i\xi \left[E(x) \frac{e^{i\xi x}}{i\xi} \right]_0^{C-F} + i\xi \int_0^{C-F} E'(x) \frac{e^{i\xi x}}{i\xi} dx \right| \\ &= \left| \int_0^{C-F} E'(x) e^{i\xi x} dx \right| \\ &\leq \frac{h^3}{24} (C-F) \sup_{x \in [0, C-F]} \left| \frac{d^4}{dx^4} \mathbb{I}\left(\frac{a - \log(1+C-x)}{b}\right) \right| \end{aligned}$$

by partial integration. Here, we have also used the fact that $x = 0$ and $x = C-F$ are points of interpolation with zero error.

A NUMERICAL INTEGRATION SCHEME

In this section, we develop a numerical integration scheme for computation of the pricing formula in Proposition 2, which uses the method for computing the characteristic functions proposed in the preceding section.

First, the real part of the integrand is even, the imaginary part is odd, and the domain of integration is symmetric such that we have

$$\begin{aligned} V_t &= e^{-r(T-t)} \left\{ F_g + A^2 \int_{-\infty}^{\infty} \text{sinc}^2 \left(\frac{\xi A}{2} \right) \times \varphi_{\tilde{R}_m}(\xi) \right. \\ &\quad \left. \times (\varphi_{\tilde{R}_n}(\xi))^{N-m} \frac{d\xi}{2\pi} \right\} \\ &= e^{-r(T-t)} \left\{ F_g + A^2 \int_0^{\infty} \text{sinc}^2 \left(\frac{\xi A}{2} \right) \times \text{Re} \{ \varphi_{\tilde{R}_m}(\xi) \} \right. \\ &\quad \left. \times (\varphi_{\tilde{R}_n}(\xi))^{N-m} \frac{d\xi}{\pi} \right\} \end{aligned} \quad (11)$$

Having to integrate the real part over only half of the domain reduces the number of computations by 75%. Since differentiating with respect to a parameter and taking real parts commute, this type of reduction extends to the computation of the greeks as well.

In order to compute the price integral numerically, an artificial upper limit of integration $\bar{\xi}$ is needed. Characteristic functions have a modulus less or equal to one which, together with the fact that $\text{sinc}^2(A\xi/2) \geq 0$, gives the following estimate of the truncation error $e(\bar{\xi})$:

$$\begin{aligned} |e(\bar{\xi}_{max})| &= e^{-r(T-t)} A^2 \left| \int_{\bar{\xi}}^{\infty} \text{sinc}^2 \left(\frac{\xi A}{2} \right) \times \text{Re} \{ \varphi_{\tilde{R}_m}(\xi) \} \right. \\ &\quad \left. \times \varphi_{\tilde{R}_n}(\xi) \frac{d\xi}{\pi} \right| \\ &\leq e^{-r(T-t)} \frac{2A}{\pi} \int_{\bar{\xi}A/2}^{\infty} \text{sinc}^2(x) dx \end{aligned} \quad (12)$$

The integral (12) is computed numerically for different values of $A\bar{\xi}/2$ and presented in Exhibit 1.

Denoting the integrand of (11) by ψ yields

$$V_t = e^{-r(T-t)} \left\{ F_g + \frac{A^2}{\pi} \int_0^{\infty} \psi(\xi) d\xi \right\}$$

This integral is then truncated at $\bar{\xi}$, which is set using Exhibit 1 and approximated with the well-known trapezoid rule of numerical quadrature.

$$\int_0^{\bar{\xi}} \psi(\xi) d\xi \approx \sum_{n=0}^{N-1} \left(\frac{\psi(\xi_n) + \psi(\xi_{n+1})}{2} \right) (\xi_{n+1} - \xi_n)$$

Instead of placing the nodes ξ_n uniformly, we try to select them such that the magnitude of the quadrature error contribution e_n from each interval $[\xi_n, \xi_{n+1}]$ is bounded by some tolerance level ϵ . Starting at $\xi_0 = 0$, this is done iteratively as follows:

According to Eriksson et al. [1996], e_n is bounded by

$$|e_n| \leq \frac{(\xi_{n+1} - \xi_n)^3}{12} \sup_{\xi \in [\xi_n, \xi_{n+1}]} |\psi''(\xi)|$$

EXHIBIT 1

Truncation Errors of the Integral

$A\bar{\xi}/2$	10	20	50	100	200	400
$\int_{\bar{\xi}A/2}^{\infty} \text{sinc}^2(x) dx$	0.0521	0.0254	0.0099	0.0040	0.0022	0.0010

If we require $|e_n| < \epsilon$, a rule for the step length $\Delta\xi_n$ can be obtained as

$$\Delta\xi_n = \left(\frac{12\epsilon}{\sup_{\xi \in [\xi_n, \xi_{n+1}]} |\psi''(\xi)|} \right)^{1/3}$$

The second derivative $\psi''(\xi)$ is approximated with

$$\psi''(\xi) \approx \frac{\psi(\xi + d\xi) - 2\psi(\xi) + \psi(\xi - d\xi)}{(d\xi)^2}$$

where $d\xi$ is some small number. We also replace $\sup_{\xi \in [\xi_n, \xi_{n+1}]} |\psi''(\xi)|$ by $|\psi''(\xi_n)|$, which is justified if the second derivative does not change too much over the interval $[\xi_n, \xi_{n+1}]$.

Although an adaptive scheme requires three times more evaluations of the function ψ than a non-adaptive scheme for a given number of steps, it compensates for this

by placing the mesh points where needed as shown in Exhibit 2. It shows that most points are placed in the center where the curvature of ψ is high and very few points are placed where the curvature is low. As a matter of fact, the adaptive integration algorithm only integrates over approximately 10% of the domain $[0, \bar{\xi}]$, which decreases the computational time significantly. Since the location of the high curvature regions of ψ vary with option and market parameters, a uniform mesh would have to be very dense in order to achieve good accuracy in all relevant cases.

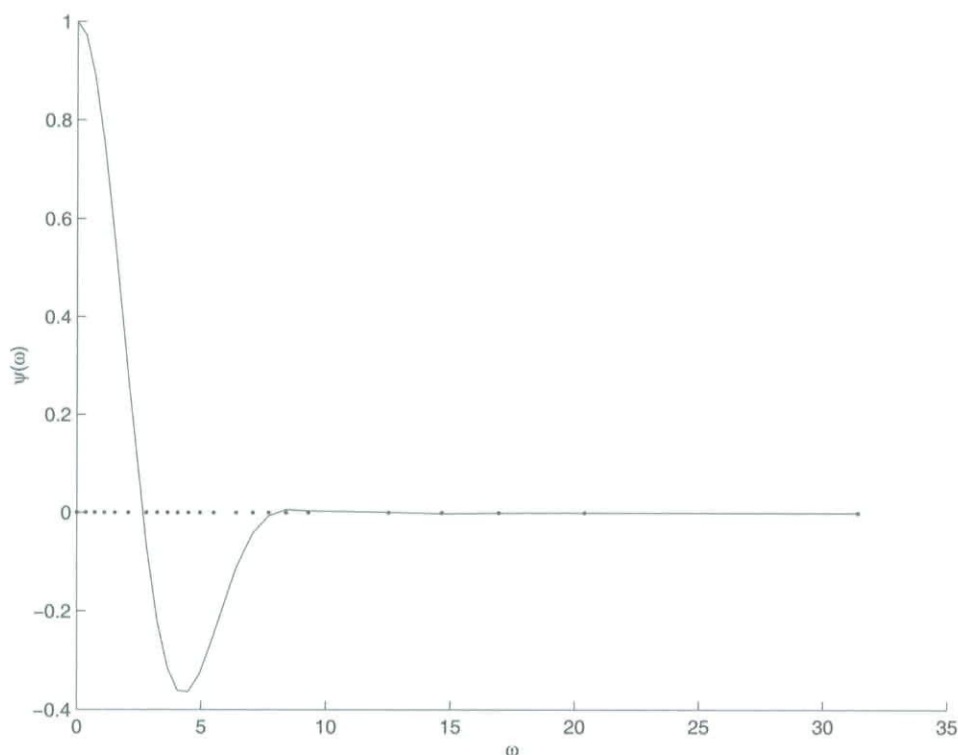
REFERENCE METHODS

In the next section, the Fourier method proposed in the four preceding sections will be compared to the following pricing methods:⁴

- 1) Monte Carlo (MC) simulation using pseudo random numbers

EXHIBIT 2

The function $\psi(\omega)$ and Mesh Points Chosen by the Adaptive Algorithm for $tol = 0.01$. Parameters: $N = 12$, $Fg = 0$, $F_l = -0.05$, $C_l = 0.05$, $r = 0.05$, $\sigma = 0.30$. Note that $\bar{x}i = 300$ for these Parameters



- 2) Quasi-Monte Carlo (QMC) simulation using a Fauré sequence
- 3) Partial Differential Equation (PDE) approach using an explicit finite difference (FD) scheme

An overview of the usage of Monte Carlo and quasi-Monte Carlo methods in the finance literature can be found in Boyle et al. [1997] and Boyle et al. [1996], respectively.

The PDE approach may need further explanation. Similar to the case of discrete Asian options, which is covered in Andreasen[1998], it can be proved that the PDE in Proposition 5 holds for the cliquet option with global floor.

Proposition 5. *The price $V_t = V(t, s, \bar{s}, z)$ satisfies the partial differential equation*

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{\sigma^2 s^2}{2} \frac{\partial^2 V}{\partial s^2} + r s \frac{\partial V}{\partial s} - rV = 0, & T_{n-1} \leq t < T_n \\ V(T_N, s, \bar{s}, z) = \max(z, F_g) \\ V(T_n^-, s, \bar{s}, z) = V(T_n, s, s, z) \\ \quad + \max(\min(s/\bar{s} - 1, C), F), & 1 \leq n \leq N \end{cases}$$

By letting $x = \log(s/\bar{s})$, a PDE with the space dimensions x and z can be derived. This equation is then solved by an explicit finite difference scheme.

NUMERICAL RESULTS

In order to rank the methods, we compare their accuracy for a given computation time for two sample derivatives, specified in Exhibit 3, both of which have existed on the Swedish market.

For each option, we compute the price, theta, and delta at $t = 0$, and insert these into the Black-Scholes PDE in Proposition 5 to obtain the gamma for free. For the Fourier method, the greeks are obtained from evaluation of the integral formulas obtained by differentiating inside the integral of the pricing formula of Proposition 2. Finite difference approximations are used to estimate the greeks in the reference methods.

We start by giving some pricing examples for different volatilities when $r = 0.05$.

All four methods in this section are implemented in the C programming language and compiled to a DLL file that is called from a test routine written in Python. Computations are made on a Dell Inspiron 8200 laptop with a 1.6 GHz Pentium® m4 processor and a 256 MB RAM.

For the Monte Carlo and quasi-Monte Carlo methods, the pseudo-random and Fauré sequences have been computed in advance and stored on the computer hard drive, which saves computations. To measure their computational accuracy, standard errors from 100 price simulations have been used. For the Monte Carlo method, we let a random number generator pick a new pseudo-random sequence for each simulation. For the quasi-Monte Carlo method, a rotation modulo one randomization is applied to the original Fauré sequence. This method is described in detail in Tuffin[1996].

The implementation of the Fourier method used in these tests allows the usage of the extended model described in endnote 2. A consequence of this is that all the N characteristic functions have to be evaluated, compared to an implementation where only two characteristic functions have to be evaluated which would be much faster. As mentioned in endnote 3, the computational

EXHIBIT 3

Characteristics of Two Sample Cliquet Options

Option	T	N	F_g	F	C
Cliquet 1	3 years	12	0	-0.05	0.05
Cliquet 2	3 years	36	0	-0.02	0.02

EXHIBIT 4

Prices and Greeks at $t = 0$ for Cliquets 1 and 2 of Exhibit 2. The Interest Rate is $r = 0.05$ Per Year

Option	σ	V	Δ	Θ	Γ
Cliquet 1	0.10	0.0952	0.4451	-0.01008	-1.484
Cliquet 1	0.30	0.0566	0.1154	-0.00167	-0.102
Cliquet 1	0.50	0.0426	0.0567	0.00385	-0.0366
Cliquet 2	0.10	0.0717	0.3339	-0.00602	-1.419
Cliquet 2	0.30	0.0401	0.0804	0.00138	-0.0755
Cliquet 2	0.50	0.0300	0.0398	0.00276	-0.0258

effort for the reference methods is not affected significantly by this extension.

The results shown in Exhibits 5 and 6 refer to the time needed to compute the price, delta, theta and gamma.

CONCLUSION

Based on the results in Exhibits 3 and 4, the Fourier integral method outperforms the Monte Carlo and quasi-

Monte Carlo methods in these two test cases. Compared to the finite difference method it seems particularly fast when $N = 36$. This holds even in the presence of dividends, reset periods of unequal length, and a time-dependent interest rate and volatility. The efficiency is achieved by converting the computation of a multi-dimensional integral into the set of one-dimensional integrals.

EXHIBIT 5

Cliquet 1: Relative Computational Errors in the Price and Gamma for the Three Methods. Flat Interest Rate $r = 0.05$

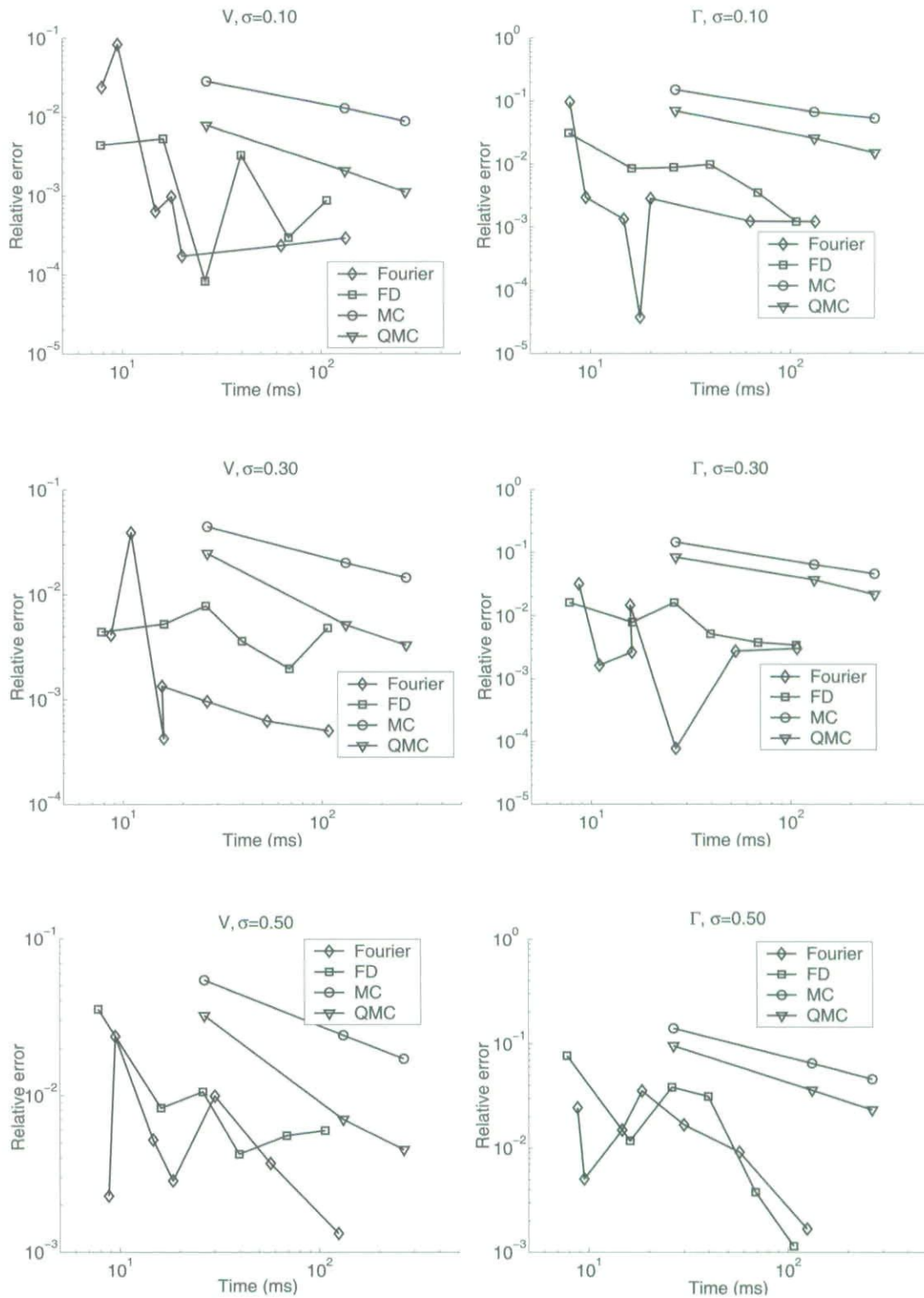
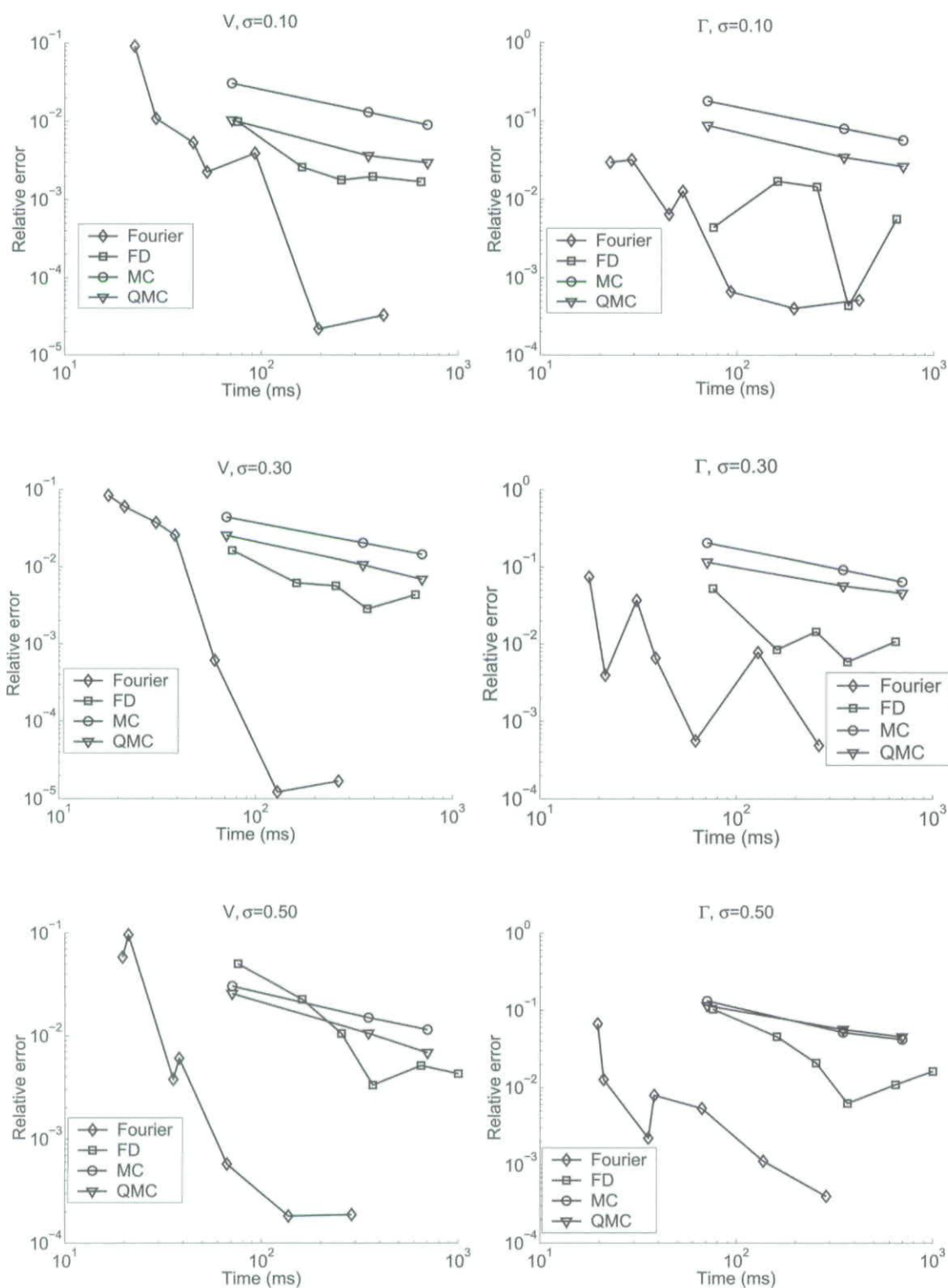


EXHIBIT 6

Cliquet 2: Relative Computational Errors in the Price and Gamma for the Three Methods. Flat Interest Rate $r = 0.05$



APPENDIX

Proof of Proposition 2

Proof. By general derivatives pricing theory (see, for example, Bingham and Kiesel [1998]), the price is given by

$$V_t = e^{-r(T-t)} E \left[\max \sum_{n=1}^N \bar{R}_n, F_g \right] | \mathcal{F}_t \quad (13)$$

$$= e^{-r(T-t)} E \left[(F_g + \max(z - F_g + \left(\bar{R}_m^t + \sum_{n=m+1}^N \bar{R}_n \right), 0)) \right] \quad (14)$$

since S_t is a Markov process. Using the relations $\bar{R}_n = C - \tilde{R}_n$, $\bar{R}_m^t = C - \tilde{R}_m^t$, and equation (8) yield

$$\begin{aligned} V_t &= e^{-r(T-t)} \left\{ F_g + E \left[\max(MC - F_g + z - \left(\tilde{R}_m^t + \sum_{n=m+1}^N \tilde{R}_n \right), 0) \right] \right\} \\ &= e^{-r(T-t)} \left\{ F_g + E \left[\max(A - \left(\tilde{R}_m^t + \sum_{n=m+1}^N \tilde{R}_n \right), 0) \right] \right\} \end{aligned}$$

By Fourier analysis (see Folland [1992] for details), we have

$$\Lambda_A(x) \equiv \max(A - |x|, 0) = A^2 \int_{-\infty}^{\infty} \text{sinc}^2 \left(\frac{\xi A}{2} \right) e^{i\xi x} \frac{d\xi}{2\pi}$$

Using this result with $x = \tilde{R}_m^t + \sum_{n=m+1}^N \tilde{R}_n$, which is non-negative by construction, gives

$$\begin{aligned} V_t &= e^{-r(T-t)} \left\{ F_g + E \left[A^2 \int_{-\infty}^{\infty} \text{sinc}^2 \left(\frac{\xi A}{2} \right) e^{i\xi(\tilde{R}_m^t + \sum_{n=m+1}^N \tilde{R}_n)} \frac{d\xi}{2\pi} \right] \right\} \\ &= e^{-r(T-t)} \left\{ F_g + A^2 \int_{-\infty}^{\infty} \text{sinc}^2 \left(\frac{\xi A}{2} \right) E[e^{i\xi \tilde{R}_m^t + \sum_{n=m+1}^N \tilde{R}_n}] \frac{d\xi}{2\pi} \right\} \end{aligned}$$

by the Fubini theorem. Independence of returns and identical distribution of $\{\tilde{R}_n\}_{n=m+1}^N$ implies that

$$V_t = e^{-r(T-t)} \left\{ F_g + A^2 \int_{-\infty}^{\infty} \text{sinc}^2 \left(\frac{\xi A}{2} \right) E[e^{i\xi \tilde{R}_m^t}] (E[e^{i\xi \tilde{R}_n}])^{N-m} \frac{d\xi}{2\pi} \right\}$$

To arrive at the formula in Proposition 2 $E[e^{i\xi \tilde{R}_m^t}]$ and $E[e^{i\xi \tilde{R}_n}]$ remain to be computed.

But

$$E[e^{i\xi \tilde{R}_n}] = e^{i\xi(C-F)} \cdot P(R_n \leq F) + \int_0^{C-F} e^{i\xi x} dP(C - R_n \leq x) + 1 \cdot P(R_n > C)$$

so

$$E[e^{i\xi\tilde{R}_n}] = e^{i\xi(C-F)} - i\xi \int_0^{C-F} \Phi\left(\frac{a - \log(1+C-x)}{b}\right) e^{i\xi x} dx$$

by (4) and partial integration. $E[e^{i\xi\tilde{R}_m}]$ is computed analogously.

ENDNOTES

¹ (a) We may replace the constants σ and r in (3) by the time-dependent but deterministic non-negative functions $r(t)$ and $\sigma(t)$. Continuous or discrete dividend yields can also be introduced. If the discrete dividend yields in reset period n are denoted $\alpha_l \in (0, 1)$, $1 \leq l \leq L$, the coefficients a and b in (5) have to be replaced by

$$a_n = \sum_{l=1}^L \log(1 - \alpha_l) + \int_{T_n}^{T_{n+1}} \left(r(t) - \frac{1}{2} \sigma^2(t) \right) dt$$

$$b_n = \sqrt{\int_{T_n}^{T_{n+1}} \sigma^2(t) dt}$$

and similarly for a_m and b_m in (6). Here, reset periods of different lengths are allowed. In this generalized model, the random variables $1 + R_n = \exp(a_n + b_n X_n)$ are still log-normal and independent, but not identically distributed.

(b) We could also let $S_t = \exp(rt + L_t)$, where L_t is a Levy-process under the chosen risk-neutral measure P . However, as noted in Cont and Tankov [2003], the monthly or quarterly returns mostly used in our context appear to be much more normally distributed than daily returns. Thus, the benefit of using a Levy-process model would be limited for this type of option.

(c) Dividends are usually paid in discrete amounts, but using a fixed-dividend model, like the one by presented in Heath and Jarrow [1988], would leave us without an explicit expression for the density of R_n . To avoid this, fixed dividends must be approximated with discrete or continuous dividend yields.

²Similar Fourier integral formulas could be derived for the extended market model discussed in Note A.1. Since the returns are not identically distributed, all $N - m$ different characteristic functions $\varphi_{\bar{R}_n}(\xi)$ would have to be evaluated, increasing the computational burden. The approach also works for the Levy-process model of Note A.2.

³An alternative approach would be to note that due to the independence of returns, the the density function of $\sum_{n=1}^N \bar{R}_n$ is given by the inverse Fourier transform of $(\varphi_{\bar{R}_n})^N$.

Knowing this density, the option price $V_t = e^{-r(T-t)} \mathbb{E}[\max(\sum_{n=1}^N \bar{R}_n, F_S) | \mathcal{F}_t]$ could be computed by numerical integration. However, $\varphi_{\bar{R}_n}$ are not known explicitly and by (10) do not go to zero as $|\xi| \rightarrow \infty$, making numerical inversion difficult.

⁴The MC, QMC, and PDE methods are directly extendable to the extended market model discussed in Note A.1, without further computational effort.

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