Barrier Options on Spot LIBOR Rates under Multi-Factor Gaussian HJM Models

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We derive approximate analytical pricing formulas for interest rate caps and floors with continuously monitored barriers in a multi-factor Gaussian HJM framework. The closed-form valuation equations are subject to a Radon-Nikodym approximation, which is similar to the proportionality assumption used by El Karoui and Rochet (1989) and to the rank 1 approximation suggested by Brace and Musiela (1994).

Monte Carlo simulation shows that the resulting analytical approximations are accurate and much faster than existing numerical and quasi-analytical pricing methods. Moreover, we provide error bounds for our solution, an innovation relative to the previous literature. These bounds provide a confidence interval for the true barrier option price, which in turn is an indicator of the precision of the proposed analytical pricing solutions.

The main purpose of this paper is to derive fast and accurate approximate analytical valuation formulas for single-barrier interest rate caps and floors, in the context of a multi-factor Gaussian Heath, Jarrow and Morton (1992) term structure model and under continuous monitoring of the underlying interest rate.

As for regular caps (floors), single-barrier caps (floors) are also portfolios of caplets (floorlets). Each caplet (floorlet) is an European call (put) option on a *nominal* money-market interest rate, i.e. on a simply compounded interest rate as, for example, the 3-month US LIBOR. However and as described, for instance, by Pelsser (2000, page 100), the component barrier caplets (or barrier floorlets) only generate their regular terminal payoff if, during the option's life, the barrier is (*knock-in options*) or is not (*knock-out options*) touched by the underlying nominal spot interest rate. As usual, the existence of a barrier rate makes such interest rate options cheaper than their regular counterparts as well as more easily tailored to the investor's expectations about the evolution of the underlying spot rate.

Although barrier options are amongst the oldest and most popular forms of exotic options,[1] the extensive treatment that they have received in the Finance literature has been almost always confined to the case where the underlying asset price follows a geometric Brownian motion (that is, typically, to barrier options on stocks, indices or currencies). To the author's knowledge, such path-dependent option pricing theory is only starting to be applied to the valuation of OTC interest rate caps and swaptions with barrier features, although the corresponding plain-vanilla contracts are commonly named as representing the largest market of interest rate options (in terms of notional amount). Given the lack of satisfactory analytical pricing solutions, barrier options on spot interest rates have been priced through numerical methods, which include both lattice versions of the HJM model as well as Monte Carlo simulation. For singlefactor and Markovian HJM specifications, the tree recombines and, therefore, the lattice discretization constitutes a viable pricing methodology. However, for non-Markovian term structure models the approximating tree will not be recombining, and the number of branches on the tree expands exponentially as the number of time steps increases. Moreover, it is well known that the computational burden grows linearly with the number of factors for the Monte Carlo technique, but more than linearly for lattice-based approaches. Nevertheless, both numerical approaches are usually too time consuming for practical purposes.

Concerning the previous literature on the analytical pricing of path-dependent interest rate options, Leblanc and Scaillet (1998) started by valuing Asian and lookback options on yields, but under a one-factor affine model (which is unable to ensure the consistency between the exotic options' prices and the initial yield curve). Chrétien and Quittard-Pinon (2000) have already priced single-barrier caps, but under a one-dimensional Gauss-Markov HJM framework and using a Fortet (1943) approximation for the first passage time density. Although such approximate solutions are faster to implement than the Monte Carlo method and benefit from reasonable asymptotic properties, they are, however, only applicable to the sub-class of Markovian (and Gaussian) HJM models and do not accommodate the existence of rebates. Moreover, such a quasi-analytical approach should be less adequate in estimating option's sensitivity parameters and more time-consuming than the closed-form pricing solutions to be proposed in the present paper. Finally, Kuan and Webber (2003) valued barrier options on zero-coupon bonds and on interest rate swaps, under the one-factor Hull and White (1990) model (as well as under a Swap Market model for the pricing of swaptions), by computing numerically the first passage time density of the short-term interest rate to a time-dependent barrier. Although a caplet (floorlet) can always be specified as an European put (call) option on a pure discount bond, the numerical solutions proposed by Kuan and Webber (2003) are not applicable to the single-barrier caps and floors considered in the present paper: while the barrier of a single-barrier cap or floor is placed over a (quoted) nominal spot rate with a constant time-to-maturity, Kuan and Webber (2003) only consider barriers on zero-coupon bonds (or, equivalently, on spot rates) with a fixed maturity date (and, therefore, with a decreasing time-to-maturity), which are not observable in the market.

The main difficulty in valuing barrier caps and floors arises from the fact that the underlying for such path-dependent interest rate options is not a forward nominal interest rate—as can be thought to be the case for regular caps or floors—but, instead, a spot nominal interest rate, which is directly observed in the market. This contract feature was also essential in choosing the valuation framework. Because the lognormal property of forward rates is not preserved for spot rates under a LIBOR Market model, a Gaussian—but not necessarily Markovian—HJM setup will be used instead, for reasons of analytical tractability. Despite the underlying restrictive assumption of a deterministic—although, possibly, time-inhomogeneous—specification for the volatility function, the model is still consistent with the initially observed term structure of interest rates. Moreover and in contrast to Chrétien and Quittard-Pinon (2000), a multi-factor formulation is adopted in order to improve the model' calibration to the market covariance matrix, as suggested, for instance, by Rebonato (1998, page 70).

Next sections are organized as follows. Section "Model description" briefly reviews the multi-factor Gaussian HJM model that will be used to derive approximate pricing solutions for the new exotic contracts covered by this paper. Section "Single-barrier caps and floors" describes and prices *single-barrier* caps and floors. For that purpose, an analytical approximation for the defective transition density of the barrier option's underlying is found in theorem 1, which constitutes the main result of the present paper. The asymptotic properties of such approximation are studied, error bounds are provided, and closed-form pricing solutions are found. Finally, section "Conclusions and further research" summarizes the main results and provides some directions for further research. Numerical examples are also presented in order to test the accuracy of the proposed pricing solutions.

MODEL DESCRIPTION

Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \geq t_0}, \mathcal{Q})$ be a filtered probability space satisfying the usual technical conditions, where $\mathbf{F}$ is the $\mathcal{Q}$ − augmentation of the natural filtration generated by a standard Brownian motion $\underline{W}^{\mathcal{Q}}(t) \in \mathcal{R}^n$ initialized at zero (at time t_0.)[2] The relative prices of all assets are assumed to be martingales under the risk neutral probability measure $\mathcal{Q}$, that is when the numéraire is taken to be the "money market account", whose time-t value is defined as:

$$\beta(t) := \exp\left[\int_{t_0}^{t} r(s)\,ds\right] \tag{1}$$

where r(t) denotes the time-t instantaneous spot rate. Therefore, i.e. following Harrison and Pliska (1981), there exists an arbitrage-free market for pure discount bond prices with maturity $T \in [t_0, \tau]$, for some fixed time τ, which are assumed to be perfectly divisible and to evolve through time according to the following stochastic differential equation:

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt + \underline{\sigma}(t, T)'.d\underline{W}^{Q}(t) \tag{2}$$

where P(t, T) represents the time-t price of a (unit face value) zero coupon bond expiring at time T, for $t \in [t_0, T]$, and . denotes the inner product in $\mathcal{R}^n$. The n-dimensional adapted volatility function $\underline{\sigma}(., T): [t_0, T] \rightarrow \mathcal{R}^n$ is assumed to satisfy the usual mild measurability and integrability requirements—as stated, for instance, in Lamberton and Lapeyre (1996, theorem 3.5.5)—as well as the "pull-to-par" condition $\underline{\sigma}(u, u) = \underline{0} \in \mathcal{R}^n$, $\forall\, u \in [t_0, T]$. Finally, and for reasons of analytical tractability, the following modelling restriction is adopted:

Assumption 1 *The volatility function $\underline{\sigma}(., T):[t_0, T] \rightarrow \mathcal{R}^n$ is deterministic.*

Of course, the main theoretical model' limitation that arises from assumption 1, that is from the normality of forward rates, is the fact that interest rates can attain negative values with positive probability (see, for instance, Rogers (1996)). However, the generalization of the pricing solutions derived in section "Single-barrier caps and floors" for more complex volatility structures (precluding the existence of negative forward rates)—using, for instance, the approach suggested by Nunes, Clewlow and Hodges (1999)—is outside the scope of the present analysis.

Equation (2), along with assumption 1, represents the Gaussian interest rate term structure model that will be used, hereafter, to derive analytical pricing solutions for European interest rate barrier options. Since the underlying of such exotic options will be spot LIBOR rates—that is interbank (risky) rates—instead of government (risk-free) rates, the Duffie and Singleton (1997) assumption of *symmetric counterpart credit risk* will be implicitly used hereafter. Meaning that the model' interest rates will be regarded not as riskless rates but rather as default- and liquidity-adjusted rates.

SINGLE-BARRIER CAPS AND FLOORS

Terminal Payoffs

This paper proposes and values four types of single-barrier caps and floors (with settlement in arrears), subject to continuous monitoring of the barrier, namely: *up-and-in caps*, *down-and-out caps*, *down-and-in floors*, and *up-and-out floors*. All these new path-dependent interest rate options share the following common appealing feature: they are only activated if the underlying spot rate evolves in a (strong enough) favorable direction; or, they expire worthless whenever the motion of the underlying spot rate is (too) adverse. Four other (less "intuitive") types of single-barrier caps and floors are also feasible—*up-and-out* and *down-and-in caps*, or *down-and-out* and *up-and-in floors*—and can be priced through the parity relation between knock-in and knock-out options (without rebates).

As for regular caps and floors, the fair price of a (forward-start) single-barrier cap or floor corresponds to the summation of the fair prices for all the component (single-barrier) caplets or floorlets. As usual, the underlying of the component (single-barrier) caplet or floorlet with settlement in arrears at time $t_{i+1} (\geq t_0)$ is the nominal time-t_i $(t_0 \leq t_i \leq t_{i+1})$ spot rate for the compounding period $\delta = t_{i+1} - t_i$, which can be defined as[3]

$$r(t_i, t_{i+1}) := \frac{1}{\delta}\left[\frac{1}{P(t_i, t_{i+1})} - 1\right] \tag{3}$$

An *up-and-in caplet* generates the same terminal payoff as a regular caplet as long as the underlying spot nominal interest rate touches (from below) the barrier rate during the life of the option. Otherwise, a rebate (possibly null) is paid on the same terminal date. Similarly, a *down-and-in floorlet* generates the same terminal payoff as a regular floorlet, if the underlying spot nominal interest rate touches (from above) the barrier rate during the life of the option, or a rebate, otherwise. More formally,

Definition 1 *The time-t_{i+1} price of a unit face value up-and-in caplet (down-and-in floorlet) on the nominal spot rate $r(t_i, t_{i+1})$, with a cap (floor) rate of r_k, subject to a barrier rate* *of $r_u(r_l)$, with a rebate rate R, and settled in arrears at time t_{i+1} is equal to*

$$IN(\theta)_{t_{i+1}}[r(t_i, t_{i+1}); r_k; r_H; R; t_{i+1}]$$

$$:= \begin{cases} [\theta r(t_i, t_{i+1}) - \theta r_k]^+ \delta \Leftarrow \exists s \in]t_0, t_i]: \\ \quad \theta r(s, s+\delta) \geq \theta r_H \\ R\delta \Leftarrow \theta r(s, s+\delta) < \theta r_H, \ \forall s \in]t_0, t_i] \end{cases} \quad (4)$$

where $r_H = r_u(> r(t_0, t_0 + \delta))$ and $\theta = 1$ for an up-and-in caplet, or $r_H = r_l(< r(t_0, t_0 + \delta))$ and $\theta = -1$ for a down-and-in floorlet.

The monitoring of the barrier is defined in terms of the observable nominal spot rate $r(s, s + \delta)$ and not in terms of the nominal forward rate $r(s, t_i, t_{i+1}) := 1/\delta[P(s, t_i)/P(t_i, t_{i+1}) - 1]$, which would be typically the underlying variable for a LIBOR Market model. This is the case because the latter interest rate is not directly observed in the money market.[4] Consequently, it is not possible to take advantage of the lognormal martingale formulation offered by a Market model, and, hence, an HJM model is used instead. Notice also that under the Kuan and Webber (2003) approach, the barrier would have to be monitored against the (fixed maturity) nominal spot rate $r(s, t_{i+1})$. However, such assumption is unrealistic since the latter spot rate would be almost never quoted in the market (because its time-to-maturity decreases every day).

A *down-and-out caplet* generates the same terminal payoff as a regular caplet as long as the underlying spot nominal interest rate never touches (from above) the barrier rate during the life of the option. Otherwise, a rebate is paid on the same terminal date. Finally, an *up-and-out floorlet* generates the same terminal payoff as a regular floorlet, if the underlying spot nominal interest rate never touches (from below) the barrier rate during the life of the option; otherwise, a rebate can be paid. That is,

Definition 2 *The time-t_{i+1} price of a unit face value down-and-out caplet (up-and-out floorlet) on the nominal spot rate $r(t_i, t_{i+1})$, with a cap (floor) rate of r_k, subject to a barrier rate of $r_l(r_u)$, with a deferrable rebate rate R, and settled in arrears at time t_{i+1} is equal to*[5]

$$OUT(\theta)_{t_{i+1}}[r(t_i, t_{i+1}); r_k; r_H; R; t_{i+1}]$$

$$:= \begin{cases} [\theta r(t_i, t_{i+1}) - \theta r_k]^+ \delta \Leftarrow \theta r(s, s+\delta) \\ \quad > \theta r_H, \ \forall s \in]t_0, t_i] \\ R\delta \Leftarrow \exists s \in]t_0, t_i]: \theta r(s, s+\delta) \leq \theta r_H \end{cases} \quad (5)$$

where $r_H = r_l(< r(t_0, t_0 + \delta))$ and $\theta = 1$ for a down-and-out caplet, or $r_H = r_u(> r(t_0, t_0 + \delta))$ and $\theta = -1$ for an up-and-out floorlet.

Because all the above mentioned options possess a single future payoff (at time t_{i+1}), their fair prices can be simply computed as the discounted expectation of such terminal payoffs, under the corresponding equivalent (forward) martingale probability measure—see, for instance, Geman, Karoui and Rochet (1995). Let $\mathcal{Q}^{t_{i+1}}$ be the t_{i+1}–forward martingale measure obtained when the t_{i+1} – maturity pure discount bond is taken as the numéraire of the economy. Assuming that measure $\mathcal{Q}^{t_{i+1}}$ exists, the relative price, with respect to the t_{i+1}–maturity zero-coupon bond, of any attainable contingent claim that settles at time t_{i+1} will be a $\mathcal{Q}^{t_{i+1}}$–martingale. Therefore, applying equation (3) to definitions 1 and 2, the time-t_0 price of the previously defined barrier options can be written not as a function of the nominal spot rate, but in terms of log-inverse pure discount prices:[6]

$$IN(\theta)_{t_0}[r(t_i, t_{i+1}); r_k; r_H; R; t_{i+1}]$$

$$= P(t_0, t_{i+1})$$

$$\mathbb{E}_{t_0}^{\mathcal{Q}^{t_{i+1}}} \Big\{ [\theta \exp(\ln P^{-1}(t_i, t_{i+1})) - \theta(1 + \delta r_k)]^+$$

$$| sup_{t_0 \leq s \leq t_i}[\theta \ln P^{-1}(s, s+\delta)] \geq \theta \ln(1 + \delta r_H) \Big\} \quad (6)$$

$$+ P(t_0, t_{i+1}) \delta R \mathcal{Q}^{t_{i+1}} \Big\{ sup_{t_0 \leq s \leq t_i}[\theta \ln P^{-1}(s, s+\delta)]$$

$$< \theta \ln(1 + \delta r_H) | \mathcal{F}_{t_0} \Big\}$$

and

$$OUT(\theta)_{t_0}[r(t_i, t_{i+1}); r_k; r_H; R; t_{i+1}]$$

$$= P(t_0, t_{i+1})$$

$$\mathbb{E}_{t_0}^{Q^{t_{i+1}}}\left\{[\theta\exp(\ln P^{-1}(t_i, t_{i+1})) - \theta(1+\delta r_k)]^+ \right.$$

$$\left. |\inf_{t_0 \le s \le t_i}[\theta \ln P^{-1}(s, s+\delta)] > \theta\ln(1+\delta r_H)\right\} \tag{7}$$

$$+ P(t_0, t_{i+1})\delta R$$

$$Q^{t_{i+1}}\left\{\inf_{t_0 \le s \le t_i}[\theta \ln P^{-1}(s, s+\delta)] \right.$$

$$\left. \le \theta\ln(1+\delta r_H)|\mathcal{F}_{t_0}\right\}$$

To convert the last two equations into analytical pricing solutions, it is necessary to derive the probability density function of terminal log-inverse pure discount bond prices with a time-to-maturity of δ-years, conditional on touching or not the barrier, and under the t_{i+1}-forward martingale measure. First, the corresponding unconditional distribution is computed.

Unrestricted density

Applying the change of measure rule contained in Nunes (2004, proposition 2.2) to equation (2), it is easy to show that

$$\ln P^{-1}(t, t+\delta) = \ln\left[\frac{P(t_0, t)}{P(t_0, t+\delta)}\right] + l(t_0, t)$$

$$+ \int_{t_0}^{t} [\underline{\sigma}(s, t) - \underline{\sigma}(s, t+\delta)]'.d\underline{W}^{Q^{t_{i+1}}}(s) \tag{8}$$

where

$$l(t_0, t) := \int_{t_0}^{t}\left\{\frac{1}{2}\left[\|\underline{\sigma}(s, t+\delta)\|^2 - \|\underline{\sigma}(s, t)\|^2\right]\right.$$

$$\left. - \left[\underline{\sigma}(s, t+\delta) - \underline{\sigma}(s, t)\right]'.\underline{\sigma}(s, t_{i+1})\right\}ds \tag{9}$$

while

$$d\underline{W}^{Q^{t_{i+1}}}(t) = d\underline{W}^{Q}(t) - \underline{\sigma}(t, t_{i+1})dt \tag{10}$$

is still a vector of standard Brownian motion increments in $\mathcal{R}^n$. Considering that

$$g(t_0,t) := \int_{t_0}^{t} \| \underline{\sigma}(s, t) - \underline{\sigma}(s, t+\delta)\|^2 ds \tag{11}$$

is finite, and using, for instance, Arnold (1992, corollary 4.5.6), it follows that the stochastic integral contained in the right-hand-side of equation (8) is normally distributed with mean zero and variance $g(t_0, t)$. Therefore,

Proposition 1 *Under the equivalent martingale measure $Q^{t_{i+1}}$, for the Gaussian HJM specification (2), and conditional on $\mathcal{F}_{t_0}$,*

$$Q^{t_{i+1}}[\ln P^{-1}(t, t+\delta) \in dx|\mathcal{F}_{t_0}]$$

$$= \phi\left\{x; \ln\left[\frac{P(t_0, t)}{P(t_0, t+\delta)}\right] + l(t_0, t), \sqrt{g(t_0, t)}\right\}dx \tag{12}$$

where the deterministic functions $l(t_0, t)$ and $g(t_0,t)$ are given by equations (9) and (11), respectively, while $\phi(X; \mu, \sigma)$ represents the probability density function of a normally distributed univariate random variable X, with mean μ and standard deviation σ.

Prices for regular caplets and floorlets can be easily obtained by integrating the corresponding terminal payoffs with respect to the unrestricted density (12). For convenience, proposition 2 restates similar analytical solutions, which follow, for example, from Brace and Musiela (1994, proposition 3.1).

Proposition 2 *Under the Gaussian HJM specification (2), the time-t_0 price of a unit face value regular caplet (floorlet) on the nominal spot rate $r(t_i,t_{i+1})$, with a cap (floor) rate of r_k, and settled in arrears at time t_{i+1} is equal to*

$$REGULAR(\theta)_t[r(t_i, t_{i+1}); r_k; t_{i+1}]$$

$$= \theta P(t_0, t_i)\Phi\left[-\theta d_0^{r_k}(t_0) + \theta\sqrt{g(t_0, t_i)}\right]$$

$$- \theta(1+\delta r_k)P(t_0, t_{i+1})\Phi[-\theta d_0^{r_k}(t_0)] \tag{13}$$

with

$$d_0^r(t) := \frac{\ln\left[\frac{(1+\delta r)P(t, t_{i+1})}{P(t, t_i)}\right] + \frac{g(t, t_i)}{2}}{\sqrt{g(t, t_i)}}$$

$\theta = 1$ *for a regular caplet,* $\theta = -1$ *for a regular floorlet, and where* Φ *represents the cumulative density function of the univariate standard normal distribution.*

Restricted Densities

This subsection provides an approximate closed-form solution for the defective densities involved in equations (6) and (7). For that purpose, a deterministic time-change will be first applied to reduce the dimensionality of the valuation problem. Then, a measure change will allow the log-inverse zero-coupon bond price to be written as a time-changed standard Brownian motion, whose restricted density is obtainable from the well known *reflection principle*. Finally, such defective density will be rewritten under the original forward martingale measure through an approximation of the corresponding Radon-Nikodym derivative.

Following, for instance, Øksendal (1995, theorem 8.20), because $g(t_0, t_0) = 0$ and assuming that the deterministic volatility function $\underline{\sigma}(., T):[t_0, T] \to \mathcal{R}^n$ is such that

$$\frac{\partial g(t_0, t)}{\partial t} > 0 \tag{14}$$

for $t_0 \leq t \leq T$, then the price process (8) can be restated as

$$\ln P^{-1}(t, t + \delta) = \ln\left[\frac{P(t_0, t)}{P(t_0, t + \delta)}\right] + l(t_0, t) + \widetilde{W}^{Q^{t_{i+1}}}_{g(t_0, t)} \tag{15}$$

where

$$\widetilde{W}^{Q^{t_{i+1}}}_{g(t_0, t)}: = \int_{t_0}^{t} [\underline{\sigma}(s, t) - \underline{\sigma}(s, t + \delta)]'.d\underline{W}^{Q^{t_{i+1}}}(s) \tag{16}$$

is a one-dimensional Brownian motion also initialized at zero, under the same martingale measure $Q^{t_{i+1}}$, but with respect to a deterministic time-change defined by $g(t_0, .):[t_0, T] \to \mathcal{R}^+$. Although the log-inverse pure discount bond price is now written in terms of a standard one-dimensional Brownian motion, it is still necessary to accommodate a time-dependent drift. That is, solving equation (15) with respect to the Wiener process would not add analytical tractability because a constant barrier imposed on the price process $\ln P^{-1}(t, t + \delta)$ would be translated into a time-dependent barrier on $\widetilde{W}^{Q^{t_{i+1}}}_{g(t_0, t)}$.

To remove the drift from the process (15), define an equivalent martingale measure Q^g such that

$$d\widetilde{W}^{Q^g}_{g(t_0, t)} = d\widetilde{W}^{Q^{t_{i+1}}}_{g(t_0, t)} + \eta(t_0, t)dg(t_0, t) \tag{17}$$

is also a time-changed one-dimensional Wiener increment, with

$$\eta(t_0, t): = \frac{\partial}{\partial g(t_0, t)}\left\{\ln\left[\frac{P(t_0, t)}{P(t_0, t + \delta)}\right] + l(t_0, t)\right\} \tag{18}$$

That is, define the change of measure

$$\frac{dQ^{t_{i+1}}}{dQ^g}\Big|_{\mathcal{F}_t} = \exp\left[\int_{t_0}^{t} \eta(t_0, v)d\widetilde{W}^{Q^g}_{g(t_0, v)} - \frac{1}{2}h(t_0, t)\right] \tag{19}$$

with

$$h(t_0, t) = \int_{t_0}^{t} \eta(t_0, v)^2 dg(t_0, v)$$

$$= \int_{t_0}^{t}\left[f(t_0, v + \delta) - f(t_0, v) + \frac{\partial l(t_0, v)}{\partial v}\right]^2 \times \left[\frac{\partial g(t_0, v)}{\partial v}\right]^{-1} dv \tag{20}$$

and assume that $h(t_0, t) < \infty$, where $f(t, T): = -\partial \ln P(t, T)/\partial T$ denotes the time-t instantaneous forward rate for date $T(\geq t)$. Integrating identity (17) between t_0 and t, and since $l(t_0, t_0) = 0$, it follows that

$$\widetilde{W}^{Q^g}_{g(t_0, t)} = \ln\left[\frac{P(t_0, t)}{P(t_0, t + \delta)}\right] + l(t_0, t) + \widetilde{W}^{Q^{t_{i+1}}}_{g(t_0, t)} + \ln P(t_0, t_0 + \delta)$$

and equation (15) can then be rewritten as a time-changed Brownian motion initialized at $\ln P^{-1}(t_0, t_0 + \delta)$:

$$\ln P^{-1}(t, t+\delta) = \widetilde{W}^{Q^g}_{g(t_0, t)} - \ln P(t_0, t_0+\delta) \tag{21}$$

Since $\ln P^{-1}(t_0, t_0+\delta)$ is a constant, a flat barrier on the process $\ln P^{-1}(t, t+\delta)$ is now translated into a constant barrier on $\widetilde{W}^{Q^g}_{g(t_0,t)}$. Therefore, taking $\theta = 1$ and $r_H = r_u(> r(t_0, t_0+\delta))$ for an up barrier, or $\theta = -1$ and $r_H = r_l(< r(t_0, t_0+\delta))$ for a down barrier,

$$\begin{aligned}
&Q^{t_{i+1}}\Big\{\ln P^{-1}(t, t+\delta) \le x,\\
&sup_{t_0 \le s \le t}[\theta \ln P^{-1}(s, s+\delta)] < \theta \ln(1+\delta r_H) | \mathcal{F}_{t_0}\Big\}\\
&= Q^{t_{i+1}}\Big\{\widetilde{W}^{Q^g}_{g(t_0, t)} \le x + \ln P(t_0, t_0+\delta),\\
&sup_{t_0 \le s \le t}[\theta \widetilde{W}^{Q^g}_{g(t_0, s)}] < \theta \ln\left[\frac{1+\delta r_H}{P^{-1}(t_0, t_0+\delta)}\right] | \mathcal{F}_{t_0}\Big\}\\
&= \mathbb{E}^{Q^g}_{t_0}\Big\{\exp\left[\int_{t_0}^{t} \eta(t_0, v)\, d\widetilde{W}^{Q^g}_{g(t_0, v)} - \frac{1}{2}h(t_0, t)\right] \qquad (22)\\
&\mathbf{1}\Big\{\widetilde{W}^{Q^g}_{g(t_0, t)} \le x + \ln P(t_0, t_0+\delta),\\
&sup_{t_0 \le s \le t}[\theta \widetilde{W}^{Q^g}_{g(t_0, s)}] < \theta \ln\left[\frac{1+\delta r_H}{P^{-1}(t_0, t_0+\delta)}\right]\Big\}\Big\}
\end{aligned}$$

where the last equality follows from equation (19) and, for instance, from Duffie (1992, equation C.3), while $\mathbf{1}_{\{\cdot\}}$ defines an indicator function.

To compute the expectation (22) explicitly, the Radon-Nikodym derivative (19) will be expressed as a function of the random variable $\widetilde{W}^{Q^g}_{g(t_0, t)}$. For that purpose, notice that the Itô integral contained in equation (19) possesses a univariate normal distribution with zero mean and variance $h(t_0, t)$. Consequently, and since $\widetilde{W}^{Q^g}_{g(t_0, t)}$ is also normally distributed with zero mean and variance $g(t_0, t)$, then

$$\int_{t_0}^{t} \eta(t_0, v)\, d\widetilde{W}^{Q^g}_{g(t_0, v)} \overset{d}{=} \sqrt{\frac{h(t_0, t)}{g(t_0, t)}}\, \widetilde{W}^{Q^g}_{g(t_0, t)} \tag{23}$$

where the symbol $\overset{d}{=}$ denotes equality in distribution. Result (23) suggests the following *proportionality approximation* for the Radon-Nikodym derivative (19).

Assumption 2 *The Radon-Nikodym derivative (19) will be approximated by*

$$\frac{dQ^{t_{i+1}}}{dQ^g}\Big|_{\mathcal{F}_t} \approx \exp\left[\sqrt{\frac{h(t_0, t)}{g(t_0, t)}}\, \widetilde{W}^{Q^g}_{g(t_0, t)} - \frac{1}{2}h(t_0, t)\right] \tag{24}$$

Notice that approximation (24) preserves the essential condition $\mathbb{E}^{Q^g}_{t_0}(dQ^{t_{i+1}}/dQ^g) = 1$, and is in the spirit of the proportionality assumption used by El Karoui and Rochet (1989, page 22) or of the rank 1 approximation suggested by Brace and Musiela (1994, equation 6.1), which have both been shown to be accurate in the context of European swaption pricing—see, for instance, Brace and Musiela (1994, table 7.5) or Pang (1996, table 6). Nevertheless, equality (24) is only exact in distribution or if $\partial\eta(t_0, v)/\partial v = 0$ for all $v \in [t_0, t]$. Therefore, all the approximate pricing solutions that will be derived hereafter will be subject to a Monte Carlo study, in order to investigate their accuracy. Moreover, subsection "Error bounds and asymptotic properties" will provide error bounds for the proposed Radon-Nikodym approximation.

The convergence in law result (24) allows equation (22) to be restated as

$$\begin{aligned}
&Q^{t_{i+1}}\Big\{\ln P^{-1}(t, t+\delta) \le x,\\
&sup_{t_0 \le s \le t}[\theta \ln P^{-1}(s, s+\delta)] < \theta \ln(1+\delta r_H) | \mathcal{F}_{t_0}\Big\}\\
&\approx \mathbb{E}^{Q^g}_{t_0}\Big\{\exp\left[\sqrt{\frac{h(t_0, t)}{g(t_0, t)}}\, \widetilde{W}^{Q^g}_{g(t_0,t)} - \frac{h(t_0, t)}{2}\right]\\
&\mathbf{1}\Big\{\widetilde{W}^{Q^g}_{g(t_0, t)} \le x + \ln P(t_0, t_0+\delta),\\
&sup_{t_0 \le s \le t}[\theta \widetilde{W}^{Q^g}_{g(t_0, s)}] < \theta \ln\left[\frac{1+\delta r_H}{P^{-1}(t_0, t_0+\delta)}\right]\Big\}\Big\}\\
&= \int_{-\infty}^{x+\ln P(t_0, t_0+\delta)} \exp\left[\sqrt{\frac{h(t_0, t)}{g(t_0, t)}}\, z - \frac{1}{2}h(t_0, t)\right] \qquad (25)
\end{aligned}$$

$$\times \left\{ \phi[z; 0, \sqrt{g(t_0, t)}] \right.$$

$$\left. - \phi\left[z; 2\ln\frac{1+\delta r_H}{P^{-1}(t_0, t_0+\delta)}, \sqrt{g(t_0,t)}\right]\right\} dz$$

where equation (25) follows, for instance, from Harrison (1985, page 9) or Musiela and Rutkowski (1998, corollaries B.3.1 and B.3.3). Differentiating the above integral with respect to x, the main theoretical result from this paper arises. This result is contained in the next theorem and provides an approximate analytical solution for the restricted density of log-inverse pure discount bond prices. It possesses a similar structure to the one found for any asset price following a geometric Brownian motion (see, for instance, Rich (1994, equations 6 and 10)).

Theorem 1 *Under the Gaussian HJM specification (2), and subject to assumptions (14) and (24),*

$$Q^{t_{i+1}}\left\{ \ln P^{-1}(t, t+\delta) \in dx, \right.$$

$$\left. sup_{t_0 \le s \le t}[\theta \ln P^{-1}(s, s+\delta)] < \theta \ln(1+\delta r_H) | \mathcal{F}_{t_0} \right\}$$

$$= \left\{ \phi[x; \ln P^{-1}(t_0, t_0+\delta) + \sqrt{h(t_0, t) g(t_0, t)}, \right. \tag{26}$$

$$\sqrt{g(t_0, t)}] - \exp\left[2\sqrt{\frac{h(t_0, t)}{g(t_0, t)}} \ln\frac{1+\delta r_H}{P^{-1}(t_0, t_0+\delta)}\right]$$

$$\times \phi[x; 2\ln(1+\delta r_H) - \ln P^{-1}(t_0, t_0+\delta)$$

$$\left. + \sqrt{h(t_0, t) g(t_0, t)}, \sqrt{g(t_0, t)}] \right\} dx$$

where $h(t_0, t)$ is given by equation (20), while $\theta = 1$ and $r_H = r_u(> r(t_0, t_0+\delta))$ for an up barrier, or $\theta = -1$ and $r_H = r_l(< r(t_0, t_0+\delta))$ for a down barrier.

Equipped with theorem 1, it is now possible to price any type of single-barrier cap or floor. Although it is outside the scope of the present paper, equation (26) is also the only ingredient required to price barrier digital caps and floors, which are described, for instance, by Pelsser (2000, page 102). Moreover, the same pricing methodology can be also applied to several structured products and Euro Medium-Term Notes whose payoff contains a trigger condition on a LIBOR rate.

Error bounds and asymptotic properties

With the purpose of characterizing the accuracy of the analytical solution proposed in subsection "Restricted densities", let

$$\varepsilon(t) := \int_{t_0}^{t}\left[\eta(t_0, v) - \sqrt{\frac{h(t_0, t)}{g(t_0, t)}}\right] d\widetilde{W}^{Q^g}_{g(t_0, v)} \tag{27}$$

define the error term implicit in assumption 2. Remember that theorem 1 was based on the adoption of approximation (24), which is equivalent to the assumption that $\varepsilon(t) = 0$. Furthermore, definition (27) allows equation (22) to be rewritten as

$$Q^{t_{i+1}}\left\{ \ln P^{-1}(t, t+\delta) \le x, \right.$$

$$\left. sup_{t_0 \le s \le t}[\theta \ln P^{-1}(s, s+\delta)] < \theta \ln(1+\delta r_H) | \mathcal{F}_{t_0} \right\} \tag{28}$$

$$= \mathbb{E}^{Q^g}_{t_0}\{\exp[\varepsilon(t)] Y(t, x, \ln(1+\delta r_H))\}$$

with

$$Y(t,x,y) := \exp\left[\sqrt{\frac{h(t_0, t)}{g(t_0, t)}} \widetilde{W}^{Q^g}_{g(t_0, t)} - \frac{h(t_0, t)}{2}\right]$$

$$\mathbf{1}\left\{ \widetilde{W}^{Q^g}_{g(t_0, t)} \le x + \ln P(t_0, t_0+\delta), \right.$$

$$\left. sup_{t_0 \le s \le t}[\theta \widetilde{W}^{Q^g}_{g(t_0, s)}] < \theta[\ln P(t_0, t_0+\delta) + y] \right\} \tag{29}$$

Based on definitions (27) and (29), next proposition offers both lower and upper error bounds for the approximation proposed in theorem 1. The length of the interval delimited by these error bounds offers an indicator of the accuracy implicit in the analytical pricing approximations proposed, which constitutes a robust innovation with respect to the previous literature. A proof is provided in appendix A.

Proposition 3 *Under the Gaussian HJM specification (2), and for any* $\rho(t) \in [-1, 1]$,

$$Q^{t_{i+1}}\Big\{\ln P^{-1}(t, t+\delta) \in dx,$$

$$sup_{t_0 \le s \le t}[\theta \ln P^{-1}(s, s+\delta)] < \theta \ln(1+\delta r_H) | \mathcal{F}_{t_0}\Big\} \quad (30)$$

$$= \Bigg\{\exp\left[\frac{V_\varepsilon(t)}{2}\right]\frac{\partial \mathbb{E}^{Q^e}_{t_0}[Y(t,x,\ln(1+\delta r_H))]}{\partial x} + \frac{\rho(t)}{2}$$

$$\frac{\sqrt{\exp[2V_\varepsilon(t)] - \exp[V_\varepsilon(t)]}\frac{\partial V_Y(t,x,\ln(1+\delta r_H))}{\partial x}}{\sqrt{V_Y(t,x,\ln(1+\delta r_H))}}\Bigg\}dx$$

where

$$V_\varepsilon(t) := \int_{t_0}^{t}\left[\eta(t_0, v) - \sqrt{\frac{h(t_0, t)}{g(t_0, t)}}\right]^2 dg(t_0, v) \quad (31)$$

is the variance of the error term (27), $(\partial \mathbb{E}^{Q^e}_{t_0}[Y(t,x,\ln(1+\delta r_H))]/\partial x)\, dx$ *is equal to the right-hand-side of equation (26),*

$$V_Y(t,x,y) := \mathbb{E}^{Q^e}_{t_0}[Y(t,x,y)^2] - \{\mathbb{E}^{Q^e}_{t_0}[Y(t,x,y)]\}^2 \quad (32)$$

$$\mathbb{E}^{Q^e}_{t_0}[Y(t,x,y)] = \Phi\left[\frac{x + \ln P(t_0, t_0+\delta) - \sqrt{h(t_0,t)g(t_0,t)}}{\sqrt{g(t_0,t)}}\right]$$

$$- \exp\left[2\sqrt{\frac{h(t_0, t)}{g(t_0, t)}}(\ln P(t_0, t_0+\delta) + y)\right]$$

$$\Phi\left[\frac{x - 2y - \ln P(t_0, t_0+\delta) - \sqrt{h(t_0, t)g(t_0, t)}}{\sqrt{g(t_0, t)}}\right] \quad (33)$$

$$E^{Q^e}_{t_0}[Y(t,x,y)^2] = \exp[h(t_0,t)]$$

$$\times \Bigg\{\Phi\left[\frac{x + \ln P(t_0, t_0+\delta) - 2\sqrt{h(t_0, t)g(t_0, t)}}{\sqrt{g(t_0, t)}}\right]$$

$$- \exp\left[4\sqrt{\frac{h(t_0, t)}{g(t_0, t)}}(\ln P(t_0, t_0+\delta) + y)\right]$$

$$\times \phi\left[\frac{x - 2y - \ln P(t_0,t_0+\delta) - 2\sqrt{h(t_0,t)g(t_0,t)}}{\sqrt{g(t_0,t)}}\right]\Bigg\} \quad (34)$$

and $\partial V_Y(t,x,y)/\partial x$ *denotes the derivative w.r.t. x of equation (32).*[7]

The parameter $\rho(t)$ can be understood as the linear correlation coefficient between the random variables exp $[\varepsilon(t)]$ and $Y(t,x, \ln(1+\delta r_H))$. Since we have no a priori information about the value of $\rho(t)$, in general, the (upper or lower) error bounds provided by formula (30) are obtained with $\rho(t) = 1$ or -1.

Based on proposition 3, it is also possible to show that the approximation proposed in theorem 1 possesses desirable asymptotic properties. Specifically, the proposed approximate density (26) can be obtained as the limit of the exact restricted density (30), as $V_\varepsilon(t) \to 0$. This means that theorem 1 should offer an approximation whose accuracy can be inferred from the magnitude of the variance of the error term (27), which is a known quantity. Actually, it is possible to prove that the error implicit in the proposed approximate density (26) is of the same order of magnitude as the standard deviation of the error term (27), that is:

$$Q^{t_{i+1}}\Big\{\ln P^{-1}(t, t+\delta) \in dx,$$

$$sup_{t_0 \le s \le t}[\theta \ln P^{-1}(s,s+\delta)] < \theta y | \mathcal{F}_{t_0}\Big\}$$

$$- \frac{\partial \mathbb{E}^{Q^e}_{t_0}[Y(t,x,y)]}{\partial x}dx = O(\sqrt{V_\varepsilon(t)}) \quad (35)$$

Pricing formulae

In order to price single-barrier caps and floors it is only necessary to compute the expectations and probabilities contained in equations (6) and (7), using the probability density offered by theorem 1. The proposed pricing solutions are summarized by the next two propositions, and all proofs are provided in appendices B and C.

Proposition 4 *Under the Gaussian HJM specification (2), and subject to assumptions (14) and (24),*

$$
\begin{aligned}
& IN(\theta)_{t_0}[r(t_i, t_{i+1}); r_k; r_H; R; t_{i+1}] \\
&= IN(\theta)_{t_0}[r(t_i, t_{i+1}); r_k; r_H; R = 0; t_{i+1}] + KIR(\theta)_{t_0}
\end{aligned} \tag{36}
$$

where

$$
\begin{aligned}
& IN(\theta)_{t_0}[r(t_i, t_{i+1}); r_k; r_H; R = 0; t_{i+1}] \\
&= REGULAR(\theta)_{t_0}[r(t_i, t_{i+1}); r_k; t_{i+1}] \\
&+ \mathbf{1}_{\{\theta r_H > \theta r_k\}}(1 + \delta r_k)P(t_0, t_{i+1}) \\
&\times \{\Phi[d_1^{r_H}(t_0)] - \Phi[d_1^{r_k}(t_0)]\} \\
&- \mathbf{1}_{\{\theta r_H > \theta r_k\}} \frac{P(t_0, t_{i+1})}{P(t_0, t_0 + \delta)} \\
&\times \exp\left[\frac{1}{2}g(t_0, t_i) + \sqrt{h(t_0, t_i)g(t_0, t_i)}\right] \\
&\{\Phi[d_1^{r_H}(t_0) - \sqrt{g(t_0, t_i)}] - \Phi[d_1^{r_k}(t_0) - \sqrt{g(t_0, t_i)}]\} \\
&+ \mathbf{1}_{\{\theta r_H > \theta r_k\}} \exp\left\{2\sqrt{\frac{h(t_0, t_i)}{g(t_0, t_i)}} \ln\left[\frac{1 + \delta r_H}{P^{-1}(t_0, t_0 + \delta)}\right]\right\} \\
&\times \Bigg\{P(t_0, t_{i+1})P(t_0, t_0 + \delta)(1 + \delta r_H)^2 \\
&\exp\left[\frac{1}{2}g(t_0, t_i) + \sqrt{h(t_0, t_i)g(t_0, t_i)}\right][\Phi(d_2^{r_H}(t_0) \\
&- \sqrt{g(t_0, t_i)}) - \Phi(d_2^{r_k}(t_0) - \sqrt{g(t_0, t_i)})] \\
&- (1 + \delta r_k)P(t_0, t_{i+1})[\Phi(d_2^{r_H}(t_0)) - \Phi(d_2^{r_k}(t_0))]\Bigg\}
\end{aligned} \tag{37}
$$

the present value of the knock-in rebate is given by

$$
\begin{aligned}
& KIR(\theta)_{t_0} = P(t_0, t_{i+1})\delta R \\
& \left\{\Phi[\theta d_1^{r_H}(t_0)] - \exp\left[2\sqrt{\frac{h(t_0, t_i)}{g(t_0, t_i)}} \ln\left(\frac{1 + \delta r_H}{P^{-1}(t_0, t_0 + \delta)}\right)\right]\right. \\
& \left. \times \Phi[\theta d_2^{r_H}(t_0)]\right\}
\end{aligned} \tag{38}
$$

with

$$
d_1^r(t) := \frac{\ln\left[\frac{P(t,t + \delta)}{(1 + \delta r)^{-1}}\right] - \sqrt{h(t, t_i)g(t, t_i)}}{\sqrt{g(t, t_i)}},
$$

$$
d_2^r(t) := \frac{\ln\left[\frac{P^{-1}(t,t + \delta)}{(1 + \delta r)^{-1}(1 + \delta r_H)^2}\right] - \sqrt{h(t, t_i)g(t, t_i)}}{\sqrt{g(t, t_i)}}
$$

and where $r_H = r_u(> r(t_0, t_0 + \delta))$ *and* $\theta = 1$ *for an up-and-in caplet, or* $r_H = r_l(< r(t_0, t_0 + \delta))$ *and* $\theta = -1$ *for a down-and-in floorlet.*

Notice that, in contrast to the previous literature, the valuation of rebates is being explicitly considered.

Proposition 5 *Under the Gaussian HJM specification (2), and subject to assumptions (14) and (24),*

$$
\begin{aligned}
& OUT(\theta)_{t_0}[r(t_i, t_{i+1}); r_k; r_H; R; t_{i+1}] \\
&= OUT(\theta)_{t_0}[r(t_i, t_{i+1}); r_k; r_H; R = 0; t_{i+1}] \\
&+ KODR(\theta)_{t_0}
\end{aligned} \tag{39}
$$

where

$$
OUT(\theta)_{t_0}[r(t_i, t_{i+1}); r_k; r_H; R = 0; t_{i+1}] \tag{40}
$$

$$= \theta \frac{P(t_0, t_{i+1})}{P(t_0, t_0 + \delta)} \exp\left[\frac{1}{2}g(t_0, t_i) + \sqrt{h(t_0, t_i)g(t_0, t_i)}\right]$$

$$\times \Phi[\theta\sqrt{g(t_0, t_i)} - \theta d_1^{\theta\max(\theta r_H, \theta r_k)}(t_0)]$$

$$- \theta(1 + \delta r_k)P(t_0, t_{i+1})\Phi[-\theta d_1^{\theta\max(\theta r_H, \theta r_k)}(t_0)]$$

$$- \theta \exp\left\{2\sqrt{\frac{h(t_0, t)}{g(t_0, t)}}\ln\left[\frac{1 + \delta r_H}{P^{-1}(t_0, t_0 + \delta)}\right]\right\}$$
$$\times \{P(t_0, t_{i+1})P(t_0, t_0 + \delta)(1 + \delta r_H)^2$$

$$\times \exp\left[\frac{1}{2}g(t_0, t_i) + \sqrt{h(t_0, t_i)g(t_0, t_i)}\right]$$

$$\times \Phi[\theta\sqrt{g(t_0, t_i)} - \theta d_2^{\theta\max(\theta r_H, \theta r_k)}(t_0)]$$

$$- (1 + \delta r_k)P(t_0, t_{i+1})\Phi[-\theta d_2^{\theta\max(\theta r_H, \theta r_k)}(t_0)]\}$$

and the present value of the deferrable knock-out rebate is given by

$$KODR(\theta)_{t_0} = P(t_0, t_{i+1})\delta R \tag{41}$$

$$\left\{\Phi[\theta d_1^{r_H}(t_0)] + \exp\left[2\sqrt{\frac{h(t_0, t_i)}{g(t_0, t_i)}}\ln\left(\frac{1 + \delta r_H}{P^{-1}(t_0, t_0 + \delta)}\right)\right]\right.$$
$$\left.\times \Phi[-\theta d_2^{r_H}(t_0)]\right\}$$

with $r_H = r_l(< r(t_0, t_0 + \delta))$ and $\theta = 1$ for a down-and-out caplet, or $r_H = r_u(> r(t_0, t_0 + \delta))$ and $\theta = -1$ for an up-and-out floorlet.

Examples

In order to test the efficiency and accuracy of the proposed pricing approximations, the HJM model under analysis has been calibrated to US market data on April 02, 2002. For simplicity, the initial spot yield curve was obtained by fitting a Nelson and Siegel (1987) function to the US LIBOR and swap rates contained in Exhibit 1, yielding an average absolute error of only 2 basis points.[8]

EXHIBIT 1

Market versus fitted US LIBOR and swap rates on April 02, 2002

LIBOR rates			Swap rates		
maturity	market	fitted	maturity	market	fitted
3M	2.04%	2.04%	2Y	4.11%	4.04%
6M	2.36%	2.39%	3Y	4.72%	4.71%
1Y	3.06%	3.06%	4Y	5.11%	5.14%
			5Y	5.40%	5.43%
			7Y	5.77%	5.77%
			10Y	6.07%	6.03%

All swap rates assume semi-annual revolving and correspond to Bloomberg mid quotes under the 30/360 day count convention. Fitted rates are obtained by estimating a Nelson and Siegel (1987) spot rate function from all market quotes and are computed through the following Nelson and Siegel (1987, equation 2) parameterization: $\beta_0 = 0.067473$, $\beta_1 = -0.50472$, $\beta_2 = -0.029139$, $\tau = 0.816930$.

Although propositions 4 and 5 are applicable to any deterministic HJM specification (even to a time-dependent and/or non-Markovian one), next examples will be cast into a simple Gauss-Markov and time-homogeneous HJM model, defined through the following volatility function:

$$\underline{\sigma}(t, T)' := \underline{G}'.\, a^{-1}.[I_n - e^{a(T-t)}] \tag{42}$$

where $I_n \in \mathcal{R}^{n\times n}$ represents an identity matrix, while $\underline{G} \in \mathcal{R}^n$ and the diagonal matrix $a \in \mathcal{R}^{n\times n}$ contain model' time-independent parameters. As shown, for instance, by Musiela and Rutkowski (1998, proposition 13.3.2), the volatility specification (42) ensures that the short-term interest rate $r(t)$ is Markovian, which will enable the comparison of the proposed approximate pricing solutions to the quasi-analytical ones offered by Chrétien and Quittard-Pinon (2000).[9]

Adopting the typical result of Principal Components Analysis—given, for instance, by Litterman and Scheinkman (1991)—of three common factors ($n = 3$), the model' parameters estimates (shown at the bottom of Exhibit 2) were obtained by minimizing the absolute percentage differences between market and model prices of at-the-money and forward-start caps for 1, 2, 3, 4, 5, 7, and 10 years. Using the Nelson and Siegel (1987) discount function provided by Exhibit 1, all the market flat yield cap volatilities (second column of Exhibit 2) were

EXHIBIT 2

Market versus fitted US caps on April 02, 2002

Cap maturity	Black volatility	A-T-M strike	cap prices market	model	% error
1Y	32.60%	3.349%	27.58	31.55	14.42%
2Y	28.10%	4.316%	108.00	110.34	2.16%
3Y	25.40%	4.938%	205.36	203.00	-1.15%
4Y	23.90%	5.333%	305.97	297.15	-2.88%
5Y	22.50%	5.591%	400.20	390.22	-2.49%
7Y	20.50%	5.892%	572.44	572.47	0.01%
10Y	19.00%	6.113%	809.48	834.32	3.07%
			Average percentage error		1.88%
			Average absolute percentage error		3.74%

All Black volatilities are Bloomberg mid quotes for forward-start and at-the-money caps. Cap prices are expressed as US$100 per basis point. Model cap prices are computed under proposition 2 and using the following estimated parameters' values:

$$G' = [-0.007413 \quad -0.012087 \quad 0.002430]$$

and

$$a = \text{diag}\{-0.743822, \quad -0.004683, \quad -1.112957\}.$$

converted into market cap prices, by applying the usual Black (1976) formula for each caplet and using as strike price the corresponding forward swap rate with quarterly compounding (third column of Exhibit 2). Model cap prices were computed under proposition 2 (with $\theta = 1$). The last column of Exhibit 2 displays the percentage differences between model and market cap prices. Although the average percentage error is under 2%,[10] the time-homogeneous volatility specification (42) generates a too large (14.42%) percentage error for the one-year cap. For practical purposes, the fit to the cap market ought to be improved using a time-dependent and/or a non-Markovian framework, such as, the piecewise constant volatility structure adopted by Brace, Gatarek and Musiela (1997, table 4.4) or the time-homogeneous but humped volatility function proposed by Mercurio and Moraleda (2000). However, the Gauss-Markov volatility function (42) will still be used in order to facilitate the comparison against the Chrétien and Quittard-Pinon (2000) quasi-analytical solutions.

Exhibit 3 (Panel A) prices a two-year (unit face value) down-and-out and zero-rebate cap with a cap rate of 4.316%, a barrier rate of 1.5% and quarterly compounding ($\delta = 0.25$), under the HJM specification offered by Exhibit 2 and using the initial yield curve contained in Exhibit 1. Exact down-and-out caplet prices were estimated through Monte Carlo simulation,[11] using the usual Euler discretization of equation (2) with 520 time steps per year, independent normal and antithetic variates generated through the Box-Muller algorithm, and 200,000 simulations. It is assumed that the barrier is monitored 260 times per year (i.e., daily monitoring). Besides the Monte Carlo price estimate, the percentage of its standard error over the mean price is also shown. Approximate barrier caplet prices were computed from proposition 5 and from a multifactor generalization of the Chrétien and Quittard-Pinon (2000) formulae,[12] while their regular counterparts were obtained from proposition 2. The corresponding percentage (pricing) errors were computed with respect to the (exact) Monte Carlo price estimate. Throughout this paper, all computations were made by running Pascal programs on an Intel Xeon 2.80 GHz processor. According to Panel A of Exhibit 3, the proposed approximate solution—proposition 5—is very fast to implement (it takes less than one second) and accurate (with pricing errors of about one standard error of the Monte Carlo estimate).

In order to further investigate the accuracy of the proposed analytical solutions, Panel B of Exhibit 3 presents an extreme example where the barrier rate (2%) of the down-and-out cap is put almost at the current level (2.04%) of the 3-month spot rate (see Exhibit 1). As a consequence, the barrier cap price differs quite substantially from the regular one (which is 228% higher) and the proposed approximate solution (proposition 5) generates an unacceptably high pricing error (-49.87%) when compared against the Monte Carlo estimate. However, one should keep in mind that two error sources are affecting the results. One is the use of approximation (24). The other one is the fact that propositions 4 and 5 assume continuous monitoring of the barrier, while the Monte Carlo price estimate (that is the proxy for the exact price) considers that the barrier is discretely monitored. And, as argued and shown, for instance, by Broadie, Glasserman and Kou (1997), "...there normal can be substantial price differences between discrete and continuous barrier options, even under daily monitoring of the barrier" . Therefore, it would be extremely important if propositions 4 and 5 could be adapted for discrete monitoring using a continuity correction similar to the one provided by Broadie, Glasserman and Kou (1997, theorem 1.1) and Broadie, Glasserman and Kou (1999, theorem 1). Alternatively, the Monte Carlo simulation could also be enhanced for continuous monitoring along the lines of Beaglehole, Dybvig and Zhou (1997) or Babsiri and Noel (1998). Such developments are left for further

EXHIBIT 3

Pricing of two-year down-and-out caps with quarterly compounding, under the HJM model of Exhibit 2 and the spot yield curve of Exhibit 1

Caplet maturity(ti) (years)	Antithetic Monte Carlo price	std. error	% std. error	Approximate analytical solution (proposition 5) price	% errors	CQP solution price	% errors	Regular cap (proposition 2) price	% errors
	Panel A: Cap with strike r_k = 4.316%, barrier r_l = 1.5% and rebate R=0%								
0.25	0.0591	1.25E-03	2.12%	0.0606	2.63%	0.0605	2.52%	0.0606	2.62%
0.5	1.9909	9.26E-03	0.47%	1.9941	0.16%	1.9827	-0.41%	2.0226	1.59%
0.75	7.1750	1.77E-02	0.25%	7.1635	-0.16%	7.0758	-1.38%	7.4609	3.99%
1	13.7871	2.20E-02	0.16%	13.8013	0.10%	13.5659	-1.60%	14.7538	7.01%
1.25	20.3570	2.39E-02	0.12%	20.3643	0.04%	19.9509	-2.00%	22.2369	9.23%
1.5	26.1346	2.58E-02	0.10%	26.1684	0.13%	25.5599	-2.20%	29.0238	11.06%
1.75	30.9524	2.77E-02	0.09%	31.0378	0.28%	30.1994	-2.43%	34.7777	12.36%
A-T-M Cap	100.4561			100.5900	0.13%	98.3951	-2.05%	110.3364	9.84%
CPU time	> 1day			< 1 second		39 minutes			
	Panel B: Cap with strike r_k=4.316%, barrier r_l = 2% and rebate R=0%								
0.25	0.0383	1.03E-03	2.69%	0.0225	-41.43%	0.0230	-40.07%	0.0606	58.08%
0.5	0.9114	6.86E-03	0.75%	0.4910	-46.13%	0.4795	-47.39%	2.0226	121.94%
0.75	2.7739	1.40E-02	0.50%	1.4558	-47.52%	1.3989	-49.57%	7.4609	168.97%
1	4.8513	2.04E-02	0.42%	2.5112	-48.24%	2.4164	-50.19%	14.7538	204.12%
1.25	6.7990	2.60E-02	0.38%	3.4404	-49.40%	3.3471	-50.77%	22.2369	227.06%
1.5	8.4491	3.06E-02	0.36%	4.1820	-50.50%	4.1313	-51.10%	29.0238	243.51%
1.75	9.7831	3.42E-02	0.35%	4.7434	-51.51%	4.7620	-51.32%	34.7777	255.49%
A-T-M Cap	33.6061			16.8462	-49.87%	16.5582	-50.73%	110.3364	228.32%
CPU time	> 1day			< 1 second		39 minutes			

Caplet prices are expressed as US$100 per basis point. CQP solution corresponds to the Chrétien and Quittard-Pinon (2000) approach, which has been generalized for multifactor Gauss-Markov models and implemented using 600 time-steps per year. Monte Carlo prices are computed through 200,000 simulations, with antithetic variates, 520 discrete time-steps per year and 260 barrier' monitoring periods per year. % std. error is the standard error divided by the Monte Carlo option price estimate. All % errors were computed against the Monte Carlo price estimates.

research. Nevertheless, Exhibit 3 (Panel B) shows that the proposed approximated prices are very close to the ones generated by the Chrétien and Quittard-Pinon (2000) approach, since both methodologies assume continuous monitoring of the barrier.

Exhibit 4 (Panel A) considers the valuation of a two-year (unit face value) up-and-in and zero-rebate cap with a cap rate of 4.316%, a barrier rate of 5% and quarterly compounding, still under the HJM specification offered by Exhibits 1 and 2. Approximate analytical barrier caplet prices were now computed from proposition 4 with an acceptable degree of accuracy: the analytical up-and-in cap price is only 6 basis points higher than the Monte Carlo estimate and it is very close to the Chrétien and Quittard-Pinon (2000) solution. Besides the proposed closed-form solution, Exhibit 4 also presents errors bounds for the barrier cap price, which were obtained by numerically integrating—through Romberg's method (see, for instance, Press, Flannery, Teukolsky and Vetterling (1994, sections 4.3 and 4.4))—each caplet terminal payoff with respect to the exact restricted density (30), with $\rho(t) = 1$ or -1 for a lower or upper price bound, respectively. Such error bounds provide a confidence interval for the true barrier option value, while the length of such interval is an indicator of the precision associated to the proposed analytical pricing solutions: for the contract under analysis, the true up-and-in cap price (with continuous monitoring) can be only 1.84% lower or 1.81% higher than the Monte Carlo estimate.

Panel B of Exhibit 4 repeats the analysis of Panel A but for a higher barrier level of 6%, which is 39% above the A-T-M strike. Notice that, because of the high level chosen for the barrier rate, the regular caplet prices are significantly different from their up-and-in counterparts (the up-and-in cap is about 35% cheaper). The pricing errors produced by proposition 4 are now larger than the ones observed in Panel A of Exhibit 4 (about -1.6% with respect to the up-and-in Monte Carlo cap price) and the percentage error bounds are too wide for practical purposes, which questions the accuracy of the proposed analytical solution for the valuation of the specific exotic contract under analysis.

Exhibit 5 highlights the fact that the accuracy of the approximate pricing solutions proposed in propositions 4 and 5 depends on the features of the barrier option under analysis (namely, its strike and barrier levels) and, also, on the HJM model under use (which is defined through the initial spot rate curve and by the specification of the volatility function). In any case, the error bounds provided by proposition 3 allow the assessment of the precision associated to such approximate analytical pricing solutions. As shown by equation (35), if those error bounds are too loose, the accuracy of propositions 4 and 5 can be improved by reducing $\sqrt{V_\varepsilon(t_i)}$, for each caplet with maturity at time $t_i + \delta$. If the specification of the volatility function is flexible enough, one possibility would be to calibrate the HJM model not only to a set of market observables (cap prices, in our examples) but also to the minimization of the quantities $\sqrt{V_\varepsilon(t_i)}$.[13] Moreover, inspection of equation (31) shows that the order of the error implicit in the proposed approximations depends (directly) on the slope of the initial yield curve. In order to test such relation, Exhibit 5 prices a two-year (unit face value) up-and-in and zero-rebate cap with a strike of 3.116%, a barrier rate of 4.331% and quarterly compounding, still under the volatility specification offered by Exhibit 2 but assuming a flat spot rate curve equal to 3.104% (which corresponds to the average of the continuously compounded spot rates contained in Exhibit 1, for maturities until two-years). The cap rate (3.116%) is again the two-years forward swap rate with quarterly compounding associated with the flat yield curve under consideration, and the barrier level (4.331%) is still defined as being 39% above the A-T-M strike. Hence, the only difference with respect to the analysis of Exhibit 4 (Panel B) concerns the slope of the initial interest rate curve. As expected, the accuracy of the proposed approximation is now much better: although the approximate price offered by proposition 4 is still 0.89% higher than the Monte Carlo estimate, the true continuously monitored price is known to be between 0.74% and 1.05% above the Monte Carlo (daily monitored) price. That is the maximum admissible absolute error implicit in proposition 4, for the contract under analysis and with respect to the true continuously monitored cap price, is equal to only 16 basis points.

Since the previous numerical experiments have only considered zero-rebate caps, Exhibit 6 (Panel A) prices a rebate with value $R = 0.1\%$ and associated with the up-and-in cap described in Exhibit 5. Equation (38) produces an excellent approximation, since the percentage difference with respect to the Monte Carlo estimate is equal to only two basis points. Additionally, the narrow error bounds generated through the integration[14] of the exact restricted density (30) show that the true rebate value lies between -0.91% and 0.88% of the Monte Carlo price. Panel B of Exhibit 6 repeats, with similar results, the previous analysis but for a higher barrier rate of 6%.

EXHIBIT 4

Pricing of two-year up-and-in caps with quarterly compounding, under the HJM model of Exhibit 2 and the spot yield curve of Exhibit 1

Caplet maturity(t_i)	Antithetic Monte Carlo		Approximate analytical solution: Proposition 4			Approximate analytical solution: % error bounds		CQP solution		Regular cap (proposition 2)	
(years)	price	% std. error	price	% errors	$\sqrt{V_E(t_i)}$	lower	upper	price	% errors	price	% errors
Panel A: Cap with strike r_k = 4.316%, barrier r_u = 5% and rebate R=0%											
0.25	0.01	6.62%	0.01	22.35%	2.41E-02	-70.34%	114.48%	0.01	18.16%	0.06	402.21%
0.5	1.37	0.66%	1.43	4.47%	3.45E-02	-3.77%	12.55%	1.44	4.82%	2.02	47.40%
0.75	6.53	0.29%	6.59	1.03%	4.23E-02	-1.41%	3.39%	6.64	1.76%	7.46	14.30%
1	13.93	0.16%	13.97	0.32%	9.40E-02	-1.73%	2.29%	14.07	1.00%	14.75	5.95%
1.25	21.63	0.10%	21.62	-0.02%	1.78E-01	-1.92%	1.75%	21.75	0.56%	22.24	2.83%
1.5	28.59	0.07%	28.55	-0.12%	2.80E-01	-1.84%	1.45%	28.70	0.38%	29.02	1.52%
1.75	34.49	0.05%	34.42	-0.22%	3.90E-01	-1.81%	1.19%	34.57	0.22%	34.78	0.83%
A-T-M Cap	106.54		106.60	0.06%		-1.84%	1.81%	107.17	0.59%	110.34	3.56%
CPU time	21.55 hours		< 1 second					39 minutes			
Panel B: Cap with strike r_k = 4.316%, barrier r_u = 6% and rebate R=0%											
0.25	1E-04	100.00%	3E-05	-76.19%	2.41E-02	-16978.44%	16788.47%	7E-05	-33.22%	0.06	55466%
0.5	0.19	2.44%	0.21	9.91%	3.45E-02	-284.61%	303.08%	0.22	12.09%	2.02	945.54%
0.75	2.59	0.65%	2.65	2.14%	4.23E-02	-50.51%	54.33%	2.73	5.43%	7.46	187.80%
1	8.48	0.33%	8.47	-0.11%	9.40E-02	-37.99%	37.01%	8.79	3.61%	14.75	74.01%
1.25	16.21	0.20%	15.98	-1.45%	1.78E-01	-34.46%	30.26%	16.58	2.28%	22.24	37.17%
1.5	23.89	0.13%	23.43	-1.91%	2.80E-01	-31.47%	25.72%	24.27	1.61%	29.02	21.51%
1.75	30.65	0.08%	29.98	-2.21%	3.90E-01	-29.42%	22.50%	30.98	1.05%	34.78	13.45%
A-T-M Cap	82.02		80.71	-1.59%		-33.19%	28.16%	83.56	1.89%	110.34	34.53%
CPU time	> 1 day		< 1 second					40 minutes			

Caplet prices are expressed as US\$100 per basis point. CQP solution corresponds to the Chrétien and Quittard-Pinon (2000) approach, which has been generalized for multifactor Gauss-Markov models and implemented using 600 time-steps per year. Monte Carlo prices are computed through 200,000 simulations, with antithetic variates, 520 discrete time-steps per year and 260 barrier' monitoring periods per year. % std. error is the standard error divided by the Monte Carlo option price estimate. All % errors were computed against the Monte Carlo price estimates. $\sqrt{V_\varepsilon(t_i)}$ is the standard deviation of the error term (27). % error bounds are obtained by integrating each caplet terminal payoff w.r.t. the exact restricted density (30), with $\rho(t)$ = 1(−1) for a lower (upper) price bound.

EXHIBIT 5

Pricing of a two-year up-and-in and zero-rebate cap with quarterly compounding, strike $r_k = 3.116\%$ and barrier $r_u = 4.331\%$, under the volatility specification of Exhibit 2 but assuming a flat continuously compounded yield curve at the 3.104% level

Caplet maturity (t_i) (years)	Antithetic Monte Carlo		Approximate analytical solution: Proposition 4			% error bounds		CQP solution		Regular cap (proposition 2)	
	price	% std. error	price	% errors	$\sqrt{V_\varepsilon(t_i)}$	lower	upper	price	% errors	price	% errors
0.25	2.14	0.58%	2.34	9.29%	2.31E-03	8.91%	9.67%	2.36	10.31%	6.77	216.80%
0.5	5.94	0.32%	6.11	2.87%	4.93E-03	2.68%	3.07%	6.21	4.52%	9.33	57.13%
0.75	8.71	0.26%	8.80	1.08%	8.21E-03	0.92%	1.24%	8.93	2.56%	11.17	28.26%
1	10.74	0.23%	10.81	0.69%	1.20E-02	0.54%	0.84%	10.95	1.96%	12.64	17.69%
1.25	12.37	0.21%	12.41	0.36%	1.62E-02	0.21%	0.50%	12.55	1.43%	13.88	12.17%
1.5	13.73	0.20%	13.75	0.16%	2.08E-02	0.02%	0.29%	13.87	1.07%	14.95	8.94%
1.75	14.89	0.19%	14.90	0.06%	2.57E-02	-0.08%	0.19%	15.01	0.84%	15.91	6.88%
A-T-M Cap	68.51		69.12	0.89%		0.74%	1.05%	69.88	2.00%	84.66	23.57%
CPU time	> 1 day		< 1 second					40 minutes			

Caplet prices are expressed as US\$100 per basis point. CQP solution corresponds to the Chrétien and Quittard-Pinon (2000) approach, which has been generalized for multifactor Gauss-Markov models and implemented using 600 time-steps per year. Monte Carlo prices are computed through 200,000 simulations, with antithetic variates, 520 discrete time-steps per year and 260 barrier' monitoring periods per year. % std. error is the standard error divided by the Monte Carlo option price estimate. All % errors were computed against the Monte Carlo price estimates. $\sqrt{V_\varepsilon(t_i)}$ is the standard deviation of the error term (27). % error bounds are obtained by integrating each caplet terminal payoff w.r.t. the exact restricted density (30), with $\rho(t) = 1(-1)$ for a lower (upper) price bound.

EXHIBIT 6

Pricing of rebates associated to two-year up-and-in caps with quarterly compounding, under the three-factor HJM model of Exhibit 5

Caplet maturity (t_i) (years)	Antithetic Monte Carlo			Approximate analytical solution: Equation (38)		% error bounds	
	price	std. error	% std. error	price	% errors	lower	upper
	Panel A: Rebate with value R = 0.1% and barrier rate r_u = 4.331%						
0.25	2.2902	9.50E-04	0.04%	2.2677	-0.99%	-1.05%	-0.92%
0.5	1.9620	1.33E-03	0.07%	1.9396	-1.14%	-1.38%	-0.89%
0.75	1.7189	1.34E-03	0.08%	1.7084	-0.61%	-1.13%	-0.08%
1	1.5390	1.27E-03	0.08%	1.5403	0.08%	-0.81%	0.98%
1.25	1.4003	1.19E-03	0.08%	1.4109	0.76%	-0.58%	2.12%
1.5	1.2889	1.13E-03	0.09%	1.3069	1.40%	-0.46%	3.30%
1.75	1.1977	1.11E-03	0.09%	1.2204	1.90%	-0.56%	4.42%
Total Rebate	11.3970			11.3942	-0.02%	-0.91%	0.88%
CPU time	> 1 day			< 1 second			
	Panel B: Rebate with value R= 0.1% and barrier rate r_u = 6%						
0.25	2.4614	1.83E-05	0.00%	2.4614	0.00%	0.00%	0.00%
0.5	2.4367	1.86E-04	0.01%	2.4359	-0.03%	-0.06%	-0.01%
0.75	2.3938	4.15E-04	0.02%	2.3921	-0.07%	-0.16%	0.02%
1	2.3356	6.18E-04	0.03%	2.3336	-0.09%	-0.29%	0.13%
1.25	2.2691	7.82E-04	0.03%	2.2677	-0.06%	-0.41%	0.32%
1.5	2.1981	9.10E-04	0.04%	2.1992	0.05%	-0.50%	0.64%
1.75	2.1276	1.00E-03	0.05%	2.1308	0.15%	-0.63%	0.99%
Total Rebate	16.2225			16.2207	-0.01%	-0.28%	0.28%
CPU time	> 1 day			< 1 second			

Rebate prices are expressed as US\$100 per basis point. Monte Carlo prices are computed through 200,000 simulations, with antithetic variates, 520 discrete time-steps per year and 260 barrier' monitoring periods per year. % std. error is the standard error divided by the Monte Carlo option price estimate. All % errors were computed against the Monte Carlo price estimates. % error bounds are obtained by integrating the exact restricted density (30), with $\rho(t) = 1(-1)$ for a lower (upper) price bound.

CONCLUSIONS AND FURTHER RESEARCH

The main purpose and contribution of this paper consisted in deriving fast and accurate approximate analytical pricing formulas, under a multi-factor Gaussian (but not necessarily Markovian) HJM framework, for single-barrier caps and floors. Contrary to the quasi-analytical pricing approaches previously found in the literature for the contracts under analysis, error bounds were also provided for the proposed approximations and rebates have been explicitly considered.

In terms of future research, a straightforward extension of the pricing methodology developed in this paper would be to price lookback caps and floors, because a closed-form expression for the probability density function of the maximum or the minimum of log-inverse discount factors can be easily obtained from theorem 1. Finally, since most of the path-dependent options traded in the financial markets do not assume continuous monitoring, it is also important to adapt the proposed pricing solutions for discrete monitoring, through the approach suggested by Broadie et al. (1997). Such correction would most probably reduce the pricing errors reported in the numerical experiments presented in this paper.

A APPENDIX: PROOF OF PROPOSITION 3

From the definition of the covariance between any two random variables, equation (28) can be restated as

$$Q^{t_i+1}\Big\{\ln P^{-1}(t,t+\delta) \leq x,$$

$$sup_{t_0\leq s\leq t}[\theta \ln P^{-1}(s,s+\delta)] < \theta\ln(1+\delta r_H)|\mathcal{F}_{t_0}\Big\}$$

$$= \mathbb{E}^{Q^e}_{t_0}\{\exp[\varepsilon(t)]\}\mathbb{E}^{Q^e}_{t_0}[Y(t,x,\ln(1+\delta r_H))]$$

$$+ \rho(t)\sqrt{\mathbb{E}^{Q^e}_{t_0}\{\exp[2\varepsilon(t)]\} - (\mathbb{E}^{Q^e}_{t_0}\{\exp[\varepsilon(t)]\})^2}$$

$$\times \sqrt{V_Y(t,x,\ln(1+\delta r_H))} \tag{43}$$

where $V_Y(t,x,y)$ is the time-t_0 variance of $Y(t,x,y)$, and $\rho(t)$ is the (unknown) linear correlation coefficient between exp $[\varepsilon(t)]$ and $Y(t,x,\ln(1+\delta r_H))$.

All the expectations involving functions of the error term $\varepsilon(t)$, and contained on the right-hand-side of equation (43), can be easily obtained from the definition of the moment generating function of a univariate normal random variable, since ε(t) is normally distributed with zero mean and variance $V_\varepsilon(t)$. Concerning the expectation of the random variables Y(t,x,y) and $Y(t,x,y)^2$, Musiela and Rutkowski (1998, corollaries B.3.1 and B.3.3) imply that

$$\mathbb{E}^{Q^e}_{t_0}[Y(t,x,y)] = \int_{-\infty}^{x+\ln P(t_0,t_0+\delta)}$$

$$\exp\left[\sqrt{\frac{h(t_0,t)}{g(t_0,t)}}z - \frac{1}{2}h(t_0,t)\right]$$

$$\{\phi[z;0,\sqrt{g(t_0,t)}] - \phi[z;2(\ln P(t_0,t_0+\delta)+y),$$

$$\sqrt{g(t_0,t)}]\}dz \tag{44}$$

and

$$\mathbb{E}^{Q^e}_{t_0}[Y(t,x,y)^2] = \int_{-\infty}^{x+\ln P(t_0,t_0+\delta)}$$

$$\exp\left[2\sqrt{\frac{h(t_0,t)}{g(t_0,t)}}z - h(t_0,t)\right]$$

$$\{\phi[z;0,\sqrt{g(t_0,t)}] - \phi[z;2(\ln P(t_0,t_0+\delta)+y),$$

$$\sqrt{g(t_0,t)}]\}dz \tag{45}$$

Equations (44) and (45) can be easily converted into the analytical solutions (33) and (34), respectively, using, for instance, Ingersoll (1987, page 14, equation 69).

Finally, differentiating equation (43) with respect to x, the error bounds (30) are obtained. ■

B APPENDIX: PROOF OF PROPOSITION 4

The pricing solution (38) follows for the present value of the knock-in rebate, after combining the second term on the right-hand-side of equation (6) with the restricted density offered by theorem 1:

$$Q^{t_i+1}\Big\{sup_{t_0\leq s\leq t_i}[\theta \ln P^{-1}(s,s+\delta)]$$

$$< \theta\ln(1+\delta r_H)|\mathcal{F}_{t_0}\}$$

$$= \int_{\mathcal{R}} \mathbf{1}_{\{\theta x<\theta\ln(1+\delta r_H)\}}\Big\{\phi[x;\ln P^{-1}(t_0,t_0+\delta)$$

$$+ \sqrt{h(t_0,t_i)g(t_0,t_i)}, \sqrt{g(t_0,t_i)}]$$

$$- \exp\left[2\sqrt{\frac{h(t_0, t_i)}{g(t_0, t_i)}}\ln\frac{1 + \delta r_H}{P^{-1}(t_0, t_0 + \delta)}\right]$$
$$\phi[x; 2\ln(1 + \delta r_H) - \ln P^{-1}(t_0, t_0 + \delta)$$
$$+ \sqrt{h(t_0, t_i)g(t_0, t_i)}, \sqrt{g(t_0, t_i)}]\Big\}dx \quad (46)$$

Concerning the expectation contained in equation (6), two cases must be distinguished.

B.1 Case 1: $\theta r_H < \theta r_k$

If $\theta r_H < \theta r_k$, then $\theta \ln P^{-1}(t_i, t_{i+1}) \geq \theta\ln(1 + \delta r_k) \Rightarrow \theta \ln P^{-1}(t_i, t_{i+1}) \geq \theta\ln(1 + \delta r_H)$. Therefore,

$$IN(\theta)_{t_0}[r(t_i, t_{i+1}); r_k; \theta r_H < \theta r_k; R = 0; t_{i+1}]$$
$$= P(t_0, t_{i+1})\int_{-\infty}^{\infty} [\theta\exp(x) - \theta(1 + \delta r_k)]^+$$
$$\times \phi\left\{x; \ln\left[\frac{P(t_0, t_i)}{P(t_0, t_{i+1})}\right] + l(t_0, t_i), \sqrt{g(t_0, t_i)}\right\}dx$$
$$= REGULAR(\theta)_{t_0}[r(t_i, t_{i+1}); r_k; t_{i+1}] \quad (47)$$

B.2 Case 2: $\theta r_H \geq \theta r_k$

In this case, the exercise region $\theta \ln P^{-1}(t_i, t_{i+1}) \geq \theta\ln(1 + \delta r_k)$ can be located below or above the barrier level $\ln(1 + \delta r_H)$. Hence, theorem 1 and proposition 1 imply that:

$$IN(\theta)_{t_0}[r(t_i, t_{i+1}); r_k; \theta r_H \geq \theta r_k; R = 0; t_{i+1}]$$
$$= P(t_0, t_{i+1})\theta\int_{\ln(1+\delta r_k)}^{\ln(1+\delta r_H)} [\theta\exp(x) - \theta(1 + \delta r_k)]$$
$$\times \left\{\phi\left[x; \ln\frac{P(t_0, t_i)}{P(t_0, t_{i+1})} + l(t_0, t_i), \sqrt{g(t_0, t_i)}\right]\right.$$
$$- \phi[x; \ln P^{-1}(t_0, t_0 + \delta) + \sqrt{h(t_0, t_i)g(t_0, t_i)},$$
$$\sqrt{g(t_0, t_i)}] + \exp\left[2\sqrt{\frac{h(t_0, t_i)}{g(t_0, t_i)}}\ln\frac{1 + \delta r_H}{P^{-1}(t_0, t_0 + \delta)}\right]$$
$$\phi[x; \ln(1 + \delta r_H)^2 - \ln P^{-1}(t_0, t_0 + \delta)$$
$$+ \sqrt{h(t_0, t_i)g(t_0, t_i)}, \sqrt{g(t_0, t_i)}]\}dx$$
$$+ P(t_0, t_{i+1})\theta\int_{\ln(1+\delta r_H)}^{\theta\infty} [\theta\exp(x) - \theta(1 + \delta r_k)]$$
$$\times \phi\left[x; \ln\frac{P(t_0, t_i)}{P(t_0, t_{i+1})} + l(t_0, t_i), \sqrt{g(t_0, t_i)}\right]dx \quad (48)$$

Adding the first and last terms on the right-hand-side of equation (48), noticing that $\theta^2 = 1$, and applying again Ingersoll (1987, page 14, equation 69), equation (37) arises. ■

C APPENDIX: PROOF OF PROPOSITION 5

The second term on the right-hand-side of equation (7) generates the pricing solution (41), after noticing that $P(t_0, t_{i+1})\delta RQ^{t_{i+1}}\{\inf_{t_0 \leq s \leq t_i}[\theta \ln P^{-1}(s,s + \delta)] > \theta \ln(1 + \delta r_H)|\mathcal{F}_{t_0}\}$ is given by equation (38), with θ replaced by $-\theta$. Concerning the expectation contained in equation (7), two cases must be considered.

C.1 Case 1: $\theta r_H < \theta r_k$

In this case $\theta \ln P^{-1}(t_i, t_{i+1}) > \theta\ln(1 + \delta r_k) \Rightarrow \theta \ln P^{-1}(t_i, t_{i+1}) > \theta\ln(1 + \delta r_H)$, and therefore, the defective probability density offered by theorem 1 yields:

$$OUT(\theta)_{t_0}[r(t_i, t_{i+1}); r_k; \theta r_H < \theta r_k; R = 0; t_{i+1}]$$
$$= P(t_0, t_{i+1})\theta\int_{\ln(1+\delta r_k)}^{\theta\infty} [\theta\exp(x) - \theta(1 + \delta r_k)]$$
$$\Big\{\phi[x; \ln P^{-1}(t_0, t_0 + \delta) + \sqrt{h(t_0, t_i)g(t_0, t_i)},$$
$$\sqrt{g(t_0, t_i)}]$$
$$- \exp\left[2\sqrt{\frac{h(t_0, t_i)}{g(t_0, t_i)}}\ln\frac{1 + \delta r_H}{P^{-1}(t_0, t_0 + \delta)}\right]$$
$$\times \phi\left[x; -\ln\frac{P^{-1}(t_0, t_0 + \delta)}{(1 + \delta r_H)^2}\right.$$
$$\left.+ \sqrt{h(t_0, t_i)g(t_0, t_i)}, \sqrt{g(t_0, t_i)}\right]\Big\}dx \quad (49)$$

Using the symmetry property $\Phi(-x) = 1 - \Phi(x)$, and following Ingersoll (1987, page 14, equation 69), equation (40) arises, for $\theta r_H < \theta r_k$.

C.2 Case 2: $\theta r_H \geq \theta r_k$

Since the *down-and-out caplet* (*up-and-out floorlet*) finishes worthless if $\ln P^{-1}(t_i, t_i + \delta) \in [\ln(1 + \delta r_k), \ln(1 + \delta r_l)](\ln P^{-1}(t_i, t_i + \delta) \in [\ln(1 + \delta r_u), \ln(1 + \delta r_k)])$, applying again theorem 1 to equation (7),

$$OUT(\theta)_{t_0}[r(t_i, t_{i+1}); r_k; \theta r_H \geq \theta r_k; R = 0; t_{i+1}]$$

$$= P(t_0, t_{i+1})\theta \int_{\ln(1+\delta r_H)}^{\theta\infty} [\theta \exp(x) - \theta(1 + \delta r_k)]$$

$$\Big\{\phi[x; \ln P^{-1}(t_0, t_0 + \delta) + \sqrt{h(t_0, t_i)g(t_0, t_i)},$$

$$\sqrt{g(t_0, t_i)}] - \exp\left[2\sqrt{\frac{h(t_0, t_i)}{g(t_0, t_i)}}\ln\frac{1 + \delta r_H}{P^{-1}(t_0, t_0 + \delta)}\right]$$

$$\Big\{\phi\left[x; -\ln\frac{P^{-1}(t_0, t_0 + \delta)}{(1 + \delta r_H)^2} + \sqrt{h(t_0, t_i)g(t_0, t_i)},\right.$$

$$\left.\left.\sqrt{g(t_0, t_i)}\right]\right\}dx \qquad (50)$$

Comparing the right-hand side of equations (49) and (50), it follows that equation (50) is equivalent to the solution of equation (49) but with $d_1^{r_k}(t_0)$ and $d_2^{r_k}(t_0)$ replaced by $d_1^{r_H}(t_0)$ and $d_2^{r_H}(t_0)$, respectively, which agrees with equation (40), for $\theta r_H \geq \theta r_k$. ■

ENDNOTES

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[1]See, for instance, the historical notes contained in Zhang (1998, page 203).

[2]Hereafter, vectors will be identified using an underscore.

[3]Without loss of generality, and in order to simplify the notation, we shall assume that $t_{i+1} - t_i = \delta, \ \forall\ i$.

[4]For a *forward rate agreement* (FRA), the start and end dates of the underlying forward compounding period change every day (by moving forward one day). Unfortunately, for the contracts under valuation, t_i and t_{i+1} are fixed.

[5]A closed-form solution can also be obtained for a nondeferrable rebate by considering the first passage time distribution of the underlying nominal spot rate, which can be derived from theorem 1. However, since nondeferrable rebates are much less common in the money-market and because of space constraints, the valuation of such rebates will be left to future research. Nevertheless, formulae is available upon request.

[6]$\mathbb{E}_t^{\mathcal{P}}(X|Y)$ denotes the expected value of the random variable X, conditional on the event Y and on $\mathcal{F}_t$, and computed under the probability measure $\mathcal{P}$. Similarly, $\mathcal{P}(\omega|\mathcal{F}_t)$ denotes the probability of event ω, conditional on $\mathcal{F}_t$, and computed under the probability measure $\mathcal{P}$. Notice also that $sup_{t_0 \leq s \leq t_i}[-\ln P^{-1}(s, s+\delta)] = -\inf_{t_0 \leq s \leq t_i}[\ln P^{-1}(s, s+\delta)]$ and $\inf_{t_0 \leq s \leq t_i}[-\ln P^{-1}(s, s+\delta)] = -sup_{t_0 \leq s \leq t_i}[\ln P^{-1}(s, s+\delta)]$.

[7]Explicit formulae available upon request.

[8]As pointed out by Björk and Christensen (1999), the choice of the functional form describing the initially " observed" yield curve should be made *consistent* with the volatility specification adopted to the HJM model (2). Nevertheless and because a single cross-section of market prices is being considered, the subsequent empirical analysis ignores the fact that there exists no HJM formulation that is *consistent* with the Nelson and Siegel (1987) family of yield curves -as shown by Filipović (1999).

[9]The explicit solutions obtained for functions $g(t, t_i)$ and $h(t, t_i)$, under the volatility specification (42), can be provided upon request.

[10]This compares very favorably with, for instance, Driessen, Klaassen and Melenberg (2003, table 7).

[11]Because the proposed analytical pricing solutions are developed for multi-factor and non-Markovian HJM formulations, the benchmark used for the proxy of the true barrier option price was the Monte Carlo price estimate instead of the lattice-based approach.

[12]The discretization of the Fortet (1943) equation for the first passage time density involved 600 time-steps per year. Pascal code available upon request.

[13]Such line of attack will not be pursued here because our parameterization is not even sufficiently rich to fit well enough the market cap prices.

[14]Analytical formulae available upon request.

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