Efficient Analytical Cascade Calibration of the LIBOR Market Model with Endogenous Interpolation

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This article presents a new method for the analytical calibration of the LIBOR Market Model. We start from the swaptions cascade calibration algorithm (CCA) introduced by Brigo and Mercurio. The CCA is very fast, leads to the perfect recovery of market quotes, and induces a direct correspondence between market volatilities and model parameters. However, tests in the previous literature are affected by numerical problems. This leads us to perform first an analysis of the effects of different correlation structures on calibration, isolating key features that impact the regularity of the outputs, and hence detecting a range of cases leading to regular results.

Then we study the link between the remaining irregularities and the use in calibration of artificial data computed via interpolation between market quotations, and introduce a new calibration algorithm. This algorithm, called Endogenous Interpolation CCA, maintains all positive characteristics of cascade calibration but is based only on directly quoted market data. Empirical results show this new algorithm to be robust and efficient, a remarkable improvement over the basic CCA. Regular and financially significant calibration outputs and diagnostic structures are now normally obtained across general market situations and model parameterizations.

Finally, we use Monte Carlo simulation to investigate the reliability of the approximation underlying cascade calibration and consider the joint calibration including caps.

We consider the calibration of the LIBOR Market Model (LMM) introduced by, among others, Brace, Gatarek and Musiela [1997]. The LMM is consistent with the Black [1976] formula used as a standard in the cap market, and this allows an immediate automatic exact calibration to cap quotes. Conversely, the LMM distributional assumptions are not compatible with the Black formula as used in the swaption market, although numerical studies have shown the discrepancy to be practically negligible and pricing approximations for swaptions have been devised.

In spite of this, most swaption calibration methodologies proposed for the LMM focus on minimizing the overall pricing error on a panel of swaption quotes (see for example Schoenmakers and Coffey [2003], Alexander [2002], part of Brigo and Mercurio [2001], and part of Brigo, Mercurio and Morini [2005]). This implies calibration errors and possibly time-consuming optimization procedures, and due to the high number of parameters the calibration problem can be largely undetermined and the solution of the optimization can be of limited financial meaning. Such methodologies are very different from the automatic inversion calibration typical of the Black formula, the market standard which market models would like to incorporate and generalize. With a market model like the LMM, often the goal is pricing exotic products as consistently as possible with reference market prices, so that calibration errors are undesirable. The "unique solution," "immediate and exact" calibration typical of the basic Black formula appears an important feature of market models.

There is no trivial way to calibrate the LIBOR Market Model to swaptions, unlike the Swap Market Model.[1] However, since the LMM is also often used as a central model for pricing swaption-related products, the possibility of keeping an "exact immediate calibration" feature when calibrating the LMM to swaptions is of real interest.

This is the goal in the approach proposed in Brigo and Mercurio [2001, 2002] and Brigo, Mercurio and Morini [2005]. This cascade calibration algorithm (CCA) is based on the analytical inversion of an industry formula proven to be reliable in pricing swaptions with the LMM. Through closed-form formulas, it gives an analytical, practically instantaneous and unique solution to the calibration problem. By means of the LMM industry swaption volatility formula coupled with the Black formula, the CCA leads to the exact recovery of market prices, without calibration errors, and induces a direct correspondence between swaption market volatilities and LMM volatility parameters. However, earlier applications in the literature were affected by numerical problems. Calibration output included nonsensical parameters such as negative volatilities.

We concentrate on detecting and eliminating the reasons for these problems. We first perform a larger empirical and numerical analysis on the behaviour of the algorithm, detecting a dependence between the choice of the exogenous correlation structure and the appearance of numerical problems. This allows us to obtain calibration results which are financially significant and imply regular patterns.

We move then to analyzing the reasons for the remaining irregularities and introduce a new calibration algorithm, the endogenous interpolation cascade calibration algorithm (EICCA). This method maintains all the positive characteristics of the previous cascade calibration, while relying only on directly quoted market data. The empirical results show this algorithm to be robust and efficient, superior to the basic CCA. Regular and financially significant outputs are now obtained across a wide range of different correlation assumptions and with different market data sets. The financial realism and regularity of the results are assessed through the analysis of various diagnostic structures such as the implied evolution of the term structure of volatility and the so-called terminal correlation.

We also assess the reliability of the approximated pricing formula underlying cascade calibration by means of Monte Carlo simulation. Finally, we briefly consider opportunities and problems in the extension of our method to a joint calibration of caps and swaptions.

We note that the cascade calibration algorithms presented in this work are designed for the popular LMM as in Brace, Gatarek and Musiela [1997], a model which assumes lognormality under the pricing measure and thus is consistent with the Black formula and a flat volatility smile curve. However, the same principle of cascade calibration can be applied, with only minor modifications, also to some alternative models which automatically imply non-flat skew and smile in the volatility curve. Two examples are the displaced diffusion libor model and the version with random parameters. This issue is investigated in Morini, Brigo and Mercurio [2006].

The possibility, given by EICCA, to calibrate our data exactly, rapidly, and under analytical tractability is highly relevant for market applications. In particular, these features render EICCA one of the few realistic approaches for applying the LMM to risk management or hybrid products, usually requiring calibration to a large number of different securities and based mostly on at-the-money quotes. Further considerations on the relevance of this algorithm in specific applications are presented in the conclusions, after the analysis contained in this work.

The article is organized as follows. We begin in Section 2 by recalling background results and definitions regarding the LMM, including the industry swaption volatility formula, and the diagnostic structures on volatility and correlation. In Section 3 we describe the CCA, presenting both intuition and explicit algorithms. In Section 4 the exogenous instantaneous correlation matrices used are described. Real market calibration cases with different correlations are presented and analyzed in Section 5. Section 6 is devoted to explaining and presenting the new EICC algorithm and its empirical calibration results. Financial diagnostics of the results and of their evolution in time are presented in Section 7. Monte Carlo tests on the accuracy of the industry swaption approximation are in Section 8. In Section 9 we consider the cap market before concluding with a summary of the results and some observations for the practical application of the methods presented.

THE LIBOR MARKET MODEL

We define a sequence of dates

$$T_0 < T_1 < \cdots < T_M < T_{M+1}$$

called the tenor structure of the model, which represents the expiry and maturity dates for the spanning forward rates in the model. The current time is $T_0 = 0$. For

$k = 0, \ldots, M$, $F_k(t)$ denotes the simple compounded forward LIBOR rate resetting at T_k (expiry) and with maturity T_{k+1},

$$F_k(t) = F(t; T_k, T_{k+1}) = \frac{1}{\tau_k}\left[\frac{P(t, T_k)}{P(t, T_{k+1})} - 1\right]$$

where $P(t, T)$ is the value at t of a zero-coupon bond with maturity T, and τ_k is the year fraction between T_k and T_{k+1}. Notice that $F_0(T_0)$ is the current spot LIBOR rate with maturity T_1 so that we are left with M forward LIBOR rates for which stochastic dynamics need to be specified.

Now we introduce the LMM dynamics, specified via the instantaneous volatility functions of the forward rates and the instantaneous correlations of the random shocks. The LMM is often equivalently specified using a vector of uncorrelated random shocks, but for the algorithms presented in this work it is clearer and more intuitive to consider correlated random shocks, so as to show separately the volatility and correlation parameters.

Let Q^{k+1} be the T_{k+1} forward adjusted probability measure which is the equivalent martingale measure associated with the numeraire $P(t, T_{k+1})$. By no-arbitrage arguments, we know that under this measure $F_k(t)$ is a martingale.

The LMM assumes lognormal dynamics for $F_k(t)$ under Q^{k+1}:

$$dF_k(t) = \sigma_k(t)F_k(t)dZ_k^{k+1}(t),\ t \leq T_k \qquad (1)$$

where $Z_k^{k+1}(t)$ is the k^{th} component of an M-dimensional standard Brownian Motion $Z^{k+1}(t)$ under Q^{k+1}.

The instantaneous correlation matrix of $Z^{k+1}(t)$ is $\rho = (\rho_{i,j})_{i,j=1,\ldots M}$, with

$$dZ_i^{k+1}(t)dZ_j^{k+1}(t) = \rho_{i,j}dt$$

The time function $\sigma_k(t)$ gives the instantaneous volatility at time t for the forward LIBOR rate $F_k(t)$. We consider in the following a general piecewise-constant parameterization (GPC) of the instantaneous volatilities, according to which $\sigma_k(t)$ is constant in-between tenor dates: $\sigma_k(t)$ takes the constant value $\sigma_{k,m}$ in the interval $T_{m-1} < t \leq T_m$. We also set the initial value $\sigma_k(0) = \sigma_{k,1}$.

The LMM Industry Swaption Formula and the SMM

A $k \times n$ ("k into n") swaption is an option to enter an n-period swap with fixed rate X at the first reset date T_k, called the maturity of the swaption. The swap rate corresponding to a swap starting at T_k (first reset date) and ending after n periods at T_{k+n} (last payment date) is denoted by $S_{k\times n}$. By no-arbitrage relationships, it can be expressed as a weighted combination of forward rates,

$$S_{k\times n}(t) = \sum_{i=k}^{k+n-1} w_i^{(k\times n)}(t)F_i(t)$$

where $w_i^{(k\times n)}(t)$ are the weights, given by the following functions of the forward rates themselves:

$$w_i^{(k\times n)}(t) = \frac{\tau_i \prod_{j=k}^{i} 1/(1 + \tau_j F_j(t))}{\sum_{s=k}^{k+n-1} \tau_s \prod_{j=k}^{s} 1/(1 + \tau_j F_j(t))}$$

By no-arbitrage arguments, we know that the swap rate $S_{k\times n}$ is a martingale under the equivalent martingale measure associated with the numeraire $A_{k\times n}(t) = \sum_{i=k}^{k+n-1} P(t, T_{i+1})\tau_i$, here called the swap measure $Q^{k\times n}$. The swap market model (SMM) assumes the following lognormal dynamics for the swap rate $S_{k\times n}$ under $Q^{k\times n}$:

$$dS_{k\times n}(t) = \sigma_{k\times n}(t)S_{k\times n}(t)dW_t^{k\times n} \qquad (2)$$

where $\sigma_{k\times n}(t)$ is the instantaneous volatility function of the swap rate $S_{k\times n}(t)$ and $W_t^{k\times n}$ is a standard Brownian Motion under $Q^{k\times n}$.

As a consequence of the lognormality assumption, in the SMM plain-vanilla swaptions are consistently evaluated with the swaption market version of the Black [1976] formula. So the price $\Pi_t^{k\times n}$ of a k into n swaption is

$$\Pi_t^{k\times n} = A_{k\times n}(t)[S_{k\times n}(t)\mathrm{N}(d) - X\mathrm{N}(d - v)]$$

$$d = \left[\ln\left(\frac{S_{k\times n}(t)}{X}\right) + \frac{v^2}{2}\right] \Big/ v$$

$$v = \sqrt{\int_t^{T_k} \sigma_{k\times n}^2(s)ds}$$

with $N(\cdot)$ = cumulative normal distribution function. When using the LMM, swaptions can be priced with the Black formula only via approximation, since according to the assumptions of the LMM the swap rates are not lognormal under their natural pricing measure, which is the swap measure. However, the approximations based on assuming a lognormal swap rate in the LMM are very accurate, since the distance of the LMM distribution of a swap rate from the lognormal one is very small (see Brace, Dun and Barton [2001] or Brigo and Liinev [2005].)

The use of the swaption Black formula in the LMM requires the computation of a swap-rate implied volatility, to be used as the input of the formula. One simple industry approximation of the swap-rate implied volatility in the LMM is based on observing first that the variability of the weights $w_i^{(k\times n)}(t)$ is practically negligible compared to the variability of the $F_i(t)$, and thus fixing the weights to their initial values:

$$S_{k\times n}(t) \simeq \sum_{i=k}^{k+n-1} w_i^{(k\times n)}(0)F_i(t)$$

The above approximated relationship allows computation via Ito's formula of the average percentage volatility of the swap rate $S_{k\times n}$ in the LMM, which for the period from 0 to T_k is given by

$$v_{k\times n}^2 = \sum_{i=k}^{k+n-1}\sum_{j=k}^{k+n-1} \frac{w_i^{(k\times n)}(0)w_j^{(k\times n)}(0)F_i(t)F_j(t)\rho_{i,j}}{T_k\left(\sum_{i=k}^{k+n-1} w_i^{(k\times n)}(0)F_i(t)\right)^2} \cdot \int_0^{T_k}\sigma_i(t)\sigma_j(t)\,dt$$

Based on the fact that the variability of forward rates in the numerator is partially offset by the variability of those in the denominator, one can now "freeze" the forward rates at their initial values, obtaining an approximated input $v_{k\times n}$ for pricing the k into n swaption with the Black formula in the LMM. Comparison with numerical prices confirms this formula to be very accurate. See Rebonato [1998] or Brigo and Mercurio [2001] for more details on both derivation and testing of the formula. Further tests on the reliability of this formula under the conditions typical of cascade calibration are in Section 8 below.

In the case of GPC parameterization, and setting $F_i = F_i(0)$, $S_{k\times n} = S_{k\times n}(0)$, $w_i = w_i^{(k\times n)}(0)$ for ease of notation, the swaption industry formula reads

$$v_{k\times n}^2 = \sum_{i=k}^{k+n-1}\sum_{j=k}^{k+n-1}\frac{w_iw_jF_iF_j\rho_{i,j}}{T_kS_{k\times n}^2}\sum_{h=0}^{k-1}\tau_h\sigma_{i,h+1}\sigma_{j,h+1} \quad (3)$$

Volatility and Correlation Structures

When calibrating a model rich in parameters like the LMM, and in particular when using non-parametric, unconstrained calibration, it is fundamental to check whether the results are financially significant. In particular, it is desirable to have regular and stable volatilities and correlations implied by the model parameters for future times. This diagnostic is often performed by analyzing the *evolution of the term structure of volatility* and the *terminal correlation*. These two concepts are briefly reviewed below.

The implied term structure of volatility (TSV) at time $t = T_a$ is the plot of, the average instantaneous volatility for the period from T_a to the expiry T_k, for each forward rate $F_k(t)$, $k = a+1, \dots, M$. Thus it is the vector

$$[V_k(T_a)]_{k=a+1,\dots,M}$$

$$V_k(T_a) = \sqrt{\frac{1}{T_k - T_a}\int_{T_a}^{T_k}\sigma_k^2(t)\,dt}$$

The evolution of the term structure of volatility is the evolution of this vector as the observation time T_a is moved forward. Once the model has been calibrated, one would like to obtain a realistic (i.e., consistent with usual market patterns), smooth and qualitatively stable evolution of the TSV. Various examples computed on market data follow.

The terminal correlation $TC_{i,j}$ is the correlation, as implied by the model, between the forward rates F_i and F_j at a future time instant T_γ. It can be computed via Monte Carlo simulation, or through efficient formulas based on the same approximations as the swaption industry formula seen above. One such formula gives the following matrix

$$[TC_{i,j}(T_\gamma)]_{i,j=1,\dots,M}$$

$$TC_{i,j}(T_\gamma)$$

$$= \frac{\exp(\int_0^{T_\gamma}\sigma_i(t)\sigma_j(t)\rho_{i,j}\,dt) - 1}{\sqrt{\exp(\int_0^{T_\gamma}\sigma_i^2(t)\,dt) - 1}\sqrt{\exp(\int_0^{T_\gamma}\sigma_j^2(t)\,dt) - 1}}$$

as from Brigo and Mercurio [2001], where tests against Monte Carlo simulation to show its accuracy are also reported.

After calibrating the model we will check if the terminal correlation is regular and consistent with the typical properties of forward-rate correlation matrices, which are reviewed in Section 4. In particular, our goal is to preserve the same properties observed for the instantaneous correlation matrix.

CASCADE CALIBRATION

We describe the cascade calibration starting with the standard $s \times s$ swaption matrix used in the market to organize the at-the-money swaption volatilities of interest to traders. Each row is indexed by the swaption maturity k and each column by the underlying swap length n. Assume, to simplify notation, that maturities and lengths are multiples of one year. An example of this matrix is shown in Exhibit 1. In the following, the volatility of a forward rate $F_i(t)$ during the year between a swaption payment date T_j and the following date T_{j+1} is often called a volatility- or σ-"bucket" as an analogy to the terminology used in term structure analysis.

Basic Cascade Calibration

Note that a triangular part of the matrix is indicated in bold in Exhibit 1. It includes swaptions $k \times n$ where $n + k \leq s + 1$. The cascade calibration is developed starting from a simple observation. In a LIBOR-based model, if we sequentially consider these bold-faced swaptions first for increasing length and then for increasing maturity, each time we move to evaluate a new swaption we need only one new piece of information on the term structure of volatility of forward rates: the volatility of a single forward rate in a single year, i.e., one volatility bucket. All remaining volatility information required for pricing this swaption has already been incorporated in the evaluation of previous swaptions.

This observation is a general fact independent of the model parameterization of instantaneous volatility. However, if volatility is chosen to be piecewise constant, or GPC, as in Section 2, the above observation implies that only one new volatility parameter is required to price one new swaption. Thus, the GPC parameterization can appear a natural choice if one is interested in "volatility bucketing," namely in precise calibration of LMM parameters to each swaption volatility quote.

This calibration should, in general, be performed via standard optimization, with all typical problems of very high dimensional optimization. However, we point out that when the industry formula of Equation (3) is used the calibration of the LMM to swaptions can be solved analytically. In fact, given correlations, the industry formula can be inverted to find the unique set of GPC volatility parameters of the LMM exactly consistent with market swaption prices. The cascade calibration performs this inversion, and finds the LMM parameters by solving an immediate cascade of second-order algebraic equations. We now describe in detail the above concepts and describe the workings of cascade calibration, before giving the detailed algorithm. Later we analyze its behaviour in an empirical application.

Notice first that the CCA requires complete swaption matrices, namely quotations for each and every maturity and length in the range considered. Therefore, the missing quotations of the Euro market (6,8 and 9 year maturities) are replaced in Exhibit 1 by linear interpolation between the adjacent market values on the same column. They are indicated in italics.

Suppose for simplicity that the swaption matrix is 3×3, like in the explanatory example of Exhibit 2. Start from the 1 into 1 swaption with implied volatility $\nu_{1\times1}$. The price of this swaption depends only on the volatility of the forward rate F_1 one year from now, namely only on $\sigma_{1,1}$. Therefore, $\sigma_{1,1}$ can be set to calibrate exactly the 1 into 1 swaption.

Next, consider swaptions of increasing length for the same maturity. When one moves to the 1 into 2 swaption the volatility of F_2 in the first year also matters for pricing. Thus the price depends on $\sigma_{1,1}$, obtained previously, and on $\sigma_{2,1}$ which can now be fixed to calibrate exactly $\nu_{1\times2}$. Analogously, for $\nu_{1\times3}$, also F_3 enters the pic-

EXHIBIT 1

Implied Volatilities of At-The-Money Swaptions (February 1, 2002)

	1	2	3	4	5	6	7	8	9	10
1	**17.90**	**16.50**	**15.30**	**14.40**	**13.70**	**13.20**	**12.80**	**12.50**	**12.30**	**12.00**
2	**15.40**	**14.20**	**13.60**	**13.00**	**12.60**	**12.20**	**12.00**	**11.70**	**11.50**	11.30
3	**14.30**	**13.30**	**12.70**	**12.20**	**11.90**	**11.70**	**11.50**	**11.30**	11.10	10.90
4	**13.60**	**12.70**	**12.10**	**11.70**	**11.40**	**11.30**	**11.10**	10.90	10.80	10.70
5	**12.90**	**12.10**	**11.70**	**11.30**	**11.10**	**10.90**	10.80	10.60	10.50	10.40
6	***12.50***	***11.80***	***11.40***	***10.95***	***10.75***	*10.60*	*10.50*	*10.40*	*10.35*	*10.25*
7	**12.10**	**11.50**	**11.10**	**10.60**	10.40	10.30	10.20	10.20	10.20	10.10
8	***11.80***	***11.20***	***10.83***	*10.40*	*10.23*	*10.17*	*10.10*	*10.10*	*10.07*	*10.00*
9	***11.50***	***10.90***	*10.57*	*10.20*	*10.07*	*10.03*	*10.00*	*10.00*	*9.93*	*9.90*
10	**11.20**	10.60	10.30	10.00	9.90	9.90	9.90	9.90	9.80	9.80

EXHIBIT 2
Swaption-Volatility Dependencies for CCA

	1y	2y	3y
1y	$v_{1\times 1}$ $\boxed{\sigma_{1,1}}$	$v_{1\times 2}$ $\sigma_{1,1}, \boxed{\sigma_{2,1}}$	$v_{1\times 3}$ $\sigma_{1,1}, \sigma_{2,1}, \boxed{\sigma_{3,1}}$
2y	$v_{2\times 1}$ $\sigma_{2,1}, \boxed{\sigma_{2,2}}$	$v_{2\times 2}$ $\sigma_{2,1}, \sigma_{2,2}, \sigma_{3,1}, \boxed{\sigma_{3,2}}$	$v_{2\times 3}$ $\sigma_{2,1}, \sigma_{2,2}, \sigma_{3,1}, \sigma_{3,2}, \boxed{\sigma_{4,1}, \sigma_{4,2}}$
3y	$v_{3\times 1}$ $\sigma_{3,1}, \sigma_{3,2}, \boxed{\sigma_{3,3}}$	$v_{3\times 2}$ $\sigma_{3,1}, \sigma_{3,2}, \sigma_{3,3}, \sigma_{4,1}, \sigma_{4,2}, \boxed{\sigma_{4,3}}$	$v_{3\times 3}$ $\sigma_{3,1}, \sigma_{3,2}, \sigma_{3,3}, \sigma_{4,1}, \sigma_{4,2}, \sigma_{4,3}, \boxed{\sigma_{5,1}, \sigma_{5,2}, \sigma_{5,3}}$

ture with its volatility in the first year $\sigma_{3,1}$, which is now the only unknown and can be used for calibrating exactly the market volatility $v_{1\times 3}$.

Now move to the next maturity, starting from swaption 2 into 1. The value of this swaption depends on the volatility of F_2 from now to the maturity of the swaption, which is in 2 years; namely, it depends on $\sigma_{2,1}$, already known, and on $\sigma_{2,2}$ which can be fixed now to match the swaption market volatility $v_{2\times 1}$. After $v_{2\times 1}$ continue considering swaptions of increasing length inside the upper part of the swaption matrix, then move to the next row, and so forth.

In Exhibit 2 we show the volatility parameters needed for pricing each swaption $v_{k\times n}$. Moving as described above inside the upper part of the swaption matrix, notice that each time a new swaption volatility $v_{k\times n}$ is considered, only a single new volatility bucket $\sigma_{k+n-1,k}$ is required for pricing. It is indicated with a box in Exhibit 2. Inverting Equation (3), $\sigma_{k+n-1,k}$ can be computed by solving a second order equation. This is what cascade calibration does. Below we detail the analytic algorithm, performing exact calibration only to the upper part of the swaption matrix. Notice that in the algorithm the index k follows the swaption maturity (the row index) while n follows the underlying swap length (the column index). Notice also that F_{k+n-1} denotes the last forward rate paid in a $k \times n$ swap.

Algorithm 1: CCA for $\mathbf{k \times n}$ swaptions with $\mathbf{k + n \leq s + 1}$.

1. Set the swaption maturity $k = 1$.

2. Set the underlying swap length $n = 1$.

3. Solve in $\sigma_{k+n-1,k}$

$$A_{k,n}\sigma^2_{k+n-1,k} + B_{k,n}\sigma_{k+n-1,k} + C_{k,n} = 0 \tag{4}$$

where

$$A_{k,n} = w^2_{k+n-1}F^2_{k+n-1}\tau_{k-1}$$

$$B_{k,n} = 2\sum_{j=k}^{k+n-2} w_{k+n-1}w_jF_{k+n-1}F_j\rho_{k+n-1,j}\tau_{k-1}\sigma_{j,k}$$

$$C_{k,n} = \sum_{i,j=k}^{k+n-2} w_iw_jF_iF_j\rho_{i,j}\sum_{h=0}^{k-1}\tau_h\sigma_{i,h+1}\sigma_{j,h+1}$$

$$+2\sum_{j=k}^{k+n-2} w_{k+n-1}w_jF_{k+n-1}F_j\rho_{k+n-1,j}$$

$$\cdot\sum_{h=0}^{k-2}\tau_h\sigma_{k+n-1,h+1}\sigma_{j,h+1}$$

$$+w^2_{k+n-1}F^2_{k+n-1}\sum_{h=0}^{k-2}\tau_h\sigma^2_{k+n-1,h+1} - T_kS^2_{k\times n}v^2_{k\times n}$$

and take the greater solution.

4. Set $n = n + 1$. *If* $k + n \leq s + 1$, *go to point 3, otherwise* $k = k + 1$;

5. If $k \leq s$ *go to point 2, otherwise stop.*

Calibrating the Entire Swaption Matrix

Notice that the above algorithm would fail if applied outside the upper part of the swaption matrix indicated in bold in Exhibit 1. In fact, when we move to the lower part, we find out that for pricing a new swaption we need more than one new volatility bucket.

For example, observe what happens in Exhibit 2 when we consider the first entry in the lower part of the swaption matrix, namely the 2 into 3 swaption. Now F_4 comes into play for the first time, and what matters is its volatility up to the 2-year maturity, namely volatility buckets $\sigma_{4,1}$ and $\sigma_{4,2}$. We have no information on these volatility buckets coming from previous available swaptions. Hence, we have two unknowns in the calibration equation and there is no market information to discriminate exactly between them. We can, however, continue the calibration by considering a relationship between them, the most natural is simply to assume they are equal. Namely, we assume that F_4 has constant volatility before the maturity of the 2 into 3 swaption, which is the first available market swaption involving F_4.

This indeterminacy is again a general fact regarding the volatility information contained in a standard swaption matrix, and as such it influences all LMM calibrations, but it is clearly revealed only in analytic calibration. The explicit assumptions we make here on the volatility of the forward rates are implicit (and often unnoticed) when calibrating with standard optimization.

With the above choice, the calibration equation for $v_{2\times 3}$ has a single unknown and can be solved exactly, finding a value for $\sigma_{4,1} = \sigma_{4,2}$ consistent with market quotations. Then we continue to move top down and from left to right, and as shown in Exhibit 2 we have multiple unknowns again when we reach $v_{3\times 3}$. We can apply the same solution used above, assuming constant volatility for F_5 when we have no market quotations allowing a precise discrimination. If we adopt analogous assumptions every time we reach a swaption in the last column in the swaption matrix, then we can extend the calibration to an entire swaption matrix of any dimension.

This analysis of the information contained in a swaption matrix about the volatility of forward rates will be repeated and extended in Section 6 for solving a different problem. It motivates the following extension of Algorithm 1 and details the assumptions that are necessary for extending analytic calibration to the lower part of the swaption matrix. The extended algorithm, that is introduced in explicit form, applies to the entire swaption matrix.

Algorithm 2: Extension of Algorithm 1 to all k $\times$ n swaptions, k,n $\leq$ s. Modify point 4 of Algorithm 1 as follows:
4. Set $n = n + 1$. If $n < s$, go to point 3; otherwise, that is, when one reaches an entry on the last column, assume that the multiple unknowns are all equal such that

$$\sigma_{k+n-1,k} = \sigma_{k+n-1,k-1} = \cdots = \sigma_{k+n-1,1}$$

Solve now for $\sigma_{k+n-1,k}$ using the equation

$$A^*_{k,n}\sigma^2_{k+n-1,k} + B^*_{k,n}\sigma_{k+n-1,k} + C^*_{k,n} = 0 \qquad (5)$$

where

$$A^*_{k,n} = w^2_{k+n-1}F^2_{k+n-1}\sum_{h=0}^{k-1}\tau_h$$

$$B^*_{k,n} =$$

$$2\sum_{j=k}^{k+n-2} w_{k+n-1}w_jF_{k+n-1}F_j\rho_{k+n-1,j}\sum_{h=0}^{k-1}\tau_h\sigma_{j,h+1}$$

$$C^*_{k,n} = \sum_{i,j=k}^{k+n-2} w_iw_jF_iF_j\rho_{i,j}$$

$$\cdot\sum_{h=0}^{k-1}\tau_h\sigma_{i,h+1}\sigma_{j,h+1} - T_kS^2_{k\times n}v^2_{k\times n}$$

Then set $k = k + 1$.

The cascade calibration is analytical and therefore has a unique, immediate and exact solution. However, no constraints can be set on the output. And the output of empirical applications in the earlier literature did not satisfy the obvious constraints of non-negativity for $\sigma_{k,m}$ and $\sigma^2_{k,m}$ $\forall$ k, m, even when calibrating only the upper triangular part of the swaption matrix.

See the examples in Exhibit 8. On the left we report the calibration results from Brigo and Mercurio [2001] where a correlation computed from previous calibrations is used and negative volatilities appear in the solution of the calibration problem. On the right, we report a different more recent calibration case in which the exogenous correlation is a historical estimate. By NN we indicate volatility parameters for which the calibration algorithm could not find a real number as the solution. In this case, the first problem arises in computing $\sigma_{7,7}$ for which the algorithm finds the imaginary value $0.112i$, where $i^2 = -1$. Clearly, either negative or imaginary volatilities simply mean that there is no acceptable volatility parameter able to accommodate market data exactly.

This could be due to misalignments, noise and inconsistencies in the input data used as suggested by Brigo and Mercurio [2001] who showed improved results after modifying and smoothing market data. However, such results could also lead to a suspicion of an unavoidable inconsistency between the model specified for cascade calibration (or even the LMM itself) and the way swaption prices are actually determined on the market. Hence, in the following, we concentrate on detecting the reasons for these problems, seeing whether it is possible to rule them out without altering market data.

The first step is testing the behavior of the algorithm with a general range of realistic exogenous instantaneous correlation matrices. Therefore, we describe now the correlation matrices typically used.

THE EXOGENOUS CORRELATION MATRIX

In cascade calibration the $M \times M$ instantaneous correlation matrix ρ is an exogenous input, so that we can use matrices consistent with market patterns, such as identified via historical estimation. The historical estimation is based on daily time series of past interest-rate curves, observed under the real world probability measure. In fact, due to the Girsanov theorem, instantaneous correlations do not depend on the probability measure under which we are specifying the dynamics. In estimating forward rate correlations for the LMM, we took into account, through interpolation, the fact that forward rates in the LMM have a fixed maturity, contrary to market quotations, where a fixed time-to-maturity is usually considered as time passes. Starting from the usual joint Gaussian approximation for $\Delta \ln F_k$, $k = 1, \ldots, M$, our estimation of correlation is based on sample mean and covariance for Gaussian variables.

A peculiar feature of forward rate correlations is decorrelation, the reduction in correlation as the distance between maturities increases. Graphically it means that the entries of the matrix columns decrease when moving downwards from the main diagonal. In our estimates this natural and intuitive tendency is clearly visible. Less apparent is a second possible feature, namely the increase in interdependency between equally spaced forward rates as their maturities increase, or sub-diagonal trend.

Parametric Forms for Correlations

Crude historical estimation of correlation matrices can be influenced by various irregularities in the time series, which can be undesirable when correlations are to be used in a financial model. As an alternative, parametric forms designed to remain regular with good properties have been proposed in the literature. Accordingly, for cascade calibration, besides crude historical estimates, we consider also the use of parametric forms approximating the main features of the estimates. The synthetic forms we use are mainly:

- *Schoenmakers and Coffey [2003] two-parameter form:*

$$\rho_{i,j} = \exp\left[-\frac{|i-j|}{M-1}\left(-\ln \rho_\infty + \eta \frac{i^2+j^2+ij - 3Mi-3Mj+3i+3j+2M^2-M-4}{(M-2)(M-3)} \right)\right]$$

where $\rho_\infty = \rho_{1,M}$ and $0 < \eta < -\ln \rho_\infty$.

This is a sub-parameterization of a general positive semidefinite form automatically featuring decorrelation and sub-diagonal trend. The next form also allows for the latter two properties, but it is not automatically positive semidefinite. Thus one has to verify a posteriori that this necessary property is satisfied.

- *Rebonato [1999] three-parameter form:*

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty)e^{[-|i-j|(\beta - \alpha(\max(i,j)-1))]}$$

To fix the parameters while taking into account the major features of the historical matrix, we use a simple method. We invert the functional form of the parametric structures, expressing the parameters as functions of specific entries of the historically estimated matrix, so that those entries will be exactly reproduced in the resulting parametric matrices. Since they are based on "pivot points" in the target matrix, outputs will be referred to as pivot matrices. We consider pivot points which are relevant in view of the two typical properties of forward rate correlations mentioned above. We show below the relationships obtained for the above forms.

- *Rebonato [1999] three-parameter form (RE3):*

$$\left(\frac{\rho_{1,M} - \rho_\infty}{1 - \rho_\infty}\right) = \left(\frac{\rho_{M-1,M} - \rho_\infty}{1 - \rho_\infty}\right)^{(M-1)}$$

to solve in ρ_∞,

$$\alpha = \frac{\ln\left(\frac{\rho_{1,2} - \rho_\infty}{\rho_{M-1,M} - \rho_\infty}\right)}{2 - M}, \; \beta = \alpha - \ln\left(\frac{\rho_{1,2} - \rho_\infty}{1 - \rho_\infty}\right)$$

with pivot points $\rho_{1,2}$, $\rho_{1,M}$ and $\rho_{M-1,M}$. They are the vertices of the lower triangle of the $M \times M$ correlation matrix, embedding basic information on decorrelation and sub-diagonal trend. For a two-parameter matrix we use $\rho_{1,2}$ and $\rho_{1,M}$, as below.

- *Schoenmakers and Coffey [2003] two-parameter form (S&C2):*

$$\eta = \frac{(-\ln \rho_{1,2})(M-1) + \ln \rho_\infty}{2}, \; \rho_\infty = \rho_{1,M}$$

If the optimal approximation of the initial matrix is not the only goal, the pivot technique has some appealing features: 1) it can be rapidly carried out by simple formulas; 2) if the pivot points are chosen appropriately, it replicates effectively the major relevant characteristics of the historical estimate; 3) it smooths the negative effects of irregularities and outliers in historical estimations; and 4) since parameters are expressed in terms of key correlation entries, they have a clear intuitive meaning. Hence, one can easily modify the pivot matrix as is often needed on the market to incorporate personal views, recent market shifts, or to carry out scenario analysis for risk management and hedging purposes. This can be done by simply modifying the value of the relevant pivot points. The matrix will vary consistently, as desired, keeping all the regularity properties of the parametric form.

In analyzing the CCA, we can use this feature to assess how results change when some basic feature of the correlation structure is altered. We also mention that when parameters are fixed through the pivot method, the approximation error, appears surprisingly close to the approximation error in an optimal fitting of the parameterization to the whole historical matrix. See the results in Exhibit 3 of approximating historical correlations computed on one year of data preceding February 1, 2002 with the Schoenmakers and Coffey [2003] parametric form with three parameters. As metrics we consider first the root mean squared error ($\sqrt{MSE}$) and then the analogous metric with errors expressed as fractions of the corresponding element of the historical matrix ($\sqrt{\%MSE}$). The latter can be particularly relevant in this context. In fact, due to the differences in magnitude of the correlation entries, the relative importance of the discrepancies can be more informative than their absolute value.

Rank Reduction Methods

Both historically estimated and freely calibrated instantaneous correlation matrices typically have full rank M, which is the number of forward rates in the model. In a model with a Gaussian structure of the stochastic shocks, like the LMM, the rank of ρ equals the number of independent stochastic factors actually driving the dynamics. When the number of factors is high, numerical and simulation methods often used to price exotic derivatives become computationally burdensome, while correct prices could often be efficiently expressed with a low factor model. Therefore, in market applications it is usually desired to replace a full rank exogenous correlation by an r-rank approximation, so as to move to a model depending on only $r < M$ independent stochastic shocks. This step is often called rank reduction.

We use two rank reduction methods. The first method is the *angles parameterization* rank reduction technique described in Rebonato and Jäckel [2000]. It is based on parameterizing the correlation matrix through trigonometric functions of a matrix Θ of angles θ. The parametric form for an $M \times M$ correlation matrix is BB', where B is $M \times r$ and given by

$$B_{i,1} = \cos \theta_{i,1}$$

$$B_{i,k} = \cos \theta_{i,k} \sin\theta_{i,1} \cdots \sin \theta_{i,k-1},\ 1 < k < r$$

$$B_{i,r} = \sin\theta_{i,1} \cdots \sin \theta_{i,r-1}$$

This parameterization guarantees BB' to be a correlation matrix of the given rank r. Using this parameterization, the rank reduction can be achieved by the unconstrained optimization

$$\min \| \rho - B'(\Theta)(B'(\Theta))' \|$$

finding the values for the angles θ that minimize a chosen approximation error. In our tests we consider the mean square error.

The second method we use is the *eigenvalue zeroing by iteration* of Morini and Webber [2006], which is given by the following two-step iteration:

1) Set to zero the $M - r$ smallest eigenvalues in the spectral decomposition of the correlation matrix ρ. The matrix obtained, $\tilde{\rho}$, is the best reduced rank approximation of ρ according to least squares, but its diagonal values may be different from 1, so it may not be a correlation matrix anymore.

2) Re-establish unit values in the diagonal of $\tilde{\rho}$ so that the resulting matrix, $\overline{\rho}$, is the closest correlation matrix approximating $\tilde{\rho}$. However, it may now be a full rank matrix. Then restart from Step 1 replacing ρ with $\overline{\rho}$, and iterate until a preferred stopping condition is true. See Morini and Webber [2006] for more details.

This method can be used to obtain in a few iterations a reduced-rank correlation matrix accurately approximating the initial matrix, being in practice more efficient than the angles parameterization method. It can be seen as alternating projections into the set of reduced rank matrices and the set of correlation matrices. An application of this principle to correlation rank reduction can be found also in Grubisic [2002].

EXHIBIT 3
Parametric Approximation of Historical Correlations

	Fitting of 171 correlations	Pivot on 3 correlations
√MSE	0.108434	0.173554
√%MSE	0.25949	0.30890

EMPIRICAL AND NUMERICAL STUDIES ON THE CCA

Here we briefly report the results of our tests on the basic CCA. Swaption and yield curve data are from February 1, 2002. Historical estimations of correlations are based on one year of data preceding February 1, 2002. These results illustrate both the potential and the limitations of CCA, and lead us to the introduction of a more efficient approach to cascade calibration in Section 6.

Our focus is the effect on upper triangular calibration of the modeling choices made on the exogenous correlation ρ. We consider different parametric configurations and ranks for the correlation matrix. We start by RE 3 pivot computed with respect to historical estimations.

The first finding is that no numerical problems appeared in tests considering low rank ($r \leq 4$), while nonsensical parameters appeared in tests considering higher rank.

Lowering the rank of a correlation matrix imposes an oscillating tendency to the columns, leading to a sigmoid-like shape for very low ranks. This is shown in Exhibit 5 for RE3. This implies, in particular, that higher rank correlation matrices feature a stronger and less smooth initial decorrelation. In matrix terms, this means that they feature columns with a steeper slope initially when beginning with the unitary element on the principal diagonal.

To verify the relevance of this feature to the appearance of σ's with weird configurations, we made further tests, using different correlation matrices and synthetic forms whose essential features can be easily modified and controlled. Results can be summarized by saying that we find numerical problems for correlations featuring stronger and faster decorrelation (initially steeper columns)—such as full rank estimated matrices—while the configurations characterized by milder initial steepness—such as matrices implying a lower-factor model—lead to real and positive volatilities. A detailed description of these tests can be found in Morini [2002] and in Brigo and Morini [2003].

What is of the main interest here is that in these tests, for specific correlations, we found calibrations with parameters which are consistent with their probabilistic meaning, i.e., real and positive, avoiding the numerical problems encountered before. However, with a model very rich in volatilities and correlation parameters like the LMM, it is important to avoid considering them as completely free fitting parameters, regardless of their configuration, internal consistency and structural implications. Therefore, in the following we check if the

EXHIBIT 4

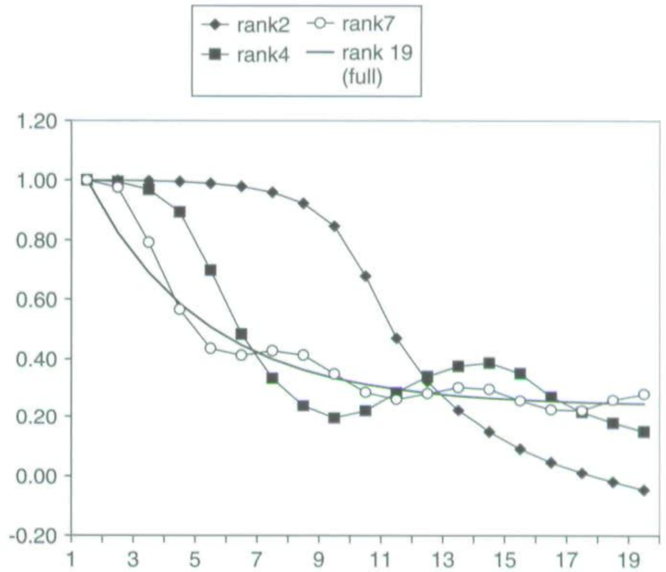
CCA Terminal Correlations. Terminal Correlations in 10 Years from Standard Cascade Calibration with Instantaneous Correlations of Rank 2 (above) and Rank 10 (below)

	10	11	12	13	14	15	16	17
10	1.00	0.93	0.90	0.92	0.86	0.85	0.93	0.92
11	0.93	1.00	0.86	0.91	0.93	0.88	0.90	0.92
12	0.90	0.86	1.00	0.92	0.91	0.91	0.88	0.94
13	0.92	0.91	0.92	1.00	0.94	0.93	0.96	0.93
14	0.86	0.93	0.91	0.94	1.00	0.95	0.92	0.93
15	0.85	0.88	0.91	0.93	0.95	1.00	0.94	0.96
16	0.93	0.90	0.86	0.96	0.92	0.93	1.00	0.96
17	0.92	0.92	0.94	0.92	0.93	0.96	0.96	1.00

	10	11	12	13	14	15	16	17
10	1.00	0.89	0.80	0.79	0.71	0.69	0.76	0.73
11	0.89	1.00	0.82	0.84	0.83	0.74	0.75	0.76
12	0.80	0.82	1.00	0.88	0.84	0.82	0.76	0.80
13	0.79	0.83	0.88	1.00	0.92	0.88	0.88	0.81
14	0.70	0.82	0.84	0.92	1.00	0.93	0.88	0.84
15	0.69	0.75	0.82	0.88	0.93	1.00	0.92	0.91
16	0.76	0.75	0.76	0.88	0.88	0.92	1.00	0.93
17	0.73	0.75	0.80	0.81	0.84	0.91	0.93	1.00

EXHIBIT 5

Correlation Matrices of Different Ranks. Plot of the First Column $\rho_{1,1}, \rho_{1,2}, \cdots \rho_{1,M}$



parameters found are also financially significant, namely reasonably regular and realistic. In particular, we analyze the diagnostic structures described in Section 2.

We see in Exhibit 6 the implied evolution of the term structure of volatility. In the plots, the line starting at year T with $T = 1, 2, \ldots, 9$ represents the TSV at year $T - 1$ for all forward rates with expiry ranging from T years to 10 years. We consider the calibration with two different exogenous instantaneous correlations. First, we consider the RE3 correlation matrix at rank 2. Considering the absence of structural constraints on the volatility function, and the exact fit to market prices, the evolution appears regular, smooth, stable over time, and overall realistic. Although it features no hump, this characteristic is consistent with the swaption data we are calibrating to.

When exogenous correlation matrices have high rank, and thus are more structured and less smooth, our tests show we can find evolutions of the TSV which are still acceptable but less satisfactory in terms of regularity and stability. See in Exhibit 6 the example referring to calibration with S&C2 at rank 10.

Now we move on to examining terminal correlations, computed after ten years at $T_\gamma = 10$. Results show, in general, that the regularity and the general features of instantaneous correlations are qualitatively preserved, as desired. In Exhibit 4 we see two examples. Above, we see the results obtained starting with instantaneous correlation S&C2 at rank 2.

The typical features of a low rank correlation matrix are clearly visible. For terminal correlations to be financially more realistic, in the sense of being more similar to market patterns, we must consider calibrations with instantaneous correlation at higher rank or with no rank reduction at all. See the example at rank 10 shown in Exhibit 4 for S&C2, where typical historical forward rates correlation patterns are more evident.

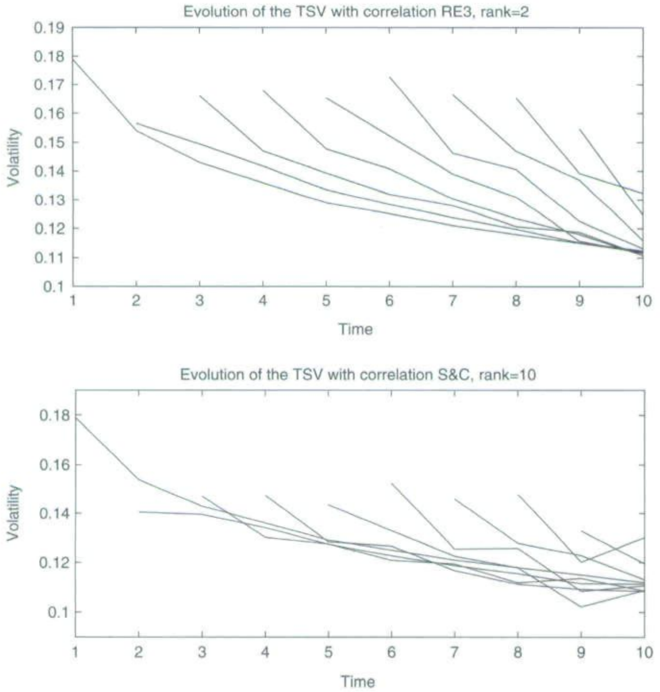
Remark 1: Results suggest that with unconstrained automatic cascade calibration and exogenous correlation, there might be a trade-off between regularity of the evolution of the TSV and realism of TC's, since they vary in opposite directions when rank is changed.

The ease in analytical calibration and the more regular TSV obtained when rank is low could be related to a practice of quoting swaption prices based on low dimensional models with simple correlation structures. This appears consistent with the analysis in Brace, Dun and Barton [2001] and De Jong, Driessen and Pelsser [2001], showing that swaption prices are well accommodated by a low dimensional LMM. However, a related investigation is outside the scope of our work, which is focused on obtaining a calibration procedure as robust as possible with respect to changes in the input structure.

We conclude this section by observing that through the CCA it is possible to obtain parameters that are acceptable as model volatilities and that imply diagnostic structures which are regular and financially realistic. This excludes cascade calibration of the LMM as inapplicable for radical misspecification. However, these satisfactory results are, in general, limited to the upper triangular calibration, and depend on particular features of exogenous correlation structures. This makes the algorithm scarcely robust and efficient. Detecting and removing the reasons for this weakness is the focus of the next part of our work.

EXHIBIT 6

CCA Evolution of the Term Structure of Volatility. TSV from Standard Cascade Calibration to Swaptions. See Section 5 for a Description



EFFICIENT ENDOGENOUS INTERPOLATION CASCADE CALIBRATION

We start with the following remark:

Remark 2: In all previous real market cascade calibration tests, negative imaginary entries occur only for σ's that depend on the artificial swaption volatilities obtained by local linear interpolation. Calibrated σ's obtained before the artificial values enter the algorithm are all real and positive (Morini [2002]).

Since the cascade of second order equations in the CCA requires a complete swaption matrix, featuring values for each and every maturity and length in the range, Brigo and Mercurio [2001] had obtained quotations for the few non-traded maturities on the Euro market by simple local linear interpolation. This modification of input data could appear minor and is not unusual in the market or in the literature (see Rebonato and Joshi [2002]). However, empirical tests show a link between these artificial values and the appearance of anomalous parameters.

As a first step, we experiment with some alternative fitting forms for the columns of the swaption matrix. An example is the following best-log-linear functional form in the maturity. For the first column of February 1, 2002, we obtain

$$\ln(\nu) = \ln(0.1785) - 0.201 \ln(T)$$

where ν denotes the swaption volatility and T the maturity. This form appears to be considerably closer to the real market pattern than a linear one. Using this more realistic interpolation for missing maturities, the negative σ entries found with locally linear interpolated inputs reduce in absolute value. Yet, acting only on the missing maturities, no exogenous interpolation we attempted eliminates the numerical problems previously found.

Better results were obtained using this realistic functional form also to replace the few market values suspected by market wisdom to be misaligned because of illiquidity for example, the 7-year maturity isolated between missing quotations. This confirms that inconsistencies in the input swaption matrix can have negative effects on calibration. However, the choice of the functional form and of the values to be considered as misaligned can become arbitrary. In addition, the need for replacing market values still brings about loss of information, and does not clarify where the inconsistencies leading to numerical problems lie.

A much more satisfactory and interesting step would be the possibility of excluding the influence of any exogenous artificial swaption values according to Remark 2, while keeping all information provided by the swaption market. This would mean developing an analytical calibration relying only on directly available market data with no need for exogenous data interpolation.

The derivation of such an algorithm follows and completes the analysis reported in Section 3 about the influence of different forward rate volatility buckets on different swaption prices.

It is now useful to note, formalizing results in Section 3, that the forward rate volatilities relevant for evaluating the swaption k into n are the volatility buckets $\sigma_{a,b}$ for $b \leq k$ and $k \leq a \leq k + n - 1$. Thus the market quotation for the k into n swaption contains information on all of these volatility buckets. Precise values for each of these volatility buckets can be computed making use of swaption volatilities $\nu_{c\times d}$ for $c \leq k$ and $k + 1 \leq c + d \leq k + n$.

To visualize them in a market swaption matrix, we point out that σ's and associated ν's influencing $\nu_{k\times n}$ represent a parallelogram, whose bottom/right vertex is given by $\nu_{k\times n}$ and extends to the first column on the left and to the first row. This is detailed in the following example, where we also show such parallelograms for different quotations.

A Simple Example

In Exhibit 7 we report an example of a 3×3 swaption matrix. We concentrate on volatilities with a 3-year maturity, situated on the third row of the matrix.

The volatility parameters $\sigma_{a,b}$ which are underlined in the table are those influencing the volatility quotation $\nu_{3\times 1}$ of the 3 into 1 swaption. Note that this is the so-called skew diagonal of the swaption matrix. Each volatility parameter $\sigma_{a,b}$ is indicated under the swaption quotation needed for computing its value consistently with the market swaption matrix in analytical calibration.

The volatility parameters and associated swaption quotations written inside boxes are those influencing $\nu_{3\times 2}$. They include those influencing $\nu_{3\times 1}$, with the addition of one skew sub-diagonal.

All volatility parameters and swaption quotations indicated in Exhibit 7 are those influencing $\nu_{3\times 3}$. They include those influencing $\nu_{3\times 1}$ and $\nu_{3\times 2}$, plus one more skew sub-diagonal. Notice that, when the lower part of the swaption matrix is considered, they can include also swaptions which are *outside* the available quotations. The recovery of this missing information using available quotations has already been discussed in Section 3, while here

EXHIBIT 7

Swaption-Volatility Dependencies for EICCA. Forward Rate Volatilities (and Corresponding Swaptions) that influence the 3 into 3 Swaption. Boxed Values influence the 3 into 2 Swaption. Underlined Values influence the 3 into 1 Swaption

	1y	2y	3y	4y	5y
1y			$\boxed{v_{1\times 3}}$ $\sigma_{3,1}$	$\boxed{v_{1\times 4}}$ $\sigma_{4,1}$	$v_{1\times 5}$ $\sigma_{5,1}$
2y		$\boxed{v_{2\times 2}}$ $\sigma_{3,2}$	$\boxed{v_{2\times 3}}$ $\sigma_{4,2}$	$v_{2\times 4}$ $\sigma_{5,2}$	
3y	$\boxed{v_{3\times 1}}$ $\sigma_{3,3}$	$\boxed{v_{3\times 2}}$ $\sigma_{4,3}$	$v_{3\times 3}$ $\sigma_{5,3}$		

we concentrate on missing maturities.

Remark 3: An additional observation of interest, which can be deduced from Exhibit 3 or 7 is that each skew subdiagonal of the swaption matrix such as $v_{m\times 1}, v_{m-1\times 2}, \ldots, v_{2\times m-1}, v_{1\times m,}$ is giving us a complete accurate volatility bucketing for the forward rate F_m, namely it gives us precise information on $\sigma_{m,1}$, $\sigma_{m,2}, \ldots, \sigma_{m,m}$, the volatility function of F_m from now to its expiry.

The relationships and dependencies analyzed here allow us to design a cascade calibration algorithm which works also in case of a missing maturity. In fact, these relationships allow us to understand which available quotations contain information that can be used to make up for missing quotations in a clever and financially significant way.

What should be done when quotations for a given maturity are missing? We see here the beginning of the procedure in a simple context. Suppose that in our matrix of Exhibit 7 the 2-year maturity quotations were missing. Due to lack of $v_{2\times 2}$ we cannot compute $\sigma_{3,2}$. However, after the above analysis, we understand that we can move to consider $v_{3\times 1}$, depending on a parallelogram containing $v_{1\times 3}$ and $v_{2\times 2}$. Since $v_{2\times 2}$ is not quoted, the corresponding forward volatility bucket $\sigma_{3,2}$ cannot be distinguished from $\sigma_{3,3}$ based on market quotations. But we can fix this indeterminacy by the simplest and most natural assumption: $\sigma_{3,2} = \sigma_{3,3}$. Now this value is computed by using the quotation $v_{3\times 1}$, which is the first available quotation depending on $\sigma_{3,2}$ and $\sigma_{3,3}$.

The Algorithm

In general, when we encounter a maturity for which market quotations are not available, we can select the first higher quoted maturity m and use the information content of the m year quotations for continuing calibration. Starting from the left, the $v_{m\times 1}$ swaption volatility will be used to recover the required volatility of the forward rate F_m consistent with market quotations, by assuming it to be constant when no data are available to infer possible changes. Moving from left to right inside the swaption matrix, we can apply the principle seen in the above example and invert the industry formula of Equation (3) even in the presence of "holes" in the swaption matrix, without the need to resort to artificial data. This is done while keeping all valuable features of cascade calibration: analytical tractability, speed, and absence of calibration errors. This method allows an exact consistent calibration based on *all* available market swaption quotes, and *only* on them. The new algorithm amounts to carrying out an endogenous interpolation, therefore it is called the endogenous interpolation cascade calibration algorithm (*EICCA*).

A few remarks are in order before presenting the explicit algorithm. Recall that s is the dimension of the swaption matrix, including non-quoted maturities, and denote by Z the set of the non-quoted maturities, namely

$$Z: = \{1 \leq z < s : v_{z\times y} \text{ missing for } 1 \leq y \leq s\}$$

Remark 4: One should by now understand that in our swaption matrix the last maturity must be quoted. Otherwise we have no market information for replacing the information content of a last missing maturity. This problem does not appear in a standard Euro market swaption matrix such as Exhibit 1, where $s = 10$ and

$$Z = \{6, 8, 9\}$$

Remark 5: It is also possible to understand, either by following the intuition given above or running the detailed algorithm below, that with this set Z the volatility buckets $\sigma_{6,6}$, $\sigma_{8,8}$, $\sigma_{9,8}$ and $\sigma_{9,9}$ are not determined by the above analytical calibration. In fact, the algorithm determines all volatility parameters related to available swaptions, but correctly skips the others. The above volatility buckets influence no market- quoted swaption. To use the terminology introduced in the above example, they are outside all "parallelograms" built on quoted swaptions. They do not affect the calibration which determines independently the other volatility buckets. When calibrating by optimization using stronger parametric assumptions on the volatility structure, this fact might be overlooked, but it is clearly revealed by precise non-parametric analytical cascade calibration. When needed, for example for presenting diagnostic structures, we use for these four buckets the following homogeneity assumption $\sigma_{6,6} := \sigma_{5,5}$, $\sigma_{8,8} := \sigma_{7,7}$, $\sigma_{9,8} := \sigma_{8,7}$ and $\sigma_{9,9} := \sigma_{8,8}$. Because we have more data when we add caps to the calibration the situation is different, as mentioned in Section 9.

Now, we present the explicit algorithm already extended for a complete calibration to the entire swaption matrix.

Algorithm 3: Endogenous interpolation cascade calibration algorithm (Morini [2003]).

1. *Set the swaption maturity* $k = 1$.
2. *Define* m^k *to be the index of the first available market quoted maturity at k, namely*

$$m^k = m = \min\{i{:}k \le i \le s,\ i \notin Z\}$$

and set

$$\sigma_{j,m} = \sigma_{j,m-1} = \cdots = \sigma_{j,k} = :\sigma_j \quad (6)$$

$$\text{for all } j : m \le j \le s + k$$

Then set $\gamma = k$ *and* $k = m$.

3. *Set the underlying swap length* $n = 1$.
4. *Solve Equation (4) in* $\sigma_{k+n-1,k}$ *with constraints of Equation (6).*
5. *Set* $n = n + 1$. *If* $n + k < s + \gamma$ *go to point 4, otherwise set*

$$\sigma_{k+n-1,k} = \sigma_{k+n-1,k-1} = \cdots = \sigma_{k+n-1,1}$$

and solve Equation (5) in $\sigma_{k+n-1,k}$. *If* $n < s$ *repeat point 5, otherwise set* $k = k + 1$.

6. *If* $k \le s$, *go to point 2, otherwise stop.*

Having discarded the influence of exogenous artificial data, we can now check how cascade calibration really works on market data.

Empirical Results

As a first example, we apply Algorithm 3 to the market data corresponding to calibrations shown in Exhibit 8. These consist of the highly problematic data set of May 16, 2000, from Brigo and Mercurio [2001] with its typical correlation matrix, and of the market data of October 10, 2003, with historical correlation at full rank.

In these cases using the standard CCA with exogenous artificial data we obtained various negative or imaginary values in calibration. The results obtained with EICCA are instead as shown in Exhibit 9. We have only real and positive σ's still allowing a perfect recovery of all market prices. Re-running earlier CCA calibration tests, using now EICCA which is based only on market quotations, all previously found numerical problems disappear.

In addition, data sets of February 1, 2002, December 10, 2002, and October 10, 2003, have been considered for general complete calibration testing, using as exogenous instantaneous correlations the corresponding historically estimated matrices and their reduced rank versions. The historical estimations have been performed using one year of data prior to the trading day used for swaption quotes. In our general calibration tests we consider reduced rank versions of all possible ranks from 2 to full rank 19 through the efficient eigenvalue zeroing-by-iteration method. We consider both upper-part calibration, which was the typical approach in earlier CCA tests and included various anomalous results, and its extension to complete calibration, which was almost always highly problematic with the earlier algorithm.

The outcome is that with EICCA, even extended to complete rectangular calibration, we find no anomalous results or numerical problems in any test output at any correlation rank.

For lower required ranks, the slower angles parameterization method has also been considered for generating low rank matrices. With this different method, for triangular calibration we find again no anomalous results or numerical problems, whereas for rectangular calibration results are similarly satisfactory with one exception. For 2002 data, when extending to the entire swaption matrix the test with rank 4 correlation, we find two negative volatilities that are very close to zero. Given that they are highly influenced by both homogeneity assumptions used for multiple unknowns, more realistic and flexible hypotheses could be used to avoid them. However, in practice, it is enough to use the eigenvalue zeroing-

EXHIBIT 8

Results from Standard Cascade Calibration. The Left-Hand Example comes from Brigo and Mercurio [2001]. The Right Hand Example uses Historical Correlations. NN Indicates Volatilities that are not Real Numbers

Standard Cascade Calibration, 05.16.2000

	1	2	3	4	5	6	7	8	9	10
F_1	0.18									
F_2	0.15	0.20								
F_3	0.13	0.16	0.23							
F_4	0.12	0.10	0.17	0.24						
F_5	0.11	0.10	0.10	0.16	0.25					
F_6	0.11	0.07	0.08	0.10	0.16	0.26				
F_7	0.10	0.10	0.05	0.07	0.11	0.16	0.26			
F_8	0.09	0.11	0.09	0.03	0.09	0.10	0.17	0.27		
F_9	0.11	0.08	0.09	0.08	0.07	0.05	0.09	0.18	0.28	
F_{10}	0.10	0.09	0.06	0.10	0.15	-0.03	0.04	0.08	0.16	0.28
F_{11}	0.09	0.09	0.08	0.04	0.03	0.21	-0.04	0.07	0.09	0.19
F_{12}	0.08	0.08	0.08	0.07	0.05	0.06	0.16	-0.01	0.07	0.09
F_{13}	0.07	0.07	0.07	0.07	0.08	0.06	0.09	0.12	-0.02	0.06
F_{14}	0.07	0.07	0.07	0.07	0.07	0.10	0.05	0.08	0.12	-0.02
F_{15}	0.07	0.07	0.07	0.07	0.07	0.07	0.10	0.04	0.06	0.12
F_{16}	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.07	0.06	0.03
F_{17}	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
F_{18}	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
F_{19}	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07

Standard Cascade Calibration, 10.10.2003

	1	2	3	4	5	6	7	8	9	10
F_1	0.29									
F_2	0.23	0.24								
F_3	0.20	0.18	0.22							
F_4	0.17	0.17	0.18	0.20						
F_5	0.17	0.12	0.17	0.18	0.18					
F_6	0.16	0.14	0.15	0.16	0.17	0.21				
F_7	0.13	0.15	0.10	0.15	0.13	0.25	NN			
F_8	0.13	0.14	0.11	0.06	0.15	0.16	NN	NN		
F_9	0.13	0.12	0.10	0.11	0.10	0.15	NN	NN	NN	
F_{10}	0.11	0.13	0.10	0.10	0.11	0.12	NN	NN	NN	NN
F_{11}	0.12	0.12	0.10	0.10	0.10	0.12	NN	NN	NN	NN
F_{12}	0.12	0.12	0.12	0.05	0.10	0.14	NN	NN	NN	NN
F_{13}	0.13	0.13	0.13	0.13	0.04	0.09	NN	NN	NN	NN
F_{14}	0.13	0.13	0.13	0.13	0.13	0.08	NN	NN	NN	NN
F_{15}	0.13	0.13	0.13	0.13	0.13	0.13	NN	NN	NN	NN
F_{16}	NN	NN	NN	NN	NN	NN	NN	NN	NN	NN
F_{17}	NN	NN	NN	NN	NN	NN	NN	NN	NN	NN
F_{18}	NN	NN	NN	NN	NN	NN	NN	NN	NN	NN
F_{19}	NN	NN	NN	NN	NN	NN	NN	NN	NN	NN

by-iteration rank reduction method, as in our main tests, or the S&C2 parametric form, to obtain positive σ's. This exception is useful because it shows that, while the macroscopic volatility structure is stable irrespective of the correlation matrix used, the details of volatility parameters have a precise dependence on the details of the correlation structure. Since instantaneous correlations are normally deemed not to have a strong influence on swaption prices, this sensitivity can appear to be a flaw.

Alternatively, it allows us to detect which correlation structures are "less consistent" with market data, in the sense that they make it difficult to match market data exactly even with full flexibility in the volatility function. More generally, this sensitivity gives us a precise indication of the influence of instantaneous correlations on calibration, an influence that is hard to detect with other methods.

FINANCIAL DIAGNOSTICS OF THE EICCA

Now that σ parameters of no statistical meaning appear to be unlikely exceptions that are easy to avoid, one has to check whether this is only related to a numerical ease of computation or corresponds to a real ability to pick out significant implied market structures. Hence, we perform extended financial diagnostics of the empirical results.

Evolution of the term structure of volatility. We analyze first, as in previous tests, the evolution of the TSV. For low-rank exogenous correlations, the TSV evolution appeared already satisfactorily realistic in the diagnostics of the basic CCA, and this is maintained with the EICCA. Exhibit 10 plots the calibration test of February 1, 2002, with an RE3 correlation matrix at rank 2, namely the same data as in the top of Exhibit 6.

Compared to our earlier result, the evolution is similar and at least as good. It is somewhat smoother and more regular, although the overall level is slightly less stationary over time.

These characteristics remain in the TSV computed with EICCA when we increase the rank of the exogenous correlation matrix. See the calibration with full rank in Exhibit 10 center panel still keeping a parametric form,

EXHIBIT 9

Results from Cascade Calibration with Endogenous Interpolation. Tests are Based on the Same Data as in Exhibit 8

Cascade Calibration with Endogenous Interpolation, 05.16.2000

	1	2	3	4	5	6	7	8	9	10
F_1	0.18									
F_2	0.15	0.20								
F_3	0.13	0.16	0.23							
F_4	0.12	0.10	0.17	0.24						
F_5	0.11	0.10	0.10	0.16	0.25					
F_6	0.11	0.07	0.08	0.10	0.16	0.25				
F_7	0.10	0.10	0.05	0.07	0.11	0.22	0.22			
F_8	0.09	0.11	0.09	0.03	0.09	0.14	0.14	0.22		
F_9	0.11	0.08	0.09	0.08	0.07	0.08	0.08	0.14	0.22	
F_{10}	0.10	0.09	0.06	0.10	0.15	0.01	0.01	0.19	0.19	0.19
F_{11}	0.09	0.09	0.08	0.04	0.03	0.06	0.06	0.14	0.14	0.14
F_{12}	0.08	0.08	0.08	0.07	0.05	0.11	0.11	0.06	0.06	0.06
F_{13}	0.07	0.07	0.07	0.07	0.08	0.08	0.08	0.05	0.05	0.05
F_{14}	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.05	0.05	0.05
F_{15}	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08
F_{16}	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.05	0.05	0.05
F_{17}	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
F_{18}	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
F_{19}	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07

Cascade Calibration with Endogenous Interpolation, 10.10.2003

	1	2	3	4	5	6	7	8	9	10
F_1	0.29									
F_2	0.23	0.24								
F_3	0.20	0.18	0.22							
F_4	0.17	0.17	0.18	0.20						
F_5	0.17	0.12	0.17	0.18	0.18					
F_6	0.16	0.14	0.15	0.16	0.17	0.18				
F_7	0.13	0.15	0.10	0.15	0.13	0.15	0.15			
F_8	0.13	0.14	0.11	0.06	0.15	0.17	0.17	0.15		
F_9	0.13	0.12	0.10	0.11	0.10	0.14	0.14	0.17	0.15	
F_{10}	0.11	0.13	0.10	0.10	0.11	0.09	0.09	0.15	0.15	0.15
F_{11}	0.12	0.12	0.10	0.10	0.10	0.09	0.09	0.12	0.12	0.12
F_{12}	0.12	0.12	0.12	0.05	0.10	0.10	0.10	0.11	0.11	0.11
F_{13}	0.13	0.13	0.13	0.13	0.04	0.07	0.07	0.09	0.09	0.09
F_{14}	0.13	0.13	0.13	0.13	0.13	0.06	0.06	0.07	0.07	0.07
F_{15}	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.07	0.07	0.07
F_{16}	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.09	0.09	0.09
F_{17}	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09
F_{18}	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11
F_{19}	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.13

S&C2, and data of the same trading day.

Due to the increased robustness of the algorithm with endogenous interpolation, which avoided numerical problems, we can now consider historically estimated correlation matrices at full rank, namely matrices with no parameterization or smoothing at all. Even in this case, we find evolutions of the TSV with characteristics of regularity and smoothness, as we can see at the bottom of Exhibit 10 for October 10, 2003.

Something is lost when increasing the rank, in particular, in terms of level stability. Although now results are smooth and regular at any rank, it appears again, consistent with our remarks in Section 5, that the standard behaviour of the TSV observed in the cap market is more easily found in the swaption market data when we assume a low-rank factor model.

Terminal Correlations. As for terminal correlations, it appears that the positive conclusions based on our previous results can be extended to the EICCA. The patterns and the properties are similar to those of the exogenous instantaneous correlation. In Exhibit 11 we see the terminal correlation referring to the October 10, 2003, full-rank calibration.

Endogenous interpolation for missing quotes. With the EICCA we can also compute the values for the missing swaption volatilities which are implicit in the parameters obtained through the cascade calibration. This can be seen as an endogenous interpolation technique, consistent by construction with the (cascade-calibrated) LMM. It is of interest to compare the configurations we obtained against market values.

Results appear generally similar across our tests. Exhibit 12 refers to the swaption volatilities from the model calibrated to market data for October 10, 2003, with historically estimated correlation at full rank. The bold-faced lines represent missing quoted maturities whose values are obtained as outputs of the EICCA, while the others are market values that are recovered exactly. The regularities typical of market quotations, such as the decreasing tendencies along maturity and along length, are generally respected by EI outputs. Endogenously interpolated volatilities appear consistent with market patterns; this is a reassuring feature in verifying the consistency between the specification of the model used for calibration and market practice.

Calibration Stability. Lastly, we briefly consider

Exhibit 10

EICCA Evolution of the Term Structure of Volatility. TSV from Endogenous Interpolation Cascade Calibration to Swaptions

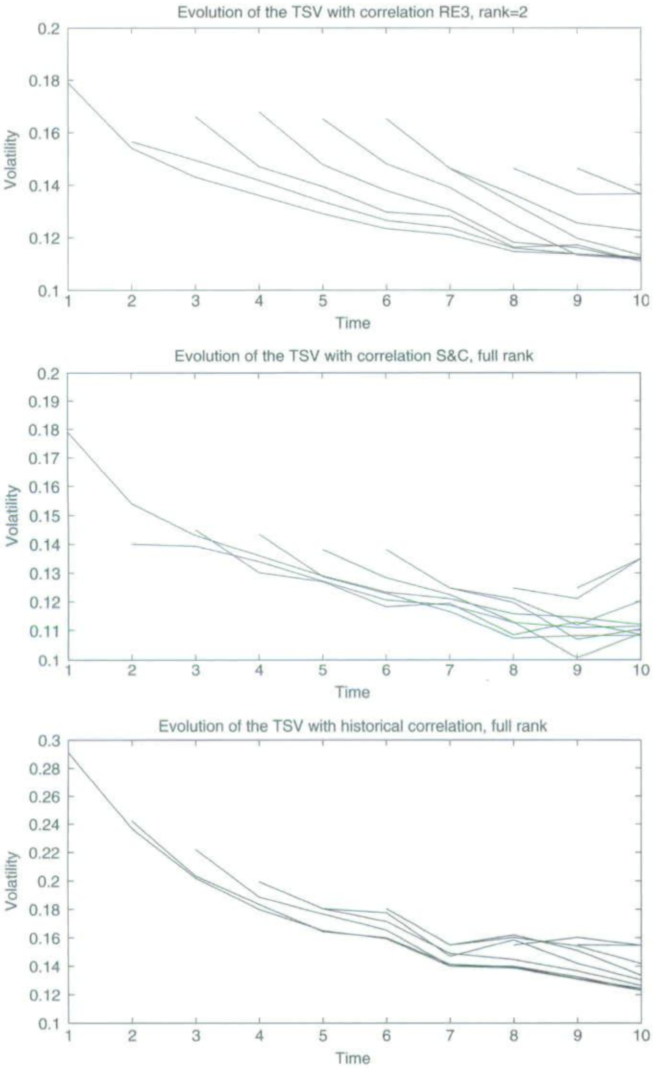


Exhibit 11

EICCA Terminal Correlations. Terminal Correlations in 10 Years from Endogenous Interpolation Cascade Calibration to Swaptions

	10	11	12	13	14	15	16	17
10	1.00	0.95	0.87	0.77	0.65	0.64	0.68	0.67
11	0.95	1.00	0.92	0.82	0.69	0.68	0.72	0.70
12	0.87	0.92	1.00	0.88	0.75	0.74	0.76	0.73
13	0.77	0.82	0.88	1.00	0.91	0.86	0.84	0.76
14	0.65	0.69	0.75	0.91	1.00	0.94	0.88	0.75
15	0.64	0.68	0.74	0.86	0.94	1.00	0.96	0.85
16	0.68	0.72	0.76	0.84	0.88	0.96	1.00	0.96
17	0.67	0.70	0.73	0.76	0.75	0.85	0.96	1.00

parameter stability over time. This is at times considered a less important requirement in modern finance, since daily recalibration is a widely accepted standard. However, if calibrated structures really extrapolate the relevant information in the market, it is reasonable to expect them not to experience sudden major changes, unless, of course, analogous market shifts take place.

We consider the results for our three data sets, each about 10-months apart, which appear similar in structure. To have a sketchy indication of their stability over time we present in Exhibit 13 a few synthetic statistics on the calibration outputs with full-rank historical correlation. The exhibit reports the average (above)

Exhibit 12

Endogenous Interpolation for Missing Maturities. Swaption Matrix Including Values from Endogenous Interpolation (bold-faced)

	3	4	5	6	7	8	9	10
3	0.172	0.161	0.150	0.144	0.139	0.135	0.132	0.130
4	0.157	0.147	0.138	0.133	0.129	0.126	0.123	0.122
5	0.143	0.135	0.129	0.125	0.122	0.119	0.117	0.116
6	**0.135**	**0.129**	**0.123**	**0.120**	**0.117**	**0.114**	**0.112**	**0.111**
7	0.127	0.120	0.116	0.113	0.110	0.108	0.107	0.106
8	**0.122**	**0.117**	**0.114**	**0.110**	**0.107**	**0.105**	**0.104**	**0.102**
9	**0.117**	**0.113**	**0.109**	**0.105**	**0.103**	**0.102**	**0.100**	**0.099**
10	0.111	0.106	0.102	0.100	0.099	0.097	0.096	0.096

and the maximum (below) variation $|\Delta\sigma|$, over all calibrated $\sigma's$ in the range, from one to another of the considered trading dates. Between brackets the analogous average and maximum variation $|\Delta v|$ for input swaption volatilities are reported.

The variations appear to be limited and reasonable, also compared to shifts occurred in the same period in the swaption market.

The results we just presented are tests and assessments that should regularly be performed for monitoring the behaviour of a calibration method. Based on the range of market cases considered, we conclude that, in spite of the lack of smoothing constraints, endogenous interpolation cascade calibrated parameters appear to show satisfactory behavior in terms of regularity and financial significance.

MONTE CARLO TESTS

The reliability of the exact swaptions cascade calibration depends also on the accuracy of the underlying

EXHIBIT 13

Calibration Stability. Average (above) and Maximum (below) Variations $|\Delta\sigma|$, and Corresponding $|\Delta\nu|$ between brackets

	Feb-02	Dec-02
Dec-02	0.018 (0.009) 0.066 (0.039)	-
Oct-03	0.029 (0.029) 0.113 (0.112)	0.023 (0.020) 0.109 (0.073)

approximation given in Equation (3). This formula has been tested previously. Here we extend the tests, based on comparing values computed with the approximated formula with those obtained via lengthy but accurate Monte Carlo simulation. We consider conditions that are typical of the cascade calibration, such as GPC volatility structure and various exogenous correlations with various ranks.

We follow testing in Brigo and Mercurio [2001], comparing analytical approximations with 98% confidence windows built around the Monte Carlo result according to the simulation standard error. In particular, we check if the approximated value is within the extremes of the Monte Carlo window, denoted by "MC inf" and "MC sup." We discretize the LMM dynamics according to the Milstein scheme, with 4 time-steps per year, and simulate 200,000 paths, "doubled" by the antithetic variates variance reduction technique.

We report in Exhibit 14 a sample of our test results, on a range of conditions specified as A, B, C and D. They refer to the same trading days considered in Section 6. Tests are indicated by swaption maturity and length in years and by the exogenous correlation used:

A) In the first tests, LIBOR volatilities have normal values, from calibrations to ordinary market conditions.

B) Then we consider a single swaption, 6×7 with S&C2 at rank 10, and test the approximation in case of anomalous values for the input σ volatilities. Calibrated volatilities are first multiplied by 1.2, and then added to 0.2. This second modification represents a more pronounced increase of the σ's. In this case we consider also Monte Carlo without variance-reduction, hence with a larger window. We test if in critical cases the price falls out of the smaller window but is contained in the larger one.[2]

C) Here we consider a 10×10 swaption with historically estimated correlation at full rank 19.

D) Finally, we consider higher input forward rates

EXHIBIT 14

Monte Carlo Tests. Testing the Accuracy of the Industry Swaption Volatility Formula, Where HE = Historical Estimate, Cal. Vol.'s = Calibrated Volatilities, MC no v.r. = Monte Carlo without variance reduction

A) Tests under normal market conditions

	MC inf	MC sup	Approx.
5 × 6 RE3 rank 2	0.10811	0.10911	0.10900
10 × 10 S&C2 rank 2	0.09753	0.09841	0.09800
5 × 6 S&C2 rank 5	0.10832	0.10933	0.10900
10 × 10 HE rank 7	0.09558	0.09643	0.09600

B) Tests under stressed instantaneous volatility (6x7, rank 10)

	MC inf	MC sup	Approx.
Cal. vol's	0.104131	0.105085	0.105000
Cal.vol's × 1.2	0.124881	0.126057	0.126000
Cal.vol's + 0.2	0.285133	0.288272	0.289777
Cal.vol's + 0.2 (MC no v.r.)	0.284671	0.290091	0.289777

C) Tests under stressed instantaneous volatility (10x10, rank 19)

	MC inf	MC sup	Approx.
Cal. vol's	0.094512	0.095363	0.095000
Cal. vol's × 1.2	0.113333	0.114372	0.114000
Cal.vol's + 0.2	0.272064	0.274788	0.277094
Cal.vol's + 0.2 (MC no v.r.)	0.270865	0.275774	0.277094

D) Tests under stressed initial forward rates

	MC inf	MC sup	Approx.
5 × 6 RE3 rank 2	0.10844	0.10941	0.10914
10 × 10 HE rank 19	0.09730	0.09813	0.09805

by adding 0.02 to all initial rates.

Results of tests considering ordinary market conditions confirm that accuracy extends also to the case of GPC volatility, with low- or intermediate-correlation rank, and even for long maturities and lengths, as in the examples given in Test A.

Considering cases as in Test B, with average maturity and length and intermediate rank of ρ, the approximated formula still works well with moderately increased volatilities. In case of very high volatilities, the approximation maintains acceptable behavior, but loses accuracy.

In case of full rank coupled with a "far" and "long" swaption, the approximation appears good for normal or slightly increased volatilities, but gets less reliable for very high volatilities. An example is Test C. Further tests confirm these observations and seem to suggest that the factors most responsible for the worsening in cases of very high volatility are elevated maturity and length, rather than high rank.

In Test D higher forward rates are considered. This feature does not seem to affect the reliability of the approximation.

In summary, tests seem to allow us to extend the generally positive judgment already given in the literature on the reliability of the industry swaption formula to configurations typical of this work, such as the GPC parameterization and different correlation structures with various ranks, ranging from 2 to 19. The formula appears less accurate in cases of very high volatility that we tested by artificially increasing market calibrated instantaneous volatilities.

CASCADE CALIBRATION AND THE CAP MARKET

Often a calibration to both caps and swaptions is the goal. Although there exists no general consensus as to the feasibility of such a joint calibration of the LMM, it is relevant to consider how information coming from cap data can be embedded in cascade calibration.

First, one has to make semi-annual caps data consistent, in terms of tenor, with annual swaptions data. This procedure is called annualization.[3] Consider three instants $0 < S < T < U$, all six-months apart, and assume that we are dealing with an $S \times 1$ swaption and with S- and T-expiry six-months caplets with Black volatilities v, v_S and v_T. The underlying forward rates are $F(t)$, $F_S(t)$ and $F_T(t)$, respectively. It is possible to relate the above volatilities by making use of no-arbitrage relationships and of the "freezing" approximation mentioned in Section 2 to obtain, in case of constant volatilities:

$$v^2 \approx u_S^2(0)v_S^2 + u_T^2(0)v_T^2 + 2\bar{\rho}u_S(0)u_T(0)v_S v_T \quad (7)$$

$$u_i(t) = \frac{1}{F(t)}\left(\frac{F_i(t)}{2} + \frac{F_U(t)F_S(t)}{4}\right), \; i = S,T$$

where $\bar{\rho}$ (infra-correlation) denotes the correlation of the semi-annual rates $F_S(t)$ and $F_T(t)$. We set $\bar{\rho} = 1$ as in Brigo and Mercurio [2001], where a detailed derivation of Equation (7) is given.

A simple way to incorporate cap data is to replace the volatilities of unitary length swaptions with the corresponding annualized volatilities that can be obtained from the semi-annual cap market via Equation (7). In theory these two quantities overlap exactly, while in practice they carry information which can be different because they come from two different markets. We obtain, for February 1, 2002, the volatilities v reported in Exhibit 15. Except for the first one, they are all higher than the swaption volatilities they are to replace. Adapting the EICCA to this situation, we can achieve a perfect fit to annualized caplet data in addition to all non-unitary length swaptions. Notice that with the addition of cap data there are no missing maturities in the first column, so that in endogenous interpolation calibration the volatility buckets mentioned in Remark 5 are not undetermined and therefore are not skipped in calibration. Through endogenous interpolation, market information is available for all volatility buckets. This can be easily seen by applying the reasoning given in Section 6 to the different data of this joint calibration.

Using historically estimated instantaneous correlations at full rank we obtain real and positive σ's. The implied evolution of the TSV is smooth, but shows a peculiar increasing tendency, as in Exhibit 16. We obtain similar results also when considering different exogenous correlations. This can arouse suspicions of inconsistency between the caps and swaptions data we mixed together, but can also be due to a bias in the approximation used and to the choice of setting infra-correlations to their maximum value. An analysis of historical or implied infra-correlation patterns could suggest different values.

Below we partially address this issue by calculating, in the above setting, the infra-correlations implied by our data via Equation (7). Thus, while before we assumed a value for $\bar{\rho}$ for replacing swaption data with annualized cap data, now we use data from the two markets for computing implied $\bar{\rho}$'s. These are the $\bar{\rho}$'s that can link and make consistent, in the sense of the approximation given in Equation (7), an annual tenor LMM exactly and automatically cascade-calibrated to swaptions with a semi-annual tenor LMM exactly and automatically calibrated to caps. Infra-correlations obtained are shown in Exhibit 15. The first value is not an acceptable correlation, and the others appear to be too low for correlations between adjacent rates. Model enthusiasts could consider similar minor anomalies as possible arbitrage opportunities to exploit, but a more cautious and critical approach is advisable in considering such simple artificial values.

First, recall that the setting used involves assumptions that may be inappropriate. Second, similar results could be explained by the existence of problems, related to market fundamentals, for which there may not be a basic congruence between the cap and swaption markets that a model can successfully detect and incorporate. Examples of these

EXHIBIT 15

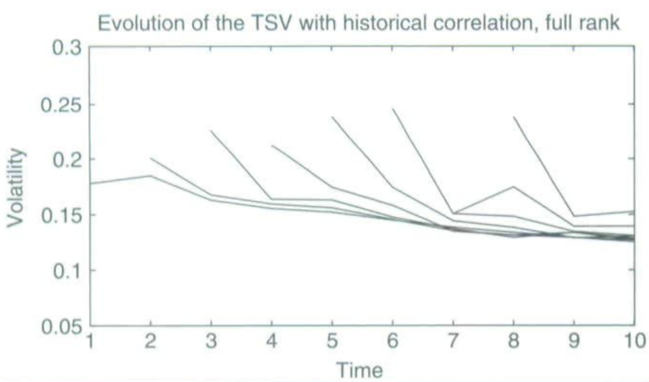
Including Caps. Annualized Caplet Volatilities ν for Different Maturities and Corresponding Swaption Implied Infra-Correlations $\overline{\rho}$

T:	1	2	3	4	5	6	7	8	9	10
v:	0.18	0.19	0.16	0.16	0.15	0.15	0.14	0.13	0.13	0.13
$\overline{\rho}$:	1.02	0.39	0.54	0.54	0.44	0.49	0.53	0.56	0.59	0.60

fundamental problems are discussed by Rebonato [2002]:

EXHIBIT 16

Evolution of the Term Structure of Volatility from Joint Cap and Swaption Calibration



illiquidity, agency problems and information costs are all obstacles to the action of those market operators, called pseudo-arbitrageurs, that are supposed to maintain fundamental consistency between markets which are related but not identical. A further warning comes from Pelsser [2000]: the losses involving several banks at the end of the 90's can be attributed to the use of models that relied on the assumption of a fixed relation between caplet and swaption volatilities that was belied by market behavior.

CONCLUSIONS

In this work we illustrated cascade calibration and explicitly introduced its extension to the entire swaption matrix. We pointed out empirical links between correlation features and the occurrence of anomalous outputs in calibration. In the process we identified correlation configurations leading to acceptable results. We showed that regular and realistic terminal correlations and a satisfactory evolution of the term structure of volatilities can be obtained even with exact calibration. We have pointed out and analyzed the role of a number of factors.

Subsequently, we analyzed interpolation for missing market quotations and then we introduced the new endogenous interpolation cascade calibration algorithm (EICCA), relying only on available market data. We can summarize the EICCA features as follows:

1. The calibration can be carried out through closed-form formulas, is practically instantaneous, and its solution is unique.
2. If the industry formula of Equation (3) is used market swaption quotes are recovered exactly.
3. The method establishes a direct correspondence between model volatility parameters σ and market swaption volatilities.
4. The instantaneous correlation matrix ρ is an exogenous input and can be chosen to be regular and realistic. Tests show these features are maintained in terminal correlation.
5. The method appears empirically efficient and robust. Tests have, in general, given forward-rate volatilities which automatically satisfy required constraints, are stable over time are consistent with market patterns if determined through endogenous interpolation, and give acceptably regular evolutions of the TSV.

Then we confirmed via Monte Carlo simulation that the underlying industry approximation of Equation (3) is accurate under the conditions typical of cascade calibration, except in the case of anomalously high volatility. We also sketched possibilities and problems in the extension of the method to a joint calibration including caps.

Cascade calibration is a rather inflexible algorithm with no smoothing or range constraints. However, the regularity and financial significance of the results obtained when, and only when, real market data are used in a fully consistent way, render cascade calibration a possible approximation of the concept of implied volatility in applying the LMM to swaptions.

The endogenous interpolation cascade calibration can also be used as a methodology to compute missing quotes consistently with available market quotes and the no-arbitrage LMM paradigm.

Last but not least, we point out the relevance of the characteristics of cascade calibration in hedging and sensitivity analysis. We recall that Piterbarg [2005] underlines two reasons that make the computation of *vegas* (sensitivities with respect to volatilities) problematic in a LMM. First, the typical presence of errors in calibrating swaptions cause vegas to be unreliable and not significant because they are "drowned in noise." Second, the usual slowness of the LMM calibration makes the computation of sensitivities, which involves recalibration, an extremely burdensome procedure. Cascade calibration is not affected by any of these problems, since it is free of calibration errors and immediate. Thus, these features, along with the direct correspondence induced between model parameters and market swaption volatilities, make it particularly appropriate for computing meaningful and precise swaptions vega bucketing.

ENDNOTES

[1]See Section 2 for a brief description of the swap market model.

[2]Smaller in any case than a typical bid-ask spread.

[3]Notice that here "annualization" refers to the corresponding tenor: annualized cap volatilities refer to caps with expiries which are 1-year apart. This is not to be confused with the usual concept of "annual volatility."

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