Credit Spread Option Valuation under GARCH

NABIL TAHANI

NABIL TAHANI
is with the School of Administrative Studies, Atkinson Faculty of Liberal & Professional Studies at York University in Toronto, Canada.
ntahani@yorku.ca

This article develops closed-form solutions for options on credit spreads with GARCH models. We extend the mean reverting model proposed in Longstaff and Schwartz (1995) and we use the Heston and Nandi (2000) GARCH specification rather than the traditional lognormal. Our model, being more general for volatility and more flexible, captures better the empirical properties of observed credit spreads. It also contains the Longstaff and Schwartz (1995) model as a special case. The GARCH coefficients are estimated using spread levels for corporate bonds.

Until recently, credit risk has been considered an unavoidable part of doing business, just like any other unhedgeable uncertainty factor. Currently, credit risk can be purchased, sold or restructured within portfolios in the same way as traditional financial risks. This is made possible thanks to credit derivatives, which are among the most important new types of financial products. These instruments offer investors an important new tool for dynamic hedging and for managing long positions in credit risk exposures. One determinant variable in credit risk is the evolution of the credit spread instead of traditional interest rates or exchange rates. Thus, we need a model for credit spreads. Basing their justification on the observed empirical properties, Longstaff and Schwartz (1995) proposed a mean reverting model for the logarithm of the credit spread. They showed that credit spreads were mean reverting in logarithm but they assumed the change in logarithm to be well approximated by the normal distribution.

This paper uses GARCH models to provide a more general framework for volatility and proposes a "closed-form"[1] valuation for European credit spread options. It is well-known that financial data-sets exhibit conditional heteroskedasticity. GARCH-type models are often used to model this phenomenon and show their ability to explain some irregularities (e.g. in equity returns) better than the traditional Geometric Brownian Motion. In the literature (see Bollerslev et al., 1992), there are many applications of ARCH models to interest rate data. But all these applications focused either on 1) Term Spread: Engle et al. (1987) and Engle et al. (1990) estimated the relationship between long- and short-term interest rates; or 2) Bond Yield Levels: Weiss (1984) estimated ARCH models on Aaa corporate bond yields and found that ARCH effects were significantly evident.

This paper estimates GARCH effects on the Credit Spread. Although most studies involving interest rates use linear GARCH models, nonlinear dependencies could possibly exist in the conditional variance. We keep the mean reversion of Longstaff and Schwartz (1995) and we use the Heston and Nandi (2000) GARCH specification in order to develop closed-form solutions for options. Being more general for volatility, the GARCH model fits observed credit spread data better than the simple mean reverting normal model.

Heston and Nandi (2000) also showed that their conditional variance process converges weakly to Heston's (1993) Stochastic Volatility model, which means that our model has a mean reverting square-root variance process as a continuous-time limit. Thus, the GARCH model contains the continuous-time Longstaff and Schwartz (1995) model as a special case.

The next section examines the mean reversion of credit spreads in our data-set. Section 2 describes the GARCH process and presents credit spread options formulas. Section 3 estimates GARCH coefficients with the maximum-likelihood method using corporate bond spread levels over treasuries; analyzes some properties of the GARCH credit spread options; and compares our model to that of Longstaff and Schwartz (1995). Finally, Section 4 provides a summary of the main results and concludes.

CREDIT SPREAD MEAN REVERSION

We examine the spread between Moody's Aaa and Baa 10- to 20-year bond indices and two US Treasury bond yields (30- and 10-year maturity). We use daily observations over the period 1986-2005. Summary statistics for the spreads are presented in Exhibit 1. We also report Skewness, Kurtosis and the Anderson-Darling Normality Test coefficients for log-spreads in Exhibit 2. These results imply that we cannot assume a normal distribution for log-spreads. Instead, we propose a different process, such as GARCH, that will take these findings into account.

We denote the logarithm of the credit spread by X_t. Exhibits 3 and 4 plot the time series of X_t for Aaa and Baa bonds over 30-year and 10-year US TBonds. We can observe that both time series display mean reversion,[2] and that the Aaa log-spreads are apparently more mean reverting than the Baa log-spreads. In order to formalize these observations and see how our data evolve over time, we regressed daily changes in the value of X_t on the value of X_t from the day before:

$$\Delta X_{t+1} = X_{t+1} - X_t = \alpha + \beta X_t + \varepsilon_t \qquad (1)$$

The regression results are reported in Exhibit 5. The slope coefficient is significantly negative in all the regressions, which implies that Aaa and Baa log-spreads are mean reverting.[3] The Aaa slope coefficients are higher (in absolute value) than those for Baas, which implies that they are more mean reverting. Note also that the logarithm of

EXHIBIT 1

Summary Statistics for Aaa and Baa Credit Spreads and Log-Spreads Over Treasuries

	Spread over 30-year TBond			Spread over 10-year TBond			Number of Observations
	Mean	Std Dev	AC	Mean	Std Dev	AC	
Aaa	0.8373	0.3514	0.0968	1.1969	0.4185	0.0971	5045
Log Aaa	−4.8570	0.3726	0.1003	−4.4841	0.3429	0.1053	5045
Baa	1.7173	0.4441	0.0955	2.0768	0.5148	0.0963	5045
Log Baa	−4.0959	0.2476	0.1016	−3.9028	0.2351	0.1047	5045

Note: Exhibit 1 presents summary statistics for the spreads and the log-spreads of both Aaa and Baa bond indices. The autocorrelation (AC) is calculated at lag 14. The Ljung-Box Q-statistic is above 300 in all cases. The corresponding Chi-square 99% quantile is 29.14.

EXHIBIT 2

Skewness, Kurtosis and Normality Test for Aaa and Baa Log-Spreads Over Treasuries

	Spread over 30-year TBond			Spread over 10-year TBond			Number of Observations
	Skewness	Kurtosis	A^2	Skewness	Kurtosis	A^2	
Log Aaa	−0.2520	2.7471	111.49	−0.3157	2.9685	16.57	5045
Log Baa	0.4238	1.8326	139.76	0.4291	2.4795	37.74	5045

Note: Exhibit 2 presents some statistics (skewness and kurtosis) for the log-spreads of both Aaa and Baa bond indices. It also reports the Anderson-Darling normality test coefficient. The spreads are computed over 30-year and 10-year US TBonds. A^2 is the Anderson-Darling normality test coefficient. All associated p-values are approximately equal to 0.000.

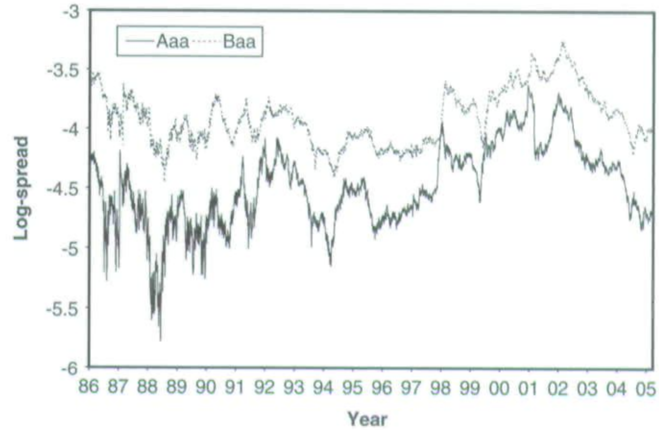
EXHIBIT 3

Aaa and Baa Log-Spreads Over the 30-year US TBond

Note: Exhibit 3 shows the log-spreads over the 30-year US TBond. The curve in top represents the Baa log-spread and that in bottom represents Aaa log-spread.

EXHIBIT 4

Aaa and Baa Log-Spreads Over the 10-year US TBond



Note: Exhibit 4 shows the log-spreads over the 10-year US TBond. The curve in top represents the Baa log-spread and that in bottom represents Aaa log-spread.

the Aaa credit spread is more volatile than that of the Baa. The standard errors of regressions using Aaa bonds are higher than those using Baa bonds. Longstaff and Schwartz (1995) found the same property with their data-set.

Given these empirical properties, we assume a mean reverting process for the logarithm of the credit spread. But unlike the Longstaff and Schwartz (1995) model and given the skewness and kurtosis analysis in Exhibit 2, we propose a GARCH framework for the volatility of the logarithm of the credit spread. We use the Heston and Nandi (2000) GARCH(1,1) specification which is asymmetric. We also use their methodology to derive a closed-form solution for credit spread options. We assume that the risk-free interest rate is constant and it is denoted by r.

EXHIBIT 5

Regression of the Daily Changes of Aaa and Baa Log-Spreads Over Treasuries

	α	β	t-ratio $\alpha^{\#}$	t-ratio $\beta^{\#}$	R^2*	SE*
Aaa-TB30y	−0.0394	−0.0181	−4.59	−4.58	0.4%	0.0466
		{−4.582}	[0.000]$^{+}$	[0.000]		
Aaa-TB10y	−0.0312	−0.0169	−4.21	−4.20	0.3%	0.04011
		{−4.204}	[0.000]	[0.000]		
Baa-TB30y	−0.0137	−0.0093	−3.13	−3.11	0.2%	0.01876
		{−3.108}	[0.002]	[0.002]		
Baa-TB10y	−0.0139	−0.0105	−3.14	−3.13	0.25	0.01894
		{−3.127}	[0.002]	[0.002]		

Note: Exhibit 5 reports the results of regressing daily changes in the value of the log-spread $(X_{t+1} - X_t)$ on the value from the day before (X_t). The regression equation is given by $\Delta X_{t-1} = X_{t+1} - X_t = \alpha + \beta X_t + \varepsilon_t$. The regression is conducted for both Aaa and Baa bond indices. The spreads are computed over 30-year and 10-year US TBonds. The following legend describes the results and the parameters presented:

* *SE* is the standard error of the regression and R^2 is the determination coefficient.

\# All the coefficients are significant at the 99% level according to the t-ratio values.

+ p-values are reported in brackets.

Dickey-Fuller test statistics are reported for the Unit Root Test. The asymptotic critical values are −3.43 at 1% and −2.86 at 5%.

THE MODEL AND THE OPTION VALUATION FORMULA

We define X_t as the value of the logarithm of the credit spread at the end of period t, and time periods are of length Δ. We assume that (X_t) follows the process given by:

$$X_{t+1} = \mu + \gamma X_t + \lambda h_{t+1} + \sqrt{h_{t+1}} z_{t+1}$$
$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 (z_t - \theta\sqrt{h_t})^2 \quad (2)$$

or equivalently, to show the mean reverting feature, by:

$$X_{t+1} - X_t = \mu + (\gamma - 1)X_t + \lambda h_{t+1} + \sqrt{h_{t+1}} z_{t+1}$$
$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 (z_t - \theta\sqrt{h_t})^2 \quad (3)$$

where h_t is the conditional variance of X_t known at time $t - 1$ and $(z_t : t \in \{1, \ldots, T\})$ is a sequence of independent standard normal random variables. As pointed out by Heston and Nandi (2000), although this specification differs from the classic GARCH models, it is quite similar to the NGARCH model of Engle and Ng (1993). This GARCH model has, however, the advantage of providing closed-form solutions for the credit spread derivatives. When β_1 and β_2 are equal to zero, our model is equivalent to the Longstaff and Schwartz (1995) model observed at discrete intervals, with a risk-premium parameter λ. This parameter was assumed to be equal to zero in Longstaff and Schwartz (1995) because their model was assumed to be risk-adjusted and the parameter μ incorporated the market price of the risk-premium. We cannot use this assumption within our model because the volatility is not constant. As in the Heston and Nandi (2000) model, the parameter θ controls the skewness or the asymmetry of the distribution of the log-spreads. If $\theta > 0$, this implies that negative z_t will raise the variance more than positive z_t. The covariance of the log-spread process and the variance process is given by:

$$Cov_{t-1}(X_t, h_{t+1}) = -2\theta\beta_2 h_t \quad (4)$$

If the kurtosis parameter β_2 is positive,[4] positive values for θ result in a negative correlation between the two processes.

As shown in Heston and Nandi (2000), the discrete time variance process h_t converges weakly to a variance process v_t that follows the square-root process of Feller (1951), Cox, Ingersoll and Ross (1985), and Heston (1993), when the time-step length Δ tends to zero (see Foster and Nelson, 1994). The log-spread process X_t will also have a continuous-time diffusion limit. Thus our model converges weakly to a mean reverting square-root variance process (X,v):

$$dX_t = (\eta - \delta X_t + \lambda v_t)\,dt + \sqrt{v_t}\,dZ_t$$
$$dv_t = (\omega - \kappa v_t)\,dt + \sigma\sqrt{v_t}\,dZ_t \quad (5)$$

where η is the long-run mean, δ is the mean reversion parameter, λ is the risk-premium parameter, and ω, κ and σ are the square-root process parameters.[5] By assuming that ω, κ, σ and λ are all zero, we get the constant-volatility risk-adjusted Longstaff and Schwartz (1995) model. Thus our valuation model will contain the Longstaff and Schwartz (1995) valuation formula with a constant risk-free rate as a special case. In order to value options, we must work under a risk-neutral probability measure.[6] Following Fong and Vasicek (1992) and Cvsa and Ritchken (2001), we assume that the risk-premium is given by $\lambda\sqrt{h_t}$.[7] Let us rewrite Equation (2) in the form:

$$X_{t+1} = \mu + \gamma X_t + \sqrt{h_{t+1}} z^*_{t+1}$$
$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 (z^*_t - \theta^*\sqrt{h_t})^2 \quad (6)$$

where

$$z^*_t = z_t + \lambda\sqrt{h_t}$$
$$\theta^* = \theta + \lambda$$

and z^*_{t+1} is a risk-neutral standard normal random variable conditional on the information available at time t.

At this point, we can derive closed-form solutions for European options using the inversion of the characteristic function technique given in Kendall and Stuart (1977). We first derive a formula for the characteristic function for the process given in Equation (2). Let $f(t, T, \phi)$ denote the moment generating function of X_T under the historical probability measure conditional on the information available at time t:

$$f(t, T, \phi) = E_t(exp(\phi\, X_T)) \quad (7)$$

where E_t denotes the time t conditional expectation operator under the historical probability measure. Motivated by the Heston and Nandi (2000) asset price model, we calculate $f(t, t+1, \phi)$ and $f(t, t+2, \phi)$ to find that f takes a log-linear form and is a function of X_t and h_{t+1}. The general form for the moment generating function is given by:

$$f(t, T, \phi) = exp(\gamma^{T-t}\phi X_t + A(t, T, \phi) + B(t, T, \phi)h_{t+1}) \quad (8)$$

where

$$A(T, T, \phi) = B(T, T, \phi) = 0 \quad (9)$$

and

$$A(t, T, \phi) = \gamma^{T-t-1}\mu\phi + A(t+1, T, \phi) + \beta_0 B(t+1, T, \phi) - \frac{1}{2}ln(1 - 2\beta_2 B(t+1, T, \phi)) \quad (10)$$

$$B(t, T, \phi) = \gamma^{T-t-1}(\theta + \lambda)\phi - \frac{1}{2}\theta^2 + \beta_1 B(t+1, T, \phi) + \frac{1}{2}\frac{\left(\gamma^{T-t-1}\phi - \theta\right)^2}{\left(1 - 2\beta_2 B(t+1, T, \phi)\right)} \quad (11)$$

Given the terminal conditions in Equation (9), $A(t, T, \phi)$ and $B(t, T, \phi)$ can be calculated recursively by Equations (10) and (11) (Appendix A derives the recursion formulas for these functions). Obviously, assuming $\gamma = 1$ leads to the Heston and Nandi (2000) GARCH(1,1) formulas. Notice that, although this formulation is given for the log-spread process under the historical probability measure, one can get the risk-neutral conditional moment generating function $f^*(t, T, \phi)$ by replacing θ and λ respectively by $\theta^* = \theta + \lambda$ and $\lambda^* = 0$.

The value of a credit spread call with maturity T and strike K is then given by:

$$C(t, T, X) = e^{-r(T-t)}E_t^*((exp(X_T) - K)^+) \quad (12)$$

where E_t^* denotes the time t conditional expectation operator under the risk-neutral probability measure and $(\cdot)^+$ denotes the positive part of a real number. We can compute the cumulative distribution function of the log-spread process by inverting its Fourier transform. Indeed, Kendall and Stuart (1977) show that for a random variable Y:

$$P(Y \geq y) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} Re\left(\frac{e^{-i\phi y}f(i\phi)}{i\phi}\right)d\phi \quad (13)$$

where $Re(\cdot)$ denotes the real part of a complex number. Using this result, we can provide a formula for the expected payoff under the risk-neutral probability measure and thus a value for the option. The credit spread call value is given by:

$$C(t, T, X) = e^{-r(T-t)}(f^*(1)P_1 - KP_2) \quad (14)$$

where

$$P_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} Re\left(\frac{K^{-i\phi}f^*(i\phi + 1)}{i\phi f^*(1)}\right)d\phi$$

$$P_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} Re\left(\frac{K^{-i\phi}f^*(i\phi)}{i\phi}\right)d\phi \quad (15)$$

P_2, which is the risk-neutral probability that the spread is greater than K at maturity, comes directly from Equation (13) and P_1 is obtained after some calculations.[8] $f^*(1)$ represents, by definition, the risk-neutral forward (or expected) value of the spread given by:

$$f^*(1) = f^*(t, T, 1) = E_t^*(exp(X_T)) \quad (16)$$

If we assume, like in Longstaff and Schwartz (1995), that the log-spread at time T is conditionally normal with mean υ and variance η^2, we have:

$$E_t^*(exp(X_T)) = exp\left(\upsilon + \frac{\eta^2}{2}\right) \quad (17)$$

that is the term present in the Longstaff and Schwartz (1995) formula which is equivalent to our $f^*(1)$ term in Equation (16). Following the same methodology as for the call, the put value is given by:

$$P(t, T, X) = e^{-r(T-t)}(K(1 - P_2) - f^*(1)(1 - P_1)) \quad (18)$$

By a simple differentiation, the call delta is given by (see Appendix B):

$$Delta = \frac{\partial C(t, T, X)}{\partial(e^X)} = e^{-r(T-t)}\gamma^T f^*(1)P_1 \times e^{-X} \quad (19)$$

The other hedging ratios can also be computed by simply differentiating Equation (14).

EMPIRICAL PROPERTIES OF CREDIT SPREAD OPTIONS

This empirical section starts with the estimation of the GARCH coefficients using the data described earlier. It proceeds to analyze the implied conditional distribution. It then presents some properties of GARCH credit spread options and compares our model to that of Longstaff and Schwartz (1995).

GARCH Estimation

For the estimation, we used daily data and set the time-step length equal to 1 day. We used both Aaa and Baa spreads over 10-year and 30-year US TBonds for the period 1986-2005. We estimated the coefficients using the maximum-likelihood method. To illustrate the mean reversion and the importance of the skewness parameter, we also estimated restricted models by setting separately $\gamma = 1$ and $\theta = 0$. These restricted models are compared to the unrestricted model using the log-likelihood ratio. Exhibit 6 reports the estimation results. In all cases, the mean reversion parameter γ was positive and significantly lower than 1.[9] The restricted model $\gamma = 1$ is strongly rejected in all cases. This reinforces the results reported in the regression analysis of the daily changes in the logarithm of the credit spread. The parameter γ was also largely significantly different from 0. We also notice that the γ for the Baa bonds are higher than those for Aaa bonds. Thus, the Aaa log-spreads are more mean reverting than Baas, which was also reported in the regression analysis. For the skewness parameter θ, the restricted model $\theta = 0$ is rejected for all bonds according to the log-likelihood ratio test; while the t-test only rejects it for the Aaa bonds. When the parameter θ is significantly positive, this implies a negative correlation between the log-spread and the volatility processes as shown in Equation (4). Log-spread processes over the 10-year TBond have more skewness than those over the 30-year TBond for both Aaa and Baa bonds (for Aaa over the 10-year TBond $\theta = 12.732$ and over the 30-year TBond $\theta = 10.438$; for Baa over the 10-year TBond $\theta = 3.587$ and over the 30-year TBond $\theta = 2.406$). The stationarity coefficient, $\beta_1 + \beta_2\theta^2$, measures the degree of the volatility mean reversion (see Heston and Nandi, 2000). The volatility processes are found to be stationary in all cases. Aaa log-spread processes have quite similar stationarity coefficients ($0.926 \cong 0.929$), while they are slightly different for Baa bonds (0.915 vs. 0.903). Exhibits 7-10 plot the volatility processes implied by the GARCH estimation for Aaa and Baa bonds respectively, as well as their long-run (unconditional) volatilities. Finally, notice that the GARCH specification (for all bonds) captures the low variance during the bond market calm period 1993-1998, preceded by higher volatility during the period around the Crash of 1987. The dot-com debacle in 2001 is evident with the highest volatility in the later part of the sample.

Implied GARCH Distribution

Using Equation (13) with the historical probability measure, we derived the implied conditional distribution of X_T. To see how this implied conditional distribution varies from the normal distribution, Exhibit 11 plots the implied conditional distribution for two different time horizons. For the 1-day maturity, the implied conditional distribution and the normal distribution coincide because the GARCH used is conditionally normal. For a 2-week time horizon, however, the implied conditional distribution is slightly negatively skewed compared to the normal distribution.

Credit Spread Option Properties

We limit our discussion to call options when we analyze the properties of our model. In order to compare our discrete-time model to the continuous-time model of Longstaff and Schwartz (1995), we define their continuous-time parameters using the coefficient estimates such that the two stationary densities of processes X_t and h_t have the same first moments. The parameters in Equation (17) are defined as follows:

$$v = X_0\gamma^T - \frac{\mu(1-\gamma^T)}{ln(\gamma)}$$

$$\eta^2 = h_{unc}(1-\gamma^{2T}) \tag{20}$$

It is clear that when T takes high values, v and η^2 have finite limits since $0 < \gamma < 1$:

$$v_\infty = \frac{-\mu}{ln(\gamma)} \approx \frac{\mu}{1-\gamma}$$

$$\eta^2_\infty = h_{unc} \tag{21}$$

Exhibit 6

Maximum-Likelihood Estimates of the GARCH Model Using Daily Data of Aaa and Baa Log-Spreads over Treasuries

	μ	γ	λ	β_0	β_1	β_2	θ	$\beta_1+\beta_2\theta^2$ #	Long-run volatility	log L	L-ratio test $\gamma=1$&	L-ratio test $\theta=0$*
Aaa-TB30y	−0.0335	0.9847	9.799e−5	2.924e−3	0.9087	1.579e−4	10.438	0.926	0.0416	14161	34.13	29.47
	(2.537) **	(163.70)	(0.202)	(2.413)	(87.646)	(21.426)	(7.094)				[0.000]	[0.000]
	[0.011]·	[0.000]	[0.840]	[0.016]	[0.000]	[0.000]	[0.000]					
		{2.552}+										
Aaa-TB10y	−0.0288	0.9866	1.084e−4	2.368e−3	0.9113	1.060e−4	12.732	0.929	0.0346	14981	43.89	36.05
	(2.074)	(149.37)	(0.174)	(2.017)	(32.843)	(21.125)	(8.891)				[0.000]	[0.000]
	[0.038]	[0.000]	[0.862]	[0.044]	[0.000]	[0.000]	[0.000]					
		{2.0256}										
Baa-TB30y	−0.0097	0.9917	8.837e−5	1.448e−3	0.9152	2.558e−5	2.406	0.915	0.0174	18284	15.87	2.76
	(2.607)	(285.96)	(0.069)	(3.681)	(82.031)	(26.156)	(0.969)				[0.000]	[0.097]
	[0.009]	[0.000]	[0.945]	[0.000]	[0.000]	[0.000]	[0.333]					
		{2.3817}										
Baa-TB10y	−0.0106	0.9884	1.038e−4	1.684e−3	0.9027	3.032e−5	3.587	0.903	0.0177	18167	18.72	4.78
	(2.280)	(250.05)	(0.082)	(1.548)	(67.223)	(18.538)	(1.626)				[0.000]	[0.029]
	[0.023]	[0.000]	[0.935]	[0.122]	[0.000]	[0.000]	[0.104]					
		{2.945}										

Note: Exhibit 6 reports the results of the maximum-likelihood estimation of the GARCH model given by $X_{t+1}=\mu+\gamma X_t+\lambda h_{t+1}+\sqrt{h_{t+1}}z_{t+1}$ and $h_{t+1}=\beta+\beta_1 h_t+\beta_2\left(z_t-\theta\sqrt{h_t}\right)^2$. The estimation is conducted for both Aaa and Baa bond indices. The spreads are computed over 30-year and 10-year US TBonds. The following legend describes the results and the parameters presented in Exhibit 6:

* For the log-likelihood ratio test for H_0: $\theta = 0$, the 95% and 99% Chi-square critical values are respectively 3.84 and 6.63.

& For the log-likelihood ratio unit root test H_0: $\gamma = 1$, the 95% and 99% critical values based on Bartlett correction, and provided in Abadir (1995), Nielsen (1997) and MacKinnon et al. (1999), are 4.13 and 6.94.

** t-ratios appear in parentheses. The t critical values at 95% and 99% confidence levels are respectively 1.645 and 2.326

Σ p-values are reported in brackets. For the log-likelihood ratio test, the p-values are those of the Chi-square distribution

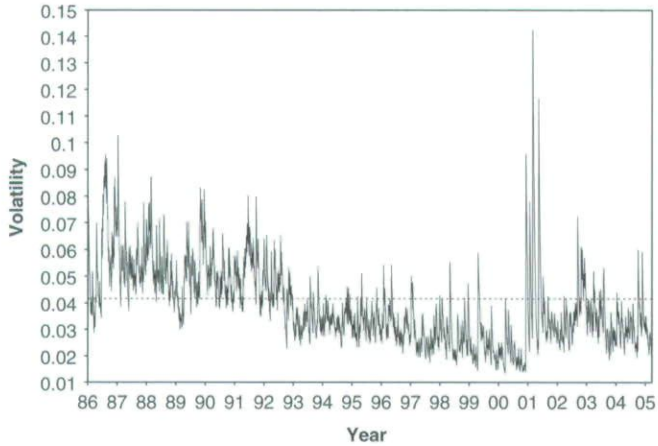
+ For γ, we also report the t-ratio for testing H_0: $\gamma = 1$ vs. H_1: $\gamma < 1$

\# The Stationarity coefficient $\beta_1 + \beta_2\theta^2 < 1$ means that the volatility process is stationary

Long-run volatility is the daily unconditional volatility given by $h_{unc}=\dfrac{(\beta_0+\beta_2)}{(1-\beta_1-\beta_2\theta^2)}$

EXHIBIT 7

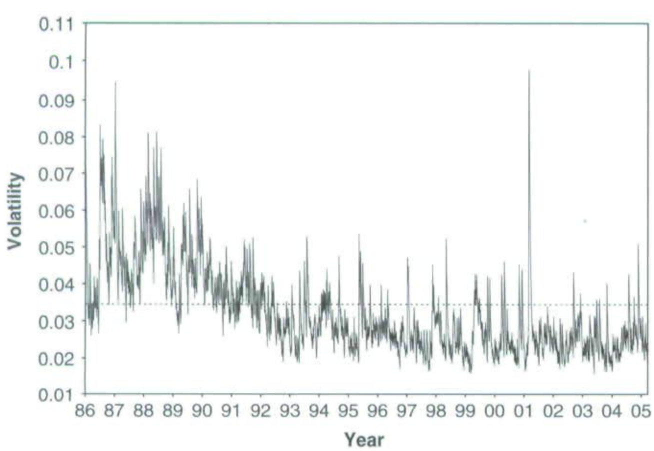
Volatility of The Aaa Log-Spread Over the 30-Year US TBond



Note: Exhibit 7 plots the log-spread volatility process of Aaa over the 30-year US TBond implied by the GARCH maximum-likelihood estimation (see line 1 in Exhibit 6). The volatility process is stationary. The horizontal line represents the long-run volatility equal to 0.0416.

EXHIBIT 8

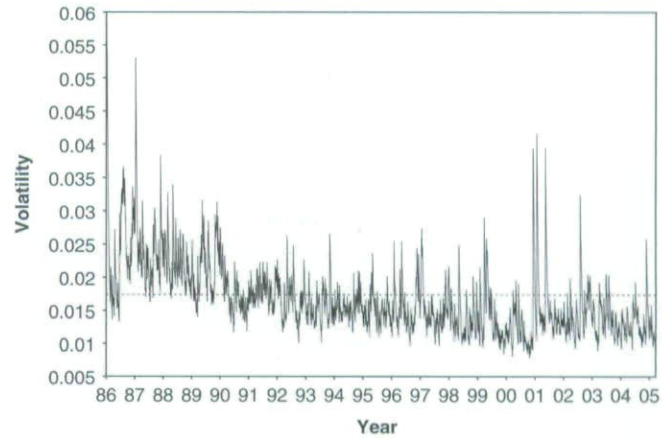
Volatility of The Aaa Log-Spread Over the 10-Year US TBond



Note: Exhibit 8 plots the log-spread volatility process of Aaa over the 10-year US TBond implied by the GARCH maximum-likelihood estimation (see line 2 in Exhibit 6). The volatility process is stationary. The horizontal line represents the long-run volatility equal to 0.0346.

EXHIBIT 9

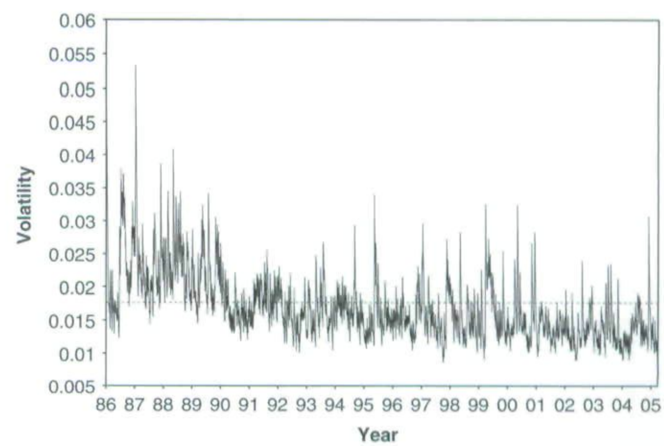
Volatility of the Baa Log-Spread Over the 30-Year US TBond



Note: Exhibit 9 plots the log-spread volatility process of Baa over the 30-year US TBond implied by the GARCH maximum-likelihood estimation (see line 3 in Exhibit 6). The volatility process is stationary. The horizontal line represents the long-run volatility equal to 0.0174.

EXHIBIT 10

Volatility of The Baa Log-Spread Over the 10-Year US TBond



Note: Exhibit 10 plots the log-spread volatility process of Baa over the 10-year US TBond implied by the GARCH maximum-likelihood estimation (see line 4 in Exhibit 6). The volatility process is stationary. The horizontal line represents the long-run volatility equal to 0.0177.

The Longstaff and Schwartz (1995) call price (hereafter LS) is evaluated using the conditionally normal distribution for X_T with mean υ and variance η^2 given by Equation (20). We use a strike $K = 1\%$, a risk-free rate $r = 10\%$ and the GARCH parameters reported in Exhibit 6 (line 1) and their risk-neutral counterparts. Exhibit 12 plots the call value as a function of the underlying credit spread for different variance ratios for $T = 2$ weeks. The LS call price and the intrinsic value are also represented. An important property of GARCH credit spread calls, already noticed by Longstaff and Schwartz (1995) within their model, is that their value can be less than the intrinsic value, which is impossible with the Black and Scholes (1973) model. This is due to the mean reversion character of the credit log-spreads. Intuitively, in-the-money calls are less likely to remain in the money over time, because the credit spread tends to decline towards its long-run mean. For variance ratios less than 1, the GARCH credit spread call prices are less than the LS price, while variance ratios greater than 1 give higher GARCH prices. Exhibit 13 plots the difference between LS and GARCH credit spread call prices for a variance ratio of 1 and $T = 2$ weeks, and shows that this difference is more important for at-the-money calls. Exhibits 14 and 15 plot credit spread call prices for 1-month and 1-year maturities. Exhibit 14 shows again that the call prices pass below the intrinsic value even when the call is only slightly in the money. Exhibit 15 gives another important property of GARCH credit spread calls. They can be concave functions of the underlying credit spread. Because of mean reversion the dynamics of the credit spread do not satisfy the first-degree homogeneity property necessary for the convexity of options (see Merton, 1973). Exhibit 15 also shows that for longer maturities, the call price is almost a horizontal line. This means that the delta of a GARCH credit spread call could be a decreasing function of the underlying credit spread and that it decreases to zero as the time to maturity increases. These delta properties can be obtained from Equation (19). Although our model is a mean reverting GARCH, these properties did not come as a surprise since Longstaff and Schwartz (1995) have found that their continuous-time model exhibits such characteristics.

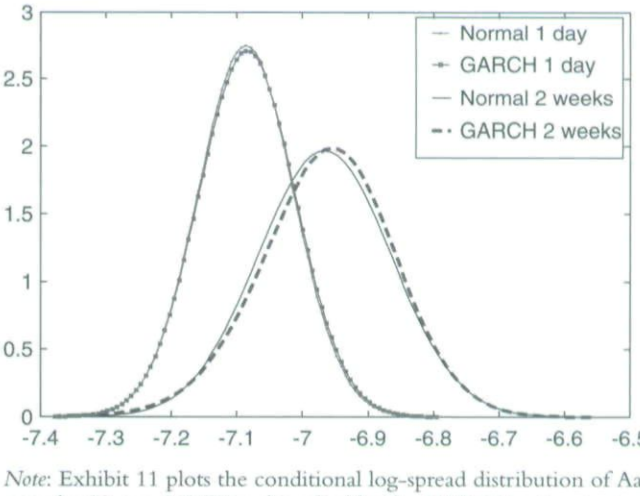
CONCLUSION

We have proposed a mean reverting GARCH model for credit log-spreads as an extension and a generalization of the Longstaff and Schwartz (1995) model which uses constant volatility. We used the Heston and Nandi (2000) GARCH specification to allow the variance process to depend on the past levels of the credit spread. Our model is then more flexible and captures the empirical properties of credit spreads in a better way than the traditional lognormal model. It also converges weakly to a continuous-time mean reverting square-root process model, which contains the Longstaff and Schwartz (1995) model as a special case. The GARCH model was estimated using the maximum-likelihood method and the important coefficients, especially the mean reversion and the skewness parameters, were found to be significant. We derived analytical solutions for European options on credit spreads using the inversion of the characteristic function technique shown in Kendall and Stuart (1977). Credit spread call prices under GARCH exhibit the same unusual properties found by Longstaff and Schwartz (1995). Indeed, the call value can be less than its intrinsic value and can be a concave function of the underlying credit spread. Finally, comparing our model to the Longstaff and Schwartz (1995) model, we have found that the difference between them is greater for at-the-money calls.

Although the closed-form solutions derived here are only for simple European calls and puts on credit spreads under GARCH; valuation expressions for other European derivatives on credit spreads, such as barrier options, could be derived with the same methodology. The main contribution of this paper is the closed-form valuation of credit spread derivatives with a stochastic volatility obtained within a GARCH framework. This implies direct pricing and hedging of derivatives using given values for the GARCH parameters; or conversely allows the underlying parameters of the model to be easily estimated using the market prices of credit derivatives.

EXHIBIT 11

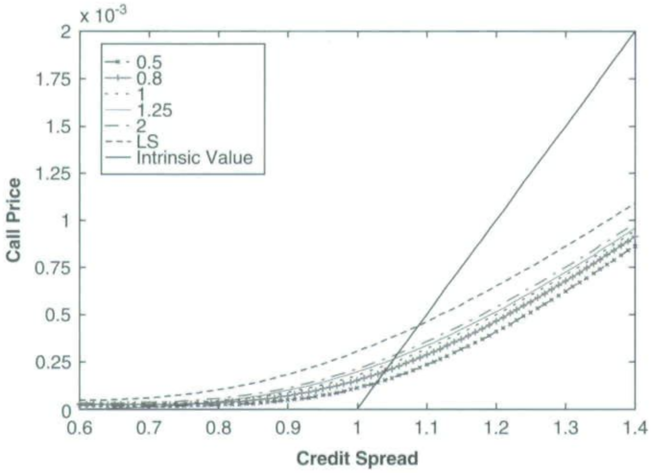
Implied Conditional Log-Spread Distribution



Note: Exhibit 11 plots the conditional log-spread distribution of Aaa over the 30-year US TBond implied by the GARCH estimates (see line 1 in Exhibit 6) for time horizons of 1 day and 2 weeks; and for a variance ratio of 1 which means that the initial variance h_0 is equal to the unconditional variance h_{unc}. It shows that the implied conditional distribution for the 1-day maturity coincides with the normal distribution; while the two distributions differ for a longer time horizon.

EXHIBIT 12

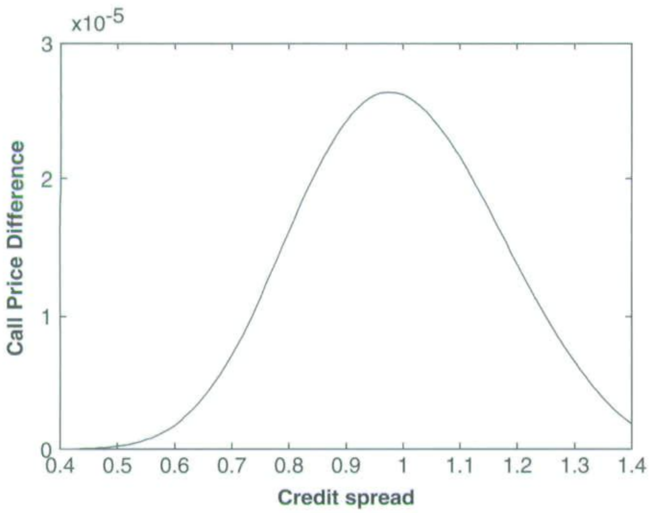
Credit Spread Call Price for Different Variance Ratios



Note: Exhibit 12 plots the Aaa credit spread call prices using the GARCH model for different variance ratios (0.5, 0.8, 1, 1.25 and 2) and the Longstaff and Schwartz (LS) model. GARCH parameters are reported in Exhibit 6 (line 1). The strike is equal to 1%, the risk-free interest rate is equal to 10% and the maturity is 2 weeks. For variance ratios less (greater) than 1, the GARCH call price is less (greater) than the LS call price. Both the GARCH and the LS call prices can be less than the call intrinsic value.

EXHIBIT 13

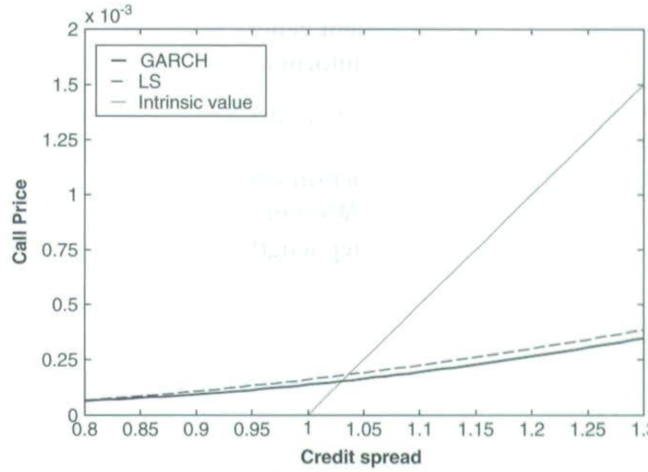
Credit Spread Call Price Difference Between the Longstaff and Schwartz (1995) and the GARCH Models for a 2-Week Maturity



Note: Exhibit 13 plots the Aaa credit spread call price difference between the Longstaff and Schwartz (LS) and the GARCH models for a variance ratio of 1. GARCH parameters are reported in Exhibit 6 (line 1). The strike is equal to 1%, the risk-free interest rate is equal to 10% and the maturity is 2 weeks. The LS at-the-money call price is equal to 2 basis points. The difference is more important at-the-money.

EXHIBIT 14

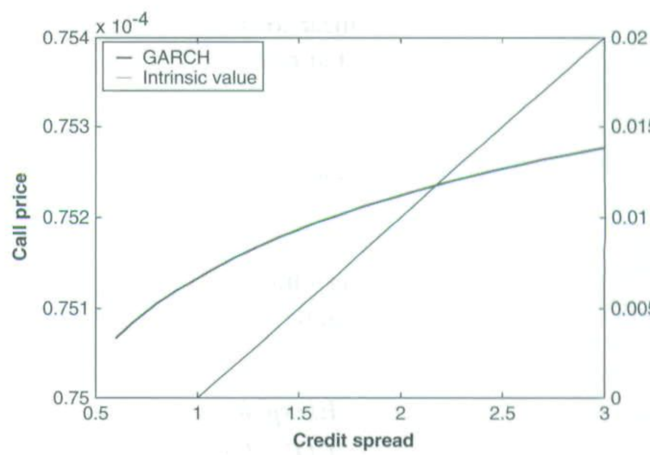
Credit Spread Call Price for a 1-Month Maturity



Note: Exhibit 14 plots the Aaa credit spread call prices using the GARCH model for a variance ratio of 1 and the Longstaff and Schwartz (LS) model. GARCH parameters are reported in Exhibit 6 (line 1). The strike is equal to 1%, the risk-free interest rate is equal to 10% and the maturity is 1 month. Both the GARCH and the LS call prices can be less than the call intrinsic value.

EXHIBIT 15

Credit Spread Call Price for a 1-Year Maturity



Note: Exhibit 15 plots the Aaa credit spread call price using the GARCH model for a variance ratio of 1. GARCH parameters are reported in Exhibit 6 (line 1). The strike is equal to 1%, the risk-free interest rate is equal to 10% and the maturity is 1 year (252 days). The GARCH call price can be a concave function of the underlying credit spread. In order to show the concavity, the scale for the axis of the call price is of 10^2 smaller than that of the axis of the intrinsic value, which implies that the call price is almost a horizontal line, and thus that the call delta is almost zero for long-term maturities.

APPENDIX A: DERIVATION OF THE MOMENT GENERATING FUNCTION

Recall that the moment generating function of X_T conditional on the time t information is given by:

$$f(t, T, \phi) = E_t(exp(\phi X_T)) \qquad \text{(A1)}$$

First, we calculate this function for $T = t + 1$ in order to get the log-linear form. Without any loss of generality, we assume that the time-step length Δ is equal to 1. We then have:

$$\begin{aligned} f(t, t+1, \phi) &= E_t(exp(\phi X_{t+1})) \\ &= E_t(exp(\phi\mu + \phi\gamma X_t + \phi\lambda h_{t+1} + \phi\sqrt{h_{t+1}}z_{t+1})) \\ &= exp(\phi\mu + \phi\gamma X_t + \phi\lambda h_{t+1})E_t(exp(\phi\sqrt{h_{t+1}}z_{t+1})) \end{aligned} \qquad \text{(A.2)}$$

We obtain a log-linear form for $f(t, t+1, \phi)$ by noting that z_{t+1} is conditionally normally distributed and that h_{t+1} is known at time t. Hence we can write:

$$f(t, t+1, \phi) = exp\left[\phi\mu + \phi\gamma X_t + \left(\lambda\phi + \frac{1}{2}\phi^2\right)h_{t+1}\right] \qquad \text{(A.3)}$$

In the same way, one can calculate $f(t, t+2, \phi)$ and find that we still obtain a log-linear form for the generating function. Let us assume that at time $t+1$, we have:

$$f(t+1, T, \phi) = exp(\phi\gamma^{T-t-1}X_{t+1} + A(t+1, T, \phi) + B(t+1, T, \phi)h_{t+2}) \qquad \text{(A.4)}$$

where $A(.)$ and $B(.)$ are deterministic functions. At time t, the generating function can be written, using the iterated expectations law:

$$\begin{aligned} f(t, T, \phi) &= E_t(exp(\phi X_T)) \\ &= E_t(E_{t+1}(exp(\phi X_T))) \\ &= E_t(f(t+1, T, \phi)) \end{aligned} \qquad \text{(A.5)}$$

Then we have by definition of $f(t+1, T, \phi)$:

$$f(t, T, \phi) = E_t[exp(\phi\gamma^{T-t-1}X_{t+1} + A(t+1, T, \phi) + B(t+1, T, \phi)h_{t+2})] \qquad \text{(A.6)}$$

Replacing X_{t+1} and h_{t+2} by their expressions as functions of X_t and h_{t+1} in Equation (2), we get:

$$f(t,T,\phi) = E_t\left\{\begin{aligned} &exp\left(\begin{aligned}\phi\gamma^{T-t}X_t + \phi\mu\gamma^{T-t-1} + A(t+1,T,\phi) \\ +\beta_0 B(t+1,T,\phi)\end{aligned}\right) \\ &\times exp\left(\beta_1 B(t+1,T,\phi)h_{t+1} + \phi\gamma^{T-t-1}h_{t+1}\right) \\ &\times exp\left(\begin{aligned}\beta_2 B(t+1,T,\phi)\left(z_{t+1} - \theta\sqrt{h_{t+1}}\right)^2 \\ +\phi\gamma^{T-t-1}\sqrt{h_{t+1}}\,z_{t+1}\end{aligned}\right)\end{aligned}\right\} \qquad \text{(A.7)}$$

The two first terms in the expectation operator are known at time t, hence the only term that we need to compute is the last one. This is where the Heston and Nandi (2000) GARCH specification gains the advantage over other GARCH models. The last term can be computed as a function of h_{t+1} using the fact that, for a standard normal variable z and constants a and b, we have:

$$E(exp(a(z+b)^2)) = exp\left(-\frac{1}{2}ln(1-2a)\right) \times exp\left(\frac{ab^2}{1-2a}\right) \qquad \text{(A.8)}$$

Under the historical probability, this can be done by developing and rearranging the last term in order to obtain a "perfect" square of z_{t+1} and h_{t+1}, added to a remaining term that depends on h_{t+1} and the GARCH parameters. Under the risk-neutral probability measure, we obtain the same calculus since z^*_{t+1} is a risk-neutral standard normal variable conditional on time t and that h_{t+1} is known and is constant conditional on time t. Finally, we get Equations (10) and (11) in the main text by rearranging all the terms in a log-linear form.

APPENDIX B: DERIVATION OF DELTA

Notice that the delta of a credit spread call is, by definition, equal to:

$$Delta = \frac{\partial C(t, T, X)}{\partial(e^X)} = \frac{\partial C(t, T, X)}{\partial X}e^{-X} \qquad \text{(B.1)}$$

Using Equation (14) that gives the call valuation expression, we can write:

$$e^{r(T-t)}\frac{\partial C(t, T, X)}{\partial X} = \frac{\partial f(1)P_1}{\partial X} - K\frac{\partial P_2}{\partial X} \qquad \text{(B.2)}$$

Notice that:

$$\frac{\partial f(t, T, \phi)}{\partial X} = \phi\gamma^{T-t} f(t, T, \phi) \quad \text{(B.3)}$$

Using Equation (15), we have:

$$\frac{\partial P_2}{\partial X} = \frac{\gamma^{T-t}}{\pi} \int_0^{+\infty} Re(K^{-i\phi} f(i\phi))\, d\phi \quad \text{(B.4)}$$

and

$$\frac{\partial f(1) P_1}{\partial X} = \gamma^{T-t} f(1) P_1 + \frac{\gamma^{T-t}}{\pi} \int_0^{+\infty} Re(K^{-i\phi} f(i\phi + 1))\, d\phi \quad \text{(B.5)}$$

In order to obtain Equation (19), we use a change of variable ($\varphi = \phi + i$) in Equation (B.4) to see that:

$$K\frac{\partial P_2}{\partial X} = \frac{\gamma^{T-t}}{\pi} \int_0^{+\infty} \mathrm{Re}(K^{-i\phi} f(i\phi + 1))\, d\phi \quad \text{(B.6)}$$

The only remaining term, the first one on the right-hand side of Equation (B.5), multiplied by the discount factor and by $exp(-X)$ gives the delta expression as in Equation (19).

ENDNOTES

[1] This is an analytical solution up to a recursive computation of the moment generating function.

[2] This observation is confirmed by the regression model estimated in Equation (1) and the estimation of the GARCH specification in Equation (2).

[3] In a previous version of this paper, and using data over the period 1986-92, we found that Aaa and Baa log-spreads were much more mean reverting with higher slope coefficients in absolute value.

[4] β_2 also controls the skewness through the term $\beta_2\theta$. But, if θ is zero, β_2 controls only the kurtosis.

[5] Notice that, unlike the GARCH model proposed here and the Heston (1993) model, the state variable X and its volatility process ν are perfectly correlated in the continuous-time diffusion limit.

[6] For the details about the risk-neutral probability measure and the preference assumptions, one can refer to Duan (1995), and more specifically Cvsa and Ritchken (2001) for non-traded assets such as interest rates.

[7] Both Fong and Vasicek (1992) and Cvsa and Ritchken (2001) make a similar assumption respectively for a continuous-time stochastic volatility and a GARCH interest rates models.

[8] A change of numeraire is needed. See Heston and Nandi (2000).

[9] As reported previously and like the regression model, using data over the period 1986-92, the GARCH specification yielded lower values for γ which means stronger mean reversion.

REFERENCES

Abadir, K.M., 1995, "The Limiting Distribution of the t Ratio under a Unit Root," *Econometric Theory*, Vol 11, 775-793.

Bollerslev, T., 1986, "Generalized Autoregressive Conditional Heteroskedasticity," *Journal of Econometrics*, Vol 31, 307-327.

Bollerslev T., R.Y. Chou, and K.F. Kroner, 1992, "ARCH Modelling in Finance: A review of the theory and empirical evidence," *Journal of Econometrics*, Vol 51, 5-60.

Cox J., J. Ingersoll, and S. Ross, 1985, "A Theory of the Term Structure of Interest Rates," *Econometrica*, Vol 53, 385-407.

Cvsa V., and P. Ritchken, 2001, "Pricing Claims under GARCH-Level Dependent Interest Rate Processes," *Management Science*, Vol 47, 1693-1711.

Duan J.C., 1995, "The GARCH Option Pricing Model," *Mathematical Finance*, Vol 5, 13-32.

Engle R.F., D.M. Lilien, and R.P. Robins, 1987, "Estimating Time Varying Risk Premia in the Term Structure: The ARCH-M Model," *Econometrica*, Vol 55, 391-407.

Engle R.F., and V.K. Ng, 1993, "Measuring and Testing the Impact of News on Volatility," *Journal of Finance*, Vol 48, 749-778.

Engle R.F., V.K. Ng, and M. Rotschild, 1990, "Asset Pricing with a Factor-ARCH Covariance Structure: Empirical Estimates for Treasury Bills," *Journal of Econometrics*, Vol 45, 213-237.

Feller W., 1966. *An Introduction to Probability Theory and Its Applications* (Vol 2), Wiley & Sons, New York.

Foster D., and D. Nelson, 1994, "Asymptotic Filtering Theory for Univariate ARCH Models," *Econometrica*, Vol 62, 1-41.

Heston S.L., 1993, "A Closed-Form Solution for Options with Stochastic Volatility, with Applications to Bond and Currency Options," *The Review of Financial Studies*, Vol 6, 327-343.

Heston S.L., and S. Nandi, 2000, "A Closed-Form GARCH Option Pricing Model," *The Review of Financial Studies*, Vol 13, 585-625.

Kendall M., and A. Stuart, 1977. *The Advanced Theory of Statistics* (Vol 1), Macmillan Publishing Co., Inc., New York.

Longstaff F.A., and E.S. Schwartz, 1995, "Valuing Credit Derivatives," *The Journal of Fixed Income*, Vol 5, 6-12.

MacKinnon J.G., A. Haug, and L. Michelis, 1999, "Numerical Distribution Functions of Likelihood Ratio Tests for Cointegration," *Journal of Applied Econometrics*, Vol 14, 563-577.

Nielsen B., 1997, "Bartlett Correction of the Unit Root Test in Autoregressive Models," *Biometrika*, Vol 84, 500-504.

Weiss A.A., 1984, "ARMA Models with ARCH Errors," *Journal of Time Series Analysis*, Vol 5, 129-143.