Executive Stock Options and Concavity of the Option Price

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The valuation of executive stock options for expensing purposes is of both practical and academic interest. There is widespread agreement that stock options should be recognized as an expense and accounting standard setters have issued guidelines on the methods that can be used to estimate their fair value. One simple method that has been proposed is based on the expected term of the option. *This method normally overstates the value of the option. This bias arises because of the non-linearity of the formula as a function of the time to maturity. For most parameter values the price is a concave function of the time to maturity. The authors show that this need not be so, and they find a set of sufficient conditions for the option price to be a concave function of time.*

The valuation of executive stock options (ESOs) for accounting purposes has been debated extensively in recent years. One of the simplified methods that has been proposed in this context leads to a bias. The nature of this bias is related to the non-linearity of the option price as a function of its time to maturity. We examine the nature of this relationship and show that the option price need not be a concave function of time to maturity and clarify some misconceptions in the literature on this point.

The first section of this article provides the context and discusses some of the issues involved in the valuation of ESOs. In particular, we discuss the *expected life of the option* and explain its significance in this context. The interpretation of expected life in some of the valuation guidelines is ambiguous and we note there are two conflicting definitions. The naive definition corresponds to the average time to exercise. It is well known that the use of this naive definition can in some circumstances introduce a bias if it is used as an input to the Black–Scholes–Merton (BSM) formula. We explore the origins of this bias in the following section and note that it is due to the non-linearity of the BSM price. The general impression seems to be that it is the concavity of the BSM price that leads to this bias.[1] We show that the option function need not be concave and examine the behavior of the second derivative of the BSM price with respect to time to maturity in the section "Concavity of the BSM price." Recent accounting changes have reduced the incentive for firms to issue options at the money and we show that this will tend to reduce the likelihood that the option price is a concave function of time to maturity. We also analyze the properties of the option when it is at the money and tidy up some statements that have appeared in the literature.

VALUATION OF EXECUTIVE STOCK OPTIONS

ESOs represent an important component of executive compensation. Over the past several years there has been a heated debate on the accounting for these instruments. For many years these options were valued at their so-called intrinsic value for accounting purposes.[2] This meant that as most options were granted with an exercise equal to the current stock price, their intrinsic value was zero.[3]

Accordingly, they were not recognized as an expense in a company's financial statements. In recent years, accounting standard setters have moved to require expensing of ESOs, where expense is based on fair value at the grant date. For example, the International Accounting Standards Board through its IFRS 2 has set guidelines for the expensing of executive stock options as has the Accounting Standards Board in Canada. In the United States, the detailed recommendations for expensing executive stock options are provided in FAS123(R). In March 2005, the SEC provided practical details on how it will interpret the standards in FAS123(R). Expensing represents a distinct improvement in income measurement as the opportunity cost of issuing ESOs (equivalently the dilution of shareholders equity that results) is brought into the financial statements proper.

However, there is typically no market value for ESOs. Consequently, fair value has to be estimated by means of an option pricing model. We should emphasize that the value required for expensing purposes represents the cost to the company and not the executive's valuation. The cost to the company will not usually be identical to the the value placed on the option by the executive. In this regard, we note a proposal by Cisco Systems Inc. (reported by Bloomberg, July 15, 2005) to base its ESO expense on the market price of a derivative financial instrument created especially for the purpose. This instrument, to be sold on the market, would have the same exercise price and five-year vesting as Cisco's ESOs and could not be traded or hedged. Any gains would be paid in Cisco shares. Its market value would be used to infer the company's ESO expense. To the extent investors are risk averse, this market value would be less than the actual ESO cost to the company. A variety of different models are available, of varying degrees of sophistication, that can be used to estimate the cost to the company. The best known is the BSM formula, which gives a closed-form expression for the value of a European option. There is evidence that with suitable modification the BSM model can produce reasonably reliable estimates of option value.[4] Given the simplicity of the BSM formula this valuation approach has obvious attractions.

The BSM formula requires six inputs:

1. The current stock price
2. The exercise price of the option
3. The risk-free rate
4. The dividend rate on the stock
5. The volatility of the stock
6. The maturity date of the option

The first three items are readily available, whereas the last three usually have to be estimated. The option value is quite sensitive to the last two.[5] The term of an ESO is uncertain, and this has to be taken into account in the valuation formula. There are detailed guidelines in FAS123(R) on how to select these inputs.

However, ESOs differ in some important ways from plain vanilla European options. In particular, ESOs can usually be exercised before their maturity, assuming that they have vested. To take account of this early exercise the BSM formula can be modified to produce an estimate of the option value. The modification consists of replacing the original time to contract maturity with a modified (shorter) time that allows for the possibility of early exercise. This modified time is often known as the *expected life* of the option. This terminology is unfortunate as its description in the guidelines admits at least two distinct interpretations.

FAS123(R) suggests a number of different methods for estimating what it calls the expected life. It is convenient to distinguish two of them, which reflect two different approaches. The first interpretation corresponds to our conventional notion of expected time to exercise. This interpretation stems from the first sentence of A27, which defines the expected life of the option as follows:

> The expected term of an employee share option or similar instrument is the period of time for which the instrument is expected to be outstanding (that is, the period of time from the service inception date to the date of expected exercise or other expected settlement).

We label this interpretation of the concept the expected life of the option, or the expected term to maturity. The accounting guidelines suggest that the real-world probabilities are to be used here and that they can be estimated from the historical experience.

To be clear, consider an example. Assume a firm has just granted an option that is fully vested after 1 year and has a full contract term of 10 years. If the probability of exercise after 1 year is 50% and the probability of exercise after 10 years is also 50%, then the expected life or expected term of the option at inception under this interpretation is 5.5 years:

$$5.5 = \frac{1}{2}1 + \frac{1}{2}10$$

The expected life may then be used as an input to the BSM model. For example, FAS123, paragraph 18b, states

> *In a closed form model the expected term is an assumption used in (or input into) the model.*

The second interpretation is to value the option using a method such as a lattice method or a Monte Carlo simulation approach that can explicitly model the exercise strategy of the executive. We will describe the mechanics of these models in more detail, but for now merely note that these models, if sensibly implemented, should be able to produce reasonable estimates of the fair value of the option. The output of a lattice model (or a Monte Carlo–based model) will be an estimate of the option price. Let us denote this value by V_0, assuming we are carrying out the valuation at time zero, the grant date. We can use this value to estimate an implied time to maturity under the BSM formula. To do this, we require assumptions about the first five inputs to the BSM model. The implied time to maturity is the time that equates the price under the BSM model with V_0. FAS123 describes this approach in paragraph A27, footnote 55. This second approach, then, starts with an estimate of the option price from a model that specifically incorporates assumptions about early exercise and then backs out an implied time to maturity that gives the same option price based on the BSM model. We label this interpretation of the concept the implied time to maturity. The implied term to maturity is computed as an output of the lattice model and then it is used again as an input to the BSM model. In this case the implied time to exercise is inferred from a model that uses the risk-neutral probabilities.

The SEC guidance notes discuss different acceptable methods of option valuation, and in particular the estimation of expected term. It is clear that the SEC views the first interpretation (our expected term to maturity) as a simple expedient that it expects will be replaced by a more sophisticated measure in future years. For example, the SEC bulletin poses the following question: "Is the staff aware of any simple methodologies that can be used to calculate expected term?"[6] Part of its response is that

> in the short term, some companies may encounter difficulties in making a refined estimate of expected term. Accordingly the staff will accept the following simplified method for plain vanilla options consistent with those in the fact set above: expected term = ((vesting term + original contractual term)/2). Assuming a ten year original contractual term and graded vesting over four years (25% of options in each grant vest annually) for the share options in the fact sheet described above the resultant expected term would be 6.25 years.

One does not have to read hard between the lines to see that the SEC does not encourage this approach. The bulletin states, "the staff does not expect that such a simplified method will be used for share option grants after December 2007."

Before proceeding we provide a brief discussion explaining how a lattice method can be used to value ESOs. Descriptions of the application of lattice methods to exchange traded options are given in standard textbooks.[7] A lattice method is based on a discretization of both the time and stock price variables and it is quite flexible in that it can reflect many of the characteristics of ESOs.[8] In particular it can incorporate the possibility of early exercise. The exercise decision can be based on the prevailing stock price or the duration of the contract or the historical experience of the company.

In the academic literature, lattice-type methods have been developed that treat the valuation of ESOs as a two-stage procedure. First a model of the executive's own subjective valuation of the option is constructed. As noted above, the executive's valuation of the option contract will not in general be equal to the company's valuation. However, the executive's valuation impacts the company's valuation, because the executive's early exercise strategy determines the future cash flows that will be payable under the contract. We can compute the company's cost by discounting these expected cash flows. The executive values the option in an expected utility framework, and this means we need to make assumptions about the executive's risk aversion and also the composition of the executive's wealth. Early papers using the expected utility approach include Huddart [1994], Kulatilaka and Marcus [1994], and Rubinsten [1995].

Recently, several papers have modified the utility-based approach through extensions to the basic model. These approaches require a lattice-based valuation framework. Chance and Yang [2004] employ an expected utility framework in a binomial setting to derive the early exercise strategy and hence compute the cost to the corporation. Cai and Vijh [2005] allow the executive to hold the market portfolio and analyze its impact on the executive's subjective valuation of the stock options. Bettis et al. [2005] note that some of the key inputs to the expected utility model, such as the executive's risk aversion and portfolio composition, are inherently unobservable. However, they are able to calibrate the model using actual exercise data and infer plausible parameters for the valuation exercise. These models are of interest as they may enable exercise patterns to be estimated based on more primitive variables than historical experience.

BIAS WHEN USING EXPECTED LIFE

In this section we discuss the implications of using these two different notions of expected life for option valuation. As mentioned, the use of the expected term can produce a bias in the resulting BSM option value, because the BSM price is a non-linear function of time. Hemmer et al. [1994] show that the use of the expected term will most likely overstate the fair value of the option to the firm. Their conclusion is based on the assumption that the BSM price is a concave function of the option price with respect to time to maturity. Although this is often the case, we will show that under some conditions the BSM function can instead be a convex function of the time to maturity, so that the use of the expected term can sometimes understate the fair value of the option.

We can illustrate the main points using an example. Suppose we have an ESO with a contract term of 10 years and that the option is fully vested after 1 year. We also assume that the expected term of this option is 5.5 years. The probability of exercise after 1 year is 50% and the probability of exercise after 10 years is also 50%. We assume the input parameters given in Exhibit 1.

EXHIBIT 1

Base Case Input Parameter Values

Parameter	Notation	Value
Current stock price	S_0	100
Exercise price	K	100
Riskfree rate	r	0.05
Dividend rate	q	0
Volatility	σ	0.50
Time	T	$1 \cdots 10.$

The option prices for these parameters are graphed in Exhibit 2. The BSM option price using a time to maturity of 5.5 years is 51.88. This corresponds to A on the graph. However, the fair value of the option assuming it is equally likely to be exercised either after 1 year or after 10 years is the average of the 1-year price and the 10-year price, which is 44.55. This is given by B in Exhibit 2. The bias arises because of the concavity of the option price. In this case the bias is 16.44% of the fair value. Hemmer et al. [1994] explain this point and propose an adjustment to remove the bias. Their formula for the fair value of the option is

$$C_{\text{vest}} + \left(\frac{C_{\text{mat}} - C_{\text{vest}}}{T_{\text{mat}} - T_{\text{vest}}} \right) (T_{\text{expterm}} - T_{\text{vest}}) \quad (1)$$

where

- T_{vest} is the time to vesting
- T_{expterm} is the expected term of the option
- T_{mat} is the time to contract maturity
- C_{vest} is the option price based on T_{vest}
- C_{mat} is the option price based on T_{mat}

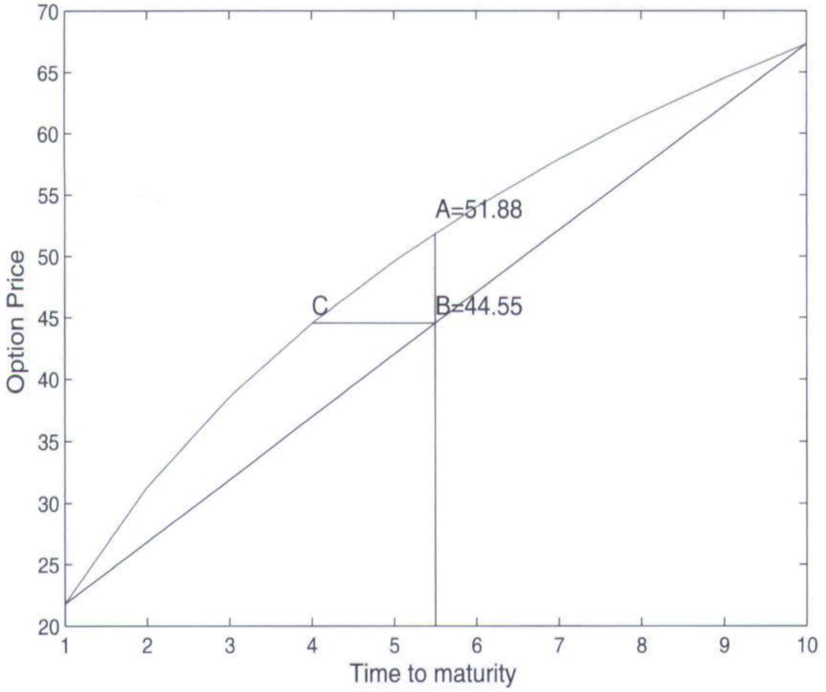
Applying this formula to the present case, we obtain the same fair value as before, namely, 44.55.

Now let us suppose that we are told that the fair value of the option price is 44.55. We can find the implied time to maturity that produces this same price under the BSM model. For this example the implied time to maturity is $4.0061 \approx 4.01$, and it produces a BSM value of 44.55. We label it point C on Exhibit 2. The expected term in this case is 5.5 years and the implied time to maturity is 4.01 years. These two concepts are different and it is misleading to refer to both of them as the expected life of the option. In effect, the use of the expected term in the BSM formula gives the fair value of an option whose contract term is exactly the expected life of the option. Using the implied time to maturity gives the expectation of the ex post value of the option. As the option price is a non-linear function of time to maturity, this latter construct is the appropriate measure of fair value.

For these parameter values the option price is a concave function of its time to maturity. A function is concave with respect to x if and only if its second derivative with

Exhibit 2

Plot of Option Prices under the BSM Model over a Region Where Price Is a Concave Function of Time to Maturity



The input stock price is 100. The strike price is 100. The time to option maturity ranges from 1 to 10 years and the volatility is 50%. Point A corresponds to the BSM option price based on a time to maturity of 5.5 years. Point C corresponds to the BSM option price based on a time to maturity of 4.01 years. Point B is the average of the 1-year option price (21.79) and the 10-year price (67.32). Point B corresponds to an option price of 44.55.

respect to x is negative. Although the second derivative of the BSM formula with respect to T is negative for many combinations of parameters, it is *not* negative for all combinations. The impression is sometimes given that it is always concave. For example, FAS123(R), paragraph A30, states

> Option value increases at a decreasing rate as the term lengthens for most if not all options. For example a two year option is worth less than twice as much as a one year option other things equal.

We will see that this statement is not true for all parameter combinations. In the next section we examine in detail the conditions under which the price function is concave. We find that for options that are either well in the money or out of the money the function is no longer concave, especially for shorter times to maturity. We illustrate this phenomenon in Exhibit 3, where the stock price is 50 and the volatility is 20% and the other parameters are as before. The option price is now a convex function of time to maturity.

In Exhibit 3 the fair value of the option is 4.51 (represented by point B), but the option price based on the expected life of the option (5.5 years as before) is just 2.90. In our terminology the ATM is 5.5 years. The implied time to maturity can be computed to be 6.78 years. In this case the fair value would be understated if we were to use the option price based on the expected term of 2.90. The degree of understatement is related to the convexity of the pricing formula. In this example the degree of understatement is almost 36% of the fair value. As we will show, the HMS adjustment, given by Equation (1), works here only because it is a very special case.

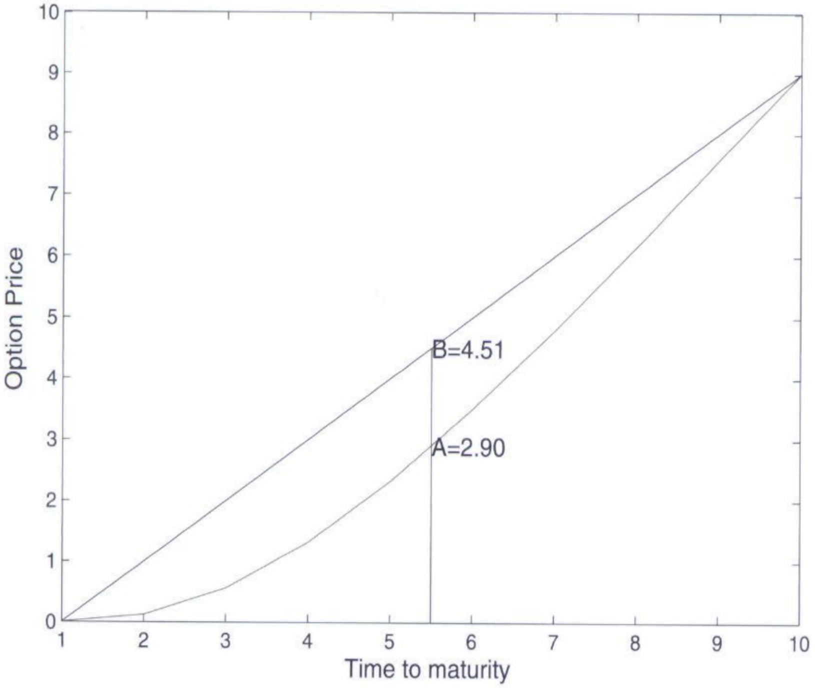
The HMS adjustment in its original form implicitly assumes there are only two relevant time points when computing the weighted option value. If we assume a more disperse distribution function, then the HMS adjustment is less.[9] The simplest way to see this is to use a numerical example. Suppose the options have a 10-year maturity and vest after 1 year. Assume that 10% of the options are exercised every year. In this case the expected term is 5.5 years as before. Assume the initial stock price is 100, the strike price is 100, and the interest rate is 5%. Exhibit 4 compares the *correct price* with the price based on the expected term and the price based on the HMS adjustment for different volatility assumptions.

Notice that in this case the expected term does a much better job than the HMS adjustment. The HMS adjustment is now less accurate because there are 10 possible exercise dates and not just 2. Although the option value based on the expected term is more accurate, it still overstates the true value of the option because for these parameter values the option is a concave function of the time to maturity.

To construct an example where the price function was convex the option would have to be either well in the money or well out of the money at issue date. Such contract specifications would have been unusual for ESOs in the past, but they are more likely to occur in future. Under the previous accounting rules there was a strong incentive

EXHIBIT 3

Plot of Option Prices under the BSM Model over a Region Where Price Is a Convex Function of Time to Maturity



The input stock price is 50. The strike price is 100. The time to option maturity ranges from 1 to 10 years and the volatility is 20%. Point A (2.90) corresponds to the BSM option price based on a time to maturity of 5.5 years. Point B (4.51) is the average of the 1 year option price and the 10 year price.

EXHIBIT 4

Comparison of Option Prices

Volatility	Correct price	BSM price based on expected term	Price using HMS adjustment
0.20	29.77	30.99	27.82
0.30	36.21	37.95	33.40
0.40	42.76	45.03	39.09
0.50	49.09	51.88	44.55

for firms to issue options at the money as this avoided any expense recognition under APB 25. However, now that the fair value is being used, there is no longer any compelling reason to always issue at-the-money options. For example, a firm that wishes to confer an immediate benefit may set the exercise price below the stock price. Alternatively, a firm may wish to increase employee motivation, and lower ESO cost, by setting the exercise price higher than the current share price. In these cases the BSM function may well be convex, in which case use of expected time to exercise will understate the fair value.

CONCAVITY OF THE BSM PRICE

This section examines the concavity of the BSM European call option price as a function of time. We show that for a range of inputs the relevant second derivative is negative, so the price is a concave function of time. The function tends to be concave for a range of stock prices that straddle the at-the-money contract and the range increases with increasing time to maturity. However, the option function is not always concave. We characterize the regions of the parameter space where the price function is convex. There is a widespread impression that the call price is always concave if the option is at the money. Although this is often true it is not always the case, and we derive a set of sufficient conditions for an at-the-money call price to be concave. We also examine the behavior of the option price as T goes to zero and show that the second derivative tends to zero unless the stock price is equal to the exercise price, in which case the second derivative tends to minus infinity.

The BSM price for a European call option is

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{2}$$

where

$$d_1 = \frac{\left(\log \frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\left(\log \frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$N(\cdot)$ denotes the standard cumulative normal density function, and the other symbols have their usual interpretation.

The first derivative of the call price with respect to T is

$$\frac{\partial c}{\partial T} = \frac{S_0 z(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT} N(d_2) \tag{3}$$

where

$$z(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

is the standard normal density function. Each term in Equation (3) is positive, and so this expression is positive. Hence the call price is an increasing function of T.

EXHIBIT 5
Parameter Values for Exhibit 6

Parameter	Value or range of values
S_0	$40 \cdots 220$
K	100
r	0.05
T	$0.5 \cdots 5$ *years*
σ	0.50

EXHIBIT 6
Second Derivative of BSM Option Price with Respect to T

Stock Price	Six months	One year	Two years	Three year	Four years	Five years
40	2.17	1.70	0.36	-0.01	-0.12	-0.15
60	5.01	0.20	-0.64	-0.56	-0.46	-0.38
80	-6.23	-3.50	-1.62	-1.01	-0.72	-0.55
100	-14.83	-5.51	-2.13	-1.25	-0.86	-0.65
120	-11.33	-5.20	-2.21	-1.33	-0.93	-0.70
140	-3.13	-3.62	-2.01	-1.30	-0.94	-0.72
160	3.06	-1.77	-1.67	-1.20	-0.91	-0.72
180	5.72	-0.22	-1.28	-1.07	-0.86	-0.70
200	5.90	0.87	-0.91	-0.93	-0.80	-0.67
220	4.92	1.53	-0.58	-0.79	-0.73	-0.64

A straightforward if tedious calculation shows that the second derivative is given by

$$\frac{\partial^2 c}{\partial T^2} = \frac{S_0 z(d_1)\sigma}{4T^{\frac{3}{2}}}(d_1 d_3 - 1) - rKe^{-rT}\left(\frac{z(d_2)d_4}{2T} + rN(d_2)\right) \quad (4)$$

where

$$d_3 = \frac{\left(\log\frac{S_0}{K}\right) - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_4 = \frac{\left(\log\frac{S_0}{K}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

The expression for the second derivative involves four terms and by inspection we cannot say whether the expression as a whole is positive or negative. We conducted a range of numerical simulations based on different input parameters. For brevity we provide just some representative results based on the parameter inputs given in Exhibit 5.

We computed the second derivative using Equation (4) for a range of current stock prices and maturity times. Specimen results are given in Exhibit 6. The second derivative is negative for the at-the-money contract. Holding the current stock price constant and increasing the time to maturity (moving to the right along the table) the second derivative is increasing. In the case of the short-term 6-month option the second derivative increases as we move away from the money (in both directions). It eventually becomes positive and then starts to decrease again as we get further away from the money.

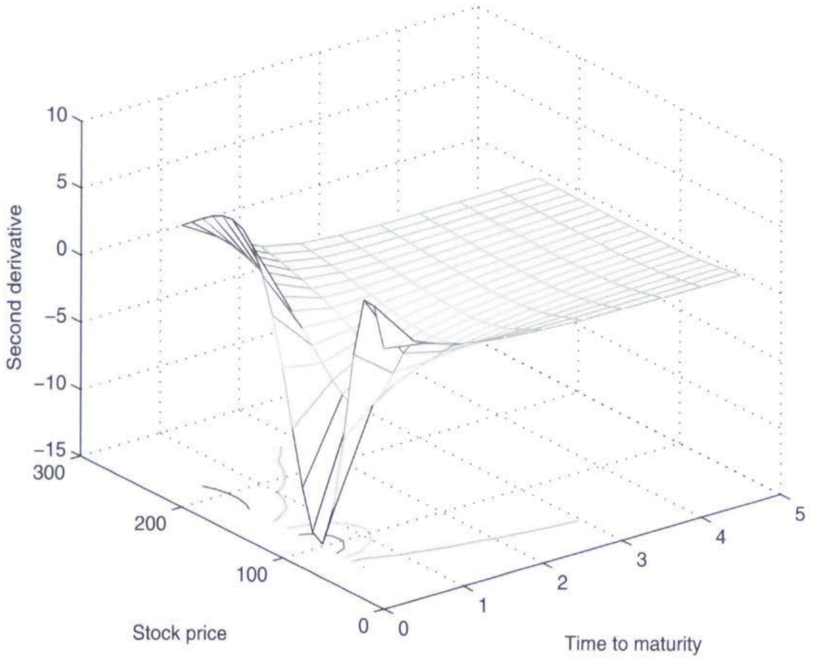
Exhibit 7 gives a three-dimensional representation of the second derivative for the numbers in Exhibit 5. We see that the derivative has a global minimum for the shortest-term option when the stock price is equal to the exercise price.

We can also summarize the information by plotting the sign of the second derivative (*Exhibit 8*).

Second Derivative when Option is at the Money

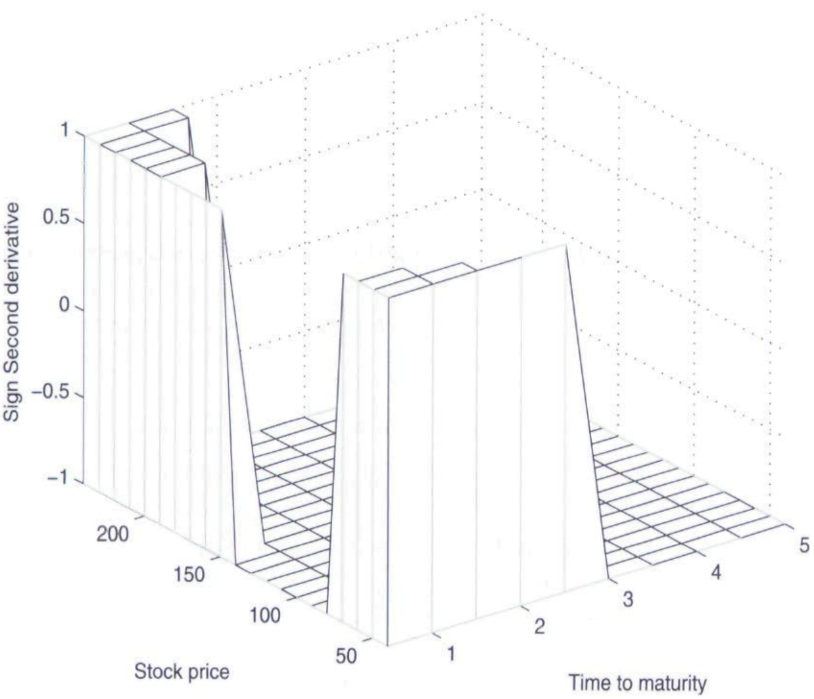
The behavior of the second derivative of the at-the-money option contract is of some special interest because in this case we are able to derive a set of sufficient conditions

EXHIBIT 7
Second Derivative of Call Price for Different Stock Prices and Maturities



Strike price = 100, risk-free rate = 0.05, σ = 0.5. Maturities range from 0.5 years to 5 years.

EXHIBIT 8
Sign of Second Derivative of Call Price for Different Stock Prices and Maturities



Exercise price = 100, risk-free rate = 0.05, σ = 0.5. Maturities range from 0.5 years to 5 years.

for the function to be concave. There appears to be a widespread impression in the accounting literature that the price function is concave when the stock price is equal to the exercise price. For instance HMS state, "Indeed when the strike equals the stock price at the grant date, as is the typical case for ESOs, the value of the call option is increasing in the time remaining to maturity at a decreasing rate." As justification for this statement reference is made to a result attributable to Henin and Ryan (1977).[10] The Henin and Ryan (HR) model is not the BSM model but it is similar. One feature of the HR model is that the risk-free rate is zero in their model. We will show analytically that the HR result as originally stated for their model is also valid for the BSM model for an at-the-money option when the risk-free rate is zero.[11] Subsequent authors have tended not to include this zero-interest-rate qualification when quoting the HR result. However, for at-the-money options the BSM formula is concave for most parameter combinations that are likely to occur in practice. In this subsection we are able to provide sufficient conditions that guarantee the function is concave.

First, we simplify the formula for the second derivative when the option is at the money. Assume that $S_0 = K$ so that

$$S_0 z(d_1) = Ke^{-rT} z(d_2)$$

In this case the second derivative in Equation (4) can be written

$$\frac{\partial^2 c}{\partial T^2} = Ke^{-rT}\left[z(d_2)\left(\frac{(r^2 - 2r\sigma^2 - \frac{\sigma^4}{4})T - \sigma^2}{4\sigma T^{\frac{3}{2}}}\right) - r^2 N(d_2)\right] \quad (5)$$

When $r = 0$, we have $z(d_1) = z(d_2)$ and so the expression on the right-hand side becomes

$$-S_0 z(d_1)\left(\frac{\left(\frac{\sigma^4}{4}\right)T + \sigma^2}{4\sigma T^{\frac{3}{2}}}\right)$$

This is clearly negative. This corresponds to the result proven by Henin and Ryan [1977] for their option model.

The sign of the second derivative is equal to the sign of the expression in square brackets in Equation (5), namely,

$$\left[z(d_2)\left(\frac{\left(r^2 - 2r\sigma^2 - \frac{\sigma^4}{4}\right)T - \sigma^2}{4\sigma T^{\frac{3}{2}}} \right) - r^2 N(d_2) \right]$$

Now this expression is negative if

$$\frac{\left(r^2 - 2r\sigma^2 - \frac{\sigma^4}{4}\right)T - \sigma^2}{4\sigma T^{\frac{3}{2}}}$$

is negative.[12] Hence a sufficient condition for the second derivative to be negative is[13]

$$\left(r^2 - 2r\sigma^2 - \frac{\sigma^4}{4}\right)T - \sigma^2 < 0$$

This is just a quadratic function of r, so we can recast the sufficient condition in terms of a range of values of r. If the risk-free rate lies in the interval

$$\sigma^2\left(1 - \sqrt{\frac{5}{4} + \frac{1}{\sigma^2 T}}\right) < r < \sigma^2\left(1 + \sqrt{\frac{5}{4} + \frac{1}{\sigma^2 T}}\right),$$

then the second derivative in Equation (5) is negative. Notice that zero lies inside this range.

There are parameter combinations for which the second derivative in Equation (5) is positive. To get these we require that the volatility is low relative to the risk-free rate. We record a combination that leads to this result (a positive second derivative) (*Exhibit 9*).

For these inputs the value of Equation (5) is 0.3759. Note the value of σ required to obtain this result is unreasonably low. Of course, if the ESO is not issued at the money, the likelihood of convexity is increased.

Sufficient Conditions for Second Derivative to be Negative

It is of interest to see whether we can find a set of conditions such that the second derivative is always negative. This suggests we let the strike price vary with time as follows

$$K(T) = S_0 e^{\lambda r T}$$

where $K(0) = S_0$ and we will let λ vary. The case when $\lambda = 0$ has already been considered in the previous subsection. Proceeding as before, we find that the second derivative of the call price can be expressed as

$$\frac{\partial^2 c}{\partial T^2} = S_0 \frac{z(d_1)}{4\sigma\sqrt{T}}\left(r^2(1-\lambda)^2 - 2r(1-\lambda)\sigma^2 - \frac{\sigma^2}{T} - \frac{\sigma^4}{4} \right) - [r(1-\lambda)]^2 K_0 e^{-r(1-\lambda)T} N(d_2).$$

When $r(1-\lambda) = 0$, the second derivative is always negative. This happens in one of two ways. Either $r = 0$, which corresponds to the case just discussed, or $\lambda = 1$. The case of $\lambda = 1$ is of interest; the strike price increases at the risk-free rate so that

$$K(T) = S_0 e^{rT}$$

This means that if the strike price is always set equal to the forward price then the option price will be a concave function of time. We can now state the following proposition.

Proposition. *If the exercise price $K(T) = S_0 e^{rT}$, then the BSM price is everywhere a concave function of time to maturity.*

Although, in theory, a firm could issue stock options with a strike price that increased at the risk-free rate this

EXHIBIT 9

Combination of Inputs That Produce a Positive Second Derivative of the BSM Option Price for an At-the-Money Contract

Parameter	Value
S_0	100
K	100
r	0.06
T	0.2
σ	0.02
$\frac{\partial^2 c}{\partial T^2}$	0.3759

does not appear likely to occur in practice. However, one can obtain the same effect by using a volatility that declines over time in a particular way. We can best illustrate this numerically. The second column of Exhibit 10 gives standard option prices based on our benchmark parameters. These are $S_0 = K = 100$, $r = 0.05$, $\sigma = 0.50$, $T = 1 \ldots 10$. The third column shows the corresponding option prices when the strike price increases at the risk-free rate. Notice that the increasing strike price reduces the value of the call option from the standard call based on constant strike price. In the case of the 10-year option the price for the standard option with a strike price of 100 is 67.316. The corresponding option contract in column (3) has a strike price of 164.872 and a price of only 57.080.

We can compute the volatility required to give a price of 57.080 for a standard 10-year call option with a strike price of 100. This works out to be 35.9%. So if one used a constant strike price and a declining volatility as shown in column (4), this would give the same option prices as column (3). These very same prices correspond to the case of constant volatility and an increasing strike price.

This result is true in general and not just for the parameters in Exhibit 10. If the term structure of volatilities is decreasing according to the exact pattern of column (4), then the option price based on a constant strike price has exactly the same behavior with respect to time to maturity as if the strike price increased at the risk-free rate and the volatility were fixed. In this case the option price will be a concave function of time to maturity.

EXHIBIT 10

Call Options Prices Based on a Constant and Increasing Strike and Implied Volatilities That Equate the Two Prices

Time to Maturity	Price of Standard Call Option $K = 100, \sigma = 0.50$	Price of Call Option with increasing exercise price $K = 100e^{.05*T}$, $\sigma = 0.50$	Volatility implied by option prices in column(3) $K = 100$
(1))	(2)	(3)	(4)
1	21.793	19.741	44.5%
2	31.328	27.633	42.6%
3	38.568	33.499	41.3%
4	44.521	38.292	40.2%
5	49.596	42.385	39.3%
6	54.015	45.971	38.5%
7	57.915	49.167	37.8%
8	61.388	52.050	37.1%
9	64.505	54.675	36.5%
10	67.316	57.080	35.9%

Behavior of the Second Derivative as T Tends to Zero

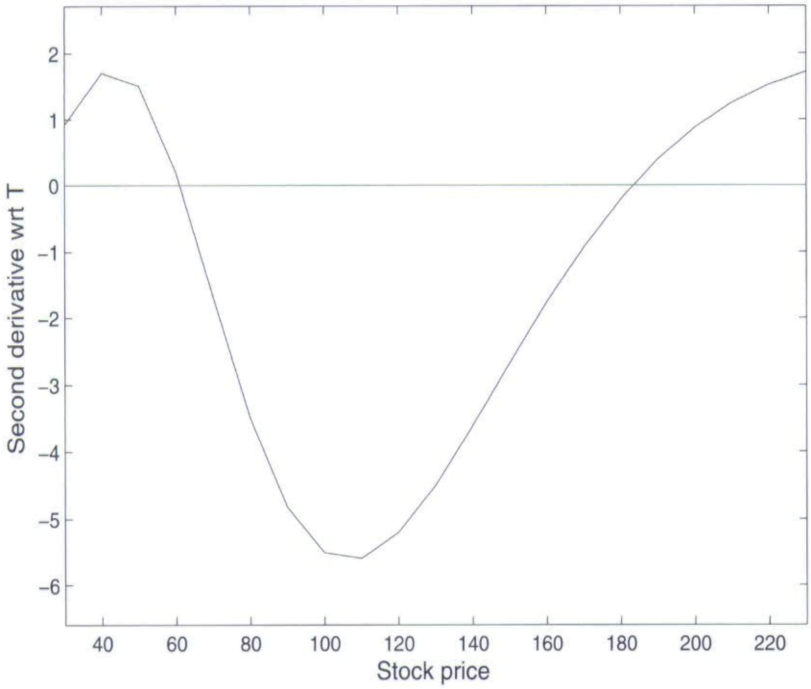
Analyzing the behavior of the second derivative as T tends to zero helps us understand the patterns we observe for non-zero T. We derive the analytical results in the appendix. Here we explore the behavior as T gets small for our benchmark example. We illustrate what is happening using graphs (*Exhibits 11–13*). In these graphs we let T get smaller and smaller and examine the behavior of $\partial^2 c/\partial T^2$ as a function of the stock price. In Exhibit 11 the time to maturity is 1 year. We see that when the stock price is equal to the strike price the second derivative is negative. As the stock price moves away from the strike price, in either direction, the second derivative increases and eventually becomes positive, reaches a maximum, and begins to decline.

Exhibit 12, where $T = 0.1$, shows the same behavior. The second derivative is large and negative when the stock price is equal to the strike price. It increases as the stock price moves away from the strike price and reaches a local maximum in either direction. Then it declines again in both directions.

Exhibit 13 shows that for very small times to maturity, the second derivative has a singularity when the stock price equals the strike price. It is this singularity that dominates the behavior as time to maturity

EXHIBIT 11

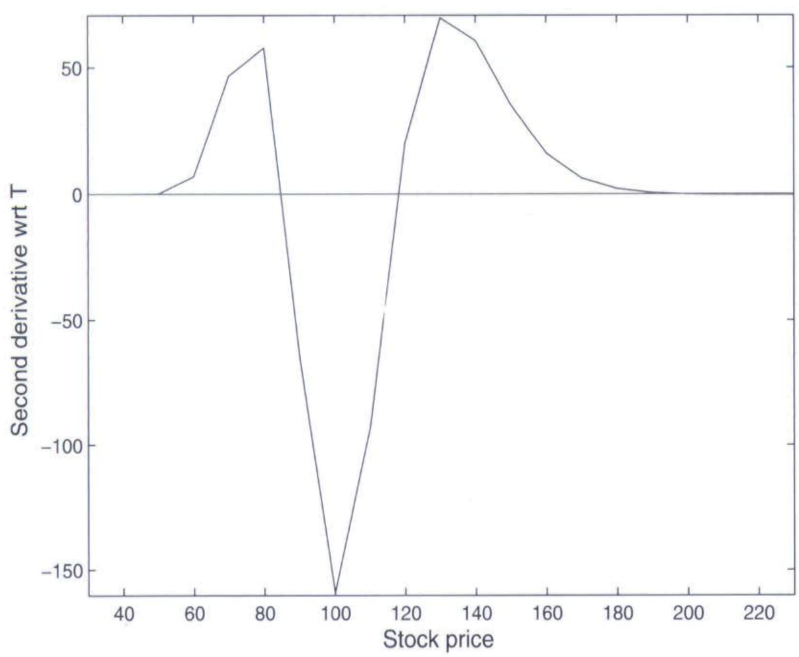
Second Derivative of Call Price with Respect to *T* As a Function of the Current Stock Price



The stock price ranges from 30 to 230. The other parameters are $K = 100$, $\sigma = 0.5$, $r = 0.05$, and $T = 1$.

EXHIBIT 12

Second Derivative of Call Price with respect to *T* As a Function of the Current Stock Price



The stock price ranges from 30 to 230. The other parameters are $K = 100$, $\sigma = 0.5$, $r = 0.05$, and $T = 0.1$.

becomes small. In addition, as we let time to maturity become larger the lowest values of the second derivative occur when the option is more or less at the money.

SUMMARY

This article examined the role of the expected term in the valuation of executive stock options for expensing purposes. We critiqued the concept of expected term and noted some confusing aspects in the current accounting guidelines. We also explored the nature of the non-linearity of the BSM option formula as a function of time to maturity. The option function tends to be concave for a range of stock prices that straddle the strike price of the contract and the range increases with increasing time to maturity. We showed that if the strike price is set equal to the forward price, the option price is a concave function of time to maturity. We analyzed the behavior of the second derivative of the BSM formula for at-the-money options and clarified and corrected some statements that have appeared in the literature.

Under the first interpretation of the concept of expected term, the concavity means that a naive application of the BSM formula overstates the value of an ESO. However, the second derivative of the option function can also be positive for certain parameter ranges. These ranges are now more likely to be relevant because the expensing of ESOs has reduced the incentive to issue at-the-money contracts. For these cases the naive use of the expected term leads to understatement of the option value. However, if we assume that exercise dates are distributed over the life of the option the bias is reduced.

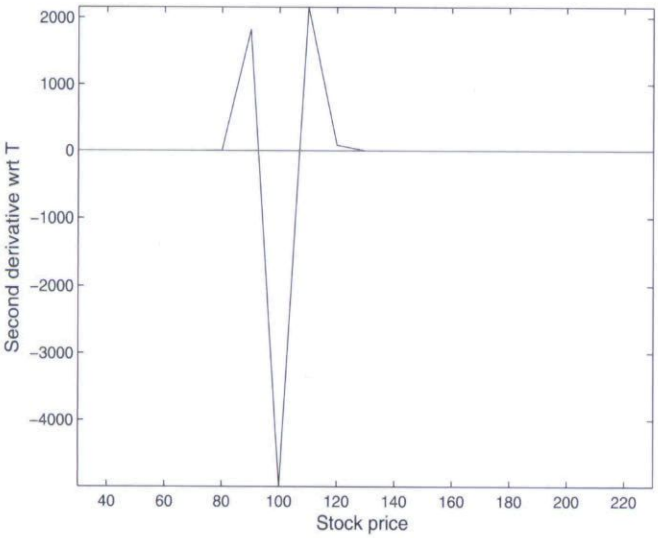
APPENDIX

Limit when Stock Price Equals Strike Price

If we let the stock price equal the exercise price and take the limit of the second derivative as T tends to zero we obtain

EXHIBIT 13

Second Derivative of Call Price with respect to T As a Function of the Current Stock Price



The stock price ranges from 30 to 230. The other parameters are $K = 100$, $\sigma = 0.5$,

$$\frac{K\sigma}{4\left(\sqrt{2\pi}\right)T^{\frac{3}{2}}}\left(-\frac{\left[r+\frac{\sigma^2}{2}\right]^2 T}{\sigma^2}-1\right)+\frac{rK\left(r-\frac{\sigma^2}{2}\right)}{2\sigma\left(\sqrt{2\pi}\right)\sqrt{T}}-r^2\frac{K}{2}$$

The dominant term here as T tends to zero comes from the second term in the first bracket. The behavior of the second derivative as T tends to infinity for $S = K$ is governed by

$$-\frac{K\sigma}{4\left(\sqrt{2\pi}\right)T^{\frac{3}{2}}} \tag{A-1}$$

Limit when Stock Price is not Equal to Strike Price

In this case let

$$x = ln\left[\frac{S}{K}\right] \neq 0$$

Recall that the expression for the second derivative is

$$\frac{\partial^2 c}{\partial T^2} = \frac{Sz(d_1)\sigma}{4T^{\frac{3}{2}}}(d_1 d_3 - 1) - rKe^{-rT}\left(\frac{z(d_2)d_4}{2T} + rN(d_2)\right)$$

where S is the stock price. This expression can be written as

$$\frac{\partial^2 c}{\partial T^2} = \frac{S\sigma}{4\left(\sqrt{2\pi}\right)T^{\frac{3}{2}}}\exp\left(-\frac{1}{2}\left(\frac{x+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{(T)}}\right)^2\right)\left[\frac{x^2-\left(r+\frac{\sigma^2}{2}\right)^2 T^2}{\sigma^2 T}-1\right]$$

$$-rKe^{-rT}\left[\frac{\frac{1}{\left(\sqrt{2\pi}\right)}\exp\left(-\frac{1}{2}\left(\frac{x+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{(T)}}\right)^2\right)}{2T}\left[\frac{x-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right]+rN(d_2)\right]$$

As T tends to zero this expression tends to

$$\frac{S\sigma}{4\left(\sqrt{2\pi}\right)T^{\frac{3}{2}}}\exp\left(-\frac{x^2}{2\sigma^2 T}\right)\left[\frac{x^2}{\sigma^2 T}-1\right]$$

$$-rKe^{-rT}\left[\frac{\exp\left(-\frac{x^2}{2\sigma^2 T}\right)}{2T\sqrt{2\pi}}\left(\frac{x}{\sigma\sqrt{T}}\right)+r\frac{1+\text{sign}(x)}{2}\right]$$

The dominant term that dictates the behavior as T tends to zero is the first one. This term can be written as

$$\frac{Sx^2}{4\sigma(\sqrt{2\pi}}\left[\frac{\exp\left(-\frac{x^2}{2\sigma^2 T}\right)}{T^{\frac{5}{2}}}\right]$$

The term in square brackets gives the dependence on T and this term tends to zero as T tends to zero.

ENDNOTES

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[1]That is, concavity with respect to time to maturity.

[2]The intrinsic value is defined to be max $(S - K, 0)$, where S is the current stock price and K is the exercise price of the option.

[3]The exercise price is also known as the strike price.

[4]See Amman and Seiz [2004], Carpenter [1998], Marquardt [2002], and Bettis et al. [2005].

[5]There are some key differences in the estimation of these parameters in the case of an ESO as compared with an exchange-traded option. As ESOs are longer-dated instruments the volatility assumption should be the future expected volatility over the term of the contract. Unlike a standard European option, where the maturity is equal to the contractual term, an ESO's term can often be less than its contractual life as an executive may wish to exercise the option early for personal reasons.

[6]Page 36, question six.

[7]See, e.g., Cox and Rubinstein [1985], Hull [2004].

[8]This flexibility is a double-edged sword, however, as it could possibly be used to manipulate the ESO valuation.

[9]We thank the referee for pointing this out and for suggesting the example.

[10]Hemmer, Matsunaga, and Shevlin state in footnote 11, page 27, that Henin and Ryan prove that for options with strike price equal to current stock price$(S = K)$ the value of a call option increases at a decreasing rate as a function of time to expiration.

[11]The HR result is that the at-the-money option price is concave.

[12]As the term $-r^2N(d_2)$ is always negative.

[13]This is not the strongest possible sufficient condition as we do not allow for the contribution of $-r^2N(d_2)$. However, we are able to get closed-form results with this simplification.

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