On the Pricing of Power and Other Polynomial Options

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Options with non-linear payoffs offer flexibility to investors, but there are few closed formulas known for European options with non-linear payoffs. An adapted decomposition of the payoff can facilitate pricing with closed-form formulas. We show that it is possible to price a polynomial option by expressing it as a combination of several power options with adapted strikes.

The general result is independent of the particular dynamics followed by the underlying. Several packages and parabolic calls are studied, along with an example of polynomial options in portfolio selection.

Options are now traded almost everywhere. As common financial tools, they have been widely analyzed. Because many financial and management decisions defend on contingent claims analysis, option pricing theory has become quite important.

Competitive markets have prompted innovations in at least two directions. First, we have come to use more realistic assumptions concerning price dynamics. Second, new types of options have appeared to meet particular demands.

The classic Black and Scholes [1973] model assumes that the price of the underlying of the option follows geometric Brownian motion. In the last three decades many models have relaxed this assumption and have produced extensions of the famous formula. Several authors since Cox and Ross [1976] and Merton [1976] have shown how to take the smile effect into account, whether suggesting a particular stochastic dynamics for volatility term or introducing jumps. Examples include Hull and White [1987]; Heston [1993]; Bakshi, Cao, and Chen [2002]; Boyarchenko and Levendorskii [2002]; and Carr et al. [2002].

The great variety of exotic options created offer interesting hedging in fund management. See Nelken [1996] and Zhang [1998], for instance. Most of the payoffs of these exotic options are linear combinations of several prices, so extension to the exact pricing obtained by Black and Scholes is still linear for the payoff.

Many instruments or projects do not generate linear cash flows, however. In fact, the potential profiles of curved payoffs are unlimited. Depending on the bargaining power of the entities in a transaction, options can be designed without the usual 45-degree profile. Examples of non-linear cash flows are squared LIBOR notes in the interest rate markets and compensation options for top management. Authors de Jager and Winsen [1992] describe a chief executive officer negotiating a European stock option that pays off 50/40 USD at maturity if the underlying price is between 90 USD and 110 USD; otherwise the option has the payoff of a standard European call with strike 100. This option with a partly polynomial payoff benefits the CEO more than a plain vanilla call.

Another example of polynomial options occurs in portfolio choice. According to Prigent [2005], investors who want a guarantee for terminal wealth of a linear type in a risky investment can express the value of optimal wealth using a polynomial option.

So what happens when a more involved contingent payoff than the traditional plain vanilla is traded? Is it possible to obtain an exact price for a call without a linear payoff? The answer is that in general there are no closed-form formulas, and solutions must be obtained through Monte Carlo simulations, trees, or other numerical procedures.

We focus on a rather general class of non-linear payoffs, particularly polynomials. Oddly enough, there is little analysis of this class of options. Certainly from a theoretical point of view, these options deserve attention.

The simplest options of this kind are, of course, *power options*, which are well known and actually traded (sometimes power options are also called *turbo options*). A powers option has the payoff of a standard European option with the price of the underlying asset raised to the power α (where α is a positive number). We show how to price power options in the Black and Scholes model, in a stochastic volatility model, and in a pure jump context.

We also study the general case of other kinds of polynomial options. Instead of trying to develop more or less sophisticated probabilistic computations, we look for an adapted decomposition of their payoff. Our main result is that polynomial options can be priced as a portfolio of power options (but with adapted strikes). As far as we know, the formulas we give are new. Furthermore, the general pricing formulas do not depend on a particular dynamics; the only requirement is to know how to price a power option.

Our first three sections are essentially theoretical; the next three are more practical. We explores the particular case of parabolic options, several packages of options, and the link between polynomial options and portfolio selection.

I. GENERAL SETTING

We consider a financial market with two primary assets: a risky asset, and a riskless asset. The latter is an instantaneous bond with return r. This return is the interest rate in the economy and is assumed to be constant. Derivative assets have the risky asset as the underlying. The market is assumed to be perfect.

We generally assume that the risky asset price, S, is modeled as the exponential of a Lévy process. The simplest case is represented by the Black and Scholes model where the Lévy process follows arithmetic Brownian motion. Thus its price follows a stochastic differential equation in the actual world:

$$\frac{dS}{S} = \mu dt + \sigma dz \tag{1}$$

where μ and the volatility σ are assumed to be constants, and the process z is a standard Brownian motion.

Mathematically speaking, we consider the probability space (Ω,A,P) and filtration $\{F_t\}_{t\geq 0}$ generated by the Brownian motion z. We refer to $(\Omega,A,\{F_t\}_{t\geq 0},P)$ as the historical or actual universe, and P is the historical probability measure. Using standard results from arbitrage theory, in the risk-neutral universe $(\Omega,A,\{F_t\}_{t\geq 0},Q)$ where Q is the risk-neutral probability measure, the dynamics for S is:

$$\frac{dS}{S} = rdt + \sigma d\hat{z}. \tag{2}$$

The process $\hat{z}$ is a standard Brownian motion under Q. The Q-measure is equivalent to P and is such that the discounted prices at rate r are Q-martingales. Q is called an *equivalent martingale measure* (EMM). In the Black and Scholes setting, this measure is unique, and the market is complete, and then the random variable $\ln(S_T)$ follows a normal low in the risk-neutral universe. Simple probabilities (when the precise low is known) no longer hold in other situations involving S_T, except in the case of (simple) power options (S_T being the price at maturity T).

II. POWER OPTIONS

Power options have been created to modify the patterns of plain vanilla options. The payoff of a power option is the difference between the underlying asset price at maturity raised to a strictly positive power and the strike price for a call (or the opposite of this difference for a put). Formally, a α-power call (where $\alpha > 0$) with maturity T and strike K is a European call with a payoff at maturity expressed as

$$\text{Payoff} = \max\left(\left(S_T\right)^{\alpha} - K, 0\right).$$

For a call where the power is higher (lower) than one, the power option has a payoff greater (smaller) than the payoff similar vanilla option. These options offer payoff patterns different from the classical linear payoff, and thus offer more flexibility.

We first establish a general formula, and then turn to the easy pricing when the underlying price is lognormal. We end by introducing two more elaborate schemes. In the first one, we show how to price these options in a stochastic volatility model; in the second one, we price the power option when the underlying price follows a pure jump process.

General Formula

Assume we have an equivalent martingale measure, Q, such that discounted prices at the constant interest rate r are Q-martingales. The α power call price is then given by:

$$C_\alpha(K,t) = e^{-r(T-t)} E^t_Q\left[\left(S_T^\alpha - K\right)^+\right].$$

Using the decomposition

$$e^{-r(T-t)} E^t_Q\left[\left(S_T^\alpha - K\right)^+\right] = e^{-r(T-t)} E^t_Q\left[S_T^\alpha 1_{S_T^\alpha > K}\right] - Ke^{-r(T-t)} Q\left(S_T^\alpha > K\right),$$

and the change of measure defined by the Radon-Nikodym derivative

$$\frac{dQ^\alpha}{dQ} = \frac{S_T^\alpha}{E_Q[S_T^\alpha]}$$

one obtains:

$$C_\alpha(K,t) = e^{-r(T-t)} E^t_Q[S_T^\alpha] \cdot Q_t^\alpha(S_T^\alpha > K) - Ke^{-r(T-t)} \cdot Q_t(S_T^\alpha > K) \qquad (3)$$

In a way, Equation (3) extends the well-known formula for a European call generally obtained via changes of numeraire (see El Karoui, Geman, and Rochet [1995]).

We use this formula in the Black–Scholes setting and in a stochastic volatility model.

Power Options in the Black and Scholes Model

It is easy to see that the power $(S_T)^\alpha$ follows a lognormal law. Indeed, as $\ln(S_T)$ is normal, then $\ln((S_T)^\alpha) = \alpha \cdot \ln(S_T)$ is normal too, but with parameters:

$$-\text{mean}: m_\alpha = \alpha \cdot \left[\ln(S_t) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right]$$

$$-\text{and variance}: v_\alpha = \alpha^2 \cdot \sigma^2 (T-t).$$

Then, using Equation (3) and after some elementary computation, one obtains the α-power call price in the Black and Scholes model:

$$C_\alpha(K,t) = e^{(\alpha-1)\cdot r\cdot(T-t) + \left[\alpha^2-\alpha\right]\cdot\frac{1}{2}\sigma^2\cdot(T-t)} \times S_t^\alpha \cdot N(d_{1,\alpha}) - K \cdot e^{-r(T-t)} \cdot N(d_{2,\alpha})$$

where

$$d_{1,\alpha} = \frac{\ln\left[\frac{S_t^\alpha}{K}\right] + \alpha \cdot \left[r + \left(\alpha - \frac{1}{2}\right)\cdot\sigma^2\right]\cdot(T-t)}{\alpha\cdot\sigma\cdot\sqrt{T-t}}$$

$$d_{2,\alpha} = \frac{\ln\left[\frac{S_t^\alpha}{K}\right] + \alpha \cdot \left[r - \frac{1}{2}\cdot\sigma^2\right]\cdot(T-t)}{\alpha\cdot\sigma\cdot\sqrt{T-t}}.$$

and $N(x)$ stands for the cumulative function at x of a standard Gaussian random variable. This formula is valid for any, $\alpha > 0$, and when $\alpha = 1$ we can express the classic Black–Scholes formula.

Power Options in a Stochastic Volatility Model

A stochastic volatility model describes an incomplete market, so we consider that an EMM Q has been chosen; see Heston [1993]. In this setting, the spot asset price S follows the diffusion, in the historical universe:

$$dS = \mu S dt + \sqrt{v(t)} S dz_1$$

$$dv = \kappa(\theta - v)dt + \eta\sqrt{v}dz_2$$

where μ, κ, and η are constants, and the processes z_1 and z_2 are two Brownian motions with correlation ρ. Here the square volatility follows a square root process, but other frameworks can be chosen, such as an Ornstein-Uhlenbeck process for the square volatility (see Stein and Stein [1991]).

Let $X = \ln(S)$ and assume, as in Heston [1993], the expression as follows for the volatility risk premium: $\lambda(S,v,t) = \lambda\sqrt{v_t}$. Then, the processes X and v follow in the risk-neutral universe the diffusions:

$$dX = (r - \frac{1}{2}v)dt + \sqrt{v}d\hat{z}_1$$

$$dv = (a - (\kappa + \lambda\eta)v)dt + \eta\sqrt{v}d\hat{z}_2$$

where $\hat{z}_1$ and $\hat{z}_2$ are two Q-Brownian motions with correlation ρ.

We assume $K^* = (\ln K)/\alpha$. To get the power option price, we use Equation (3). As the probabilities of the event $X_T > K^*$, under Q and Q^α, are not immediately available in closed form, following Heston [1993], we introduce characteristic functions and use the formulas:

$$Q_t(S_T^\alpha > K) = Q_t(X_T > K^*)$$
$$= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\exp\left\{\frac{-iuK^* \hat{f}(u)}{iu}\right\}\right]du \quad (4)$$

$$Q_t^\alpha\left(S_T^\alpha > K\right) = Q_t^\alpha\left(X_T > K^*\right)$$
$$= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\exp\{-iuK^*\}\hat{f}^{(\alpha)}(u)}{iu}\right]du \quad (5)$$

where $\hat{f}$ and $\hat{f}^{(\alpha)}$ are the Fourier transforms of X_T under the measures Q and Q^α, or more precisely:

$$\hat{f}(u) = E_Q^t\left[e^{iuX_T}\right]$$
$$\hat{f}^{(\alpha)}(u) = E_{Q^\alpha}^t\left[e^{iuX_T}\right] = \frac{1}{E_Q^t\left[S_T^\alpha\right]} E_Q^t\left[e^{(\alpha+iu)X_T}\right].$$

In order to use Equation (3), the only remaining step is to compute the expectation of S_T^α, but we should note that:

$$E_Q^t\left[S_T^\alpha\right] = \hat{f}(-i\alpha)$$

The characteristic function can be obtained in closed form, and the integrands in Equations (4) or (5) are smooth functions, so the numerical integration presents no difficulties.

Power Options in a Pure Jump Lévy Model

Let $X = \{X_t\}_{t\geq 0}$ be a one-dimensional Lévy process in a probability space. The characteristic function of the random variable X_t can be represented in the form $E\left[e^{i\xi X_t}\right] = \exp\{-t\psi(\xi)\}$. The function ψ is called the characteristic exponent of X.

The Lévy-Khintchine formula is written as:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\gamma\xi + \int_0^\infty\left(1 - e^{i\xi x} + i\xi x 1_{[-1,1]}(x)\right)\nu(dx)$$

where $\sigma \geq 0$ and $\gamma \in \mathbb{P}$ are constants, and ν is a measure on $\mathbb{R} - \{0\}$ with $\int_{-\infty}^{+\infty}\min(1,x^2)\nu(dx) < \infty$. The triple (σ^2,γ,ν) is the generating triplet. The measure ν is called the Lévy measure. If $\sigma = 0$, the Lévy process is a pure non-Gaussian process, without a diffusion component.

For a general presentation of Lévy processes, see Sato [1987] or to Bertoin [1998].

Here we consider again an economic model where there are two primary assets: a risk-free asset with return r, and a risky asset whose price now is the exponential of a Lévy process X. From general results due to Delbaen and Schachermayer [1994], an EMM is equivalent to the no-arbitrage condition. Except for the Brownian motion and the Poisson process, this measure is not unique.

We look for an EMM of the Esscher type, defined by:

$$\frac{dQ}{dP} = \frac{\exp\{\theta X_t\}}{E_P\left[\exp\{\theta X_t\}\right]}$$

where θ is a constant, and we choose the Esscher parameter, θ, so that the discounted prices at rate r are Q-martingales. It follows that the EMM chosen is such that:

$$\psi^Q(\xi) = \psi^P(\xi - i\theta) - \psi^P(-i\theta).$$

This equation in θ has a unique solution that gives the desired EMM. We now assume that the underlying price is a CGMY ("Carr-Geman-Madan-Yor") process. For a detailed presentation of this process see Carr et al. [2002]. Such a process is defined for $C > 0$, $G \geq 0$, $M \geq 0$, $Y < 2$, and $Y > 0$, by:

$$\nu(x) = C\left(\frac{e^{Gx}}{(-x)^{1+Y}}\cdot 1_{\{x<0\}} + \frac{e^{-Mx}}{x^{1+Y}}\cdot 1_{\{x>0\}}\right).$$

and the characteristic exponent is written as:

$$\psi(\xi) = -C\Gamma(-Y)\left[\left(M - i\xi\right)^Y - M^Y + (G + i\xi)^Y - G^Y\right].$$

The function Γ is the usual gamma function.

Now consider a derivative of a European type with a terminal payoff $g(X_T)$ at expiration T. Let us define

$$\hat{g}(\xi) = \int_{-\infty}^{+\infty} e^{-iy\xi} g(y)dy.$$

Then the time t price of the derivative can be written:

$$F(S,t) = \frac{1}{2\pi}\int_{-\infty+i\varsigma}^{+\infty+i\varsigma} e^{ix\xi - \tau(r+\psi(\xi))}\hat{g}(\xi)d\xi$$

where $\tau = T - t$, and the constant ς is chosen to insure convergence.

In the case of a power call, the price can be expressed as:

$$C_\alpha(K,t) = -\frac{K}{2\pi}\int_{-\infty+i\varsigma}^{+\infty+i\varsigma} \frac{\alpha \exp\left\{i\xi \ln(S_t / K^{1-\alpha}) - \tau(r+\psi(\xi))\right\}}{\xi(\xi+i\alpha)} d\xi.$$

for any $\zeta \in (-M, -\alpha)$. This integral can be computed using the fast fourier transform algorithm.

III. POLYNOMIAL OPTIONS

We describe a way to price a standard call with a polynomial payoff, that is, a European call whose payoff represents the difference between a polynomial expression in the final price and a strike level. As far as we know, these options have not yet been studied and valued using closed formulas. They can take a number of trans and thus offer great flexibility. Moreover, if one knows how to deal with such payoffs, theoretically one can consider other continuous payoffs when they can be approximated by polynomials.

We assume an EMM has been chosen. In the usual models, it is clear it is possible to price with a (quasi-closed) form formula any payoff of the form:

$$\max\left(\left(S_T\right)^n - K, 0\right) \quad \text{where } n \geq 1.$$

But how is it possible to price with an exact formula a European call with another payoff, such as:

$$\max\left(\left(S_T\right)^3 + S_T - K, 0\right)?$$

A difficulty arises even in the simplest case of a lognormal price because although $(S_T)^3$ and S_T are separately driven by geometric Brownian motions, this is not the case for their sum: $(S_T)^3 + S_T$.

And, more generally, how do we price a call with a payoff such as:

$$\max\left(A(S_T) - K, 0\right),$$

where $A(x)$ is a real polynomial such as $A(0) = 0$.

The usual answer, as expressed in Ravindran [1996], for instance, is that for this type of payoff, as it is impossible to obtain the law of $A(S_T)$, it would be impossible to find a closed formula. Thus one must use classic numerical methods, such as tree procedures or Monte Carlo simulations.

We shall see that there are at least several cases of polynomial payoffs, where we can obtain closed-form formulas or quasi-explicit formulas, and this is suitable because we do not try to obtain the law of $A(S_T)$. We first prove the result as follows:

Theorem 1. *Let $A(x) = \sum_{j=1..n} a_j \cdot x^j$ be a polynomial with degree n, and K a positive real number satisfying:*

- *the polynomial $B(x) = A(x) - K$ has exactly one strictly positive root λ*
- *$B(x) \leq 0$ for $0 \leq x \leq \lambda$ and $B(x) \geq 0$ for $x \geq \lambda$.*

Then the European call $C_A(K,T)$ with maturity T and payoff:

$$\max\left(B(S_T) - K, 0\right)$$

can be priced by the formula:

$$C_B(K,T) = \sum_{j=1..n} a_j \cdot C_j(\lambda^j, T) \tag{6}$$

where $C_j(\lambda^j, T)$ represents the call power-j with strike λ^j and maturity T.

Proof. As λ is assumed to be the unique positive root for P, we can distinguish two separate cases:

1. Assume $S_T < \lambda$. Then $(S_T)^j \leq \lambda^j$ and therefore $\max((S_T)^j - \lambda^j, 0) = 0$ for every $j = 1..n$, for every $j = 1..n$, and we have:

$$\sum_{j=1..n} a_j \cdot \max\left((S_T)^j - \lambda^j, 0\right) = 0.$$

On the other hand, with the assumptions made for B, and as $S_T < \lambda$, we have: $B(S_T) \leq 0$, which is also: $A(S_T) \leq K$, Then it is also possible to write:

$$\max(A(S_T) - K, 0) = 0.$$

Thus, in this case we have the equality:

$$\sum_{j=1..n} a_j \cdot \max\left((S_T)^j - \lambda^j, 0\right) = \max\left(A\left(S_T\right) - K, 0\right).$$

2. Assume $S_T \geq \lambda$. Then $(S_T)^j \geq \lambda^j$ and therefore $\max((S_T)^j - \lambda^j, 0) = (S_T)^j - \lambda^j$ for every $j = 1..n$, Then we have:

$$\sum_{j=1..n} a_j \cdot \max\left((S_T)^j - \lambda^j, 0\right) = \sum_{j=1..n} a_j \cdot \left((S_T)^j - \lambda^j\right)$$
$$= A(S_T) - \boldsymbol{A}(\lambda)$$
$$= A(S_T) - K$$

because the definition of λ induces $A(\lambda) = K$.

As a consequence of the assumptions as to B, we also have $B(S_T) \geq 0$ because here $S_T \geq \lambda$, which means that we also have: $A(S_T) - K \geq 0$, Thus we may write:

$$\max\left(A\left(S_T\right) - K, 0\right) = A\left(S_T\right) - K.$$

In this case also we have the same equality:

$$\sum_{j=1..n} a_j \cdot \max\left((S_T)^j - \lambda^j, 0\right) = \max\left(A\left(S_T\right) - K, 0\right).$$

Finally, for every $S_T \geq 0$ this equality always holds, and then the expectation of the payoff $\max(A(S_T) - K, 0)$ can be calculated as a combination of several expectations, each corresponding to a power option, and Equation (6) is proved.

Thus we have proven that a call with a polynomial payoff can be priced by a portfolio of power options, although we did this with a certain constraint over the polynomial. Actually, it is possible to extend the class of polynomial payoffs that can be priced in this way, but we judge it useful to explicate the previous result, as it makes clearer the usual case (of standard calls) where the polynomial A is just $A(S) = S$, and when the unique root of the polynomial $B(S) = A(S) - K = S - K$ is simply K.

Theorem 2 is an extension:

Theorem 2. *Let* $A(x) = \sum_{j=1..n} a_j \cdot x^j$ *be a polynomial with degree* n*, and* K *a positive real number, satisfying:*

- *The polynomial* $B(x) = A(x) - K$ *has exactly two strictly positive roots,* λ *and* μ*, such as* $\lambda < \mu$*,*
- $B(x) \leq 0$ *for* $0 \leq x \leq \lambda$
  $B(x) \geq 0$ *for* $\lambda \leq x \leq \mu$
  $B(x) \leq 0$ *for* $\mu \leq x$.

Then the European call $C_A(K, T)$ *with maturity* T *and payoff:*

$\max(A(S_T) - K, 0)$

can be priced by the formula:

$$C_A(K,T) = \sum_{j=1..n} a_j \cdot \left(C_j(\lambda^j, T) - C_j(\mu^j, T)\right) \qquad (7)$$

where $C_j(H, T)$ *represents the call power-j with strike* H *and maturity* T.

Proof. We shall distinguish three cases this time.

1. Assume $S_T < \lambda$. Then $(S_T)^j \leq \lambda^j$ for every $j = 1..n$, and as $\lambda < \mu$, this requires that $(S_T)^j < \mu^j$, Then we have:

$\max((S_T)^j - \lambda^j, 0) = 0$ and $\max((S_T)^j - \mu^j, 0) = 0$ for every $j = 1..n$.

Therefore we obtain:

$$\sum_{j=1..n} a_j \cdot \max\left((S_T)^j - \lambda^j, 0\right) - \sum_{j=1..n} a_j \cdot \max\left((S_T)^j - \mu^j, 0\right) = 0$$
$$= \max(A(S_T) - K, 0)$$

exactly as in case 1 of Theorem 1. Then, by taking the expectations, we obtain for this case Equation (7).

2. Assume $\lambda \leq S_T < \mu$. Then $\lambda^j \leq (S_T)^j < \mu^j$ for every $j = 1..n$, and thus the equalities: $\max((S_T)^j - \lambda^j, 0) = (S_T)^j - \lambda^j$ and $\max((S_T)^j - \mu^j, 0) = 0$ hold for every $j = 1..n$, Then it is possible to write:

$$\sum_{j=1..n} a_j \cdot \max\left((S_T)^j - \lambda^j, 0\right) - \sum_{j=1..n} a_j \cdot \max\left((S_T)^j - \mu^j, 0\right) =$$
$$= \sum_{j=1..n} a_j \cdot \left((S_T)^j - \lambda^j, 0\right) - 0 = A\left(S_T\right) - A\left(\lambda\right) =$$
$$= A\left(S_T\right) - K = B\left(S_T\right).$$

But with the assumptions of B we know that $B(S_T) \geq 0$ when $\lambda \leq S_T < \mu$, so the value $B(S_T)$ is also: $B(S_T) = \max(A(S_T) - K, 0)$, We finally have:

$$\sum_{j=1..n} a_j \cdot \max\left((S_T)^j - \lambda^j, 0\right) - \sum_{j=1..n} a_j \cdot \max\left((S_T)^j - \mu^j, 0\right)$$
$$= \max\left(A\left(S_T\right) - K, 0\right),$$

and as previously, by taking the expectations (still in a risk-neutral universe), we obtain in this case too Equation (7).

3. Assume $\lambda < \mu \leq S_T$. Then $\lambda^j < \mu^j \leq (S_T)^j$ for every $j = 1..n$, and thus the equalities: $\max((S_T)^j - \lambda^j, 0) = (S_T)^j - \lambda^j$ and $\max((S_T)^j - \mu^j, 0) = (S_T)^j - \mu^j$ hold for every $j = 1..n$, Then it is possible to write:

$$\Delta = \sum_{j=1..n} a_j \cdot \max\left((S_T)^j - \lambda^j, 0\right) - \sum_{j=1..n} a_j \cdot \max\left((S_T)^j - \mu^j, 0\right) =$$
$$= \sum_{j=1..n} a_j \cdot \left((S_T)^j - \lambda^j, 0\right) - \sum_{j=1..n} a_j \cdot \left((S_T)^j - \mu^j, 0\right) =$$
$$= -\sum_{j=1..n} a_j \cdot \left(\lambda^j\right) + \sum_{j=1..n} a_j \cdot \left(\mu^j\right) = -A(\lambda) + A(\mu)$$

and as λ and μ are assumed to be roots of the polynomial $B = A - K$, we have:

$A(\lambda) = K = A(\mu)$,

Finally, we have obtained $\Delta = 0$.

On the other hand, as here we suppose $\mu \leq S_T$, then with the assumptions as to B, we have: $B(S_T) \leq 0$ which gives: $\max(B(S_T), 0) = 0 = \max(A(S_T) - K, 0)$.

Thus, even in this case, we have:

$$\sum_{j=1..n} a_j \cdot \max\left((S_T)^j - \lambda^j, 0\right) - \sum_{j=1..n} a_j \cdot \max\left((S_T)^j - \mu^j, 0\right)$$
$$= \max\left(A(S_T) - K, 0\right),$$

and as before, by taking expectations, we obtain again Equation (7).

Finally, whatever the value of S_T, our decomposition of the payoff is always correct, and as it leads to the formula in Equation (7), Theorem 3 is proved.

Remark 1. Two important facts should be emphasized. First, our decomposition of a polynomial option in a combination of several power options is independent of the dynamics followed by the underlying. Second, we could achieve this result precisely because we did not focus on the law of $A(S_T)$, so this theorem shows that a change of numéraire is not always the unique way to find closed formulas for options with complex payoffs.

Remark 2. It is obvious that using the same path it is easy to extend this theorem but retain the same hypotheses. Thus we can express:

Theorem 3. *Let* $A(x)=\sum_{j=1..n} a_j \cdot x^j$ *be a polynomial with degree n, and K a positive real number, satisfying:*

- *the polynomial* $B(x) = A(x) - K$ *has exactly p strictly positive roots,* $\lambda_1, \lambda_2, \ldots \lambda_p$, *such as* $\lambda_1 < \lambda_2 < \ldots < \lambda_p$,
- $B(x) \leq 0$ *for* $0 \leq x \leq \lambda_1$

and B alternates its sign between two consecutive roots.

Then the European call $C_A(K, T)$ *with maturity T and payoff:*

$$\max(A(S_T)-K, 0)$$

can be priced by the formula:

$$C_A(K,T)=\sum_{j=1..n} a_j \cdot \left(\sum_{k=1..p} (-1)^{k+1} C_j\left((\lambda_k)^j, T\right) \right) \tag{8}$$

where $C_j(K,T)$ *represents the call power-j with strike H and maturity T.*

Remark 3. From the proofs of Theorem 1 or Theorem 2, we also see that we do not have to restrict these theorems to "classic" polynomials, that is to say to polynomials with only integer powers (usual polynomials). Indeed, the proofs remain true, with the same assumptions, when we use Müntz polynomials. This makes our theorems more useful as they actually apply to a broader class of payoffs than classic polynomials.

A Mûntz polynomial is a function f written:

$$f(x)=a_0 x^{\delta_0}+a_1 x^{\delta_1}+\ldots+a_n x^{\delta_n},$$
$$\text{where } 0 \leq \delta_0 < \delta_1 < \ldots < \delta_n,$$

and where the coefficients $0 \leq \delta_0 < \delta_1 < \ldots < \delta_n$ are not necessary integers.

To use a Müntz polynomial f instead of a classic polynomial A (in our theorems), it is necessary to have $\delta_0 > 0$.

Knowing how to price any polynomial payoff is a theoretically interesting result because a well-known mathematical theorem indicates that any continuous function may be approximated by a polynomial in every compact set (Stone-Weierstrass theorem). Then for payoffs limited as to underlying and size, this would provide a way for finding valuations.

There is another classic approximation theorem that concerns non-integer exponents. The Müntz-Szaz theorem claims:

whenever the family of powers

$$\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$$

where

$$0 < \lambda_1 < \lambda_2 < \ldots,$$

is defined on (compact) interval I (of the real axis), if:

$$\sum_{k\geq 1} \frac{1}{\lambda_k} = +\infty,$$

then the closure of the linear combinations over $\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ is precisely the space of continuous function over I.

Finally, as it is possible to approximate in a compact set any continuous payoff by using polynomials or Muntz polynomials, it is then useful to have closed formulas for options with such payoffs. We have shown how to obtain such formulas for some kinds of polynomials using adequate decompositions of their payoffs.

IV. PARABOLIC OPTIONS

We can now give an explicit formula for some different European options with polynomial payoffs. First we investigate a polynomial payoff with degree 2. Such a payoff is part of a parabolic curve, so we call these options *parabolic options.* We also add an example of their influence when they are mixed with vanilla options.

Parabolic Options

We describe a new class of European options with non-linear payoffs.

Definition: We shall name a parabolic call:

– *with bases K_1 and K_2, where $K_1 < K_2$,*
– *with maturity T*
– *with multiple h, where $h \geq 0$,*

the European call written on an asset with price S_T at maturity with the payoff:

$$\max\left(h\left(S_T - K_1\right)\left(K_2 - S_T\right), 0\right). \tag{9}$$

We denote by $C_p(K_1, K_2, h, T)$ such a call. Using Equation (7) and Theorem 2, we obtain directly a closed-form formula for these calls:

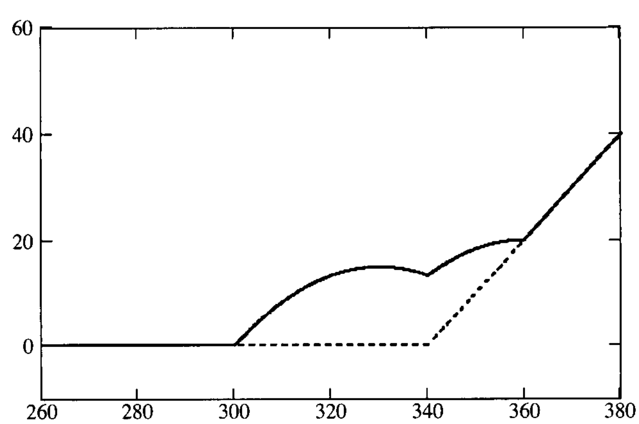
$$C_P\left(K_1, K_2, h, T\right) = h \cdot \left\{ C_2\left(\left(K_2\right)^2, T\right) - C_2\left(\left(K_1\right)^2, T\right) + \left(K_1 + K_2\right) \cdot \left[C_1\left(K_1, T\right) - C_1\left(K_2, T\right)\right]\right\}.$$

It is interesting to set the multiple h = $1/(K_2 - K_1)$ in order to standardize such an option. This way a parabolic option be completely defined by its strike bases and its maturity.

Example of a Combination

To illustrate the influence of a parabolic call in a simple package, we assume a particular combination of a standard call mixed with a parabolic call. We start by considering a single standard call with strike 340, a one-year maturity, and a volatility parameter $\sigma = 0.25$.

EXHIBIT 1
Plot of Two Payoffs



Dotted line: Payoff of Single Call with Strike 340. Continuous line: Payoff of the Package (same cell with parabolic call added).

Now, consider a package made up of a standard call with strike 340 (the same as the first one) and a parabolic call with strike bases 300 and 360. This way, both options are set at a one-year maturity (250 days) and a volatility parameter $\sigma = 0.25$. The risk-free rate is assumed to be 5%.

It is easy to obtain exact prices in the Black and Scholes framework for these options. Say, for instance, that the underlying is 350. For the standard call we obtain a price of 48.2 and for the combination (with its multiple of 1/60), we obtain a price of 50.8. Thus there is a difference of only around 5% between these two prices.

Exhibit 1 plots the two payoffs. Above 360, the two options appear almost equivalent. But notice that at maturity, when the underlying is near the strike 340 and drops suddenly under this level, the standard call is less effective than the simple combination that includes the parabolic call. Indeed the combination continues to earn some money until the strike 300, and this advantage is obtained for only a little extra cash.

V. SOME BEST OPTIONS

A *best option* is an option that allows its owner to choose the best of several payoffs. Typically their payoffs rise under both down and up events, and often the payoffs are of the same kind. In the case of vanilla options, the payoff for such a classic best option with strike K is:

$$\textit{pay-off} = \max\left(S_T - K, K - S_T\right)$$

where S_T represents the price at maturity of the underlying. This best option is easily priced as it is equivalent to a usual straddle. Its payoff is also linear and symmetric for up and down cases.

We describe three examples of best options, each with a non-symmetric and non-linear payoff.

Example 1. Assume an option has a payoff as follows:

$$\textit{pay-off} = \max\left[\left(S_T - K\right)\left(L - S_T\right), a\left(K - S_T\right)\right] \tag{10}$$

where $K < L$ and $a \geq 0$.

Let us see how such a payoff could be priced. According to the general property:

$$\max\left(x, y\right) = \max\left(x - y, 0\right) + y, \tag{11}$$

one can notice:

$$\max\left[\left(S_T-K\right)\left(L-S_T\right),a\left(K-S_T\right)\right]=$$
$$=\max\left(SL-S^2-KL+KS-aK+aS,0\right)+aK-aS$$
$$=\max\left(-S^2+\left(K+\left[L+a\right]\right)S-K\left[L+a\right],0\right)+aK-aS$$
$$=\max\left(\left(K-S\right)\left(S-\left[L+a\right]\right),0\right)+aK-aS,$$

This shows that this best option is a package of the following:

- A parabolic call with bases K and $L + a$,
- A long position on a cash amount with value, aK.
- A short position of aS on the underlying.

As we have obtained a closed formula for a parabolic option, the best option can be readily valued.

Example 2. A second case with a non-symmetric and non-linear payoff and an underlying with a lognormal dynamic also allows an exact pricing. Assume payoff:

$$\textit{pay-off}=\max\left[a\left(S_T-K\right),\left(\sqrt{K}-\sqrt{S_T}\right)\cdot\sqrt{S_T}\right] \tag{12}$$

where $a \geq 0$.

Exhibit 2 plots two payoffs. In the dotted line there is a classic straddle: *pay–off* = max[$0.5(S_T - 200)$, $0.5\ (200 - S_T)$]. In the continuous line the payoff is:

$$\textit{pay-off}=\max\left[0.5\left(S_T-200\right),\left(\sqrt{200}-\sqrt{S_T}\right)\cdot\sqrt{S_T}\right],$$

That is, we consider the payoff of Equation (12) with $K = 200$ and $a = 0.5$.

One can see that the two payoffs are similar over $K = 200$. In the down case, obviously the payoff of Equation (12) earns less than the classic straddle, but for a range of 100 points under the strike 200 it still earns an adequate amount. As generally it costs less than a regular straddle, this option appears best assuming complete protection until zero is not wanted.

When S_T is assumed to be lognormal, then $Z_T = \sqrt{S_T}$ is also lognormal. Thus the payoff may be written as:

$$\textit{pay-off}=\max\left[a\left(\left(Z_T\right)^2-K\right),\left(\sqrt{K}-Z_T\right)\cdot Z_T\right],$$

It is easy to see that with the help of Equation (11), we can write:

$$\max\left[a\left(\left(Z_T\right)^2-K\right),\left(\sqrt{K}-Z_T\right)\cdot Z_T\right]=$$
$$=\max\left[-\left(1+a\right)Z_T{}^2+\sqrt{K}\cdot Z_T+K,0\right]+a\left(S_T-K\right)$$

which shows that this best option is a package of:

- A parabolic option.
- A long position of aS on the underlying.
- A short position on a cash amount with value aK.

Thus once again as we can price a parabolic option exactly, the payoff of Equation (12) can be valued using a closed formula.

Example 3. A *balloon option* is a special case of a best option that offers a non-symmetric payoff and that improves the natural up-side evolution of the underlying over a fixed strike. We describe a simple balloon option.

Consider the payoff:

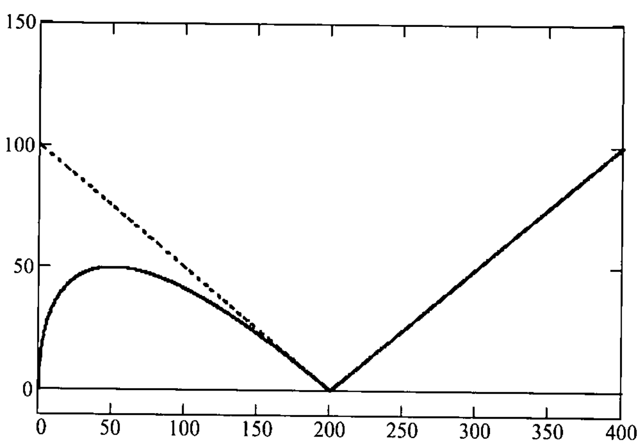
$$\textit{pay-off}=\max\left[S_T,\left(S_T\right)^\alpha-R\right] \tag{13}$$

(with $R \geq 0$ and $\alpha \geq 1$). Thus, beyond a particular value of S_T say L, greater than R, the part $(S_T)^\alpha - R$ of Equation (13) earns more than the simple underlying. We have a balloon effect after L.

Now, the payoff of Equation (13) may be expressed as follows:

$$\max\left(S_T,\left(S_T\right)^\alpha-R\right)=\max\left[\left(S_T\right)^\alpha-S_T-R,0\right]+S_T$$

EXHIBIT 2
Plot of Two Payoffs



Dotted line: Classic straddle. Continuous line: Payoff

In this form, we can see that it is completely priced as a portfolio

- with one share of the underlying.
- with another option with payoff $= \max[(S_T)^\alpha - S_T - R, 0]$ which may be valued using our Theorem 1.

VI. POLYNOMIAL OPTIONS AND PORTFOLIO SELECTION

Prigent [2005] shows that polynomial options appear in problems related to guaranteed funds. We explicate his asset allocation model for European exercise. This is a portfolio optimization problem under constraints, which guarantees a minimum value for an investor's terminal wealth.

We consider a Black and Scholes economy with two assets: a risky asset that follows a geometric Brownian motion, and a riskless asset with constant return. An investor wants to determine the best allocation between these assets by maximizing the expected utility of terminal wealth. The notation is:

- W = the wealth process.
- T = the horizon.
- u = the utility function
- θ = the self-financing portfolio.

The program faced by the investor is: $\max_\theta(E[u(W_T)])$ with $W_0 = w_0$.

This well-known problem can be solved either by the dynamic approach or by the martingale approach of Cox and Huang [1989].

Now consider the constraint program:

$$\max_\theta(E[u(W_T)])$$

with

$$WT \geq F(T)$$

and

$$W_0 = w_0$$

The floor F(T) can be a constant or a function of the risky asset at T. This problem is common in portfolio management, particularly for pension funds. Prigent [2005] shows that the solution of the constrained problem is given by the unconstrained problem where the initial wealth is appropriately modified. Let us denote by W^u the optimal wealth in this problem and by W^* the optimal wealth associated with the portfolio choice with guarantee. Then:

$$\begin{aligned} W_T^* &= W_T^u + \max\left(F(T) - W_T^u, 0\right) \\ &= F(T) + \max\left(W_T^u - F(T), 0\right) \end{aligned}$$

With a CRRA utility function, the terminal wealth W_T^u can be written $\lambda \cdot S_T^m$ where λ and $\mathbf{m}$ are appropriate constants. Then the optimal terminal wealth for our guaranteed fund can be written:

$$W_T^* = K + \max\left(\lambda \cdot S_T^m - K, 0\right)$$

in the case of a constant guarantee F(T) = K. This is clearly the payoff of a power option.

Consider now for our guaranteed fund a protection of a linear type $F(T) = aS_T + b$, where a and b are positive constants. Then the optimal terminal wealth with guarantees is written:

$$W_T^* = aS_T + b + \max\left(\lambda \cdot S_T^m - aS_T - b, 0\right)$$

This is obviously a polynomial option. Using our formulas obtained above, we can price the option precisely and therefore obtain the cost of the guarantee.

When the risky asset does not follow a geometric Brownian motion, the terminal wealth is no longer a power type in S, but we still can hope to reach a solution by approximating the option part of the optimal terminal wealth by a polynomial option using Bernstein expansion.

VII. CONCLUSION

We have extended the classic Black and Scholes formula to power and polynomial payoffs. We consider different processes to describe the underlying price movements and obtain some closed-form formulas for different polynomial payoffs. This is clearly a supplementary result in the pricing of exotic (and here also non-linear) options.

A second result is that an adequate decomposition of the payoff provides a way to price some polynomial payoffs as portfolios of several power options with specific strikes. This result is even independent of the dynamics of the underlying.

One result of our work deserves emphasis. Both changes of numéraire, one of the most powerful tools in finance, and some adapted decompositions of the payoffs let us find closed-form formulas (or quasi-explicit formulas) for pricing purposes. This point on methodology is worthy of noting because there are plenty of options with simple payoffs that are useful in fund management for which closed or quasi-explicit formulas are not available.

Options with non-linear payoffs offer great flexibility. Of particular note they allow a large variety of combinations into best options. But they are also related to investment payoffs with option characteristics that may not feature a 45-degree elbow. That we can price them exactly is useful also in real options theory.

Other applications of polynomial options can be found in portfolio management. Beyond potential use in derivatives markets, compensation options for top management represent another useful application.

For all these reasons polynomial options are important tools both in theory and practice. We provide a general way to price them that could be useful for further research and other applications.

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