

# Static Hedging of Barrier Options under General Asset Dynamics: Unification and Application

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*We propose a new method for static hedging of barrier options under general asset dynamics. It unifies previous approaches and nests their extensions. Using a finite set of hedge instruments the method is directly implementable and we show how to operationalize the hedge in a jump-diffusion model with correlated stochastic volatility. The performance of the hedge is thoroughly studied and generic sources of hedge errors are addressed.*

In this article, we introduce a new method for static hedging of barrier options in the presence of leverage, correlated stochastic volatility, and jumps in the dynamics of the underlying. As the method uses a finite but otherwise arbitrary set of hedge options we obtain results that are directly implementable. The method has, as special (limiting) cases, virtually every static hedge method previously described in the literature, e.g. the calendar-spreads of Derman et al. [1995], the strike-spreads from Carr and Chou [1997] and Carr et al. [1998], the jump and local volatility extensions in Andersen et al. [2002] and the stochastic volatility case in Fink [2003].

The basic idea is to discretize the state space consisting of time, stock price, volatility, and jump-induced overshoot. The method constructs a portfolio from a finite set of hedge instruments such that the value of this portfolio matches the value of the barrier option as closely as possible at expiry and at the nodes in the discretized state space. At each matched point in time, the hedge portfolio consists of

options with concurrent or later expiries and (all but one) strikes beyond the barrier.

An advantage of the new method is that it allows us to fully incorporate model and/or market information in order to optimize hedge performance. Technically, the method works with both real-world and risk-neutral parameters, and we demonstrate that as the models are incomplete and hedges inherently imperfect, this distinction is important; it matters which are used where. The method is thoroughly tested on a hard problem (a “discontinuous” barrier option; the up-and-out call) and compares favorably with traditional hedge methods.

The rest of the article is structured as follows. The unifying static hedge method is developed under general asset dynamics in the section A Unifying Static Hedge. First, we review methods previously derived in the literature. Second, we formulate the hedge. Third, we operationalize the hedge under a parsimonious jump-diffusion model with correlated stochastic volatility. The section Hedging Performance is a comprehensive simulation study of the performance of the hedge, including comparisons with dynamic strategies. The final section discusses extensions and concludes.

## A UNIFYING STATIC HEDGE

Empirical evidence suggests that the dynamics of assets exhibit both stochastic

volatility and discontinuities as found e.g. by Bates [1991] Bakshi et al. [1997] Eraker et al. [2003] and Eraker [2004]. In this section we develop a method for constructing static hedges of barrier options under such asset dynamics by unifying and extending previous methods suggested in the literature.

Most of the existing literature has developed theoretically perfect hedges under various assumptions on the dynamics of the underlying. In many cases, the resulting hedges require positions in a continuum or even a double continuum of plain vanilla options. Although elegant and theoretically satisfying, this is clearly inappropriate for practical applications. To apply the results, the hedges have to be approximated by finite hedge portfolios and previous work shed little light on this link. To avoid this, we work directly in an applied setting and construct an approximate hedge using a finite set of hedge instruments. This means that the hedge is directly implementable and the performance results are interpretable.

The hedging method is described under the following general asset dynamics (under a real-world probability measure  $\mathbb{P}$ )

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma(S(t), t, V(t)) dW_s(t) + J(t) dN(t), \quad (1)$$

where  $V(t)$  is a non-negative Ito process that may be correlated with  $dW_s(t)$  and  $dN(t)$  is an independent compensated Poisson process. Depending on the choices of  $\sigma(S(t), t, V(t))$ , the process for instantaneous variance<sup>1</sup>  $V(t)$  and the distribution of the jump size  $J(t)$ , this model specification encompasses most of the popular models of asset dynamics.

Being able to price your hedge instruments is necessary for any nontrivial static hedge method to work. Our hedge instruments will be vanilla call options. For many special cases of the general model, such prices are available in semi-closed form, and in the following we let

$$C(s, t, v; K, T)$$

denote the call-price function, i.e.  $C(S(t), t, V(t); K, T)$  is the time- $t$  price of a strike- $K$ , expiry- $T$  call given that the stock price is  $S(t)$  and the instantaneous variance is  $V(t)$ . For a particular model, say  $\mathbf{M}$ , we denote the call-price by  $C^{\mathbf{M}}(s, t, v; K, T)$ . Note that, as the model in Equation (1) is generically (very) incomplete, the call-price function depends non-trivially on risk-neutral (“ $\mathbb{Q}$ ”) parameters.

The static hedging method is developed using a 0-rebate up-and-out call with strike- $K$  and barrier  $B = B^{\text{Upper}}$  expiring at time  $T$ . This is done to ease the

exposition, but the method can easily be adapted to all other single barrier options.

The overall idea behind static hedging of barrier options is to construct a portfolio of plain vanilla options that matches the barrier option’s value at (and above) the barrier level and at expiry. This portfolio is bought at time 0, held until the barrier option is knocked out or expires, at which time it is liquidated.

## Previous Methods

To introduce ideas and notation, we briefly review (operationalized versions of) methods previously introduced in the literature.

Suppose first that we have a standard Black–Scholes model (so  $\mathbf{M} = \text{BS}$ ), i.e. in the notation of Equation (1) we set  $\sigma(\cdot, \cdot, \nu) \equiv \sqrt{\nu}$ ,  $V(t) \equiv \sigma_{BS}^2$  and  $J(t) \equiv 0$ .

The calendar-spread approach of Derman et al. [1995] starts by including the underlying call-option in the hedge. Next, the value of the barrier option at the barrier is matched at  $N^M$  match points in time before  $T$ , i.e.  $t_j^M < T$ ,  $j = 1, \dots, t_{N^M}^M$ . This is done by taking a position in the strike- $B$ , expiry- $T$  call option such that the value of the portfolio is 0 at time  $t_{N^M}^M$ , if the asset is at the barrier, i.e. by solving

$$\gamma_{N^M} C^{BS}(B, t_{N^M}^M, \sigma_{BS}^2; B, T) = -C^{BS}(B, t_{N^M}^M, \sigma_{BS}^2; K, T).$$

As both options so far in the hedge portfolio expire at the same time as the barrier option, we say that they constitute an expiry hedge and denote this by  $\Lambda(S(t), t, \sigma_{BS}^2; K, T) = C^{BS}(S(t), t, \sigma_{BS}^2; K, T) + \gamma_{N^M} C^{BS}(S(t), t, \sigma_{BS}^2; B, T)$ . Working back through calendar-time at each match point, say  $t_j^M$ , take a position in a call expiring no later than the next match point, say at time  $t_j^M < t_j^H \leq t_{j+1}^M$ , such that the value of the portfolio is again 0 at time  $t_j^M$  if the asset is at the barrier. This amounts to solving

$$\begin{aligned} \gamma_j C^{BS}(B, t_j^M, \sigma_{BS}^2; B, t_j^H) &= -\Lambda(S(t), t, \sigma_{BS}^2; K, T) \\ &\quad - \sum_{k=j+1}^{N^M-1} \gamma_k C^{BS}(B, t_j^M, \sigma_{BS}^2; B, t_k^H), \quad j = N^M - 1, \dots, 1. \end{aligned}$$

Note, that the hedge portfolio weights  $\gamma_j$  are found by solving simple  $1 \times 1$  linear systems recursively.

The strike-spread hedge of Carr and Chou [1997] is achieved by a more careful/elaborate construction of the expiry hedge portfolio  $\Lambda$ . Specifically, think of  $N^M$  match points,  $t_j^M$ s, as given, and suppose  $N^M$  expiry- $T$  calls with strikes,  $K^j$ s, beyond the barrier are to be included in the expiry hedge. Let  $\gamma_i$  be the number of

units used of the  $i$ th hedge instrument. By first including the underlying option, a reasonable choice of  $\gamma$  is the solution to

$$A\gamma = \Pi,$$

where  $\Pi_i = -C^{BS}(B, t_j^M, \sigma_{BS}^2; K, T)$  and  $A_{j,i} = C^{BS}(B, t_j^M, \sigma_{BS}^2; K^i, T)$ . This means that  $\Lambda(S(t), t, \sigma_{BS}^2; K, T) = C^{BS}(S(t), t, \sigma_{BS}^2; K, T) + \sum_{k=1}^{N^M} \gamma_k C^{BS}(S(t), t, \sigma_{BS}^2; K^k, T)$  matches the payoff of the barrier option at expiry when  $S(T) \leq B$ , and at the  $t_j^M$  points along the barrier. Because  $\Lambda$  is a simple  $T$ -claim, it is characterized by its payoff function. Letting  $N^M$  tend to infinity, this (purely numerically determined) function converges to the (analytically determined) adjusted payoff function of Carr and Chou [1997],  $(S(T)/B)^{(1-2R/\sigma_{BS}^2)} \times (B^2/S(T) - K)^+$ , see Exhibit 1.

Supposing now that we consider a stochastic volatility extension of the Black-Scholes model. That is, let  $V(t)$  be a non-negative Ito process. A natural way to augment the calendar-spread approach, as suggested in Fink [2003] is to include (for each  $t_j^M$ ) a number,  $N_j^V$ , of hedge options in the portfolio such that the portfolio value equals that of the barrier option for  $N_j^V$  levels for the instantaneous variance  $V(t_j^M)$ . That is, at a match point the instantaneous variance is discretized and a number of variance levels, say  $V_j^k$ ,  $k = 1, \dots, N_j^V$  is hedged. Choosing the levels is model-

dependent and operationalization is addressed in the section Hedge Levels in a Parsimonious Model, but take the levels as given for now. Naturally, each variance level requires a separate hedge instrument, so  $N_j^V$  different hedge options are needed in total for the match point  $t_j^M$ . These may have different expiries, but cannot expire later than the next match point. As the number of variance hedge levels, the associated hedge strikes, and expiries may vary from match point to match point according to market availability, it is convenient to introduce the following notation:

**Hedge strike**  $K_j^i$  is the strike of the  $i$ th hedge option used to match the portfolio at the match point  $t_j^M$ . Let  $I_j$  be the index set of the hedge strikes associated with the match point  $t_j^M$ . The barrier may well be included in these hedge strikes.

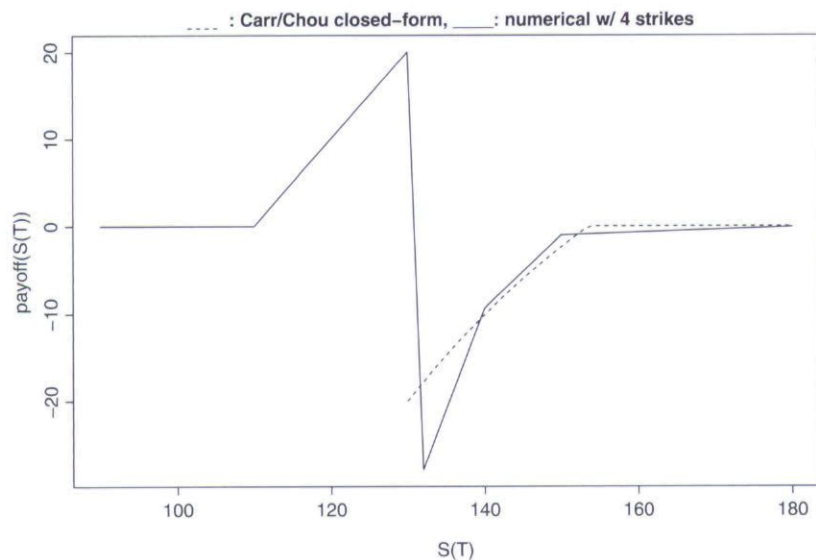
**Hedge expiry**  $t_{j,i}^H$  is the expiry of the  $i$ th hedge option used to match the portfolio at match point  $t_j^M$ . The hedge expiries must be no later than the next match point,  $t_i^M < t_i^H \leq t_i^{\Lambda}$

Now, the hedge weights related to match point  $t_j^M$  are found by solving

$$A_j \gamma_j = \Pi_j. \quad (2)$$

## EXHIBIT 1

Payoff Function of the Expiry Hedge Portfolio,  $\Lambda$ , for an up-and-out Call Option with Strike 110 and Barrier-130



$\Lambda$  is constructed with four evenly spaced match points and strikes 130, 132, 140, and 150. The dotted line is Carr/Chou's analytical adjusted payoff function;  $S(T)/B^{(1-2R/\sigma_{BS}^2)} \times (B^2/S(T) - K)^+$  above the barrier.

Here  $A_j$  is an  $N_j^V \times N_j^V$  matrix with entries  $A_{j,k,i} = C^{SV}(B, t_j^M, V_j^k, K_j^i, t_{j,i}^H)$ . It is the values of the hedge instruments at match point given that the asset is at the barrier for a number of different levels of the instantaneous variance. As in the Black-Scholes case,  $\Pi_j$  in Equation (2) is the value of the portfolio already formed to match the value at later match points. The total hedge portfolio is again found by working backwards through time.

The case of deterministic volatility and jumps is considered by Andersen et al. [2002] Essentially, those results hedge against all possible levels beyond the barrier to which the asset may jump. The deterministic volatility allow for calculation of the hedge portfolio value for all asset levels and hence for the calculation of hedge portfolio weights. Incorporating an operationalized version of those results directly in our method would be a cumbersome digression. However, the idea of forming a (model-dependent) hedge

portfolio to match the value of the barrier option at asset levels beyond the barrier is useful.

## Static Hedging under General Dynamics

In the general model the state variable is two-dimensional and the jump component means that the barrier may be passed discontinuously, resulting in an overshoot beyond the barrier. Thus, to construct a perfect hedge, we need to match payoffs for a double continuum of state variable values at each point in time until expiry. In effect, we need to match payoffs in a three dimensional grid of state variable and time combinations, as depicted in Exhibit 2. However, we have (at most) only a double continuum of hedge instruments available, namely vanilla options distinguished by strike and expiry. This shows that, even in theory, a perfect hedge cannot be constructed, there are simply not enough distinct hedge instruments available.<sup>2</sup> Indeed, the previous literature is characterized by constraining one of the state variables to be known at the first passage time and thus

having enough hedge instruments available. This is done by Andersen, et al. [2002], where jumps induce an overshoot, but the volatility is a deterministic function of time and asset level and e.g. by Fink [2003] where the volatility is stochastic, but continuity of the asset process means that the underlying is exactly on the barrier at the knock-out time.

The key idea for construction of an approximate hedge of a barrier option against both stochastic volatility and overshoot is to use a discretization of the cube of time, instantaneous variance and overshoot and hedge against the most likely points of the resulting grid. Choosing the grid points is model-dependent, an example is considered in the section Hedge Levels in a Parsimonious Model.

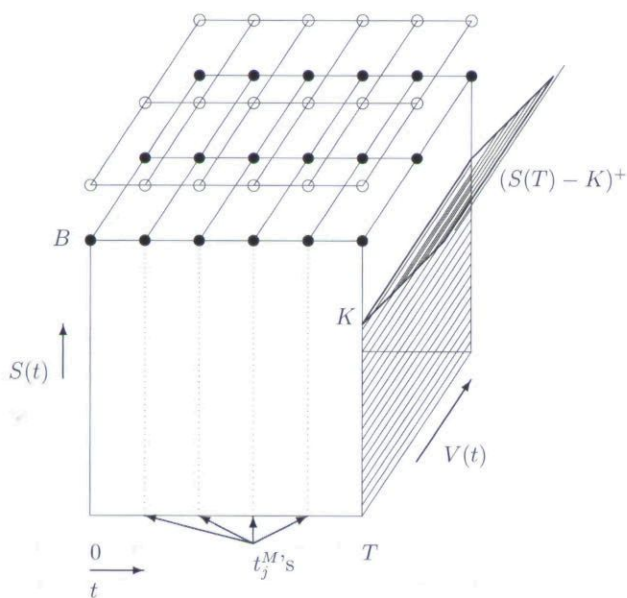
The construction of the hedge is a two-step procedure, where the steps relate to the strike-spread approach and the calendar-spread approach, respectively.

The first step constructs a hedge, denoted  $\Lambda(T)$ , related to the expiry of the barrier option. Naturally, the underlying option is included to replicate the payoff on those paths that do not cross the barrier. The key insight of the strike-spread approach is that a portfolio of the underlying option and options with strikes at and beyond the barrier can be constructed, such that its value resembles that of the barrier option along the barrier. Under the Black-Scholes model and certain special cases of stochastic volatility, a theoretically perfect hedge can be constructed. This is not possible in the general model. But the idea is important and can be used in the following ways. Take as given a set of strikes for expiry- $T$  options. Then, either take the portfolio specified by the strike-spread approach or use it as inspiration for a model-independent expiry hedge (as in the section Hedging Performance). For now we, assume that the expiry hedge has been found, i.e. that  $\Lambda(T)$  has been specified.

The second step is then working backwards as in the calendar-spread approach. At a given match date  $t_j^M < T$  a portfolio will have been formed to match the value at the next match point  $t_{j+1}^M$  or expiry- $T$  if  $j = N^M$ . This portfolio will have some value at time  $t_j^M$ . Now, positions in a number of new hedge options are added to the portfolio to match the barrier option and the portfolio values. The stochastic volatility component is hedged by matching values  $N_j^V$  for the variance levels for  $S(t_j^M) = B$  as discussed in the section Previous Methods. Again, choosing which levels to hedge is model-dependent and an example will be addressed later. The number of variance levels that may be hedged, is restricted by the availability of hedge instruments, each level requiring a distinct

### EXHIBIT 2

The Shaded Area is the Expiry Payoff Function for the Barrier Option (an up-and-out Call in this Case)



The aim of the static hedge construction is to design a portfolio, whose value is 0 on the grid; the bullets thus represent ((time) match points, asset (jump) hedge levels, variance hedge levels).

hedge instrument. So far, the second step has been similar to the method in Fink [2003], our contribution being increased flexibility and allowing for more general dynamics.<sup>3</sup>

We now consider hedging the jump component as well. To this end, the asset value beyond the barrier is discretized at each match point. As for the stochastic volatility component, the choice of levels is exemplified in the section Hedge Levels in a Parsimonious Model. Basically, one would like to hedge the most likely levels to which the asset may jump. Say that  $N_j^J$  asset levels beyond the barrier are hedged at a match point  $t_j^M$ . Call these jump hedge levels and denote them by  $S_j^k$ ,  $k = 1, \dots, N_j^J$ . To each jump hedge level we must also hedge against a variance level, corresponding to choosing a grid point in Exhibit 2. By expanding the index set  $I_j$ , these variance levels are included in  $\{V_j^k\}_{k=1}^{N_j^V+N_j^J}$ . Strictly speaking, the jump component means that we should have  $t_{j,i}^H < t_{j+1}^M$  as cases with  $t_{j,i}^H = t_{j+1}^M$  may give a non-zero payoff, if the asset jumps across the barrier exactly at time  $t_{j+1}^M$ . Such a jump, however, occurs with probability 0, so we allow for  $t_{j,i}^H \leq t_{j+1}^M$ .

At each match point  $t_j^M$ ,  $j = N^M, \dots, 1$ , the corresponding portfolio weights can now be found by solving the following  $(N_j^V + N_j^J) \times (N_j^V + N_j^J)$  system of linear equations

$$A_j \gamma_j = \Pi_j, \quad (3)$$

where

$$A_j = \begin{bmatrix} C(B, t_j^M, V_j^1; K^1, t_{j,1}^H) & \dots & C(B, t_j^M, V_j^1; K_j^{N_j^V+N_j^J}, t_{j, N_j^V+N_j^J}^H) \\ \vdots & \ddots & \vdots \\ C(B, t_j^M, V_j^{N_j^V}; K^1, t_{j,1}^H) & \dots & C(B, t_j^M, V_j^{N_j^V}; K_j^{N_j^V+N_j^J}, t_{j, N_j^V+N_j^J}^H) \\ C(S_j^1, t_j^M, V_j^{N_j^V+1}; K^1, t_{j,1}^H) & \dots & C(S_j^1, t_j^M, V_j^{N_j^V+1}; K_j^{N_j^V+N_j^J}, t_{j, N_j^V+N_j^J}^H) \\ \vdots & \ddots & \vdots \\ C(S_j^{N_j^J}, t_j^M, V_j^{N_j^V+N_j^J}; K^1, t_{j,1}^H) & \dots & C(S_j^{N_j^J}, t_j^M, V_j^{N_j^V+N_j^J}; K_j^{N_j^V+N_j^J}, t_{j, N_j^V+N_j^J}^H) \end{bmatrix}$$

$$\Pi_j = \begin{bmatrix} \Pi(B, V_j^1) \\ \vdots \\ \Pi(B, V_j^{N_j^V}) \\ \Pi(S_j^1, V_j^{N_j^V+1}) \\ \vdots \\ \Pi(S_j^{N_j^J}, V_j^{N_j^V+N_j^J}) \end{bmatrix}.$$

The elements in  $\Pi$  are the values of the options in the hedge that have already been included, i.e.

$$\Pi(x, V_j^i) = \begin{cases} -\Lambda(x, t_j^M, V_j^i; K, T) & j = N^M \\ -\Lambda(x, t_j^M, V_j^i; K, T) - \sum_{n=j+1}^{N^M} \sum_{k \in I_n} \gamma_n^k C(x, t_j^M, V_j^i; K_n^k, t_{n,k}^H) & j < N^M \end{cases} \quad (4)$$

Thus, given a discretization of the state space the hedge construction only requires calculation of option prices and solution of systems of linear equations.

Note, that all the portfolio weights can be found simultaneously by combination of the above recursive sequence of linear equation systems into a block matrix structure. Furthermore, the requirement that  $t_{j,i}^H \leq t_{j+1}^M$  can be relaxed if the hedge weights are found simultaneously and the coefficient matrix is allowed to be dense. The details are straightforward, but notationally cumbersome.

This approach extends naturally to the case where each jump hedge level is hedged using more than one variance hedge level, but to bound an already horrendous notation we leave it out.

If the  $A_j$  is not square, then an approximate solution can be obtained by standard methods. We note, that even if the matrix is square, an approximate solution may be appropriate in applications. This is investigated further in the section Hedge Levels in a Parsimonious Model.

The methods from the section Previous Methods are obtained as special cases as follows. Using only the first step with  $N_j^V = 1$  and  $N_j^J = 0$  at all match points, is the strike-spread approach. Using only the second step, again with  $N_j^V = 1$  and  $N_j^J = 0$  at all match points and only the barrier as hedge strike, is the calendar-spread approach. Allowing for more variance hedge levels, but using the same number of levels at all match points,  $N_j^V = N^V \geq 1$ ,  $j = 1, \dots, N^M$  gives the method of Fink [2003].

### Hedge Levels in a Parsimonious Model

In principle, finding the hedge described in the section Static Hedging under General Dynamics is as easy as computing vanilla option prices and solving a system of linear equations. The difficulty lies in the specification of variance and jump hedge levels, and in the choice of hedge instruments, but as discussed we would like to include “everything the market offers” as hedge instruments, and then have “the method” determine which are to be used.

In the following we make explicit choices of hedge levels under the slightly restricted asset dynamics of the

stochastic volatility jump-diffusion (SVJ) model used by Bakshi et al. [1997] among others. The model has the form

$$\begin{aligned} dS(t) &= S(t) \left( \mu dt + \sqrt{V(t)} dW_S(t) + J(t) dN(t) \right) \\ dV(t) &= (\theta_V - \kappa_V V(t)) dt + \sigma_V \sqrt{V(t)} dW_V(t) \\ dW_S(t) dW_V(t) &= \rho dt \\ dN_t &\sim P(\lambda dt), \\ \ln(1 + J(t)) &\sim N \left( \log(1 + \mu_J) - \frac{\sigma_J^2}{2}, \sigma_J^2 \right), \end{aligned}$$

where all parameters are constant and  $P(\cdot)$  is the Poisson distribution function. Furthermore, we assume the short rate is constant,  $R$ .<sup>4</sup> In the SVJ model prices of European options are known in semi-closed form, see appendix A. Default parameter values, chosen as those estimated in Eraker [2004], are also reported in Exhibit 3. Note that jumps are fairly frequent (typically one every 2 years), negative on average (but just slightly so, even the  $\mathbb{Q}$ -mean that might reflect a “fear of jumps” is only  $-2\%$ ), and have standard deviation of about 7%.

A reasonable first jump hedge level to include is the expected overshoot

$$\begin{aligned} S_j^1 &= E_{t_0}^{\mathbb{P}} \left[ S_{t_j^M-}(1 + J_{t_j^M}) \mid S_{t_j^M-}(1 + J_{t_j^M}) > B, \tau_B = t_j^M \right], \\ j &= 1, \dots, N^M. \end{aligned} \quad (5)$$

## EXHIBIT 3

### Default Parameter Values, Implementation Choices, and Contract Specifications

| Parameter          | Value                            | Parameter                          | Value  | 99% confidence interval |
|--------------------|----------------------------------|------------------------------------|--------|-------------------------|
| $S(0)$             | 100                              | $\kappa_V^{\mathbb{Q}}$            | 2.772  | (2.5200, 3.0240)        |
| $K$                | 110                              | $\kappa_V^{\mathbb{P}}$            | 4.788  | (1.2600, 8.5680)        |
| $R$                | 0.02                             | $\theta_V / \kappa_V^{\mathbb{P}}$ | 0.042  | (0.0384, 0.0448)        |
| $V(0)$             | 0.042                            | $\sigma_V$                         | 0.512  | (0.4712, 0.5494)        |
| $T$                | 1                                | $\rho$                             | -0.586 | (-0.652, -0.526)        |
| $\Delta K$         | 1                                | $\mu_J^{\mathbb{Q}}$               | -0.020 | (-0.061, 0.020)         |
| $B^{\text{Upper}}$ | 130                              | $\mu_J^{\mathbb{P}}$               | -0.004 | (-0.086, 0.076)         |
| $B^{\text{Lower}}$ | 80                               | $\sigma_J$                         | 0.066  | (0.050, 0.095)          |
| $\mu^{\mathbb{Q}}$ | $R - \lambda \mu_J^{\mathbb{Q}}$ | $\lambda$                          | 0.504  | (0.2520, 0.7560)        |
| $\Delta t$         | 1/3150                           | $\mu^{\mathbb{P}}$                 | 0.066  | (0, 0.000, 0.1260)      |

In principle, this can be calculated from the joint distribution of  $S_{t_j^M-}$ ,  $V_{t_j^M-}$ ,  $J_{t_j^M}$ , and  $\tau_B$ , which determines the distribution of the overshoot  $S_{t_j^M-}(1 + J_{t_j^M}) - B > 0$ . This, however, is not known in closed-form. It would essentially be equivalent to having closed-form expressions for barrier option prices. Recent work in this direction is presented in Schoutens [2003], Kou and Wang [2004], and Jacobsen [2005], but the results are not yet powerful enough to cover the SVJ model specification. Therefore, we use simulation to estimate the conditional expectation in Equation (5). Our studies indicate that relatively few paths are needed to obtain reliable estimates, especially when the simulations are combined with fitting techniques. More specifically, sample the conditional expectation at a number of points in time and then interpolate by e.g. splines or a polynomial to obtain an estimate of the expectation for all relevant times.

Similar to the jump hedge levels, the expected instantaneous variance at the match point conditional on the asset passing the barrier for the first time is the most reasonable first variance hedge level to use. Again, the conditional expectation can in principle be calculated from the joint distribution of  $S_{t_j^M-}$ ,  $V_{t_j^M-}$ ,  $J_{t_j^M}$ , and  $\tau_B$ , but this is not known in closed form, so we use simulation.

The simulation approach easily extends to cases, where we use quantiles of the distribution of the overshoot or the variance at the first hitting time. (In the following, when we report results for multiple variance hedge levels these are the 5, 25, 75, and 95 percentiles of the distribution.)

To put it as a catch-phrase, “hedging is done in the real world”, so the performance of imperfect hedges should be evaluated under  $\mathbb{P}$ . This point is rarely addressed in the literature on static hedging of barrier options. Most of the previous approaches formulate a hedge by constructing a portfolio that matches the barrier option value at each point on the barrier and at expiry. This means that the price of the hedge portfolio and the barrier option must be equal by no-arbitrage. Hence, these results are correctly obtained under the pricing measure. Unfortunately, these results typically involve an infinite number of hedge options. To operationalize the results, the hedges must be discretized and a finite number of options are used. This means that the hedge is no longer perfect, hence the values are no longer necessarily equal. Thus, the pricing measure is no longer the relevant measure for the conditional distributions of state variables. To illustrate the relevance of using the correct measure, the differences of the estimated conditional expectations of the

variance at the first hitting time, obtained under  $\mathbb{P}$  and  $\mathbb{Q}$  are shown in Exhibit 4. From the figure it may be seen that if we use  $\mathbb{Q}$ -estimates, then this corresponds to hedging against an instantaneous variance, which is off by up to about 4 percentage points on average in the case of the lower barrier.

A problem we encounter is that the  $A_j$ -matrix in Equation (3) quickly becomes ill-conditioned, when  $N_j^V$  increases even in the case without jumps.  $A_j$  is close to, although not exactly, singular, so while a direct solution for  $\gamma_j$  is possible, its entries, i.e. the portfolio weights, become unreasonably large for practical purposes. Without some sort of regularization the portfolio price and hedge accuracy can even diverge. As an alternative to direct solution we use truncated singular value decomposition (SVD). This method works by decomposing the ill-conditioned matrix as

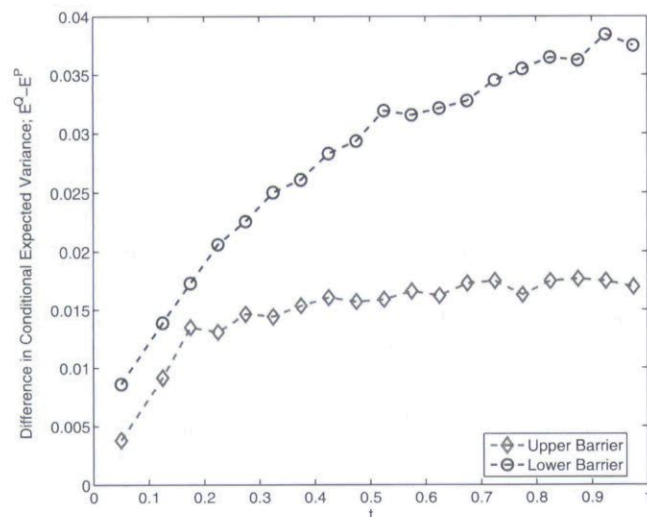
$$A_j = U W V^T,$$

where  $W$  is a diagonal matrix with the singular values of  $A_j$  as elements, and  $U$  and  $V$  are orthogonal matrices; see Press et al. [1992] for details. The direct solution to Equation (3) is, then

$$\gamma_j = A_j^{-1} \Pi_j = V W^{-1} U^T \Pi_j.$$

## EXHIBIT 4

Differences in Conditional Expected Instantaneous Variances at the First Hitting Time of the Upper ( $B = B^{\text{Upper}}$ ) and Lower ( $B = B^{\text{Lower}}$ ) Barriers



$$E^{\mathbb{Q}}[V(t) | S(t) = B \wedge \tau_B = t] - E^{\mathbb{P}}[V(t) | S(t) = B \wedge \tau_B = t].$$

From this it is clear why small singular values may well lead to very large hedge weights. The SVD method overcomes this as follows. Working with  $W^{-1}$  an approximation is formed by zeroing the inverse,  $1/w_i$ , of singular values that are less than some threshold,  $\varepsilon$ . That is, if  $w_i < \varepsilon$  then set  $1/w_i = 0$ . This gives a modified diagonal matrix  $\hat{W}^{-1}$ . The SVD solution of Equation (3) is then

$$\hat{\gamma}_j = V \hat{W}^{-1} U^T \Pi_j.$$

In effect the SVD method computes an approximate solution by ignoring precisely those equations that lead to very large hedge weights.

The use of SVD corresponds to minimizing positions in the hedge instruments at the cost of an approximate solution, i.e. an expected hedge error. Given that any hedge using a finite set of hedge options will not be perfect, an expected error is an acceptable trade-off for consistently small hedge positions. This singularity problem is also noted by Fink [2003], who suggests another regularization approach: the volatility hedge levels and the strikes of the hedge instruments are chosen to maximize the determinant of  $A_j$ . While the connection to SVD is not entirely clear to us, we think it is more natural to let hedge levels depend on the probabilistic nature of the model, let the hedge instruments be what the market offers, and then do the adjustments on what the hedger actually chooses, namely his portfolio.

We have experimented with the SVD approach and found it to be far superior to direct solution, when the number of volatility hedge levels increases; Exhibit 5 reports results for the pure stochastic volatility case. The number of match points is  $N^M = 10$ , the reflected butterfly (described in detail in Hedging Performance) is used for construction of the  $\Lambda(T)$  portfolio and for the SVD solution, we use  $\varepsilon = 0.2$  as cut-off point for the singular values.<sup>5</sup> From Exhibit 5 it is clear that using a direct solution method is inappropriate. The size of the hedge positions explodes as more variance hedge levels are included, and prices and hedge accuracy diverge. This is because large positions mean large errors when the barrier is hit at a point that is not on the grid of time points and variance hedge level. Similar results (worse, in fact for direct solution) hold, when there is a jump component in the model.

## HEDGING PERFORMANCE

In this section, the performance of the static hedge is investigated. First, we consider the impact of different

## EXHIBIT 5

### Performance of Static Hedges Constructed Using Direct Solution or Truncated Singular Value Decomposition

| $N^V$ | DIRECT SOLUTION    |       |       |          | SINGULAR VALUE DECOMPOSITION |       |       |          |
|-------|--------------------|-------|-------|----------|------------------------------|-------|-------|----------|
|       | $\max \gamma_j^t $ | price | $\mu$ | $\sigma$ | $\max \gamma_j^t $           | price | $\mu$ | $\sigma$ |
| 1     | 0.5                | 101.1 | -0.1  | 40.8     | 0.5                          | 101.1 | -0.1  | 40.8     |
| 3     | 6.2                | 100.7 | 3.8   | 37.3     | 3.0                          | 99.2  | 2.4   | 37.7     |
| 5     | 1172.4             | 123.9 | 38.7  | 221.6    | 1.5                          | 99.3  | 2.0   | 37.2     |

Varying numbers of volatility hedge levels. By  $\mu$  and  $\sigma$  we denote, respectively, the (sample) mean and standard deviation of (discounted) hedge errors. The largest absolute hedge weight (apart from those in  $\Lambda$ ) is denoted by  $\max|\gamma_j^t|$ . All numbers, except hedge weights, are given as a percentage of the barrier option price, which is 1.3150.

expiry hedges. Next, we investigate the effects from varying the number of match points, variance hedge levels and jump hedge levels. Then, we compare the performance of the static hedge to that of two dynamic hedging strategies. All investigations consider the case of an up-and-out call.

The investigations of the performance of the static hedge are conducted as simulation studies with this experimental design:

1. Initialize the hedge strategies.
2. Simulate the asset dynamics. If the asset price crosses the barrier, then record passage time and hedge values. If the barrier is crossed by a jump, then the post-jump asset level is used when liquidating the hedge portfolios, otherwise the barrier is used as level. If dynamic strategies are considered, then only simulate to the next rebalancing point.
3. At the next rebalancing time, adjust the dynamic hedges.
4. Iterate until expiry and record terminal values.

The simulations are made using an Euler discretization of the asset dynamics, see Appendix A for details.

Each simulation path runs from time 0 to time  $\tau$ , which is either expiry or the first time the barrier is crossed, whichever comes first. At time  $\tau$  all strategies are liquidated and we record the discounted hedge error, by which we mean

$$\varepsilon_m^k = \exp(-R\tau_m)(\text{Value of hedge portfolio}_k(\tau_m) - \text{Value of barrier option}(\tau_m)).$$

for the  $k$ th hedge strategy and the  $m$ th stock-price path. The tables in Exhibits 7–12 report the sample mean ( $\mu$ ) and sample standard deviations ( $\sigma$ ) of the discounted hedge errors relative to the barrier option price. The latter is a parsimonious measure of hedge accuracy, although other measures, such as value-at-risk inspired quantiles or expected short-fall, could be analyzed.

To operationalize the hedge construction described in the previous sections we need to make some further choices. First of all, we choose to use equidistant match points unless otherwise stated. We let the expiry points of the hedge options associated with a match point be that match point with  $\Delta t^M$  added. An example illustrating this corresponds to monthly expiries being available. With  $N^M = 10$  the match points are then  $t_j^M = j/12, j = 1, \dots, 10$  and the corresponding expiry points are  $t_j^H = (j+1)/12, j = 1, \dots, 10$ . Next, we assume equidistant strikes for the hedge options and use the same strikes at each expiry point. We include the barrier in the set of hedge strikes and set  $K^1 = B$ , but write  $B$  for clarity.

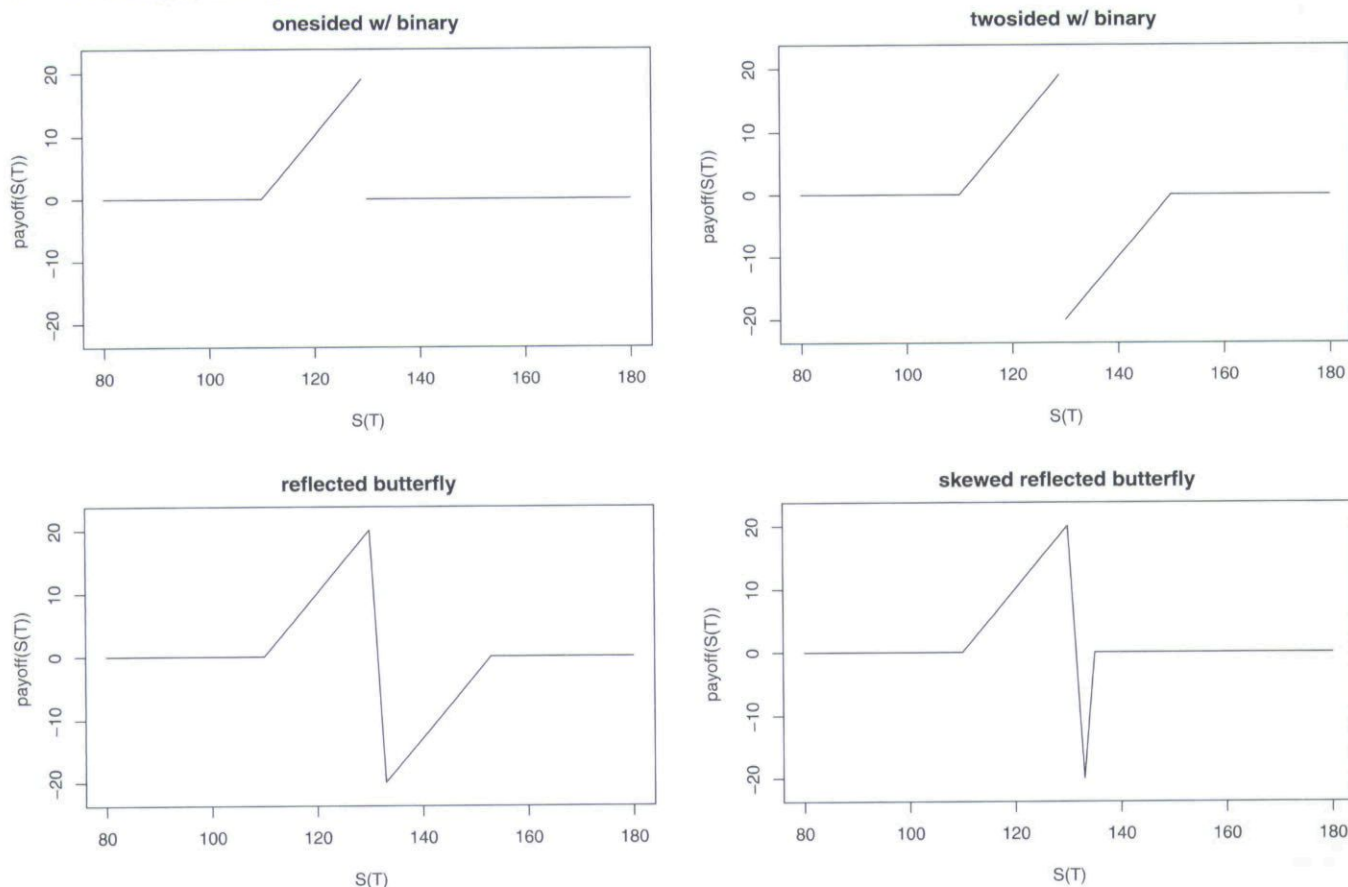
### Static Hedge Performance

Hedging options with discontinuous payoffs is hard and thus interesting. Therefore, we consider an up-and-out call option which has exactly this feature.

First, we investigate the effect of different expiry hedges, i.e. different choices of  $\Lambda(T)$ , as depicted in Exhibit 6. A natural candidate is a portfolio consisting of a long position in the underlying call, a short position of  $B - K$  binary calls (ignoring briefly that binaries are not exchange-traded) and a short position of one strike- $B$  call. We term this the perfect onesided hedge. And while this portfolio is a perfect match of the barrier option's payoff at expiry, it is not the best choice in a static hedging context. This is because the portfolio has a strictly positive value at the barrier at time points prior to expiry. And the barrier option could knock out at any of these. This can be remedied by adding to the onesided hedge payoff a negative component above the barrier. The work by Carr shows that in a Black-Scholes model there are (simple, but non-linear) closed-form expressions for this component that ensure that the resulting portfolio value is exactly 0 along the barrier. In more complicated models, it is not possible to achieve a perfect match, but we use the insight in the following (model independent) way: use a portfolio consisting of the underlying call, a short position of  $2(B - K)$  in a

## EXHIBIT 6

### Different Expiry Hedges



(Or: Choices of  $\Lambda$ .)

## EXHIBIT 7

### Hedge Performance for Varying Expiry Hedges

| Hedge               | $N^V = 0$ |          | $N^V = 1$ |          |
|---------------------|-----------|----------|-----------|----------|
|                     | $\mu$     | $\sigma$ | $\mu$     | $\sigma$ |
| Perfect Onesided    | 80.0      | 159.9    | 24.7      | 114.0    |
| Reflected butterfly | -13.6     | 54.9     | -1.6      | 45.4     |
| Perfect Twosided    | -21.2     | 51.3     | -8.0      | 39.9     |

$N^M = 10$  and  $N^I = 0$ . All numbers are given as a percentage of the barrier option price, which is 1.2547.

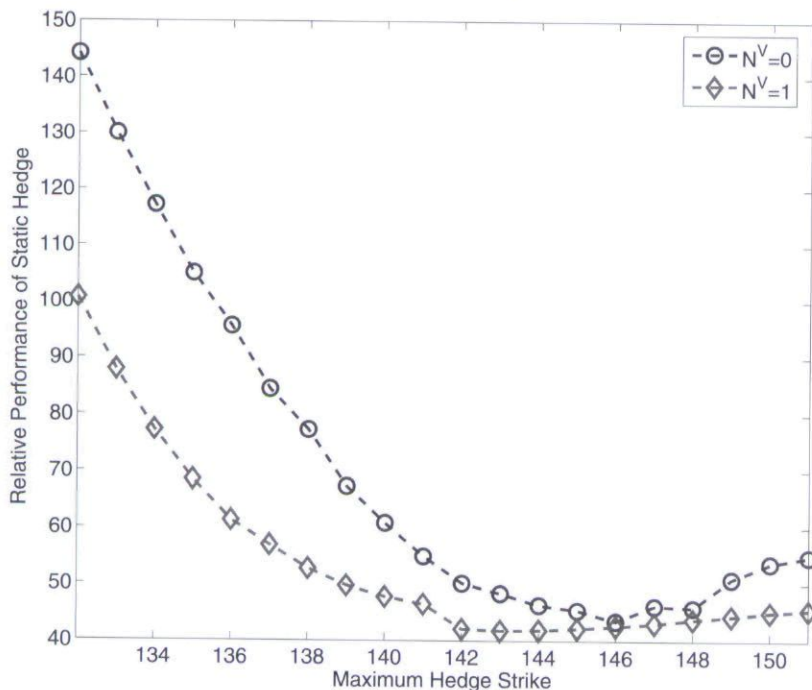
strike- $B$  binary call and a short position of one in the strike- $(2B - K)$  call. We term this the perfect twosided hedge. If binaries are not available, then one way to approximate this is to construct a reflected butterfly position. The underlying call,  $\alpha = (2K - B - K^2) / (K^2 - B)$

strike- $B$  calls,  $\beta = (-(K^3 - K) - \alpha(K^3 - B)) / (K^3 - K^2)$  strike- $K^2$  calls and a short position of one strike- $K^3$  call, where  $K^2 = B + \Delta K$  and  $K^3 = K^2 + (B - K)$ .

Exhibit 7 shows the relative performance of the three choices of expiry hedging. From the exhibit, we conclude that, indeed, the choice of  $\Lambda(T)$  is important. A problem with the twosided expiry hedges, in practice, is that the expiry hedge option with the highest strike is very deeply out-of-the-money. We can remedy this with a skewed version of the reflected butterfly expiry hedge, where  $K^3$  varies (and chosen in practice to be the highest available strike in the market). Apart from  $\alpha$  and  $\beta$  as above, this requires a position of  $\delta = -1 - \alpha - \beta$  in the strike- $K^3$  call. The accuracy of the static hedges is shown in Exhibit 8. Several interesting points may be seen from the graphs. First, the performance of a hedge based solely on  $\Lambda(T)$  performs almost as well as one that

## EXHIBIT 8

Accuracy of Static Hedges for Varying Maximum Hedge Strike, When  $\Lambda(T)$  is a Skewed Reflected Butterfly



The accuracy is the standard deviation of the discounted hedge errors given as a percentage of the barrier option price, which is 1.2547.

## EXHIBIT 9

Performance of Static Hedges of an up-and-out Call

| $N^J$ | $N^M$ | $N^V$ | $\mu$ | $\sigma$ | $N^J$ | $N^M$ | $N^V$ | $\mu$ | $\sigma$ |
|-------|-------|-------|-------|----------|-------|-------|-------|-------|----------|
| 0     | 10    | 1     | 30.4  | 86.6     | 1     | 10    | 1     | 37.5  | 90.0     |
|       |       | 3     | 30.4  | 87.5     |       |       | 3     | 21.2  | 81.8     |
|       |       | 5     | 25.3  | 81.8     |       |       | 5     | 25.7  | 81.9     |
|       |       | 1     | 13.3  | 61.0     |       |       | 1     | 28.4  | 76.4     |
|       |       | 3     | 10.9  | 61.6     |       |       | 3     | 10.32 | 65.1     |
|       | 24    | 5     | 11.5  | 59.0     |       |       | 5     | 13.6  | 62.5     |
|       |       | 1     | 5.2   | 59.0     |       |       | 1     | 26.6  | 72.0     |
|       |       | 3     | 4.7   | 52.4     |       |       | 3     | 6.9   | 67.5     |
|       |       | 5     | 5.4   | 51.2     |       |       | 5     | 8.4   | 58.3     |

Varying numbers of match points and volatility hedge levels, with and without hedge of the jump component. All numbers are given as a percentage of the barrier option price, which is 1.2547.

hedges one variance level along the barrier provided the expiry hedge is chosen appropriately. Thus, decomposing the performance of the entire hedge, we can view the expiry hedge and hence the strike-spread component as

being of primary importance. The skewed reflected butterfly has not been constructed based on specific model assumptions, but resembles the adjusted payoff in the strike-spread approach. Our investigation thus illustrates and explains the robustness of the strike-spread approach to model misspecification, see Nalholm and Poulsen [2006] for an investigation of this. Second, we observe that the performance does not improve below a certain threshold for the two hedges, which we consider here. Lowering this threshold is possible by inclusion of more hedge options.

On October 6, 2005, the maximum strike available for a call on the S&P 500 index at CBOE was 33.8% OTM.<sup>6</sup> In our setting this corresponds to a maximum hedge strike of about 134. Scaling the available strikes means that  $\Delta K = 1$  is a conservative choice. In practice, a barrier option will not have a barrier beyond the range of tradable strikes, as the writer typically needs to hedge at least some of the exposure.

In the interest of practical relevance of our study, we limit our investigations to using the skewed reflected butterfly with a maximum strike of  $K^3 = 135$ . With  $\Delta K = 1$  this means that at any expiry point, we have six hedge strikes available, including the barrier. Thus,  $(N^V = 5, N^J = 1)$  will be the maximum number of hedge levels used.

Next, the performance of the static hedge for varying specifications is investigated. Specifically, we vary the number of match points, the number of volatility hedge levels and whether the jump component is hedged. The results are reported in Exhibit 9.

From Exhibit 9, we see that taking stochastic volatility into account, when constructing static hedges does matter. This is in line with the results for dynamic hedging found by Bakshi et al. [1997]. Furthermore, we see that the hedge performance does not improve as more volatility levels are included and that it actually deteriorates, when hedging the jump component. This is a combined effect stemming from the use of SVD and the small average overshoot, when the barrier is crossed by a jump. Hedging the jump component means taking positions in more hedge options. As the overshoot is small on average for the default parameters, the  $A$  matrices in Equation (3) become closer to singular and larger

approximations are made. When other state variable values than those hedged against are realized, then a larger hedge portfolio contributes to the hedge error. Other parameter values could induce a larger overshoot and make hedging of the jump hedge component necessary. For instance, our results (not reported) indicate that if jumps are typically (very) negative, as some sources argue they are, then a down-and-out put is primarily knocked out by jumps, and using one or two jump hedge levels has a visible effect.

A closer look at the simulations that gave us the results in Exhibit 7, reveals that the main contribution to the

hedge errors and their variation arises from barrier passages in the interval from the last expiry point to expiry, i.e.  $\tau_B \in ]t_{N^M}^H, T]$ . The reason is that only the expiry hedge is in effect in this interval. A natural way to address the concentration of large hedge errors close to expiry is to change the assumption of uniform match and expiry points and move the last expiry point closer to  $T$ . To investigate this, we fix  $N^V = 1$  and  $N^J = 0$ . The match points are placed at  $t_j^M = j/12, j = 1, \dots, 10$  and the expiry points at  $t_{j,1}^H = t_{j+1}^M, j = 1, \dots, 9$  and  $t_{10,1}^H = T - x$ . We then vary  $x \in ]0, 1/12]$  and report the performance of the hedge in Exhibit 10. This investigation is in the spirit of Geiss [2002], who considers non-equidistant placement of time points for improvement of convergence rates in simulations of stochastic integrals. From Exhibit 10 it may be seen that the performance is improved relative to the earlier experiments. The closer the last expiry point is to  $T$ , the better the hedge performs. In practice, however, exchange-traded options are usually available with equidistant expiries. Therefore, we continue to use uniform match and expiry points, but note that if trading in options expiring closer to expiry of the barrier option is possible, then it can improve the hedge performance significantly.

For perfect, continuous-time hedges, drift does not matter. Depending on background, one can call this the fundamental insight/result of Black, Scholes and Merton, or Girsanov's theorem. For plain vanilla options, a healthy simulation exercise shows that even with discrete adjustments, drift matters little (and if "little is too much", then Wilmott ([1998] Chapter 20) suggests adjustments). For discrete static hedging of barrier options, the hedge errors stem exclusively from barrier passages, and the probabilities of such passages depend a great deal on drifts. Hence the real-world drift matters for hedge performance, for the order of magnitude of residual risk. Exhibit 11 illustrates this.

### Comparison with Dynamic Strategies

Having investigated the performance of the static hedge under different specifications, we now compare it with the performance of dynamic hedge strategies.

We consider two different dynamic strategies. These are usual delta and delta-vega

## EXHIBIT 10

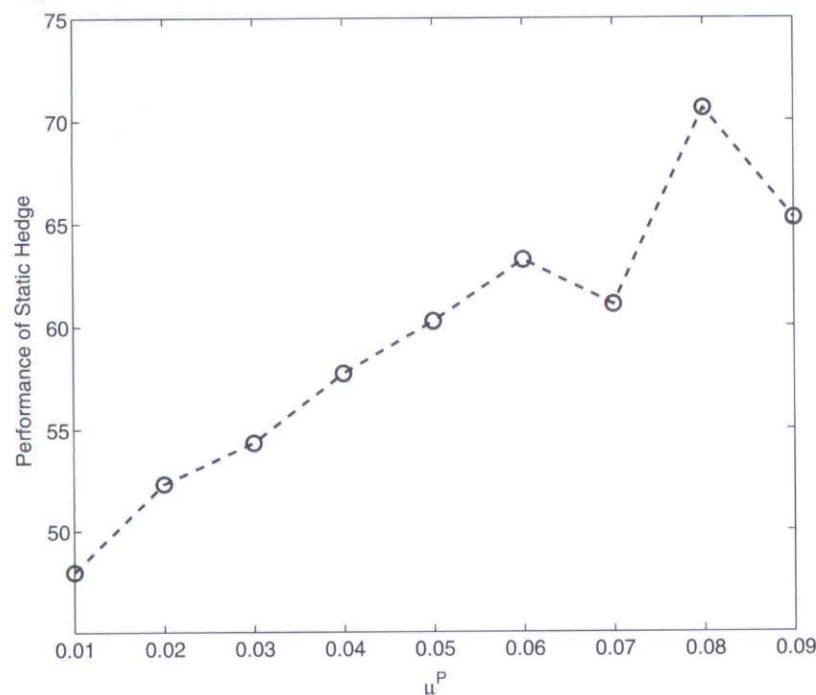
### Effect of Position of Last Expiry Point

| $x$   | $\mu$ | $\sigma$ |
|-------|-------|----------|
| 1/24  | 5.3   | 57.3     |
| 1/52  | 4.9   | 51.6     |
| 1/252 | 4.3   | 49.7     |

All numbers are given as a percentage of the barrier option price, which is 1.2547.

## EXHIBIT 11

### Performance of Static Hedge based On $\mathbb{Q}$ -expectations with $N^M = 10, N^V = 1$ , and $N^J = 0$ , When the Drift of the Real-World Dynamics is Varied



hedges with the added twist that the option underlying the barrier option is included in each strategy.

**delta** This hedge uses the underlying option and a delta-hedge of the residual. This improves the traditional delta-hedge by removing the kink in the payoff function of the underlying option, a feature known to affect the performance of dynamic

strategies adversely. The residual exposure is calculated based on the closed-form greeks for options under the Black–Scholes model. The implied volatility of the underlying option,  $\sigma_{\text{imp}}$ , is used.

**delta-vega** This strategy is like the delta-hedge, but uses the strike-  $K^{\text{vega}}(t) = S(t) \exp((R + 1/2 \sigma_{\text{imp}}^2)(T - t))$  call to hedge the residual vega exposure. As the  $K^{\text{vega}}(t)$  changes with  $S(t)$  and  $\sigma_{\text{imp}}^2$ , this means that positions are taken in a different call at each rebalancing point. The strike  $K^{\text{vega}}(t)$  call is used because it has the maximal vega and hence induces the smallest hedge positions.

To avoid unrealistic hedge strategies we truncate the interval of possible delta's and vega's to  $[-10, 10]$ . Rebalancing the dynamic strategies every 2 days and comparing them with a static hedge with  $N^M = 10$ ,  $N^V = 1$  and  $N^J = 0$ , the results are reported in Exhibit 12. The performance of the static hedge is much better than what is found for the dynamic strategies. This result is

encouraging, as the hedge used in the comparison is an almost minimal implementation of our approach. Even more so, because the dynamic strategies we have compared with are not entirely naive. A somewhat surprising finding, is that the vega-neutral dynamic strategy does not perform better than the just delta-neutral strategy. This effect can be attributed to two sources. Firstly, just before expiry the positions in the strike-  $K^{\text{vega}}(t)$  call become large on some asset paths, resulting in large errors at expiry. Once again, it is the *curse of discontinuities* in the form of exploding greeks, that shows up close to  $(B, T)$ . Secondly, the vega used in the dynamic hedge is the Black–Scholes vega, and hence the wrong exposure is hedged. Taken together, this leads to the poor performance of the delta-vega-hedge.

In Exhibit 13 we decompose the performance of each hedge, by considering the standard deviation of the hedge errors resulting from first passage times in bins centered at the markers. The performance of the static hedge is better than or equal to that of the dynamic strategies along the barrier. The performance of the static hedge can be improved by increasing  $N^V$  or  $N^M$ . Improving on the dynamic strategies is less obvious. What

## EXHIBIT 12

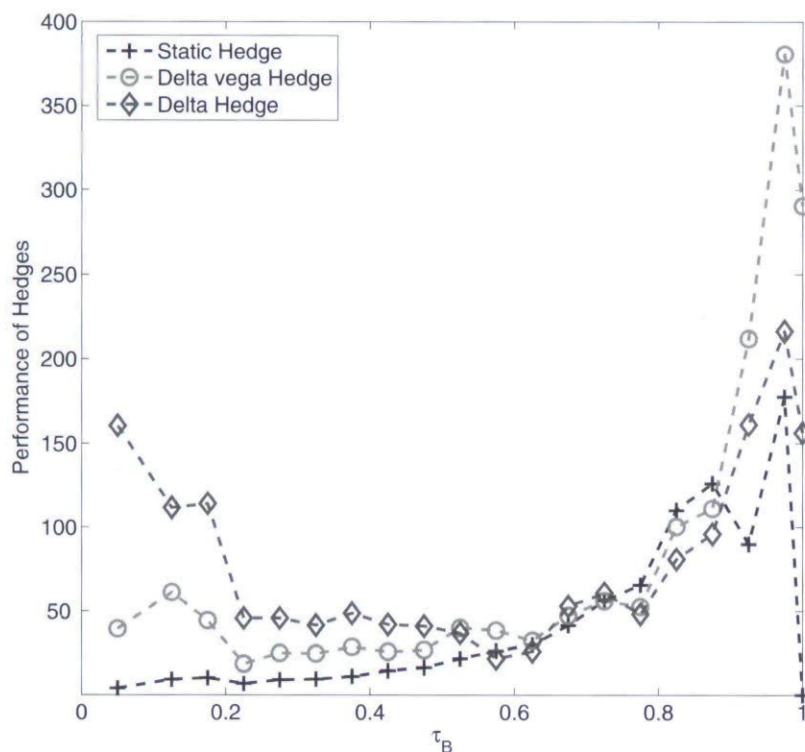
### Performance of Static Hedge and Dynamic Hedges

| Hedge                        | $\mu$ | $\sigma$ |
|------------------------------|-------|----------|
| Static hedge                 | 13.2  | 67.5     |
| Dynamic $\Delta$ -hedge      | -23.8 | 145.9    |
| Dynamic $\Delta$ -vega-hedge | -26.5 | 264.1    |

All numbers are given as a percentage of the barrier option price, which is 1.2547.

## EXHIBIT 13

### Performance of Static and Dynamic Hedges Along the Barrier



Standard deviations of discounted hedge errors resulting from first passage times in bins centered at the markers, are given as a percentage of the barrier option price, which is 1.2547.

really makes the difference in performance, is the perfect hedge at expiry in the static hedge. Most paths stay below the barrier, so a perfect expiry hedge is crucial. Contrary to the dynamic strategies, the static hedge does not have a residual hedge portfolio to unwind at expiry. This is one of the real benefits of static hedges compared with dynamic strategies, when applied in a real world situation with an incomplete market and discrete hedging.

## CONCLUSION AND EXTENSIONS

We proposed a new method for constructing static hedges of barrier option under general asset dynamics. The method extends the literature by considering general asset dynamics and by explicitly addressing the need for simultaneous use of both the real-world and pricing measures, when applying static hedging approaches in incomplete models. Thorough investigations of the hedge have identified sources of hedge errors common to static hedges of barrier options, decomposed the performance to find the strike-spread component being of primary importance, and shown that the hedge performs much better than traditional hedge strategies.

The formulations and investigations made simplifying assumptions for ease of exposition. In the following, we mention how to relax some of these.

At the cost of more notation, the method can be extended to incorporate time-varying rebates and barriers. To incorporate time-varying rebates,  $H(t)$ , we just need to match that value at each match point. We need to add  $H(t_j^M)$  to the right-hand side of Equation (3), i.e. to Equation (4). To incorporate a time-varying barrier,  $B(t)$ , we continue to require the hedge strikes to be beyond the barrier at the corresponding expiry point. For example, for an up-and-out contract we require  $\{K_j^k\}_{k \in I_j} > B(t_{j,i}^H)$  with  $K_j^1 = B(t_{j,1}^H)$ . The rest of the method carries through. Naturally, a time-varying barrier affects the conditional densities of the instantaneous variance and the overshoot conditioned on the first passage time. This makes the operationalization of the hedge more cumbersome, but using simulations and fitting techniques, as we have done, remains feasible.

The barrier feature is present in a number of more complicated products, multiple barrier options, and look-backs for instance. Although we have not elaborated on this, extending our method to these options is feasible. Essentially, it can be applied to all the contracts dealt with in the literature as special cases.

For most real-life options, barriers are discretely observed, and this case is not as trivial as one might think, see Broadie et al. [1999]. But still, the ideas underlying our method can be adapted. Specifically, at each monitoring date, the asset levels beyond the barrier and the variance dimension are discretized as in our method. Now however, the trigger time need not be a first passage time. This affects the conditional distributions used for choosing the hedge levels. In particular, more asset hedge levels should be included, as not only an overshoot needs to be hedged, but also all the paths that have gone beyond the barrier level since the last monitoring time.

If there are non-negligible transaction costs associated with trading in the hedge options, then these should be reflected in the static hedge. Taking transaction costs into account is a non-trivial extension of our method, but the principle carries through. For barrier options with discontinuous payoff functions, the expiry hedge related to the strike-spread approach consists of relatively large positions. Thus, there is a trade-off between deteriorating hedge accuracy from smaller positions in the expiry hedge and smaller cost of setting up the hedge. One should then optimize this trade-off. The results are very dependent on the contract specifications and the cost structure faced by the hedger. Investigations of this type are left for future research.

## APPENDIX A

### Pricing Formula, Monte Carlo Scheme and Parameters

In the SVJ model European call prices are known in semi-closed form. The expressions, derived by Bakshi et al. [1997], are repeated here for completeness:

$$C^{SVJ}(S(t), t, V(t); K, T) = S(t) \Pi_1(t, \tau, S(t), R, V(t)) - KB(t, \tau) \Pi_2(t, \tau, S(t), R, V(t))$$

where  $\tau = T - t$ ,  $B(t, \tau) = e^{-R\tau}$ ,

$$\Pi_j(t, \tau, S(t), R, V(t)) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left( \frac{e^{-i\phi \ln(K)} \hat{f}_j(t, \tau, S(t), R, V(t), \phi)}{i\phi} \right) d\phi, \quad j = 1, 2,$$

and

$$\begin{aligned}
\hat{f}_1 &= \exp(i\phi(\ln(S(t)) - \ln(B(t, \tau))) \\
&\quad - \frac{\theta_V}{\sigma_V^2} \left[ 2 \ln \left( 1 - \frac{(\xi_V^Q - \kappa_V^Q + (1+i\phi)\rho\sigma_V)(1-e^{-\xi_V^Q\tau})}{2\xi_V^Q} \right) \right. \\
&\quad \left. + (\xi_V^Q - \kappa_V^Q + (1+i\phi)\rho\sigma_V)\tau \right] \\
&\quad + \lambda\tau \left( (1+\mu_J^Q)((1+\mu_J^Q)^{i\phi} e^{i\phi(1+i\phi)\sigma_J^2/2} - 1) - i\phi\mu_J^Q \right) \\
&\quad + \frac{i\phi(i\phi+1)(1-e^{-\xi_V^Q\tau})}{2\xi_V^Q - (\xi_V^Q - \kappa_V^Q + (1+i\phi)\rho\sigma_V)(1-e^{-\xi_V^Q\tau})} V(t) \Big) \\
\hat{f}_2 &= \exp(i\phi(\ln(S(t)) - \ln(B(t, \tau))) \\
&\quad - \frac{\theta_V}{\sigma_V^2} \left[ 2 \ln \left( 1 - \frac{(\xi_V^* - \kappa_V^Q + (1+i\phi)\rho\sigma_V)(1-e^{-\xi_V^*\tau})}{2\xi_V^*} \right) \right. \\
&\quad \left. + (\xi_V^* - \kappa_V^Q + (1+i\phi)\rho\sigma_V)\tau \right] \\
&\quad + \lambda\tau \left( (1+\mu_J^Q)((1+\mu_J^Q)^{i\phi} e^{i\phi(1+i\phi)\sigma_J^2/2} - 1) - i\phi\mu_J^Q \right) \\
&\quad + \frac{i\phi(i\phi+1)(1-e^{-\xi_V^*\tau})}{2\xi_V^* - (\xi_V^* - \kappa_V^Q + (1+i\phi)\rho\sigma_V)(1-e^{-\xi_V^*\tau})} V(t) \Big) \\
\xi_V &= \sqrt{(k_V^Q - (1+i\phi)\rho\sigma_V)^2 - i\phi(i\phi+1)\sigma_V^2} \\
\xi_V^* &= \sqrt{(k_V^Q - (1+i\phi)\rho\sigma_V)^2 - i\phi(i\phi-1)\sigma_V^2}.
\end{aligned}$$

We use standard quadrature methods to compute the integrals. It is possible to use fast Fourier transform techniques (see Chourdakis [2004]), but as we only require the pricing of few options (differing by strike) at any one time, there is little reason to do so.

For use in the Monte Carlo studies, we discretize the model using the following Euler scheme. For pricing of the barrier option, we simulate using  $\mathbb{Q}$ -parameters, while for the hedging studies, we use the  $\mathbb{P}$ -parameters, where these differ.

$$\begin{aligned}
S(t+\Delta t) &= S(t) \left( 1 + \mu^M \Delta t + \sqrt{V(t)} \Delta W_S(t) \right) / \left( \exp(M(t)) - 1 \right), \quad \mathbb{M} = \mathbb{P} \vee \mathbb{Q} \\
M(t) &= \begin{cases} 0, & \Delta N(t) = 0 \\ (\ln(1 + \mu_J^M) - \sigma_V^2/2) \Delta N(t) & \Delta N(t) > 0, \end{cases} \quad \mathbb{M} = \mathbb{P} \vee \mathbb{Q} \\
&\quad + \sigma_J \Delta W_J(t), \\
V(t+\Delta t) &= \max \left( 0, V(t) (1 + (\theta_V - k_V^M V(t)) \Delta t) \right. \\
&\quad \left. + \sigma_V \sqrt{V(t)} \Delta W_V(t) \right), \quad \mathbb{M} = \mathbb{P} \vee \mathbb{Q} \\
\Delta W_S(t) &\sim N(0, \Delta t), \quad \Delta W_V(t) \sim N(0, \Delta t), \quad \Delta W_S(t) \Delta W_V(t) = \rho \Delta t \\
\Delta N(t) &\sim \text{Poisson}(\lambda \Delta t), \quad \Delta W_J(t) \sim N(0, \Delta N(t))
\end{aligned}$$

As 0-boundary condition for  $V$  we use  $V(t+\Delta t) = \max(0, V(t+\Delta t))$ . Reasons for not using a reflecting boundary are investigated in Asmussen et al. [1995]. We are aware, that using a discretized process introduces both a bias in the paths and in the frequency of barrier crossings; however, by using a small time step this should be negligible.

The default parameter values used in the investigations are reported in Exhibit 3 along with contract specifications and implementation choices. The parameter estimates are taken from Tables III and IV in Eraker [2004] and reproduced here for convenience.

## ENDNOTES

We thank Martin Jacobsen and Mads Stenbo Nielsen for useful comments.

<sup>1</sup>We refer to  $V$  simply as instantaneous variance. So we do not add "of returns" or "of log-increments" each and every time, and one should note that  $V(t) = \lim_{\Delta t \rightarrow 0} \text{var}_t(\ln S(t+\Delta t)) / \Delta t$  only when  $\sigma(\cdot, \nu) \equiv \sqrt{\nu}$  and there are no jumps.

<sup>2</sup>This conclusion was reached by a different line of reasoning in an appendix by Andersen et al. [2002]. A theoretical way to remedy this is to assume availability of volatility derivatives.

<sup>3</sup>In Fink [2003], only the Heston model is considered and the same number of variance levels are used at each match point.

<sup>4</sup>The full model from Bakshi et al. [1997] allows for stochastic interest rates, but they find the effects of this on (stock) option hedge performance to be of minor importance, so we leave it out.

<sup>5</sup>Other choices are possible and one could even optimize to satisfy a certain tolerance for the expected hedge error.

<sup>6</sup>On this arbitrarily chosen date, the index level was 1196 while the maximum strike available was 1600. Data from [www.cboe.com](http://www.cboe.com).

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