

Delivery Options and Treasury-Bond Futures Hedge Ratios

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A widely used approach for calculating hedge ratios for Treasury futures contracts assumes that the contract will be settled with the currently cheapest-to-deliver note or bond. With that single-deliverable assumption, the futures' PVBP (price value of a basis point) is the converted, forward PVBP of \$100,000 par of the cheapest to deliver. In reality, however, as market yields fluctuate, the identity of the cheapest-to-deliver bond may change. The authors derive the PVBP for futures contracts using an exchange option model and show that the futures' PVBP accounting for the uncertainty in the ultimate delivery bond is typically very different from the PVBP implied by a single-deliverable model. Explicitly accounting for the fair value of the futures price eliminates the discontinuity and instability that otherwise would characterize the interest rate sensitivity of the futures contract. Finally, the two-deliverable model allows for futures hedge ratios to exhibit negative convexity that is not possible in a single-deliverable model.

Derivatives such as futures contracts on Treasury bonds (T-bonds) and notes are tailor-made for hedging interest-rate risk, and in principle, computation of an optimal hedge ratio should be easy. The risk-minimizing number of contracts is obtained by dividing the price value of a basis point (PVBP) of the underlying cash position (i.e., the change in dollar value resulting from a 1 basis point change in yield) by the PVBP of the futures contract.

While the PVBP of the cash position is straightforward for parallel yield curve shifts, the PVBP of the futures contract can be more

difficult to calculate because of delivery options in the contract. Any Treasury bond with maturity or time to first call of at least 15 years is eligible for delivery against the T-bond contract. Similarly, the T-note contract may be settled with any note originally issued with 10 years to maturity with remaining maturity at least $6\frac{1}{2}$ years. However, because the conversion factors used to adjust delivery terms for the actual bond or note delivered do not perfectly reflect relative value, there is almost always one unique "cheapest-to-deliver" (CTD) bond. If the identity of that bond could be determined in advance, then the PVBP of the contract would also be easy to compute as the converted, forward PVBP of \$100,000 par of the CTD. Some sources employing this methodology are surveyed in Rendleman [1999].

However, in practice, the identity of the ultimately cheapest-to-deliver bond is currently unknown, and in empirically important situations, e.g., when market rates are in the vicinity of 6%, the CTD can flip between the highest and the lowest duration bond in the set of eligible delivery bonds. This gives rise to considerable instability in hedge ratios computed from methodologies limited to the analysis of the currently cheapest-to-deliver bond.

In this article, we derive hedge ratios for T-bond and T-note contracts (and by extension, other contracts) that give short positions delivery or "quality" options. We demonstrate that methodologies that do not explicitly account for such options will result in

incorrect hedge ratios—ratios that are highly unstable (in fact, discontinuous) at the point where the cheapest-to-deliver asset changes. In contrast, the properly computed hedge ratio is relatively flat and continuous at all yields. Moreover, hedge ratios computed without regard to delivery options necessarily are convex in yields, whereas properly computed hedge ratios actually exhibit mild negative convexity much of the time, a property that cannot be obtained without allowing for delivery options. Finally, we show that an informal approach sometimes advocated for the problem of hedging in the presence of delivery options—namely, to use a probability-weighted average of hedge ratios derived from the consideration of each eligible delivery bond in turn—offers considerable improvement over a single-delivery analysis, but still is not the full solution to the hedging problem.

In the next section, we show how the conventional analysis of the hedging problem results in unstable and incorrect hedge ratios. In Section II, we derive hedge ratios that account for delivery options and compare the properties of these hedge ratios with the conventional ones. In Section III, we consider the empirical import of our results. Section IV concludes.

I. HEDGE RATIOS

As noted, the hedge ratio is the PVBP of the underlying cash position divided by that of the contract:¹

$$\text{Hedge ratio} = \frac{\text{PVBP cash portfolio}}{\text{PVBP contract}} \quad (1)$$

For example, if the PVBP of a cash position is \$1,300 and the PVBP of a 10-year futures contract is \$65, then fully hedging the interest risk of the cash position requires shorting $\$1,300/\$65 = 20$ contracts.

The PVBP of a cash position is simply the sum of individual cash bond PVBPs and that sum, in turn, is the PVBP per \$100 par times the par amount of each bond in the cash position. PVBP per \$100 par is defined as:²

$$\text{PVBP} = \text{Modified duration} \times \text{full price}/10,000 \quad (2)$$

Finding the PVBP of a futures contract is a much thornier task. A widely used solution to the PVBP of a futures contract assumes that the deliverable bond or note will be the one that currently would be cheapest to deliver. This procedure, however, can lead one astray. To see why,

remember that the conversion factor used to adjust for the relative value of each eligible delivery bond is the price (as a percentage of par value) the bond would have if the yield curve were flat at 6%. Since the actual yield curve will always depart from this simple case, the ratio of actual value to conversion factor will vary across bonds. The cheapest-to-deliver issue at contract maturity has the lowest ratio of price to conversion factor. Before maturity, the CTD is considered to be the bond with the highest implied repo rate.

If the yield curve is flat and shifts are parallel, there are only three possibilities for CTD:

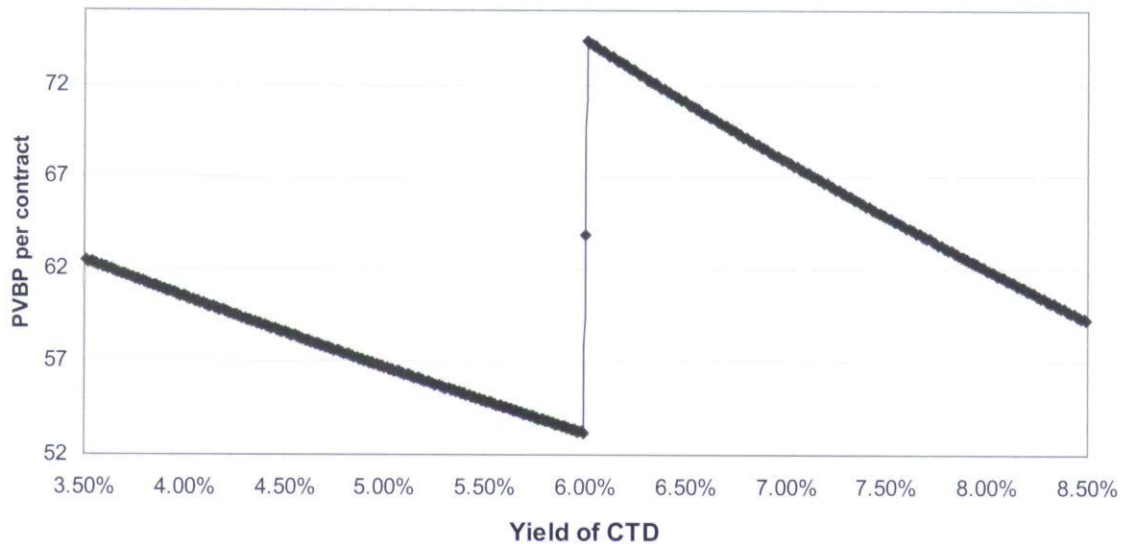
1. If the actual yield curve is flat at 6%, conversion factors match market prices, the entire basket is equally cheap to deliver, and we may choose one issue arbitrarily as CTD.
2. For yields above 6%, all issues cheapen, but the longest duration deliverable cheapens most, making it CTD.
3. For yields below 6%, all issues richen, but the shortest duration richens least, making it CTD.

The upshot is that either the shortest or the longest duration of all eligible bonds or notes will be selected as cheapest to deliver.³ One might then be tempted to employ Equation (1) to derive a hedge ratio, substituting the currently CTD bond when estimating the denominator of the right-hand side.

The problem one immediately encounters when using this approach is that the hedge ratio is highly discontinuous at yields of 6%. Exhibit 1, which plots the PVBP of the Treasury note contract, illustrates the problem. We assume that the shortest duration note has a 6% coupon and a $6\frac{1}{2}$ year maturity (the shortest eligible maturity), whereas the longest duration note also pays a 6% coupon but has maturity of 10 years (the longest maturity note). To the left of 6%, the short duration note is cheapest to deliver; to the right, the long duration note is CTD. Durations and PVBPs decline as yields increase, but at 6% where the optimal delivery note switches, the PVBP takes a discrete jump upward, from about \$53 to \$74, which is nearly a 50% increase. Correspondingly, Equation (1) implies that the hedge ratio would discretely jump downward. If actual hedge ratios are in fact this unstable, then the contract would be a poor hedging vehicle in any region near 6% market yields, or more generally, whenever there is a meaningful probability that the identity of the CTD might change.

EXHIBIT 1

Price Value of a Basis Point for T-note Futures Contract Computed Using the Currently Cheapest-to-Deliver Bond



At 6% yield, the identity of the cheapest-to-deliver bond changes, resulting in an apparent discontinuity in the contract PVB

II. ACCOUNTING FOR THE DELIVERY OPTION

Delivery (or quality) options have value because they allow the short position to select the delivery asset that maximizes its profits. Not surprisingly, then, these delivery options will result in a reduction to equilibrium futures prices. For example, Gay and Manaster [1984], who studied the quality option in CBOT wheat futures contracts, concluded that if there are only two relevant delivery assets, the equilibrium value of the futures price at time t for delivery at time T , $F_t(T)$, is

$$F_t(T) = e^{r(T-t)}[X_{1t} + S_t(T) - W_t(T, X_{1t}, X_{2t})] \quad (3)$$

where X_{1t} is the spot price of the currently CTD asset, $S_t(T)$ is the present value of storage costs over the life of the contract, and $W_t(T, X_{1t}, X_{2t})$ is the value of an exchange option to switch from the current CTD, X_{1t} , to a new CTD, X_{2t} , if the cheapest to deliver asset changes. In our application, "storage costs" equal the difference between the repurchase rate and the current yield and may be negative. Equation (3) is simply the familiar spot-futures parity relationship adjusted downward for the value of the short position's quality option.

The value of the quality option, $W_t(T, X_{1t}, X_{2t})$, may be derived given the stochastic properties of the prices of the underlying deliverable assets. For example, if both follow geometric Brownian motion, one can use Margrabe's [1978] two-asset exchange option pricing formula. Hemler [1990] extended Gay and Manaster to more than two deliverables and applied the model to price Treasury futures contracts.

These articles focus on the pricing as opposed to hedging implications of delivery options. However, the hedging properties fall out naturally from the pricing functions since the PVB of the contract can be computed by taking the derivative of $F_t(T)$ in Equation (3) with respect to a change in the interest rate. This calculation automatically accounts for the impact of rate changes on the choice of the optimal delivery issue through the impact of the derivative of $W_t(T, X_{1t}, X_{2t})$. Thus, even if one wishes to use alternative option pricing models (e.g., one that does not assume geometric Brownian motion, or one that allows for multiple delivery assets), the approach we lay out easily generalizes to whatever option pricing model is employed.

We will apply Gay and Manaster's model with two deliverable assets to Treasury note and bond futures. While this may seem overly simplistic in light of the many bonds that are eligible for delivery, we have seen that when yield

curves are flat, only two bonds are relevant—the longest and shortest duration eligible bonds. Even if the term structure is not perfectly flat, one or the other of the two extreme-duration bonds often turn out to be cheapest to deliver. Therefore, this simplification actually sacrifices little generality.⁴

To obtain a closed-form solution for $W_t(T, X_{1t}, X_{2t})$, we make the following simplifying assumptions.⁵ We assume that all notes or bonds in the deliverable basket have 6% coupons. This assumption is innocuous since the important point is to allow the deliverable basket to contain bonds of varying duration. By considering only 6% coupon bonds, all conversion factors are identically equal to 1.0, and we may avoid the distractions that arise from calculating and carrying conversion factors in the calculations. A quirk of how conversion factors are calculated otherwise gives rise to a coupon effect in determining CTD (see Grieves and Mann [2004]). In turn, that coupon effect influences the duration and convexity of futures contracts. But the effect is minimal and leads us away from the focus of this article, which is the impact of delivery options. By introducing a small amount of additional noise in the terms of delivery, considering other coupon levels would, if anything, slightly increase the value we calculate for the exchange option. In practice, however, reinstating conversion factors makes nearly no difference in our results. As in Exhibit 1, we assume that the relevant delivery notes (i.e., the shortest and longest duration notes) have maturities of $6\frac{1}{2}$ and 10 years, respectively.

We also assume that deliveries take place on a single date within the delivery month. This is common practice when valuing quality options (e.g., Hemler [1990]), and allows us to abstract from “wild card” or timing options (see Kane and Marcus [1986]).

As noted, we assume that the yield curve is flat and shifts are parallel. This assumption allows us to refer to (and calculate derivatives with respect to) “the interest rate.” We could relax this assumption if we were willing to specify how changes on yields of bonds of different maturities were related, e.g., if we had a “yield beta,” but this would add little insight to the model.

Finally, we assume that bond prices follow geometric Brownian motion. This enables us to obtain a Black-Scholes style solution as in Margrabe’s [1978] model. While bond prices cannot literally follow geometric Brownian motion (because the bond price must equal par value at maturity), this assumption is not a severe problem for short-dated options on long-maturity bonds where the pull to par value is negligible over the life of

the option. Geometric Brownian motion for short-dated options on bond prices also is used in Hemler [1990].

In our application, Equation (3) for the equilibrium value of the T-bond futures price simplifies. Define X_{1t}^* as the price for forward delivery of the currently cheapest-to-deliver eligible bond at contract expiration, and similarly, define X_{2t}^* as the price for forward delivery of the other eligible bond.⁶ We can rewrite Equation (3) as

$$F_t(T) = X_{1t}^* - W_t(T, X_{1t}^*, X_{2t}^*) \quad (4)$$

In other words, the T-bond futures price equals the forward price for delivery of the specific bond that is currently CTD minus the value of the short’s option to switch the delivery bond.

Because bond prices are assumed to follow geometric Brownian motion, the value of the exchange option is a modified version of the Black-Scholes formula:

$$W_t(T, X_{1t}^*, X_{2t}^*) = X_{1t}^* N(d_1) - X_{2t}^* N(d_2) \quad (5)$$

where

$$\begin{aligned} d_1 &= \frac{\ln(X_{1t}^*/X_{2t}^*) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \\ \sigma^2 &= \text{Var}[\ln(X_{1t}^*/X_{2t}^*)] \end{aligned}$$

Our assumption of parallel shifts for the yield curve means that we can differentiate Equation (3) with respect to “the” bond yield, γ . Taking that derivative produces:

$$\partial F_t(T)/\partial \gamma = (\partial X_{1t}^*/\partial \gamma)[1 - N(d_1)] + (\partial X_{2t}^*/\partial \gamma) N(d_2) \quad (6)$$

Notice that the partial derivatives of F_t , X_{1t}^* , and X_{2t}^* with respect to γ are all PVBPs. If the exchange option to switch delivery bonds is deep out of the money (i.e., if X_{1t}^* is sufficiently below X_{2t}^* that it is highly unlikely that the CTD will switch before contract expiration), then both $N(d_1)$ and $N(d_2)$ are close to zero, and the hedge ratio for the futures contract is virtually identical to that derived from the currently cheapest-to-deliver bond. However, when the exchange option is near the money, the hedge ratio must be computed as a blend of the hedge ratios corresponding to each delivery bond. Moreover, the derivative in Equation (6) is continuous. Even as yields cross the 6% threshold where the identity of the CTD

switches, the hedge ratio does not take the sort of discrete jump observed in Exhibit 1.

Practitioners concerned with the unpredictability of the delivery bond sometimes say, "the PVBP of a futures contract is the probability-weighted average of the converted-forward PVBPs of the deliverable basket." Equation (6) demonstrates that while there is a germ of truth in this statement, it is not precisely correct. The PVBP of the futures contract with two deliverables is $[1 - N(d_1)]$ times the converted forward PVBP of the CTD plus $N(d_2)$ times the converted forward PVBP of the other deliverable. While this formulation resembles a weighted average of PVBPs, it is worth noting that these "weights" do not sum to 1.0, and additionally, that neither "weight" equals (or is necessarily even close to) the probability that any particular bond will be cheapest to deliver. $N(d_2)$ is the risk-neutral probability that the option to switch delivery bonds will be exercised, but this may differ substantially from the objective probability of exercise, especially when the switching option is near the money.

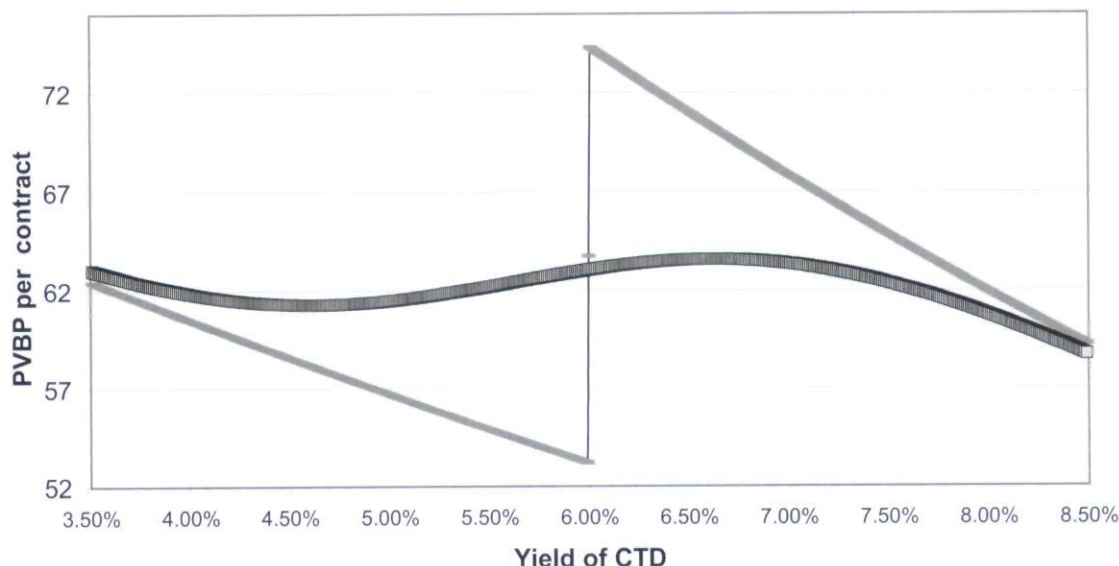
III. NUMERICAL RESULTS

Exhibit 2 displays the PVBP of T-note contracts derived from Equation (6) overlaid with the contract PVBP derived from the single-deliverable model (from Exhibit 1). The volatility measure for the exchange option embedded in the futures contract is the volatility of the returns of the exchangeable assets with respect to one another, i.e., the variance rate of $\ln(X_{1t}^*/X_{2t}^*)$.⁷ The exchange option is evaluated using the shortest ($6\frac{1}{2}$ year) and longest (10 year) issues as the two eligible deliverables.

Notice that in contrast to the single-deliverable approach in Exhibit 1, once we recognize the value of the delivery option, the calculated PVBP for the contract is smooth as a function of market yield. The implied hedge ratio is also quite stable: the contract PVBP plots as a relatively flat function of market yield. If one were to take the single-delivery model seriously, portfolio managers hedging fixed income portfolios would suddenly have to buy or sell many extra contracts just because yields moved from 5.99% to 6.01%. This counterfactual

EXHIBIT 2

Price Value of a Basis Point for T-Note Futures Contract Overlaid with the Contract PVBP Derived from the Single-Deliverable Model



PVBP computed incorporating the value of the delivery option is given by the curved black line. The PVBP is now continuous and relatively flat with respect to changes in market yields. The gray downward sloping lines are PVBPs derived from the single-deliverable model.

implication is absent from a model in which the quality option is properly incorporated.

In addition, as in Burghardt et al. [1994], the PVBP of the futures contract exhibits negative convexity for a wide range of values, which no single-deliverable model could possibly find. When the coupon on the hypothetical underlying Treasury was lowered from 8% to 6% in March 2000, some controversy arose about reintroducing negative convexity into futures contracts—at least as long as yields remained near 6%. That controversy would have been unintelligible using a single-deliverable model of futures contracts.

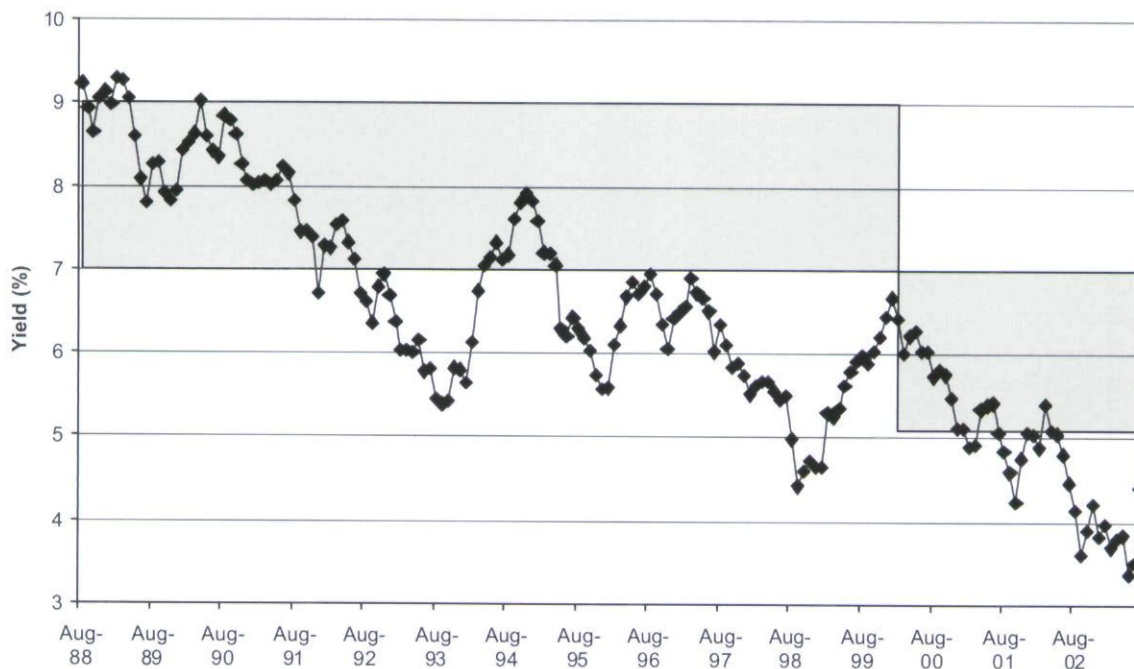
Not surprisingly, the difference in hedge ratios derived from a single-deliverable versus exchange option approach is greatest when the exchange option is near the money. At yields far from 6%, the identity of the eventual CTD is clear, and the option to switch the delivery bond is unimportant. The right to switch the delivery issue makes the biggest difference compared to the single-deliverable model when the option is at the money. For example, in Exhibit 2, when the market yield is 5%, the difference between the two-deliverable PVBP and the

one-deliverable PVBP is nearly 7.8%. That difference grows to 17.8% at a yield of 6%. As yields continue to increase from 6%, however, the difference diminishes, falling to just over 7% for yield levels of 7%. The upshot is that within one percentage point of the hypothetical coupon on the Treasury underlying the futures contract, hedge ratios are significantly different when the quality option is included in the two-deliverable model.

Finally, we consider whether all of this matters, other than as an intellectual exercise. The quality option has important empirical bearing on the hedge ratio only if the option to switch bonds is apt to be exercised, which, as noted earlier, is when market yields are near the coupon rate on the hypothetical note underlying the contract. Exhibit 3 plots yield levels for on-the-run 10-year T-notes over the past 15 years. Until February 2000, the hypothetical note underlying the contract had a coupon of 8%. For these years, the region between 7% and 9% is shaded. Starting in March 2000, when the coupon rate on the hypothetical note was reduced to 6%, we have shaded the region between 5% and 7%. Exhibit 3 illustrates that a large fraction of the total observations do in

EXHIBIT 3

Yields of On-the-Run 10-Year Treasury Notes



Boundaries of shaded regions are drawn at levels equal to the coupon rate of the hypothetical note underlying the contract plus and minus 100 basis points. Until February 2000, that coupon rate was 8%; starting in March 2000, it was reduced to 6%.

fact lie within the shaded area. This suggests that our hedge ratio is likely to be substantially more accurate than one derived from a single-deliverable approach.

IV. CONCLUSION

In this article, we derived the PVBP for futures contracts using Gay and Manaster's [1984] exchange option model. We show that futures' PVBP from a two-deliverable model is very different from futures' PVBP from a single-deliverable model for a wide range of yields. Explicitly accounting for the quality option in the derivation of the futures price eliminates the discontinuity and instability that otherwise seems to characterize the interest rate sensitivity of the futures contract. Finally, the two-deliverable model allows for futures hedge ratios to exhibit negative convexity, which is not possible in a single-deliverable model.

ENDNOTES

We are extremely grateful to Mark Griffiths, Steven V. Mann, and J. Clay Singleton for helpful comments.

¹Equation (1) assumes no basis risk between the cash and futures price.

²Notice that modified duration is multiplied by full (or dirty) price. Accruing interest to a bond shortens its duration. At constant yields, the PVBP a day before a coupon payment is nearly identical to the PVBP the day after; in contrast, duration "jumps" when the cash is paid out (see Kopprasch [1985]).

³This conclusion is a bit facile. With non-flat yield curves and non-parallel shifts, it is possible for other than the longest or shortest duration bond to emerge as CTD, but even in this case, the longest and shortest duration issues often will be the optimal delivery vehicles. We will maintain the flat yield curve assumption for simplicity. If the assumption were violated to the degree that other bonds might be CTD, the approach we pursue would still go through except that the valuation equations derived below would have to allow for multiple-asset options. This would increase the numerical complexity of the analysis considerably without shedding additional light on the economics of the problem.

⁴As noted earlier, if other bonds are in fact viable delivery candidates, we could simply expand the option valuation function $W_i(T, X_{1i}, X_{2i})$ to allow for other eligible bonds. Ours is a stylized world with a flat term structure.

⁵As we have emphasized earlier, one can extend our model to generalize beyond these simplifying assumptions by employing

more complex option valuation functions. For example, $W_i(T, X_{1i}, X_{2i})$ can be valued using numerical methods such as an interest-rate tree model, which allows for considerably greater generality in modeling assumptions. Results from these models are extremely close to the ones we present later (as we have demonstrated in unreported analysis). Our PVBPs "match" those derived from numerical methods in the stylized flat curve world. Most important, hedge ratios derived from all of these models will exhibit the stability and continuity that characterize our approach.

⁶Remember that because all eligible delivery bonds have 6% coupons, all conversion factors are identically 1.0 and therefore do not appear explicitly in Equation (4).

⁷We used a relative return volatility that resulted in the same value for the quality option as would a binomial tree with 20% short-rate volatility. The implied relative return volatility is 3.884%. We used the same volatility for all yield levels.

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