# Stocks Paying Discrete Dividends: Modeling and Option Pricing

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*In the Black-Scholes model, any dividends on stocks are paid continuously, but in reality dividends are always paid discretely, often after some announcement of the amount of the dividend. It is not entirely clear how such discrete dividends are to be handled; simple perturbations of the Black-Scholes model often fall into contradictions. The authors' approach here is to recognize the stock price as the net present value of all future dividends, and to model the (discrete) dividend process directly. The stock price process is then deduced and various option-pricing formulae derived. The Black-Scholes model with continuous dividend payments results as a limit as the time between dividend payments goes to zero.*

## I. INTRODUCTION

In finance, stock prices are typically modeled directly[1] without referring to the economic value of the payments obtained by possessing the stock, i.e., the cash flow generated by the future dividends. Often the existence of dividend payments is simply ignored, even though in option pricing the validity of a result may depend crucially on the absence of dividends (a good example is the price equality between European and American calls on a non-dividend paying stock in the presence of a non-negative interest rate).

Arbitrage pricing theory (APT) tells us that the price of a share in a company should be equal to the present value of the future dividend payments (for more on present value relations between stock prices, returns, and dividends see for example chapter 7 of Campbell, Lo, and MacKinlay [1997]). Thus, modeling of the share price properly requires modeling of the dividend payment stream (and of a suitable discounting process). Though a direct modeling of the stock price process may turn out to be consistent with the APT representation of the stock price, attempts to write down the joint distribution of dividends and stock price in some "nice" way frequently lead to inconsistency (see, e.g., Bos, Gairat, and Shepeleva [2003], Bos and Vandermark [2002], and the references therein). An overview of so-called dividend corrections of the Black-Scholes formula that are in the same spirit can be found in chapters 9 and 13 of Jarrow and Rudd [1987]. However, all those corrections admit deficiencies, such as the assumption of known dividends or of dividend yield payments, neither being in line with real applications.

Such inconsistencies become most evident when we try to model a stock with a discrete set of dividend payments at dates $t_1 < t_2 < \cdots$. This is a very real problem (*all* stocks have dividend payments at discrete dates!), but there has so far been no entirely satisfactory way to handle it. In this article, we propose to begin by *modeling the dividend payments*; the stock price process is then a *consequence* of the model assumed for the dividends. We shall use our model to derive expressions for prices of options, taking care to treat separately the case where the next dividend has been announced, and where it has not yet been announced.

## II. THE GENERAL APPROACH

Suppose that the share pays dividend $D_i$ at time $t_i$. Then assuming that we are working in the appropriate pricing measure, we have the APT expression:

$$S_t = E_t\left(\sum_{\{t_m > t\}} \beta(t_m) D_m\right) / \beta(t) \tag{1}$$

for the ex-dividend price of the stock at time $t$, where we write

$$\beta(t) \equiv \exp(-\int_0^t r_s ds)$$

for the discount factor. Depending on the form of the dividends and their relation to the riskless interest rate, this expression for $S$ can be considerably simplified. We will look at some more specific examples later where such features as non-coincidence between dividend announcement and dividend payment time, changing dividend payment policy, or option pricing are dealt with explicitly.

REMARKS

1. Note first of all that the above stock price is only finite if the dividend price process satisfies suitable growth conditions. Otherwise the infinite series in Equation (1) might not be integrable.
2. It is of course possible that some of the future dividends may be known at time $t$.
3. The classical Gordon growth model (see Gordon [1962] or chapter 7 of Campbell, Lo, and MacKinlay [1997]) is a particular case of our model. Although it is well known in the econometric literature on asset prices it has to our knowledge not yet been used as the basis for option pricing in the presence of dividends.

## III. OPTION PRICING WITH DIVIDENDS

From now on, except where indicated, we suppose that the riskless rate is constant, and that the dividends are of the form:

$$D_j = \lambda X(t_j)$$

where $X$ is an exponential Lévy process and $\lambda$ is some positive constant. We also suppose that

$$EX_t / X_0 = e^{\mu t} \tag{2}$$

for some $\mu < r$. We also assume that for some fixed $h > 0$, the times of dividend payments are multiples of $h$:

$$t_m = mh, \quad m = 1, 2, \ldots$$

From Equation (1) we obtain

$$\begin{aligned} S_t &= \sum_{m \geq k} e^{-r(mh-t)} \lambda E_t X_{mh} \\ &= \sum_{m \geq k} e^{-r(mh-t)} \lambda X_t e^{\mu(mh-t)} \\ &= \frac{\lambda X_t e^{-(r-\mu)(kh-t)}}{1 - e^{-(r-\mu)h}} \end{aligned} \tag{3}$$

for $t \in ((k-1)h, kh)$ if we assume that announcement and payment time for the dividends coincide. In particular, we have

$$\begin{aligned} S(h) &= S(h-) - \lambda X(h) = \lambda X(h)\left(\frac{1}{1 - e^{-(r-\mu)h}} - 1\right) \\ &= S(h-) e^{-(r-\mu)h} \end{aligned} \tag{4}$$

Note that this relation has the important consequence that in our model the *absolute* size of the dividend payment is random, but not its *relative* size. Consequently, we have the following result.

***Proposition 1.*** *The time-0 price of a call option with strike K and expiry $T \in (kh, (k+1)h)$ is*

$$e^{-rT} E\left[(S_0 e^{-k(r-\mu)h} e^{(r-\mu)T} X_T / X_0 - K)^+\right] \tag{5}$$

*In the special case where $X_t = \exp(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t)$ the option price is given by the (Black-Scholes) formula:*

$$\tilde{S}_k \Phi(d_1) - K e^{-rT} \Phi(d_2) \tag{6}$$

*with*

$$\tilde{S}_k = S_0 e^{-k(r-\mu)h}$$

$$d_1 = \frac{\log(\tilde{S}_k / K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

REMARKS

1. The analogue (6) of the Black-Scholes formula is just the same as the usual Black-Scholes formula, once we correct the initial stock price for the proportional dividends paid out before expiry. Compare with the market practice of subtracting the *already known* dividend from the stock price and using the resulting difference as input for the Black-Scholes formula in place of the stock price (e.g., Bos and Vandermark [2002] and Bos, Gairat, and Shepeleva [2003]); in our approach above we do not yet know the dividend payment exactly, but we shall presently modify our results to take care of this feature. It is worth emphasizing that *because we have begun by modeling the process of dividends, we never fall into the kind of inconsistencies that bedevil many common industry approaches.*
2. The above analysis deals with European calls; for American calls, one has to take into account the fact that an exercise of the option just before the payment of a dividend might be favorable!
3. In the Brownian case, the market remains complete as the filtration is the filtration of the log Brownian motion $X$. Note further that between the jumps at the dividend dates $kh$ the stock price satisfies the familiar stochastic differential equation:

   $$dS_t = S_t\,(rdt + \sigma dW_t)$$

   under the pricing measure.
4. Formula (1) shows that under the pricing measure the discounted stock price is a supermartingale which is a martingale between dividend payment times and only decreases after a dividend payment.

### Dividends Announced in Advance

Here, we still assume that dividends are paid at times $h, 2h, \ldots$ However, the amount that is paid at these times is announced at times $\varepsilon h$, $(1 + \varepsilon)h, \ldots$, respectively, and equals

$$\theta X_{(k+\varepsilon)h}$$

with $0 < \varepsilon < 1$ and $\varepsilon$ a fixed positive and known constant. From the time of the announcement up to the time of the dividend payment, the share price contains a deterministic component, the present value of the (now known) next dividend payment. We therefore think of the share price as the sum of the ex-dividend price and the evolution of the present value of the next dividend payment.

If we interpret the announcement of the dividend as the payment of its present value at the announcement time, then with

$$\lambda = \theta e^{-r(1-\varepsilon)h}$$

we can use the earlier analysis to obtain the ex-dividend share price as

$$S_t^{ex} = \lambda X_t \frac{e^{-(r-\mu)\{(k+1+\varepsilon)h-t\}}}{1-e^{-(r-\mu)h}} \tag{7}$$

and the cum-dividend price as

$$S_t^{cum} = S_t^{ex} + \lambda X_{(k+\varepsilon)h} e^{r(t-(k+\varepsilon)h)} \tag{8}$$

for $t \in ((k + \varepsilon)h, (k + 1)h)$. For $t \in (kh, (k + \varepsilon)h)$, the stock price is simply given by

$$S_t = \frac{\lambda X_t e^{-(r-\mu)((k+\varepsilon)h-t)}}{1-e^{-(r-\mu)h}} \tag{9}$$

as before.

Computing the price of a European call is very similar to the first case (where announcement and payment coincide) provided the expiry $T$ of the option is in some interval of the form $(kh, (k + \varepsilon)h)$. However, if the expiry falls in some interval of the form $((k + \varepsilon)h, (k + 1)h)$, then things are a little more complicated, since the cum-dividend price of the stock involves the value of $X$ at two different times. The following result summarizes what happens.

***Proposition 2.*** *The time-0 price of a European call option with strike $K$ and expiry $T \in ((k + \varepsilon)h, (k + 1)h)$ is*

$$e^{-rT} E\left[\left(\frac{\lambda X_T e^{(\mu-r)((k+1+\varepsilon)h-T)}}{1-e^{-(r-\mu)h}} + \lambda X_{(k+\varepsilon)h} e^{r(T-(k+\varepsilon)h)} - K\right)^+\right] \tag{10}$$

### Changing Dividend Policy

Suppose that at time $T_a = (k + \varepsilon)h$ the firm announces that the next dividend will be $b\theta X(T_a)$ rather than $\theta X(T_a)$. This could be interpreted as the firm altering its policy toward investment in the production process, perhaps reducing the next dividend so as to invest more in future production. Just prior to $T_a$, we have

$$S(T_a-) = \frac{\lambda X(T_a)}{1-e^{-(r-\mu)h}}$$

and just after $T_a$ we would have an ex-dividend price

$$S^{ex}(T_a) = \frac{\lambda X(T_a)e^{-(r-\mu)h}}{1-e^{-(r-\mu)h}}$$

if the original dividend policy were followed, but

$$\begin{aligned} S(T_a-)-b\lambda X(T_a) &= \frac{\lambda X(T_a)(1-b+be^{-(r-\mu)h})}{1-e^{-(r-\mu)h}} \\ &\equiv \frac{b'\lambda X(T_a)e^{-(r-\mu)h}}{1-e^{-(r-\mu)h}} \end{aligned} \tag{11}$$

in the light of the modified dividend announced at $T_a$. Here,

$$b' = b + (1 - b)e^{(r-\mu)h}$$

If the firm changes the announced dividend in this manner, we shall assume that its intention is for all *subsequent* dividend announcement times to announce $b'\theta X_{(m+\varepsilon)h}$, so that the form of the dynamics after $(k + 1)h$ is the same as it would have been, only scaled by the factor $b'$.

## Pricing American Options

One of the main practical problems (besides calibration of the parameters) is that in reality we have to deal with American options and not with European ones. As there are dividends, we no longer have equality between the prices of European and American calls. This requires explicit consideration of various different cases if the American call matures after the next dividend payment (for simplicity we assume that it matures before the second next payment and also that the second next payment will not be known before maturity of the call):

- If the height of the dividend payment is already known (i.e., if the payment has already been announced) then a straightforward application of the Geske-Roll-Whaley formula (see e.g., Appendix 12B of Hull [2003]) yields the required option price.
- If the height of the dividend payment is not yet known, then no explicit formula is available and numerical integration together with solving a nonlinear equation a number of times is needed (details are not complicated given the idea of the proof of the Geske-Roll-Whaley formula but are notationally cumbersome).

None of these issues is difficult to deal with, though they do require attention to detail; it is intended that a subsequent article will carry out such a study.

## Calibration of the Parameters

To calibrate the relevant parameters for our model note that the volatility can be calibrated in one of the standard ways (i.e., implicitly via call prices or via historical estimation based on observing the stock price changes). The parameters $h$, $r$, $a$, and $\lambda$ can simply be observed (or set to 1). For obtaining $\mu$ there are (at least) two different possibilities: one is through a calibration via a Black-Scholes type formula of Proposition 1 (in particular via Equation (6)) as one can calibrate $\tilde{S}$ with the help of (European) call prices and from that obtain $\mu$. Another opportunity is via the relation:

$$e^{-(r-\mu)h}S(t-) = S(t) = S(t-)-\Delta \tag{12}$$

i.e.,

$$1-e^{-(r-\mu)h} = \frac{\Delta}{S(t-)} \tag{13}$$

exactly at the dividend payment time, which should be estimated from those quotients at past dividend payment dates.

## Portfolio Optimization in the Presence of Dividends

Besides option pricing problems, one could also consider a classical continuous-time portfolio problem in the framework of this article. However, it does not take long to realize that this question quickly collapses to the original problem.

To explain why, we restrict ourselves here to the situation when dividend payment and announcement dates coincide and we assume that the Lévy process $X$ is

Brownian motion with drift, so between dividend jumps, the stock price evolves as

$$dS_t = S_t(bdt + \sigma dW_t)$$

for some constant $b$. Assuming a constant riskless rate $r$, the conclusion of Merton [1969] is that the investor with constant relative risk aversion coefficient $R$ maintains a constant proportion:

$$\pi_{\text{opt}}(t) = \frac{b-r}{\sigma^2 R} \tag{14}$$

of his wealth in the risky asset. But with dividends, nothing is changed! Indeed, between dividend payments, his investment in the risky asset evolves exactly as it did in the original Merton problem, and at the moments that dividends are paid, the value of his portfolio is unaltered; all that has happened is that the portfolio has shifted a little toward cash. But then the extra cash will be immediately re-invested in stock, and the evolution of wealth and consumption is as for the original Merton problem.

## CONCLUSIONS AND FURTHER ASPECTS

We have proposed a simple model for an asset that pays discrete dividends, based on modeling the dividend process itself as a log-Lévy process. This approach has several advantages:

- Apart from the dividends, the price dynamics are simple and conventional.
- There is no difficulty in dealing with dividends that are announced before they are paid.
- The standard (Black-Scholes-Merton) model results if we make the times between dividend payments shrink to zero, while similarly reducing the size of the payments.
- The model is based on arbitrage-pricing principles and is completely consistent.

There are many further directions that could be studied:

- We could incorporate the possibility of supply shocks, modeled as an independent log-Lévy process multiplying the stock price (the mathematics would not be changed by this, but the interpretation would).
- The dividend ratio $\lambda$ could be random. As long as this is independent of the $X$-process we should have no computational problems, though the issue of incompleteness now arises.
- While pricing of European puts and calls in our dividend modeling framework does not cause any particular problems (compared to the standard non-dividend setting), the occurrence of the dividend jumps of the stock price process turns the pricing of various exotic options into interesting problems. An obvious class where the pricing is a challenging problem is barrier options. This, however, has to be expected, and even more, it is clear that a simple adjustment of a Black-Scholes formula is not the appropriate way to care for discrete dividends when pricing barrier options.
- Indeed, any exotic equity option studied under standard Black-Scholes assumptions will behave differently under the discrete-dividend model developed here, and deserves fresh study.

## ENDNOTES

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[1]The Black-Scholes model and the Cox-Ross-Rubinstein binomial model are examples of this approach.

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