Asian Options Can Be More Valuable Than Plain Vanilla Counterparts

GEORGE L. YE

GEORGE L. YE
is an assistant professor of finance in the Sobey School of Business at Saint Mary's University in Nova Scotia, Canada.
george.ye@smu.ca

It is commonly accepted that Asian options are cheaper than their plain vanilla counterparts. By deriving and analyzing the boundary conditions of options as volatility goes to zero, it is shown that this proposition may be violated for dividend-protected put options. The effect of this violation is quite significant even for reasonable parameters.

Asian option payoffs depend on the average price of the underlying asset over a period of time. Although there are many variations, Asian options can be classified into two broad categories: average price options, and average strike options. It has been generally accepted that an average price option is less valuable than a similar plain vanilla European option. Such a proposition has appeared in research such as Chance [2003, p. 524], Haug [1997], Hull [2002, p. 443], Kemna and Vorst [1990], Nelken [1999, pp. 190-192], and Nielsen and Sandmann [2003].

The primary argument underlying this proposition is that the average price of a stock over a period is less volatile than the stock price at a particular time. As a result, average price options have lower prices. This argument ignores a critical condition: option price is a monotonic increasing function of volatility only if all other price determinants remain unchanged.

For an Asian option, the underlying state variable is the average price of a stock over a period of time, but the stock price at maturity and the average price of the stock over a period of time are two different random variables following different probability distributions. While it is true that the average price of a stock is less volatile than the stock price at maturity, their means may also differ, even in a risk-neutral world.

In particular, the average price of a non-dividend-paying stock is expected to grow more slowly than the stock price itself. Thus, for put options on a non-dividend-paying stock, or for dividend-protected stock puts, while lower volatility is a force driving the price of an average price put down, the lower expected growth drives the put price up. Once the latter effect dominates the former, the average price put will be more expensive than its plain vanilla European counterpart.

Nielsen and Sandmann [2003] provide a proof of the proposition that an Asian option is worth less than the equivalent European option with respect to dividend-protected call options. They also claim their result on Asian calls can be carried over to Asian puts via put-call parity. Yet put-call parity for Asian options cannot lead to the desired result.*

There are occasional arguments in the literature against this proposition. For example, Turnbull and Wakeman [1991] show that an average price option can be more valuable than its plain vanilla European counterpart when the maturity of the option is shorter than the average period.

We examine this proposition on put

options. Like Nielsen and Sandmann [2003], we assume that the options are dividend-protected. We consider Asian options whose averaging period falls within the life of the option. These are more common than the options discussed in Turnbull and Wakeman [1991].

Our approach is distinctive in the way it examines the proposition by analyzing the boundary conditions of those option prices. The lower bounds on options are derived by letting the volatility of the underlying stock price go to zero. One of the advantages of a lower bound derived in this way over the conventional lower bounds derived by using a no-arbitrage argument is that it ensures continuity of the option price in volatility near the bounds.

The lower bounds on Asian options are compared to the lower bounds on their plain vanilla European counterparts. If the lower bound on an Asian option is higher than the lower bound on its plain vanilla European counterpart, the Asian option will be more valuable than its plain vanilla counterpart for a range of volatilities. Thus, the proposition is violated.

I. ASSUMPTIONS AND LOWER BOUNDS OF OPTIONS

Assume there is a non-dividend-paying stock in an economy. Let S_t denote the stock price at time t, σ the volatility of the stock price, and r the risk-free rate of interest with continuous compounding. We assume r is constant and positive.

In the economy there are average price puts and plain vanilla European puts written on the stock. Let X and T be the strike price and the time to maturity of an option.

For average price puts, the underlying state variable, or the average price, is defined as:

$$S_{ave} = \frac{1}{n}\sum_{i=1}^{n} S_{t_i} \tag{1}$$

with $0 \le t_1 < t_2 < \ldots < t_n = T$.

Let $P(Y, X, T)$ be the payoff function of a put option, where Y is the value of the underlying state variable at maturity time T, with $Y = S_T$ for vanilla options and S_{ave} for average price options. The subscripts A and E are used to distinguish an average price option from a plain vanilla European option.

The payoff functions of these two types of options are given by:

$$P_E(S_T, X, T) = \max(X - S_T, 0) \quad \text{(European put)} \tag{2}$$

$$P_A(S_{ave}, X, T) = \max(X - S_{ave}, 0) \quad \text{(Average price put)} \tag{3}$$

For convenience, the notation is defined:

$$R = e^{rT} \tag{4}$$

$$Q = \frac{1}{n}\sum_{i=1}^{n} e^{rt_i} \tag{5}$$

Since the prices of both the plain vanilla European options and the average price options decline monotonically with reduced volatility of the underlying stock price, the limit price of an option as volatility goes to zero provides a lower bound on the option price, which is determined only by observable variables. To distinguish these from the lower bounds derived by Merton [1973] using a no-arbitrage argument, we call the lower bounds derived in this way the *volatility lower bounds*.

If the volatility of a stock is reduced to zero, the stock price is certain. Therefore, the expected return on the stock must be equal to the risk-free rate. Thus, it follows:

$$\lim_{\sigma \to 0} S_t = S_0 e^{rt} \quad \text{for any } t > 0 \tag{6}$$

From Equations (1), (5), and (6), it follows that:

$$\lim_{\sigma \to 0} S_{ave} = S_0 Q \tag{7}$$

Given the certainty of the stock price, the limit payoff of an option is also certain, which results in the equations:

$$\lim_{\sigma \to 0} P_E(S_T, X, T) = P_E(S_0 R, X, T) = \max(X - S_0 R, 0) \tag{8}$$

$$\lim_{\sigma \to 0} P_A(S_{ave}, X, T) = P_A(S_0 Q, X, T) = \max(X - S_0 Q, 0) \tag{9}$$

Because these payoffs are certain, the price of an option as $\sigma -> 0$, or the lower bound of the option price, must be equal to the present value of the limit payoff discounted at the risk-free rate. Let $LB_A(S_0, X, T)$ and

$LB_E(S_0, X, T)$ denote the volatility lower bound of an Asian put option and the volatility lower bound of a vanilla European put option. This results in volatility lower bounds on those options as follows:

$$LB_E(S_0, X, T) = \max(X/R - S_0, 0) \qquad (10)$$

$$LB_A(S_0, X, T) = \max(X/R - S_0Q/R, 0) \qquad (11)$$

Note that our lower bound on the plain vanilla European option is the same as the "no-arbitrage" lower bound derived by Merton [1973].

II. ASIAN PUTS VERSUS VANILLA EUROPEAN PUTS

We can compare the volatility lower bound on an average price put given by Equation (11) to the lower bound on a plain vanilla European put given by Equation (10). Since $1 < e^{rt_i} < e^{rT}$ for any $r > 0$, $T > t_i > 0$, $i = 1, 2, \ldots, n - 1$, from Equations (4) and (5) we have $1 < Q < R$. Thus, Equations (10) and (11) yield:

$$LB_{AP}(S_0, X, T) \geq LB_{EP}(S_0, X, T)$$

for any S_0, but:

$$LB_{AP}(S_0, X, T) > LB_{EP}(S_0, X, T)$$

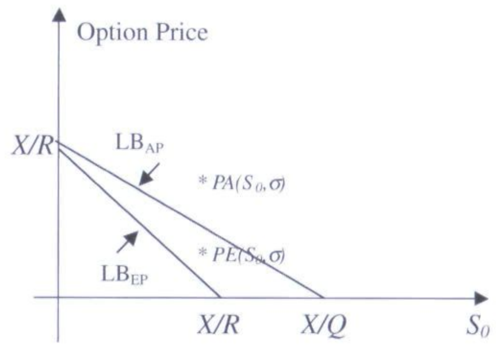
for $S_0 < X/Q$. This indicates that the volatility lower bound on the average price put is above the volatility lower bound on the plain vanilla European put when $S_0 < X/Q$, while both bounds are equal to zero when $S_0 \geq X/Q$. These bounds are plotted in Exhibit 1.

Assume an option price is continuous in σ. Given any $S_0 < X/Q$, since the volatility lower bound on the plain vanilla European put is the limit price of the option as $\sigma -> 0$, and is lower than the lower bound on the average price put, setting the price of the plain vanilla European put equal to the lower bound on the average price put must imply a positive volatility, denoted by $\sigma^*(S_0)$. Because option prices are monotonic in volatility, for any $\sigma < \sigma^*$, the plain vanilla European put price is lower than the corresponding lower bound on the average price put, and so is lower than the corresponding average price put price.

To formalize the argument, let $P_E(S_0, \sigma)$ and $P_A(S_0, \sigma)$ denote the plain vanilla European put price and average price put price when the stock price is S_0 and the volatility

EXHIBIT 1

Lower Bounds on Average Price Puts and Plain Vanilla Puts



is σ. Then, for any $S_0 < X/Q$, from (10) and (11) it follows that:

$$LB_{EP}(S_0, X, T) < LB_{AP}(S_0, X, T)$$

Thus, there is a positive volatility, $\sigma^*(S_0) > 0$, such that for any $0 < \sigma < \sigma^*(S_0)$:

$$P_E(S_0, \sigma) < P_E[S_0, \sigma^*(S_0)] = LB_{AP}(S_0, X, T) \leq P_A(S_0, \sigma)$$

This implies:

$$P_E(S_0, \sigma) < P_A(S_0, \sigma)$$

for any $S_0 < X/Q$ and $0 < \sigma < \sigma^*(S_0)$.

Thus, this contradicts the proposition that an Asian option is cheaper than its plain vanilla counterpart.

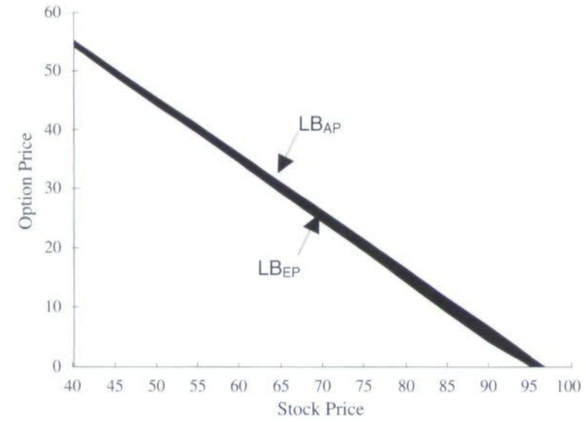
III. NUMERICAL EXAMPLE

A numerical example shows that this violation may occur even for a reasonable set of parameters. Consider an average price put and a plain vanilla European put on a non-dividend-paying stock. Both options have a strike price of \$100 and one year to maturity. Assume the continuously compounded risk-free interest rate is $r = 6\%$ per year. The average price is defined as the average of monthly closing prices, including the initial price:

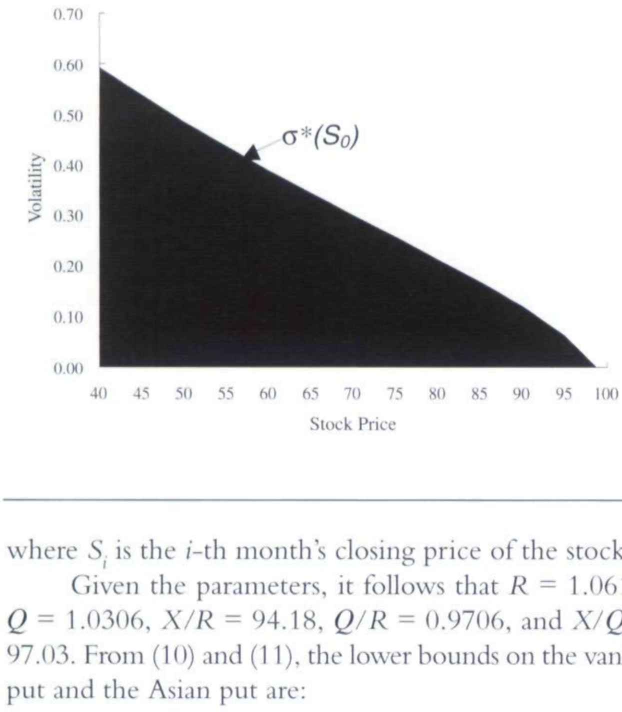
$$S_{ave} = \frac{1}{13}\sum_{i=0}^{12} S_i$$

EXHIBIT 2
Area Where Proposition is Violated

A. Lower Bounds and Violated Area in Option Price—Stock Price Plane



B. Violated Area in Volatility—Stock Price Plane



where S_i is the i-th month's closing price of the stock.

Given the parameters, it follows that $R = 1.0618$, $Q = 1.0306$, $X/R = 94.18$, $Q/R = 0.9706$, and $X/Q = 97.03$. From (10) and (11), the lower bounds on the vanilla put and the Asian put are:

$$LB_E(S_0, 100, 1) = \max(94.18 - S_0, 0) \quad (12)$$

$$LB_A(S_0, 100, 1) = \max(94.18 - 0.9706S_0, 0) \quad (13)$$

respectively. These bounds are plotted in Exhibit 2, Panel A.

From Panel A it can be determined that for any $S_0 < 97.03$, the lower bound on the average price put is above the lower bound on the plain vanilla European put. Thus, there is a positive volatility, $\sigma^*(S_0)$, for any $S_0 < 97.03$ such that for any $0 < \sigma < \sigma^*(S_0)$ and $S_0 < 97.03$, the average price put price is higher than the corresponding plain vanilla European put price. That is, for any $S_0 < 97.03$, $\sigma^*(S_0)$ defines a range of volatility within which the average price put is more expensive than the vanilla put.

Note that $\sigma^*(S_0)$ is the volatility implied by setting the plain vanilla European put price equal to the lower bound of the average price put, given the stock price S_0. That is, $\sigma^*(S_0)$ is defined by:

$$P_E[S_0, \sigma^*(S_0)] = LB_{Asian}(S_0)$$

From Equation (13), we have:

$$P_E[S_0, \sigma^*(S_0)] = 94.18 - 0.9706S_0 \quad (14)$$

for any $S_0 < 97.03$.

A plain vanilla European put pricing model must be used to solve Equation (14) to determine $\sigma^*(S_0)$. For instance, suppose the Black-Scholes option pricing model is used. With the parameters above, we solve Equation (14) numerically for $S_0 < 97.03$, and obtain a curve, $\sigma^* = \sigma^*(S_0)$, on the volatility-stock price plane, as plotted in Exhibit 2, Panel B.

Although the band between the two lower bounds is narrow in Panel A, its corresponding area in the volatility-stock price plane is quite wide in Panel B. This implies that the Asian option is more expensive than its plain vanilla counterpart for quite a wide range of volatility. For example, when the stock price is \$70, the result is $\sigma^*(S_0) = 30\%$. Thus, for any volatility below 30%, which is the case for many stocks, the average price put will be more expensive than its plain vanilla counterpart.

Next, Panel B shows that the deeper the options are in the money, the wider the range of volatility in which the Asian option is more expensive than its plain vanilla counterpart. If the puts are near or out of the money, it may be true that an average price put is cheaper than its plain vanilla counterpart, but if the puts are deep in the money, the average price put will be more expensive than its plain vanilla counterpart, even when volatility is quite high.

IV. SUMMARY

We have examined a commonly accepted proposition that Asian options are cheaper than their plain vanilla counterparts. A boundary analysis indicates that this proposition is actually a misconception, at least for average price puts.

It is further proven that an average price put can be more valuable than a plain vanilla European put with the same strike price and maturity when the option is in the money with a certain depth, and the volatility is below a certain level, depending on the stock price. An example illustrates that the proposition can be violated for a reasonable set of parameters.

ENDNOTES

The author thanks an anonymous referee for many helpful comments and suggestions.

*According to Vorst [1996], put-call parity for average price options that are dividend-protected can be written as:

$$P_A = C_A + X/R - (Q/R)S_0$$

where P_A and C_A are average price put and call prices. Let C_E and P_E denote plain vanilla European call and put prices. According to Nielsen and Sandmann [2003], the average price call is cheaper than the similar plain vanilla European call: $C_A \leq C_E$.

From the put-call parity formula for plain vanilla European options, it follows that:

$$C_E = P_E - X/R + S_0$$

By combining these equations and inequalities, we have:

$$P_A \leq P_E + S_0(1 - Q/R)$$

Since $Q/R < 1$, this inequality does not lead to the desired result that an average price put is cheaper than its plain vanilla counterpart, $P_A \leq P_E$.

REFERENCES

Chance, D. *An Introduction to Derivatives & Risk Management*, 6th ed. Cincinnati: South-Western, 2003.

Haug, E. *The Complete Guide to Options Pricing Formulas*. New York: McGraw-Hill, 1997.

Hull, J. *Options, Futures, and Other Derivatives*, 5th ed. Englewood Cliffs, NJ: Prentice-Hall, 2002.

Kemna, A., and A. Vorst. "A Pricing Method for Options Based on Average Asset Values." *Journal of Banking and Finance*, 14 (1990), pp. 113-129.

Merton, R. "Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science*, 4 (1973), pp. 141-183.

Nelken, I. *Pricing, Hedging, and Trading Exotic Options*. New York: McGraw-Hill, 1999.

Nielsen, J., and K. Sandmann. "Pricing Bounds on Asian Options." *Journal of Financial and Quantitative Analysis*, 38 (2003), pp. 449-473.

Turnbull, S.M., and L. Wakeman. "A Quick Algorithm for Pricing European Average Options." *Journal of Financial and Quantitative Analysis*, 26 (3)(1991), pp. 377-389.

Vorst, T. "Average Options." Chapter 6 in I. Nelken, ed., *The Handbook of Exotic Options*. New York: McGraw-Hill, 1996.