Derivative Pricing Models with Regime Switching: *A General Approach*

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To develop a general method for incorporating a change in regime into stochastic price processes, the assumption is two states of the world, a volatile state, and a regular state, and the stochastic process that asset prices follow is different depending on the state of the world. The method uses a two-state Markov chain in continuous time; time in the volatile state is a stochastic time change of the volatile state process. Facilitating the regime-switching behavior this way, option pricing solutions are readily obtainable for various stochastic price processes.

Much of the option pricing literature has focused on stochastic processes in order to make up for the empirical biases of the Black-Scholes [1973] option pricing model. The most notable bias is the volatility smile (or skew or smirk) that results because Black-Scholes underprices options that are away from the money. Common alternative processes are affine jump- diffusion and pure jump models.

Hamilton [1990] introduces an alternative way to model price processes. In this case, the stochastic process followed by prices is different in different states of the world. This kind of price process is called a *regime-switching model.*

For example, in a two-state regime-switching model asset prices may follow a process with relatively low volatility in the first state and relatively high volatility in the second state. The price process switches between the two states over time (the process that determines the current state of the world is itself a random process), so there are periods of stable random behavior with intermittent periods of high volatility.

From an economic perspective, regime-switching behavior captures the changing preferences and beliefs of households and investors regarding asset prices as states of the world change, such as from a bull market to a bear market. Surprise events like earnings announcements, scandals, or changes in macroeconomic variables signal to investors new information that can cause asset price processes to follow completely different dynamics (switch regimes).

I demonstrate a general method of incorporating regime-switching behavior into asset price processes in order to price options. There are two regimes. The first state is the *common state,* the relatively low-volatility regime, and the second state is the *volatile state*, the relatively high-volatility state.

The log asset price process in the volatile state is the normal state process plus an additional random process driving excess volatility:

State 1 process:	Common process
State 2 process:	Common process + volatile process

The option pricing method applies to a volatile state process in the class of Lévy processes. This includes a wide range of common processes in finance, such as Brownian motion, Poisson processes, and gamma processes.

Research dealing with regime-switching behavior and option pricing includes (in continuous time): Naik [1993], Chourdakis [2001a], and Konikov and Madan [2002]; and (in discrete-time): Duan, Popova, and Ritchken [1999], Bollen, Gray, and Whaley [2000], Chourdakis and Tzavalis [2000], Campbell and Li [2002], and Chourdakis [2001b]. My regime-switching models differ from these models with respect to the stochastic processes considered for regime switching, and also with respect to the method used to incorporate regime switching into the asset price process.

The approach is general and may be used with many different classes of stochastic processes. This represents an advance over model-specific derivations of option pricing formulas that cannot be readily extended to alternative price processes. Such added flexibility in modeling regime-switching behavior is a major contribution.

I. CONTINUOUS-TIME MARKOV CHAIN

Assume there are two states of the world and a continuous-time Markov chain process $\{X_t\}_{t\geq 0}$, which takes on values of 1 and 2, so that:

$$X_s = \begin{cases} 1 & \text{if the process is in state one at time } t = s \\ 2 & \text{otherwise} \end{cases}$$

The transition probability function, $P_{ij}(t)$, gives the probability that, conditional on the process being in state i at time s, it will be in state j at time $t + s$. For $s, t \geq 0$, $i, j \in \{1, 2\}$, $P_{ij}(t)$ is given by the conditional probabilities:

$$\begin{aligned} P(X_{t+s} = 2|X_s = 1) &= P_{12}(t) \\ P(X_{t+s} = 1|X_s = 1) &= P_{11}(t) = 1 - P_{12}(t) \\ P(X_{t+s} = 1|X_s = 2) &= P_{21}(t) \\ P(X_{t+s} = 2|X_s = 2) &= P_{22}(t) = 1 - P_{21}(t) \end{aligned}$$

which are independent of s.

Define the instantaneous transition rate λ_1 as the rate at which the process makes a transition from state 1 into state 2, and define λ_2 as the rate at which the process moves from state 2 to state 1. Then, for a minuscule time interval of length dt and $i, j \in \{1, 2\}$:

$$P_{ij}(dt) = \lambda_i dt + o(dt)$$

and:

$$P_{ii}(dt) = 1 - \lambda_i dt + o(dt)$$

The governing properties of a continuous-time Markov chain are completely determined by the λ_i. A continuous-time Markov chain can be defined as a process that moves from state to state according to a discrete-time Markov chain, but the amount of time spent in state i before transiting to another state is distributed exponentially with mean $1/\lambda_i$.

Over an interval of length t:

$$P_{ij}(t) = \frac{\lambda_i}{\lambda_i + \lambda_j}\left[1 - e^{-(\lambda_i + \lambda_j)t}\right] = 1 - P_{ii}(t)$$

Define the probability distribution of the initial state of the process at $t = 0$ as:

$$\begin{aligned} P(X_0 = 1) &= p_1 \\ P(X_0 = 2) &= p_2 = 1 - p_1 \end{aligned}$$

which need not depend on λ_1 or λ_2^2. Conditioning on the process being in the initial state i implies $p_i = 1$ and $p_{j\neq i} = 0$.

Define the random variable:

$$I_s = \begin{cases} 1 & \text{if } X_S = 2 \\ 0 & \text{if } X_S = 1 \end{cases}$$

The occupation time, $\{T_t\}_{t\geq 0}$, is the amount of time the Markov chain process spends in state 2, and over the time interval $[0, t]$, T_t is given by:

$$T_t = \int_0^t I_s ds$$

The moment-generating function of T_t is (see Darroch and Morris [1968]):

$$\psi_t(u) = E\left(e^{uT_t}\right) = \mathbf{p}' e^{t(\mathbf{R}+u\mathbf{D})}\mathbf{1} \tag{1}$$

where:[1]

$$\mathbf{p} = \begin{bmatrix} p_2 \\ p_1 \end{bmatrix}$$

is the vector of initial state probabilities at $t = 0$, and $\mathbf{1}$ is a vector of ones:

$$\mathbf{R} = \begin{bmatrix} -\lambda_2 & \lambda_2 \\ \lambda_1 & -\lambda_1 \end{bmatrix}$$

and

$$\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Because $\mathbf{R}$ and $\mathbf{D}$ are matrices, $e^{\mathbf{A}}$ (where $\mathbf{A}$ is an $n \times n$ matrix,) represents the exponential matrix, defined as:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{A}^n$$

Numerically, $e^{\mathbf{A}}$ can be efficiently calculated using

$$e^{\mathbf{A}} \approx \left(\mathbf{I} - \mathbf{A}\frac{1}{n}\right)^{-n} = \left[\left(\mathbf{I} - \mathbf{A}\frac{1}{n}\right)^{-1}\right]^n$$

Choosing n to be a high power of 2, say, $n = 2^k$, $e^{\mathbf{A}}$ can be approximated by calculating the inverse of $\mathbf{I} - \mathbf{A}\ (1/n)$ and then performing k multiplications (see Ross [2000, p. 351]).

II. OPTION PRICING

Over the time interval $[0, t]$ the asset price process is assumed to switch randomly between states 1 and 2 according to the evolution of the two-state Markov chain process defined above. Define $\{T_t\}_{t\geq 0}$ as the time the asset price process spends in the second state, and $\{C_t\}_{t\geq 0}$ as the common process, which characterizes the risk-neutral asset price process in state 1 and which also obtains in state 2. The volatile process driving excess price volatility in the second (volatile) state will be denoted by the *time-changed* process $\{V_t\}_{t\geq 0}$, which is a Lévy process, and the occupation time of the asset price process, $\{T_t\}_{t\geq 0}$, is the *directing process*, or the *operational time* of V. This means that the volatile process V operates or evolves only when the Markov chain process is in the second state. It is assumed that $\{C_t\}_{t\geq 0}$, $\{V_t\}_{t\geq 0}$, and $\{T_t\}_{t\geq 0}$ are independent.

The asset price at time t is constructed as

$$S_t = S_0 e^{(r-q)t} e^{C_t} e^{V_{T_t}} \tag{2}$$

where r is the risk-free rate, and q is a continuous dividend yield (for a stock index) or the foreign risk-free rate (for a currency).

The asset price return under this construction has three components: the instantaneous drift $(r - q)t$, the general stochastic component C_t, and the state 2 excess volatility component V_{T_t}. In this framework, C_t models the normal stochastic component governing the asset price behavior, while V_{T_t} models erratic and irregular shocks to the asset price process when the Markov chain is in the second state.

To facilitate arbitrage-free pricing with respect to the risk-neutral pricing measure, it is important to construct the state processes so as to satisfy the martingale condition; that is, so that $E(e^{C_t}) = 1$ and $E(e^{V_{T_t}}) = 1$, in order to ensure that $E(S_t) = S_0 e^{(r-q)t}$, where $E(\cdot)$ is the time zero mathematical expectations operator under the risk-neutral pricing measure.

The process $\{V_t\}_{t\geq 0}$ is defined as a Lévy process in order to facilitate derivation of the characteristic function of $\ln S_t$. A Lévy process is an adapted right-continuous stochastic process with $V_0 = 0$ and the property that non-overlapping increments are independent and stationary (see Epps [2000, p. 131]).[2] That is, the increments $V_t - V_s$ and $V_r - V_q$ are independent for $q < r \leq s < t$ (the two time intervals do not overlap), and the probability distribution of the increment $V_t - V_s$ depends only on the length of the time interval $t - s$ (increments with time intervals of the same length are identically distributed).

A useful result based on these properties is that the characteristic function of V_t may be expressed as:

$$\theta_t(u) = \theta_1(u)^t \tag{3}$$

As a consequence of (3), deriving the characteristic function of the time-changed process V_{T_t} is conveniently straightforward. Define the characteristic function of V_t as $E(e^{iuV_t}) = \theta_t(u)$. Then the characteristic function of the time-changed process is:

$$\begin{aligned} E\left(e^{iuV_{T_t}}\right) &= E\left[E\left(e^{iuV_\tau} \mid T_t = \tau\right)\right] \\ &= E\left(\theta_1(u)^{T_t}\right) \\ &= E\left(e^{T_t \ln\theta_1(u)}\right) \\ &= \psi_t(\ln\theta_1(u)) \end{aligned} \tag{4}$$

where $\psi_t(\cdot)$ is the moment-generating function of the state 2 occupation time, T_t, given in Equation (1).

Therefore, the characteristic function of the time-changed process V_{T_t} is just the moment-generating function of the directing process with respect to $\ln\theta_1(u)$ (i.e., with respect to the logarithm of the characteristic function of the Lévy process at $t = 1$, also called the characteristic exponent). This result is a general property of time-changed Lévy processes (see Epps [2000, p. 136] and Carr and Wu [2004]).

With the moment-generating function of T_t given in (1), the characteristic function of V_{T_t} in (4) may be expressed as:

$$\psi_t(\ln\theta_1(u)) = \mathbf{p}'e^{t(\mathbf{R}+\ln\theta_1(u)\mathbf{D})}\mathbf{1} \tag{5}$$

From the asset price construction in (2), the log asset price can be written as

$$\ln S_t = \ln S_0 + (r - q)t + C_t + V_{T_t}$$

Then, using the result in (5), the characteristic function for the log asset price is:

$$\begin{aligned}\phi_t(u) &= E(e^{iu\ln S_t}) = E(e^{iu\ln S_0 + iu(r-q)t + iuC_t + iuV_{T_t}}) \\ &= e^{iu(\ln(S_0 + (r-q)t)}E(e^{iuC_t})E(e^{iuV_{T_t}}) \\ &= S_0^{iu}e^{iu(r-q)t}\omega_t(u)\mathbf{p}'e^{t(\mathbf{R}+\ln\theta_1(u)\mathbf{D})}\mathbf{1} \\ &= S_0^{iu}e^{iu(r-q)t}\omega_t(u)\psi_t(\ln\theta_1(u))\end{aligned} \tag{6}$$

where $E(e^{iuC_t}) = \omega_t(u)$ and $E(e^{iuV_1}) = \theta_1(u)$ are the characteristic functions of C_t and V_1.

Given the risk-neutralized asset price process, this characteristic function can be used to price derivatives. Since the work of Heston [1993], characteristic functions have been widely used in pricing derivatives (see Carr and Madan [1998], Bakshi and Madan [2000], and Duffie, Singleton, and Pan [2000], for detailed treatments). For example, a European put option on an asset whose price follows the general regime-switching behavior described above can be priced using the formula:

$$C(S_0, t; X) = \frac{1}{2}S_0e^{-qt} + e^{-rt}\int_{-\infty}^{\infty}\frac{e^{-iux}}{2\pi u(i-u)}\phi_t(u-i)du$$

(avoiding, of course, $u = 0$ when integrating numerically), where X is the strike price, q is the continuous dividend yield (for a stock index) or the foreign risk-free rate (for a currency), r is the risk-free rate of interest, and x is the log strike price (see Edwards [2003], Epps [2003], and Lee [2004]). European call options can be priced using this formula and put-call parity.

There are many different stochastic processes that can be used to model the common state C_t process. Possibilities include processes in the affine jump-diffusion class (see Duffie, Singleton, and Pan [2000]), which encompasses many of the most popular option pricing models (Merton [1976], Heston [1993], and Bates [1996]), or pure jump models (Madan, Carr, and Chang [1998], Carr and Wu [2003]). For the volatile state process, V, processes must be of the Lévy class.

The first common state process I consider is Brownian motion:

$$C_t = \sigma W_t - \frac{\sigma^2}{2}t \tag{7}$$

with characteristic function

$$\begin{aligned}\omega(u) &= E\left(e^{iuC_t}\right) \\ &= e^{-u\frac{\sigma^2}{2}t(u+1)}\end{aligned} \tag{8}$$

The $-\frac{\sigma^2}{2}t$ term in (7) is a drift adjustment that ensures that the martingale condition $E(e^{C_t}) = 1$ holds. This common state process specification corresponds to the stochastic component of the Black-Scholes model.

Another possible specification of C_t is to allow the volatility of the process, σ_t, to follow a mean-reverting diffusion as in Heston [1993]:

$$\begin{aligned}dC_t &= \sigma_t dW_{at} \\ d\sigma_t^2 &= (\alpha - \beta\sigma_t^2)dt + \varphi\sigma_t(\rho dW_{at} + \bar{\rho}dW_{bt})\end{aligned}$$

where $\bar{\rho} = \sqrt{1-\rho^2}$, and W_{at} and W_{bt} are independent Brownian motions.

If volatility evolves according to this diffusion process, the exact functional expression of C_t is not known, although its characteristic function is given by:

$$\omega_t(u) = e^{A(t)+B(t)\sigma_0^2}$$

where

$$A(t) = -\frac{\alpha t}{\varphi^2}\ln(1 + \frac{1}{2}(\rho\varphi iu - \beta - g)\frac{1 - e^{gt}}{g2})$$

$$B(t) = \frac{iu + u^2}{\rho\varphi iu - \beta + g\frac{1 + e^{gt}}{1 - e^{gt}}}$$

$$g = \sqrt{(\rho\varphi iu - \beta)^{\rho} + \varphi^2(iu + u^2)}$$

From the pure jump class of models, C_t may be specified as in Carr and Wu [2003] as a *finite moment log stable* (FMLS) process.[3]

The specification is

$$C_t = \sigma L_t^{\alpha,\beta} + \sigma^{\alpha}\sec\left(\frac{\pi\alpha}{2}t\right)$$

where $L_t^{\alpha,\beta}$ denotes an α-stable process with tail index $\alpha \in (1, 2)$ and skewness parameter $\beta = -1$. The term

$$\sigma^{\alpha}\sec\left(\frac{\pi\alpha}{2}t\right)$$

is a drift adjustment to satisfy the martingale condition.[4]

For the volatile state process, V, potential Lévy process models include Brownian motion and the FMLS process. For an example, I consider compound Poisson processes of the construction:

$$V_s = \sum_{j=1}^{N_s} Y_j - \lambda s(\varphi_Y(-i) - 1) \qquad s \geq 0 \tag{9}$$

where $N_s \sim Poisson(\lambda s)$, the Y_j are i.i.d. random variables with finite mean and variance, and $\varphi_Y(-i) = E(e^Y)$ is the characteristic function of Y evaluated at $-i = -\sqrt{-1}$.

The construction in (9) ensures that V_s is a martingale under the risk-neutral pricing measure. The general form of the characteristic function for a compound Poisson process constructed this way is:

$$\theta_s(u) = e^{\lambda s[\varphi_Y(u) - iu\varphi_Y(-i) - 1 + iu]} \tag{10}$$

The time-changed compound Poisson process that evolves only in the second state is then given by:

$$V_{T_t} = \sum_{i=1}^{N_{T_t}} Y_i - \lambda T_t(\varphi_Y(-i) - 1) \tag{11}$$

Using (10) and applying the property of time-changed Lévy processes given in (4), the characteristic function for the process in (11) is given by (5).

Constructed this way, the compound Poisson process V_{T_t} drives excess volatility in the volatile state of the world. When the Markov process is in the second state, the stock price process S is randomly shocked by jump sizes drawn from the distribution of Y at Poisson rate λ, thereby driving excess volatility in the second state.

To construct V_{T_t} the random variable Y may be chosen from a number of distributions, such as, for example, normal $Y \sim N\left(\ln(1 + v) - \xi^2/2, \xi^2\right)$, with:

$$\varphi_Y(u) = e^{iu\ln(1+v) - \xi^2/2 - u^2\xi^2/2} \tag{12}$$

or uniform $Y \sim \text{Uniform}(\varepsilon, \vartheta)$, with:

$$\varphi_Y(u) = \frac{e^{iu\vartheta} - e^{iu\varepsilon}}{iu(\vartheta - \varepsilon)}$$

The possibility of bankruptcy occurring in the volatile state can be modeled as well by defining Y such that $Y = 0$ (and $\varphi_Y(-i) = 1$) with probability one if a Poisson event (bankruptcy) occurs (i.e., if $N_{T_t} > 0$, the firm will have gone bankrupt during the time interval $[0, t]$).

Modeling the volatile state as a compound Poisson process in this way lets us model a variety of volatile and erratic behavior and even bankruptcy in the undesired state of the world.

Any of these common state processes can be combined with any of the volatile state processes to model regime-switching asset price processes with characteristic functions of the form given in (6). For example, combining the specifications in (7), (11), and (12), with

$$C_t = \sigma W_t - \frac{\sigma^2}{2}t,\ V_{T_t} = \sum_{i=1}^{N_{T_t}} Y_i - \lambda v T_t,\ Y \sim i.i.d.$$

$$N\left(\ln(1 + v) - \xi^2/2, \xi^2\right)$$

and

$$N_s \sim Poisson\ (\lambda s)$$

the stock price process is

$$S_t = S_0 e^{(r-q-\frac{\sigma^2}{2})t+\sigma W_t} \prod_{j=0}^{N_{T_t}} Y_j e^{-\gamma \nu T_t} \tag{13}$$

In this case, the first state (common state) process follows Black-Scholes [1973] dynamics, and the second (volatile) state process follows the Merton [1976] jump-diffusion model dynamics. The stock price process switches between the Black-Scholes and the Merton models according to the two-state Markov chain process described earlier. The Poisson jump process drives the excess price volatility in the second state. Besides modeling two-state behavior, by incorporating the regime switch between the Merton jump-diffusion and Black-Scholes models, the model avoids the rapid convergence to normality inherent in the Merton model by itself.

The characteristic function of $\ln S_t$ is, from Equations (6) and (8):

$$\begin{aligned}\phi_t(u) &= S_0^{iu}\omega_t(u)\psi_t(\ln\theta_1(u)) \\ &= e^{iu[\ln S_0+(r-q-\sigma^2/2)t]} e^{-u^2\frac{\sigma^2}{2}t} \mathbf{P}' e^{t(\mathbf{R}+\ln\theta_1(u)\mathbf{D})}\mathbf{1}\end{aligned}$$

with $\ln\ \theta_1(u) = \gamma(e^{iu(\ln(1+v)-\xi^2/2)-u^2\xi^2/2} - (1 + iuv))$.

III. CONCLUSION

My primary contribution is the introduction of a general way to incorporate regime switching in financial models of asset returns through a stochastic time change with the volatile state process directed by the time spent in the volatile state over an interval $[0, t]$. As demonstrated above, this method is flexible in terms of the pricing models it may be applied to. Lévy processes such as Brownian motion, log-stable processes, gamma processes, and subordinated Lévy processes may be used to drive excess volatility in the second state.

ENDNOTES

[1]If p_1 and p_2 are chosen according to the limiting probabilities:

$$p_i = \lim_{t\to\infty} P_{ij}(t) = \frac{\lambda_i}{\lambda_i + \lambda_j}$$

the unconditional probability that the process is in state i at any time t is the same as p_i.

[2]The properties of independent and stationary increments of Lévy processes imply that for each t the distribution of V_t is *infinitely divisible*, which means that V_t can be expressed as the sum of n i.i.d. random variables, $\{V_{t/n}\}_n$, for any positive integer n. Therefore, the characteristic function of V_t can be expressed as $\theta_t(u) = \theta_{t/n}(u)^n$. Taking logs, one can write

$$\ln\theta_{t+1/n}(u) - \ln\theta_t(u) = \ln\theta_{1/n}(u) = n^{-1}\ln\theta_1(u)$$

Multiplying by n and taking the limit as $n \to \infty$, one can derive a simple ordinary differential equation whose solution yields $\theta_t(u) = e^{t\ln\theta_1(u)}$, and hence the result in (3).

[3]The main contribution of the FMLS model is that it allows for infinite moments of the return distribution for any moments greater than one, but with all finite moments of the asset price level. This causes a failure of the central limit theorem as the maturity horizon lengthens, thus capturing important observed properties of the volatility term structure.

[4]The restrictions on α and β ensure that the moments of the asset price level are finite, and hence enable option pricing.

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