Default Correlation Dynamics with Business Cycle and Credit Quality Changes

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The structural model for default correlations developed here can deal with temporal and cross-sectional differences. Corporate debt and first passage-time models are extended to incorporate systematic risk factors in the asset value dynamics. Using the extended models, we analyze the relationship between default probabilities and default correlations, credit quality, and business cycle fluctuations. These extensions provide an easily implementable approach to default probabilities and default correlations that uses both firm-specific and macroeconomic information and allows theoretical justification for recent empirical findings.

A few potential uses of the model include pricing credit derivatives subject to market risk and multiple defaults, and managing credit portfolio risk.

Default correlation has become an important factor in pricing credit derivatives and managing credit portfolios. There is increasing evidence of temporal and cross-sectional differences in defaults and their potential impact on default correlations. Ignoring such differences may introduce a substantial bias in valuation. Gersbach and Lipponer [2000], for example, have demonstrated that adverse macroeconomic shocks can increase default correlations, and this effect may account for more than 50% of the increase in credit risk.

Another aspect of importance is cross-sectional differences in default correlations. Das, Fong, and Geng [2001] provide some evidence that default correlations among high-grade issuers are higher than among low-grade issuers. Findings by Nagpal and Bahar [2001], however, point the other way; default correlations among high-grade issuers are found to be lower. Analytic clarification is in order.[1]

Duffie and Singleton [1999], Jarrow and Yu [2001], and Yang [2002] are among those who have tried to incorporate macro-dependencies and cross-sectional differences in a reduced-form framework. Although the market-based reduced-form approach has many desirable properties, the absence of specification of a default-triggering mechanism and the model's dependence on data make a companion structural form model a necessity.

The structural approaches to date, however, fall short of explaining dependencies and differences in the default correlation structure. While Zhou [2001] extends Merton's [1974] model by allowing local correlations in firm value dynamics, the result is not adequate to accommodate the issues at hand, particularly in addressing persistent macro-dependencies.

We develop a structural model for default correlation dynamics that is capable of dealing with the observed macro-dependencies and cross-sectional differences. The classic Merton [1974] model and the first-passage-time model of Zhou [2001] are both extended to incorporate stochastic macroeconomic factors in the asset value dynamics. These extensions allow default probabilities and default correlations to be explicitly dependent on macroeconomic shocks.

Using these analytical models, we investigate the features of default correlations induced by changes in the market environment and their implication. The macro-dependence we introduce produces a clustering effect, as it is designed to do; it makes both default probability and default correlation high in a bearish market. Cross-sectional differences in default correlations can also be induced by the impact of systematic risk factors on asset values. Our model reveals that default correlations of high-grade firms with less idiosyncratic risk tend to be greater than those of low-grade firms.

Our primary contributions are as follows. First, we suggest an easily implementable approach to default probabilities and default correlations that uses both firm-specific and macroeconomic information. Second, we provide a theoretical justification for various default correlation dynamics that appear in the empirical literature.

I. MODEL

We first present a basic default correlation structure and develop an extended Merton model and an extended first-passage-time model for default correlation dynamics, accounting for macroeconomic shocks.

Assumptions

Assumption 1. *The macroeconomic state variables Y_n have correlated mean-reverting Gaussian distributions:*

$$dY_n = \kappa_n(\theta_n - Y_n)dt + \delta_n dW_{Y_n}, \quad 1 \leq n \leq N \tag{1}$$

where κ_n, θ_n, and δ_n are positive constants, and W_{Y_n} is a standard Brownian motion under the spot probability measure. The macroeconomic state variables are allowed to be correlated; two Brownian motions W_{Y_i} and W_{Y_j} $(1 \leq i,j \leq N)$ are correlated with instantaneous correlation coefficients $\rho_{Y_iY_j}$.

This assumed dynamics of the state variables is the same as the instantaneous spot interest rate process of Vasicek [1977]. Since economywide state variables, such as the default-free spot interest rate, production growth rate, and inflation rate, are mean-reverting, it is reasonable that state variables are assumed to follow an Ornstein-Uhlenbeck process.[2]

Assumption 2. *The asset value V_m of firm m has a lognormal diffusion process:*

$$\frac{dV_m}{V_m} = \left(a_m^T \cdot Y(t) + b_m\right) dt + \sigma_m dW_{V_m}$$

$$\text{for } 1 \leq m \leq M \tag{2}$$

where a_m is an $N \times 1$ vector of constants; a_m^T is the transpose of the vector a_m; $Y(t)$ is an $N \times 1$ vector; b_m is a constant; σ_m is the standard deviation of asset returns; and W_{V_m} is a standard Brownian motion under the spot probability measure. The asset values are assumed to be correlated with each other as well as with the macroeconomic state variables: $E[dW_{V_i}dW_{V_j}] = \rho_{V_iV_j}dt (1 \leq i, j \leq M)$, and $E[dW_{V_m}dW_{Y_n}] = \rho_{V_mY_n}dt$ $(1 \leq m \leq M,\ 1 \leq n \leq N)$.

The elements of the vector $a_m^T = (a_{m1}, a_{m2}, \ldots, a_{mN})$ are expected asset return sensitivities of firm m to common risk factors $Y(t)$, and the constant b_m is a firm-specific expected asset return portion not affected by common risk factors. To incorporate macroeconomic shocks into the stochastic asset value process, the instantaneous expectation of the asset return is assumed to have a level dependence on stochastic common risk factors.

Although the instantaneous variance of asset returns may also be subject to economywide shocks, we do not incorporate this dependence here because the assumption on business cycle-adjusted stochastic asset value process is sufficient to illustrate the impact of the business cycle on asset value dynamics. In addition, even if all variables are assumed to follow a multivariate normal distribution, the assumption does not necessarily make the tail-dependence of firm returns go to zero. In a model such as Zhou [2001], where default correlations are induced only by joint movements of the local changes of Gaussian type, tail-dependence would go to zero as correlations go farther into the tail.

In our model, however, the macro-dependence could induce non-negligible extreme tail-dependence except when the macroeconomic variable goes to the extreme tail at the same time. In other words, the extreme tail-dependence of two different firms could still be quite high, depending on the state of the macroeconomic variable and the sensitivity of each firm's returns to the macroeconomic variable

The relation between systematic risk factors and asset values is modeled in two ways: through dependence at the expectation level, and through correlated shocks. To be specific, dependence of expected asset returns on the macroeconomic state is the first way, and instantaneous correlation between the unexpected changes in asset returns and the state variables, $\rho_{V_mY_n}$ is the other.

Assumption 3. *The default of a firm is triggered by a decline in its asset value. For each firm m, there is a time-dependent value $K_m(t)$ such that the firm continues to operate and meets its contractual obligations as long as $V_m(t) > K_m(t)$.*

We measure credit quality (or credit grade) by leverage, V/K. The default boundary can be defined in various forms. For example, Black and Cox [1976] take an exponential form in time t because the expected debt value usually takes this form. In practical applications, the default boundary is a weighted average of the firm's long-term and short-term liabilities. The default boundary can also be stochastic, moving together with firm value and the business cycle.

Default Correlation Structure

Suppose there are two random variables, $D_1(t)$ and $D_2(t)$, that describe the default status of two firms over a given horizon t:

$$D_m(t) = \begin{cases} 1 & \text{if firm } m \text{ defaults by } t, \ \mathrm{m} \in \{1,2\} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

It is reasonable to assume that when one firm defaults, the other firm may have a higher likelihood of defaulting. Because both firms are subject to the general economy, their industry, or their region, they are likely to have a positive default correlation. This phenomenon is what Zhou's [2001] structural default correlation model tries to address.

In Zhou's model, however, the default correlation between two firms is caused only by the local asset return correlation. The default correlation between two firms with a trivial local asset return correlation would thus be almost zero, yet the firms may exhibit a high correlation during economic downturns despite their trivial asset return correlation. Addition of common economywide risk factors to the stochastic asset value movements would be necessary to account for such dependence.

The default correlation $\rho_{DC}(t)$ over a given horizon t is defined as:

$$\rho_{DC}(t) = \frac{E[D_1(t) \cdot D_2(t)] - E[D_1(t)]E[D_2(t)]}{\sqrt{Var[D_1(t)] \cdot Var[D_2(t)]}} \tag{4}$$

Because $D_m(t)$ is a Bernoulli binomial random variable, we have:

$$\begin{aligned} E[D_m(t)] &= P(D_m(t) = 1), \\ Var[D_m(t)] &= P(D_m(t) = 1) \cdot [1 - P(D_m(t) = 1)] \end{aligned}$$

where $P(A)$ is the probability that event A occurs.

The default correlation is useful in determining joint default probability and in evaluating the probability that either firm defaults. The joint default probability is:

$$\begin{aligned} &P(D_1(t) = 1 \ and \ D_2(t) = 1) = \\ &P(\tau_1 \le t \ and \ \tau_2 \le t) = E[D_1(t) \cdot D_2(t)] \end{aligned} \tag{5}$$

where τ_m $(= \min_{t\ge0}\{t \mid V_m(t) \le K_m(t)\})$ is the first time that the firm m asset value reaches its default threshold level $K_m(t)$. The probability that either firm defaults is:

$$\begin{aligned} &P(D_1(t) = 1 \ or \ D_2(t) = 1) \\ &= P(D_1(t) = 1) + P(D_2(t) = 1) - P(D_1(t) = 1 \ and \ D_2(t) = 1) \\ &= E[D_1(t)] + E[D_2(t)] - E[D_1(t) \cdot D_2(t)] \end{aligned} \tag{6}$$

One can easily see that, given the individual default probabilities, positive default correlation reduces the probability that either firm will default.

Extended Merton Model of Default Correlation

We first investigate a simple case of two firms that can default only at maturity, as in Merton's model. Proposition 1 describes the joint default probability with one systematic risk factor, which is easily extended to the case with multiple systematic risk factors.

Proposition 1. *With deterministic default threshold level $K_m(t)$, the joint default probability with maturity t is:*

$$\begin{aligned} &P(V_1(t) \le K_1(t) \ and \ V_2(t) \le K_2(t)) \\ &= \int_{-\infty}^{d_1(t)} \int_{-\infty}^{d_2(t)} \frac{1}{2\pi\sqrt{1-\rho_{AR}(t)^2}} \exp\left(-\frac{x^2 - 2\rho_{AR}(t)xy + y^2}{2(1-\rho_{AR}(t)^2)}\right) dxdy \\ &= BVN(d_1(t), d_2(t), \rho_{AR}(t)) \end{aligned}$$

where for m = 1, 2:

$$\begin{aligned}
d_m(t) &= \frac{\ln K_m(t) - M_m(t)}{S_m(t)}, \quad \rho_{AR}(t) = \frac{S_{1,2}(t)}{S_1(t)S_2(t)} \\
M_m(t) &= \ln V_m(0) + \left(b_m - \frac{1}{2}\sigma_m^2\right)t + \\
&\quad a_m\left[B(t;\kappa)(Y_0 - \theta) + \theta t\right] \\
S_m^2(t) &= \sigma_m^2 t + 2a_m\left(\frac{\rho_{V_mY}\sigma_m\delta}{\kappa}\right)\left[t - B(t;\kappa)\right] + \\
&\quad a_m^2\left(\frac{\delta}{\kappa}\right)^2\left[t - 2B(t;\kappa) + B(t;2\kappa)\right] \\
S_{1,2}(t) &= a_1a_2\left(\frac{\delta}{\kappa}\right)^2\left[t - 2B(t;\kappa) + B(t;2\kappa)\right] + \\
&\quad \frac{a_1}{\kappa}(\rho_{V_2Y}\sigma_2\delta)\left[t - B(t;\kappa)\right] + \\
&\quad \frac{a_2}{\kappa}(\rho_{V_1Y}\sigma_1\delta)\left[t - B(t;\kappa)\right] + \sigma_1\sigma_2\rho_{V_1V_2}t \qquad (7) \\
B(t;\kappa) &= (1 - e^{-\kappa t})/\kappa
\end{aligned}$$

$BVN(a, b, \rho)$ = *cumulative probability distribution function for a standard bivariate normal variable.*

The covariance between asset returns $S_{1,2}(t)$ has three components. The first component [the first term in Equation (7)] represents the common dependence of expected asset returns on the systematic risk factor. The second component (the second and third terms in the equation) represents the correlated shocks between asset returns and common risk factors, and the third component (the fourth term in the equation) corresponds to the local asset return correlation. The first two components reflect the interdependence among firms caused by economywide fluctuations.

Given the default probability of each single firm and the joint default probability of both firms, default correlations can be easily obtained using Equation (4). The Merton model of default correlation (the standard structural model) is obtained as a special case when expected asset return has zero sensitivity to systematic risk factors.

Extended First-Passage-Time Model of Default Correlation

Because the assumption that a risky bond may default only at its maturity is restrictive and unrealistic, the first-passage-time model has been developed and widely used. Notable examples include Black and Cox [1976], Leland and Toft [1996], Longstaff and Schwartz [1995], and Zhou [2001]. Of these, Zhou [2001] is the first to develop a model for two firms, while all other models are for a single firm.

Generalizing Zhou's approach, we develop a first-passage-time model with common economywide risk factors. The joint default probability of two firms with a single risk factor is expressed in Proposition 2. This approach can be easily extended to a case with multiple systematic risk factors and multiple firms (*see Appendix A*).

Proposition 2*. Discretize time into* n_t *equal intervals, and define date* $t_j = jt/n_t = j\Delta t$ *for* $j \in (1, 2, \ldots, n_t)$. *Similarly discretize Y-space into* n_Y *equal intervals between some chosen minimum* $\underline{Y}$ *and maximum* $\overline{Y}$, *and define* $Y_i = \underline{Y} + i\Delta Y$ *for* $i \in (1, 2, \ldots, n_Y)$, *where* $\Delta Y = (\overline{Y} - \underline{Y})n_Y$. *With a deterministic default threshold level* $K_m(t)$ *for* m = 1, 2, *the joint default probabilities during time period t under the first-passage-time model are:*

$$P(D_1(t) = 1 \; and \; D_2(t) = 1) = \sum_{j=1}^{n_t}\sum_{i=1}^{n_Y} q(Y_i, t_j) \qquad (8)$$

where

$$\begin{aligned}
q(Y_i, t_1) &= \Delta Y \Psi(Y_i, t_1) \; \forall i \in (1, 2, \ldots, n_Y) \\
q(Y_i, t_j) &= \Delta Y \left[\Psi(Y_i, t_j) - \sum_{v=1}^{j-1}\sum_{u=1}^{n_Y} q(Y_u, t_v)\psi(Y_i, t_j | Y_u, t_v)\right] \\
&\quad \forall i \in (1, 2, \ldots, n_Y), \; \forall j \in (2, \ldots, n_t)
\end{aligned}$$

$$\begin{aligned}
\Psi(Y, t) &= \pi(Y_t, t | Y_0, 0) \times \\
BVN &\left(\frac{\mu_1(Y_t, t | l_0^1, Y_0, 0)}{\Sigma_1(Y_t, t | l_0^1, Y_0, 0)}, \frac{\mu_2(Y_t, t | l_0^2, Y_0, 0)}{\Sigma_2(Y_t, t | l_0^2, Y_0, 0)}, \Theta(Y_t, t | l_0^1, l_0^2, Y_0, 0)\right)
\end{aligned}$$

$$\begin{aligned}
\psi(Y_t, t | Y_s, s) &= \pi(Y_t, t | Y_s, s) \times \\
BVN &\left(\frac{\mu_1(Y_t, t | l_s^1 = 0, Y_s, s)}{\Sigma_1(Y_t, t | l_s^1 = 0, Y_s, s)}, \frac{\mu_2(Y_t, t | l_s^2 = 0, Y_s, s)}{\Sigma_2(Y_t, t | l_s^2 = 0, Y_s, s)},\right. \\
&\left. \Theta(Y_t, t | l_s^1 = 0, l_s^2 = 0, Y_s, s)\right)
\end{aligned}$$

Here, $\pi(Y_t, t | Y_s, s)$ *is the transition density for the systematic risk factor;* l_t^m *is the logarithm of the leverage ratio* $K_m(t)/V_m(t)$; $\mu_m(Y_t, t | l_s^m, Y_s, s)$ *and* $\Sigma_m(Y_t, t | l_s^m, Y_s, s)$ *are the expectation and standard deviation of* l_t^m, *respectively, conditional upon the values* $\{Y_t, l_s, Y_s\}$. $\Theta(Y_t, t | l_s^1, l_s^2, Y_s, s)$ *is the correlation between* l_t^1 *and* l_t^2 *conditional upon* $\{Y_t, l_s^1, l_s^2, Y_s\}$.

Proof: See Appendix B for proof of the proposition and for the definitions of μ_m, Σ_m, and Θ.

Using the result of Collin-Dufresne and Goldstein [2001], we can easily obtain the default probabilities of firms exposed to fluctuation of systematic risk factors. The default correlation can then be derived using Proposition 2. Our extended first-passage-time model of default correlation nests Zhou's [2001] model as a special case when asset values are not affected by the systematic risk factor.

II. IMPLICATIONS OF THE MODEL

We can analytically show the relation between default probabilities and business cycles. We also show that changes in default correlation are related to both credit qualities and economic conditions.

Default Probability Depending on Business Cycle

Crouhy, Galai, and Mark [2000, 2001], Das, Fong, and Geng [2001], Erlenmaier and Gersbach [2001], and Fridson, Garman, and Wu [1997] show default probabilities to be dependent on the credit quality of each issuer cross-sectionally and to vary with the business cycle over time. A company's ability to pay its bondholders depends on its ability to generate profits, which may be sharply impaired in a recession. In fact, many empirical results show high default probabilities in a recessionary economy.

Current structural models, however, are not equipped to explain such regularities. In the structural models we have to date, the macroeconomic impact enters the system only implicitly through firm value, and there is no way to isolate the macro effect. An explicit representation of this dependence in the system is necessary.

The default probability of an issuer in the extended Merton model is:

$$P(V_m(t) \leq K_m(t)) = N(d_m(t)) \tag{9}$$

where

$$d_m(t) = \frac{\ln K_m(t) - M_m(t)}{S_m(t)} \tag{10}$$

$$\begin{aligned} M_m(t) &= \ln V_m(0) + \left(b_m - \frac{1}{2}\sigma_m^2\right)t + \\ &\quad a_m\left[B(t;\kappa)(Y_0 - \theta) + \theta t\right] \end{aligned} \tag{11}$$

$$\begin{aligned} S_m^2(t) &= \sigma_m^2 t + 2a_m\left(\frac{\rho_{V_mY}\sigma_m\delta}{\kappa}\right)[t - B(t;\kappa)] \\ &\quad + a_m^2\left(\frac{\delta}{\kappa}\right)^2[t - 2B(t;\kappa) + B(t;2\kappa)] \end{aligned} \tag{12}$$

$$B(t;\kappa) = (1 - e^{-\kappa t})/\kappa \tag{13}$$

Equation (9) can be used to explain the observed phenomenon of default clustering. Assume, for example, that the economywide variable Y represents the production growth rate. If the instantaneous asset return is directly proportional to the production growth rate, the expected asset return, $M_m(t)$, is greater in a bullish market ($Y_0 > \theta$) than in a bearish market ($Y_0 < \theta$) as shown in Equation (11).

Equation (12) shows that the variance of asset return, $S_m^2(t)$, is composed of two parts. One is the firm-specific (diversifiable) risk represented by the first term of the asset return variance, and the other is the economywide (systematic) risk associated with the second and third terms of the asset return variance. The systematic uncertainty of the asset return is related to the sensitivity of the asset return to the market condition, a_m, and the correlation between macroeconomic shock and asset value, ρ_{V_mY}. While the Merton model cannot tell the difference between two firms when they have the same asset return variance, the extended model can make finer distinctions by decomposing the total variance into idiosyncratic and systematic parts.

The default probability of a firm in the extended Merton model is the same as in Merton's default model if the asset value does not depend on the business cycle (i.e., $a_m = 0$). Firm value, however, fluctuates with the state of the economy in general (i.e., $a_m \neq 0$), and so does the default probability.

The sensitivity of the default probability to Y_0, representing the current economic condition, is

$$\frac{\partial N(d_m(t))}{\partial Y_0} = n(d_m(t))\left[-a_m\frac{B(t;\kappa)}{S_m(t)}\right] \tag{14}$$

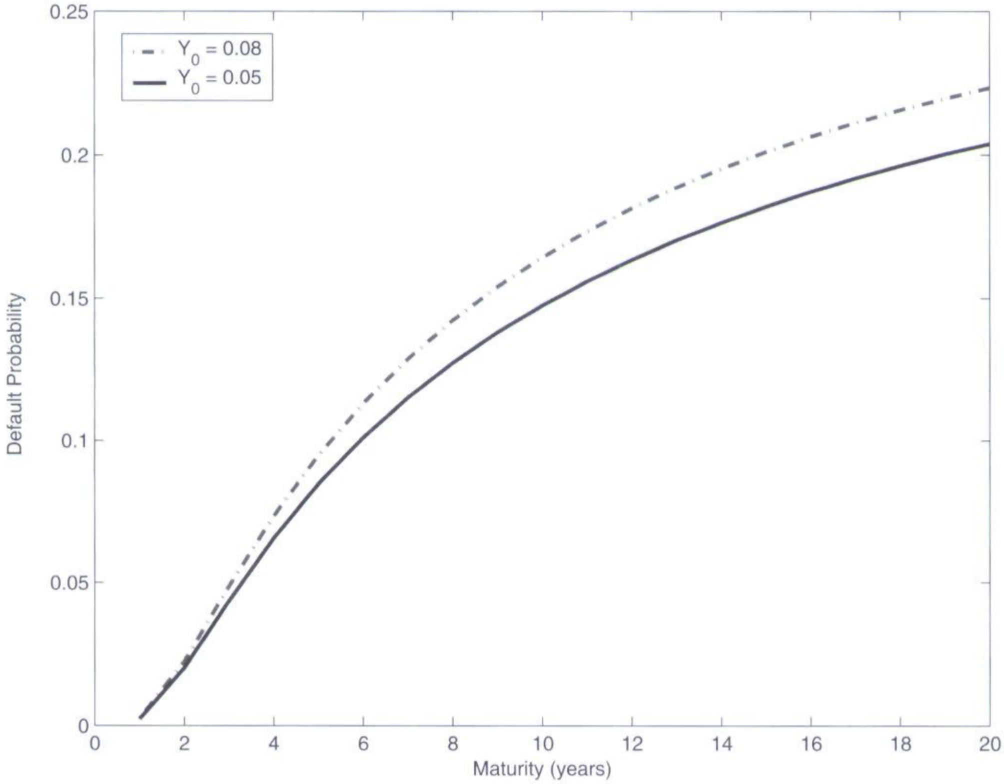
As shown in Equation (14), the partial derivative of the default probability with respect to an economywide shock has the opposite sign from a_m regardless of investment horizon t.[3]

When asset returns are positively related to the economic state variable (in other words, when sensitivity a_m is positive), asset values increase with the level of the state variable and, as a result, default probabilities decline.

Exhibit 1 shows that negative (positive) macroeconomic shocks increase (reduce) default probabilities. It also shows that differences in default probabilities induced by market conditions increase with the investment horizon. The average default probability across different maturities is 11.04% higher in downturns than in normal times. In addition, there is a greater difference in default

EXHIBIT 1

Relation Between Default Probability and Market Environment in Extended Merton Model



Y_0 is implicitly assumed to be a negative market indicator such as the unemployment rate. Parameter values: $V/K = 2$, $a = -0.3$, $\sigma = 0.25$, $\kappa = 0.1$, $\theta = 0.08$, $\delta = 0.03$, and $\rho_{VY} = -0.1$.

probabilities when firm values have more systematic risk exposures.

Default Correlation Depending on Credit Quality

Recent credit risk research shows that default correlations between high-grade firms are quite different from those between low-grade firms, regardless of investment horizon. For example, Das, Fong, and Geng [2001] show that the default probabilities of higher-grade firms are more correlated and have greater higher-order moments, using a monthly time series of one-year default probabilities for 7,000 non-financial U.S. public firms rated by Moody's. Default correlations are obtained from the correlation of hazard rates estimated from the Moody's default probability data. The results indicate that the correlation of default probabilities depends on the credit quality of issuers.[4]

The default risk of a credit portfolio, in fact, is made up of two parts. One part is the default probability of each individual issuer, and the other part is the default correlation between issuers, which is the source of the diversification effect in portfolio value. Accordingly, as the default correlation increases, the diversification effect is diminished, and there is a greater probability of a large loss due to multiple defaults. This association makes accurate estimation of default correlation a necessity in managing the default risk of portfolios.

Exhibit 2 presents the correlation of default probabilities and changes in default probabilities. It shows that correlations of default probability among high-grade firms are almost six times higher than those among low-grade firms.

Other authors suggest higher default correlations among low-grade issuers than high-grade issuers. Barnhill and Maxwell [2002] find that, as credit quality deteriorates, systematic risk exposure (as measured by the equity beta) increases, which in turn raises the default correlation. Gersbach and Lipponer [2000] and Zhou [2001] also find that default correlations decline as credit quality improves. Zhou [2001] explains the cross-sectional variation in the

EXHIBIT 2
Average Correlation Within Class

Average correlation within class	Aaa/Aa	A	Baa	Ba	B	Caa/Ca/C
Default Probability Levels	0.5033	0.3628	0.1696	0.0888	0.0945	0.0869
Default Probability Changes	0.1656	0.0810	0.0572	0.0313	0.0268	0.0588

Monthly time series of probabilities of default for U.S. non-financial public firms over January 1987–October 2000 obtained from Moody's Risk Management Services.

default correlation with fixed local asset return correlation as determined by the stock return correlation, regardless of the credit quality. His approach assumes that the systematic risk exposure is equal for all the firms, but that the distance-to-default differs in explaining the cross-sectional default correlation dynamics.[5]

Although there are contrary results in the literature, a consistent conclusion is that default correlations are positively related to the systematic risk exposures of firms, regardless of credit quality, which cannot be explained in the structural approaches such as Merton [1974] and Zhou [2001].

It can be shown that default correlations rise as the systematic risk exposures of issuers increase, using Proposition 3, which is based on the extended Merton model of default correlation. When two firms have equal firm-specific parameters, we can analytically show the dependence of the default correlation on the systematic risk exposures of issuers.[6]

Proposition 3. *The partial derivative of the default correlation ρ_{DC} with respect to systematic risk exposure ρ_{VY} is:*[7]

$$\frac{\partial \rho_{DC}}{\partial \rho_{VY}} = \frac{\partial \rho_{DC}}{\partial \rho_{AR}} \cdot \frac{\partial \rho_{AR}}{\partial \rho_{VY}} \tag{15}$$

and the partial derivative of the asset return correlation ρ_{AR} with respect to the correlation between asset value and macroeconomic shock ρ_{VY} has the same sign as a:

$$\frac{\partial \rho_{AR}}{\partial \rho_{VY}} = \frac{1}{S^4(t)}\left[2a(1-\rho_{V_1V_2})\sigma^2 t\left(\frac{\sigma\delta}{\kappa}\right)(t-B(t;\kappa))\right] \tag{16}$$

where ρ_{VY} is used as a proxy for systematic risk exposure, $S^2(t) = Var_0[\ln V(t)]$, and $B(t;\kappa) = (1-e^{-\kappa t})/\kappa$.

Proof: See Appendix C.

When a is positive (i.e., when the systematic risk factor has a positive effect on expected asset return), asset return correlation ρ_{AR} increases with ρ_{VY} using Equation (16). Therefore, default correlation ρ_{DC} is higher when the level of systematic risk exposure is high, as shown in Equation (15).[8]

The first two graphs in Exhibit 3 show that the default correlation in the Merton model declines, the farther the firm is from default. Panel A plots the relation between distance-to-default (and Panel B the relation between default correlations) and the investment horizon for different credit qualities.

Panel B illustrates that default correlations always drop with improving credit quality in the Merton model, which is not always consistent with empirical findings. By Proposition 3, however, default correlation can increase with credit quality, where firms in a high-quality class are more subject to systematic risk than those in a low-quality class (*see Panel D of Exhibit 3*).

The last two graphs in Exhibit 3 illustrate the possibility that default correlations can increase even though asset values are farther away from the specific default threshold. For example, default correlations among high credits with $\rho_{VY} = -0.5$ ($a = -0.5$) are 22.62% higher than among low credits with $\rho_{VY} = 0$ ($a = 0$) for a ten-year maturity.

In fact, high-grade firms producing export goods can be more exposed to foreign exchange risk than low-grade firms producing mainly domestic consumer goods. Revaluation of the domestic currency causes asset values of the high-grade exporters to deteriorate, thereby causing more frequent defaults among the high-grade exporters. Consequently, default correlations among high-grade firms can be higher than among low-grade firms.

Of course, when all firms are equally exposed to systematic risk, low-grade firms may simultaneously default more easily than high-grade firms with negative market shocks. This phenomenon is often observed in a real world because of the higher default probabilities of the low-grade firms.

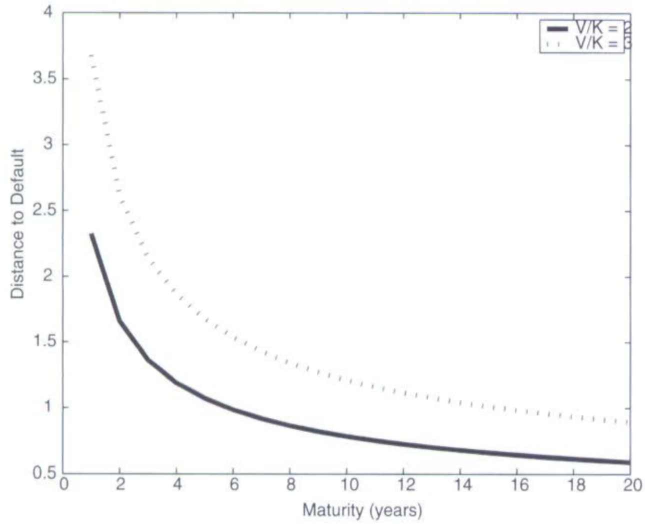
Default Correlation Depending on Business Cycle

Default correlations vary with time and depend on prevailing economic states as empirical findings show.

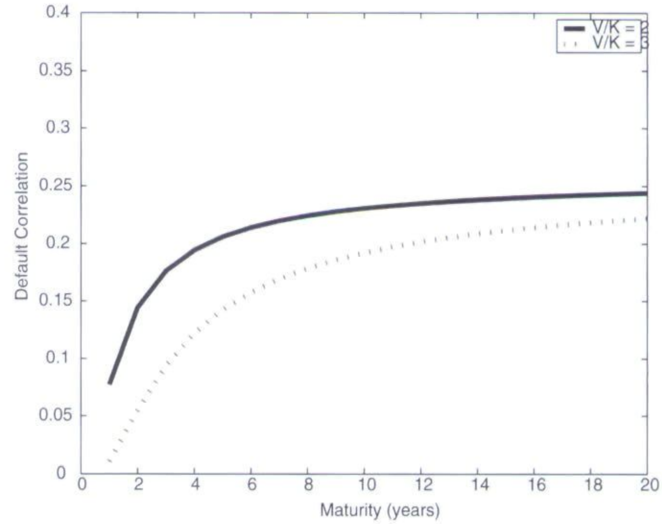
EXHIBIT 3

Relative Business Cycle Effects on Asset Values with Different Credit Qualities in Extended Merton Model

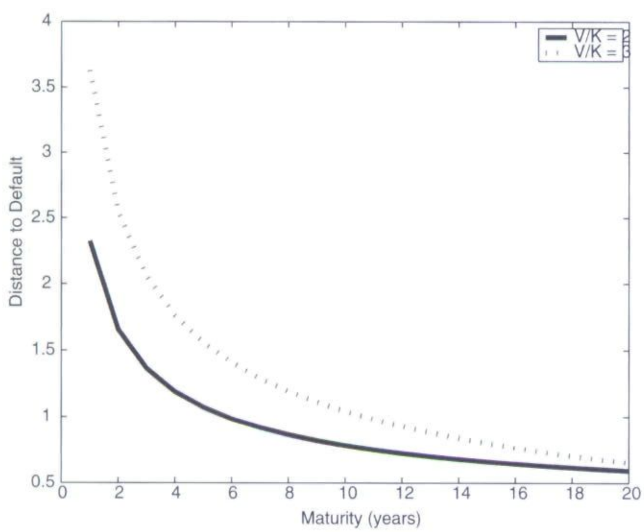
Panel A. Term Structure of Distance to Default— No Systematic Risk Exposures



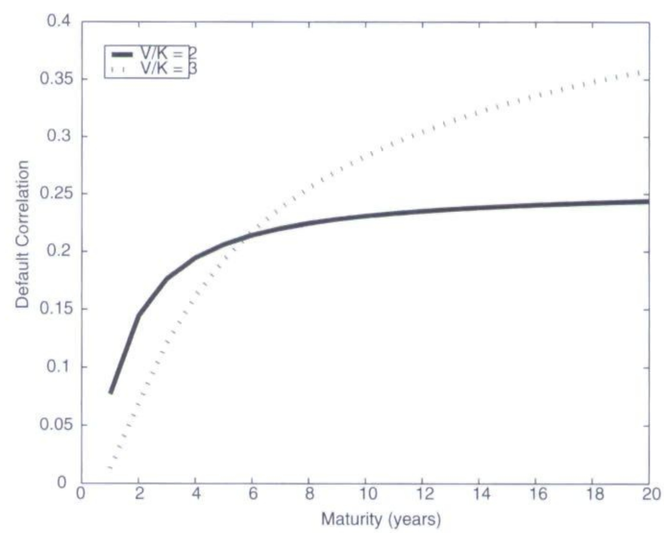
Panel B. Term Structure of Default Correlation— No Systematic Risk Exposures



Panel C. Term Structure of Distance to Default— With Different Systematic Risk Exposures



Panel D. Term Structure of Default Correlation— With Different Systematic Risk Exposures



In Panels A and B, the sensitivities of the low-grade (high-grade) asset values to a systematic risk factor are $a_L = 0$ ($a_H = 0$). In Panels C and D, the sensitivities of the low-grade (high-grade) asset values to a systematic risk factor are $a_L = 0$ ($a_H = -0.5$). The other parameter values are $\sigma_1 = \sigma_2 = 0.3$; $\rho_{V_1V_2} = 0.5$; $\kappa = 0.1$; $\theta = 0.08$; $Y_0 = 0.05$; and $\delta = 0.03$.

They tend to be greater in stressful markets than under normal conditions. That is to say, default correlations are greater when default probabilities are high.

Using a structural model, Barnhill and Maxwell [2002] show that asset values are more sensitive to macroeconomic conditions in downturns since average credit quality declines as economic conditions deteriorate. Gersbach and Lipponer [2000] also find that negative macroeconomic shocks heighten default correlations, thereby engendering cyclical effects as portfolio diversification benefits decline.

Using reduced-form models, Das et al. [2001] and Das, Fong, and Geng [2001] also suggest a relation between the default correlation and the business cycle. Their results parallel those of structural models in that default correlations are estimated to be higher when markets move down.

Although this effect is frequently mentioned in the default correlation-related empirical research, theoretical analysis has not been yet developed in the structural models.

With the introduction of a variable Y_0 representing the current market environment, it can be shown that the direction of default correlation movement is related to economywide conditions, as indicated in Proposition 4. We use in deriving it the fact that default correlations are positively related to the asset return correlation.

Proposition 4. *By the chain rule, the partial derivative of the default correlation ρ_{DC} with respect to the current business condition Y_0 is:*[9]

$$\frac{\partial \rho_{DC}}{\partial Y_0} = \frac{\partial \rho_{DC}}{\partial \rho_{AR}} \cdot \frac{\partial \rho_{AR}}{\partial \rho_{AL}} \cdot \frac{\partial \rho_{AL}}{\partial Y_0} \tag{17}$$

and the partial derivative of the asset level correlation ρ_{AL} with respect to Y_0 has the opposite sign of the sum of the a_m as follows:[10]

$$\frac{\partial \rho_{AL}}{\partial Y_0} = -2\rho_{AL}(a_1 + a_2)B(t;\kappa) \tag{18}$$

where $B(t:\kappa) = (1 - e^{-kt})\kappa$.

Proof. See Appendix D.

For example, suppose the unemployment rate reflects a negative economywide shock. Then the default correlation for an unemployment rate of 5% is 12.9%, while the default correlation for an unemployment rate of 8% is 13.5% for parameter values of $a_1 = a_2 = -0.3$; $V_1/K_1 = V_2/K_2 = 2$; $\rho_{V_1 V_2} = 0.5$; and $t = 1$. This difference in default correlations caused by the business cycle can be more easily seen using the first-passage-time approach. In fact, the impact of a macroeconomic shock on default correlation strengthens in a recession when credit-risky portfolio management is more necessary, so consideration of the business cycle in determining default correlation is essential.

When financial market stress contracts the investment or production of all firms, profits decline and firms have diminished capacity to pay their liabilities. Many firms may default simultaneously due to general market shocks although their interdependence is small. On the other hand, interfirm dependencies may have a material impact on defaults even when macro-dependence is relatively low.

Interrelatedness created by the *chaebol* system in Korea is a clear case. This interdependence can, of course, be explained by the local asset return correlation alone. Both types of dependencies are needed for a full explanation of the default dynamics.

III. COMPARISONS WITH THE EXTENDED FIRST-PASSAGE-TIME MODEL

We compare the extended first-passage-time model first with Merton [1974] and then with Zhou [2001].

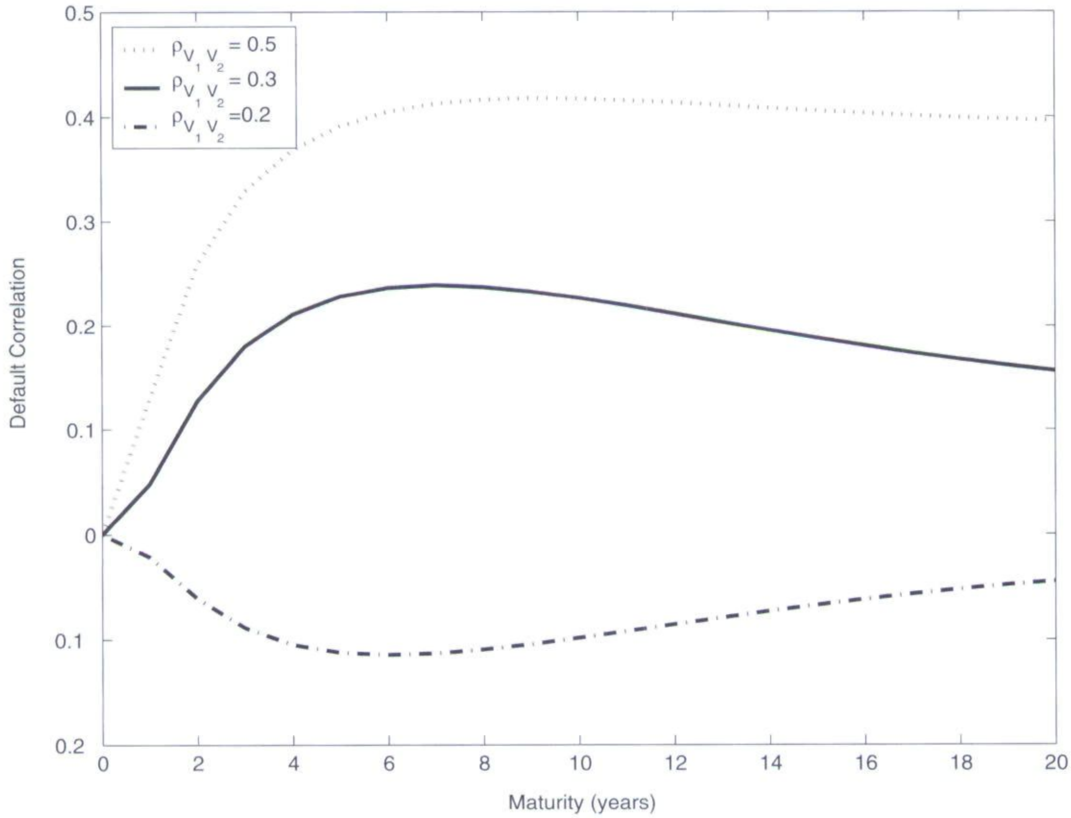
Extended First-Passage-Time Model versus Extended Merton Model

The extended Merton model can be used to explain empirical evidence on the properties of default probabilities and default correlations. The sign of the relationship between default correlations and business cycles and credit quality in the extended Merton model is, in general, the same as in the extended first-passage-time model. The business cycle effect on default correlation dynamics is stronger in the extended first-passage-time model, however, because it recognizes that many firms may default at different times over the investment horizon, due to economywide shocks.

With our parameter values, the percentage increment in default correlations due to the market environment can be almost four times higher in the extended first-passage-time model than in the extended Merton model, particularly when asset value dynamics are highly sensitive to market conditions. For example, the percentage increment of the default correlation with a one-year matu-

EXHIBIT 4

Relation Between Default Correlation and Time for Given Local Asset Return Correlations



rity in bearish market conditions over that in normal times is 2.54% in the extended Merton model, while it is 8.83% in the extended first-passage-time model.[11]

Extended First-Passage-Time Model versus Zhou Model

Zhou's default correlation model explains joint default events by the local asset return correlation between two firms. Exhibit 4 illustrates the relation between default correlation and time for given local asset return correlations when asset values are not exposed to general economic conditions. It shows that default correlations increase with local asset return correlations and converge over time.[12]

Zhou's model cannot explain the default clustering phenomenon that has been observed in recent financial crises in East Asia and the United States, and it also cannot be used to obtain the joint default probability of multiple firms. Exhibit 5 plots the relation between default correlations and the investment horizon under varying degrees of dependence of asset values on market shocks.

Implications of this dependence are that:

- Default correlations persist longer when asset values are more related to negative macroeconomic shocks. When there is no systematic risk exposure in asset values, default correlations initially increase and then slowly decline with time, which is the same as the results in Zhou's approach. Default correlations among issuers with negative systematic risk exposure increase persistently with time, while default correlations among different credit classes with positive systematic risk exposure decline more rapidly than those without systematic risk exposure for longer time horizons. Hence, joint default events over a long enough time horizon can occur because of negative economywide shocks.
- Default correlations are not always lower than local asset-level correlations because default events are affected by macroeconomic fluctuations and not just local asset-level correlation. This result is consistent with the default clustering phenomenon observed across firms in different industries or in different regions.

- The difference in default correlations due to systematic risk exposure is very sensitive to the level dependence of asset values on macroeconomic risk factors. For example, the default correlation between firms with $a = -0.1$ is almost 20% to 30% higher than for matching firms with $a = 0$ for long investment horizons.
- Generally, default correlations between firms exposed to positive economic shocks are lower than when the firm values do not have macro-dependencies.

When multiple defaults due to economywide shocks are ignored, default correlations may decline as the time horizon lengthens. This phenomenon can be explained by the fact that multiple defaults over a long horizon are caused mainly by idiosyncratic factors in such a model. But when market shocks are considered, the default correlation term structure may change. Under assumed macro-dependence, however, the cumulative effects of the comovement of asset values during economic downturns would result in higher default correlations over a long time horizon than for firms without it.

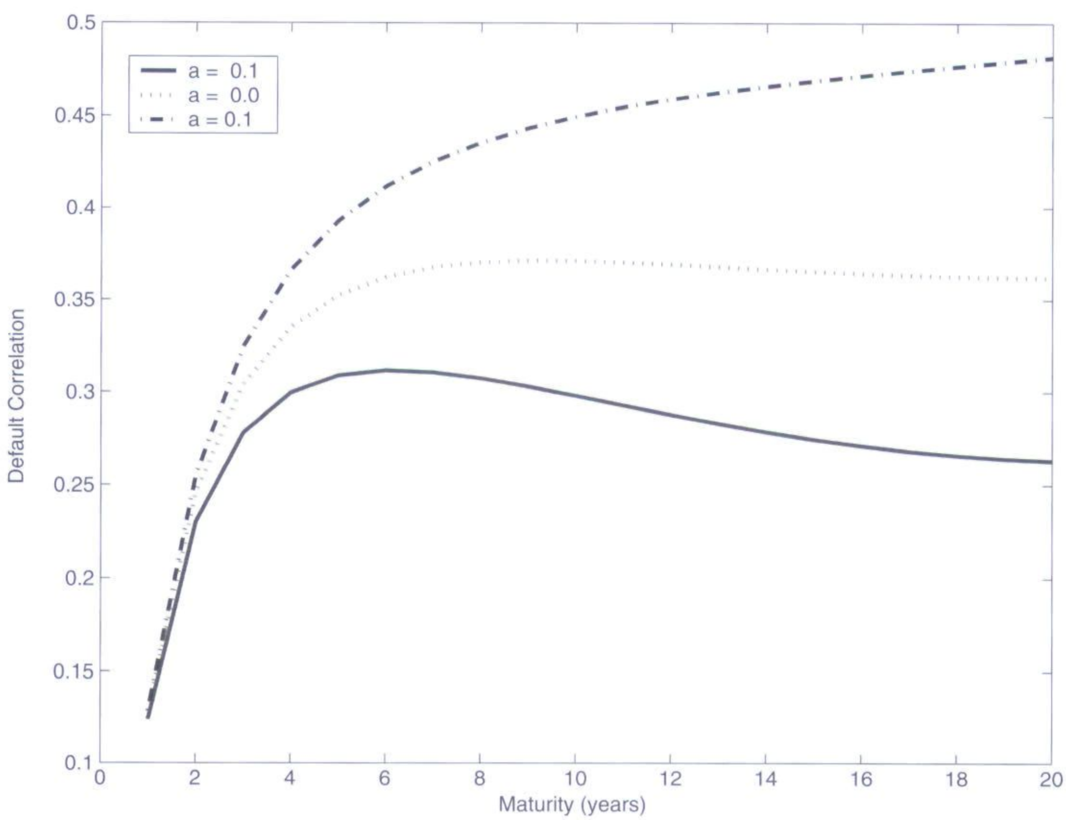
Exhibit 6 depicts the relation between default correlations and the investment horizon for firms with different credit qualities and different systematic risk exposures. The average default correlation among high-grade firms ($V/K = 4$) with higher systematic risk exposure is nearly 29.4% higher than average default correlation among low-grade firms ($V/K = 2$) with lower systematic risk exposure.[13]

Undoubtedly, when all the firms in various credit classes are assumed to have equal systematic risk exposure, default correlations among firms in a specific credit class rise as the credit qualities of firms decline, which is consistent with the results of Barnhill and Maxwell [2002], Crouhy, Galai, and Mark [2000, 2001], Erlenmaier and Gersbach [2001], Gersbach and Lipponer [2000], and Zhou [2001]. Therefore, our extended first-passage-time model covers the two contrary opinions on the relation of default correlations and credit quality.

Another noteworthy indication of the superiority of the extended first-passage-time model over the Zhou

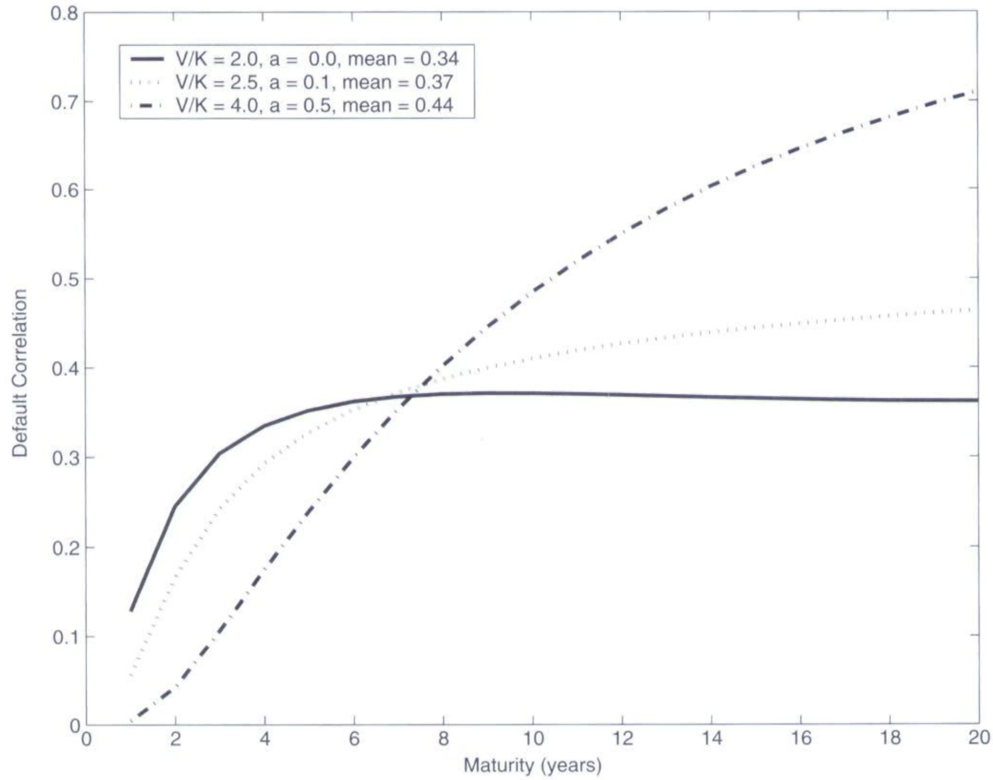
EXHIBIT 5

Relation Between Default Correlation and Level-Dependence of Asset Values on Market Shock in Extended First-Passage-Time Model



E X H I B I T 6

Relation Between Default Correlation and Credit Grades with Different Systematic Risk Exposures in the Extended First-Passage-Time Model



model lies in its power to explain the business cycle-adjusted default correlation dynamics. Zhou contends there are dynamic cyclical effects on default correlations because the credit qualities of firms are dynamic. This indirect reflection of the business cycle, however, cannot be used in explicitly forecasting default correlations based on prevailing market conditions.

Exhibit 7 plots the relation between default correlations and the investment horizon for various states of the economy. Default correlations among firms with higher macro-dependence are nearly 20% higher in downturns than in normal times (see Panel D). The difference in default correlations due to differing market environments sharply increases initially and then declines as the investment horizon lengthens, and it also increases with systematic risk exposure (see Panels C and D).[14]

Although Zhou provides an analytical formula for calculating joint default probabilities, his solution cannot be applied to cases involving more than three firms. Another drawback to Zhou's model occurs when the instantaneous expected asset return is not equal to the growth rate of the firm's liabilities, in which case his results depend on the double integral of Bessel functions, a burdensome calculation. Our approach can be preferable to derive the joint default probabilities of multiple firms with general deterministic default threshold functions.[15]

IV. APPLICATIONS

In current structural models default correlations are analyzed with only firm-specific information. Without considering general economic conditions—which have been shown to critically influence the probabilities of multiple default events as well as a single default event—we cannot correctly price credit derivatives and manage the credit risk of bond portfolios. The business cycle-adjusted default correlation analysis has some useful applications here.

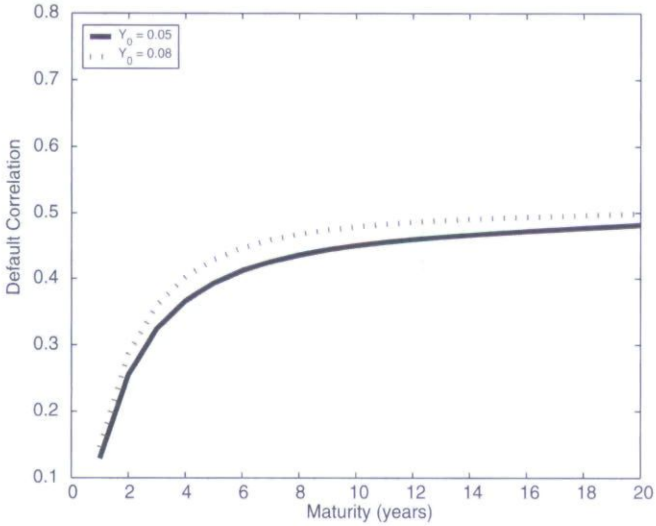
Credit Derivatives Pricing

Among the fastest growing over-the-counter derivatives products are credit derivatives, of which the credit default swap is the most actively used product. The market

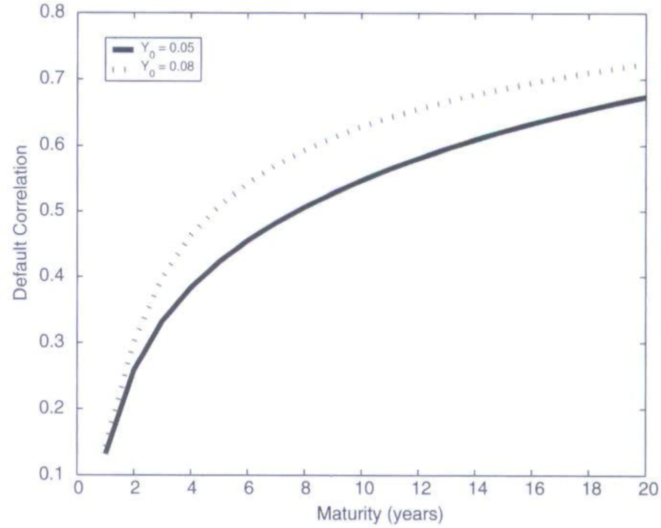
EXHIBIT 7

Relation between Default Correlation and Market Environment in Extended First-Passage-Time Model

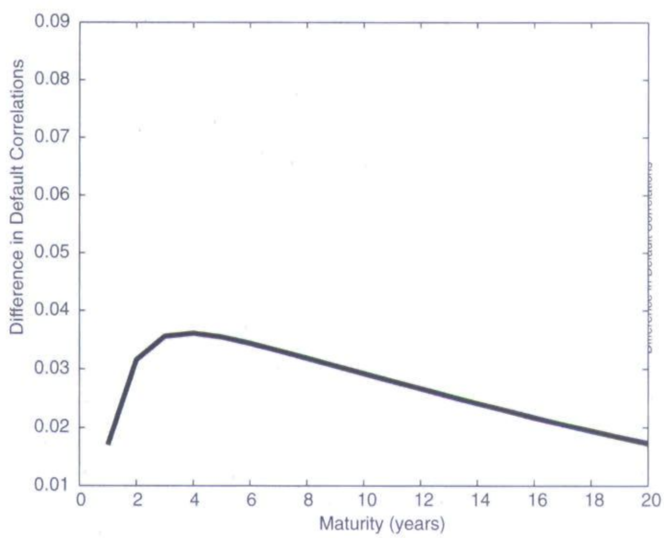
Panel A. Term Structure of Default Correlations—Lower Macro-Dependencies



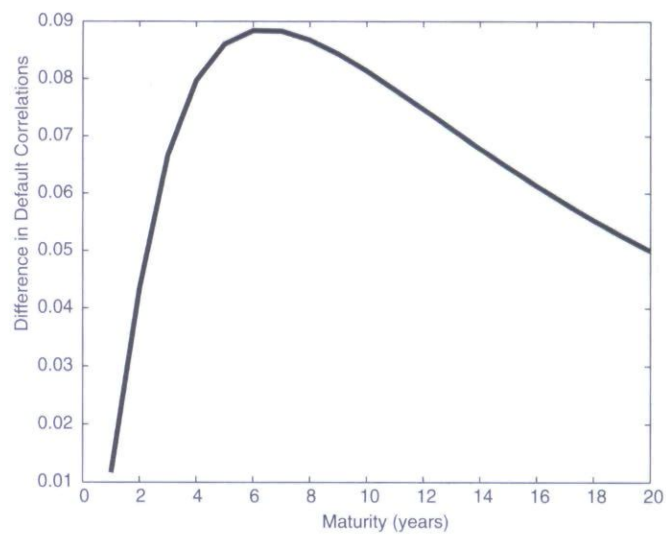
Panel B. Term Structure of Default Correlations—Higher Macro-Dependencies



Panel C. Term Structure of Difference in Default Correlations—Lower Macro-Dependencies



Panel D. Term Structure of Difference in Default Correlations—Higher Macro-Dependencies



In Panels A and C $a = -0.1$ and $\rho_{VY} = -0.1$. In Panels B and D $a = -0.3$ and $\rho_{VY} = -0.3$. The other parameters are $V_1/K_1 = V_2/K_2 = 2$; $\sigma_1 = \sigma_2 = 0.3$; $\kappa = 0.1$; $\theta = 0.08$; $\delta = 0.03$; and $\rho_{V_1V_2} = 0.5$.

for credit default swaps, as measured by the notional amount of contracts traded per year, has grown from about $919 billion in 2001 to over $8,422 billion in 2004. The valuation of credit default swaps has attracted much interest both practically and theoretically. Default correlation analysis is important in the valuation of credit default swaps with counterparty default risk and basket credit default swaps. Ignoring the impact of general economic shocks on default probabilities or default correlations can be a critical error in determining a default swap premium.

Another straightforward application of business-cycle-adjusted default correlations is the valuation of corporate bonds secured by another firm with a higher credit grade. For example, a letter of credit (LOC)-backed security transfers the risk of default from the debtholder to the LOC bank. If both the LOC bank and the debt obligor default, the LOC-backed security must be priced lower than when we don't consider the business cycle. In particular, the LOC-backed security must be priced much lower if we incorporate the default correlation between the LOC bank and the debt obligor into determining joint default probability, and it needs to be priced much lower in recessionary periods than in normal times.

Besides credit default swaps and LOC, the macro-dependencies of default correlations must be incorporated into pricing a portfolio of debt such as collateralized bond obligations and collateralized debt obligations.

Credit Risk Management

The problems of analyzing portfolios with credit risk are well known. The most critical and difficult part of aggregate credit analysis is estimating default correlations. During the portfolio holding period, however, constant use of the average default correlation across investment horizons induces fatal error in managing credit portfolio risk, especially when macroeconomic conditions are stressful. In fact, the impact of default correlations on credit portfolio risk is more critical during recessions.

Stochastic macroeconomic factors have an impact on credit portfolio loss distribution and credit value at risk (VaR). As economywide environments fluctuate, default probability, default correlation, and loss given default also change, which in turn changes the shape of a portfolio loss distribution. Under our model, when firm value moves with the business cycle, default correlations increase during economic downturns. In this case, there is also a greater probability of a large loss, and a portfolio loss distribution will have a fatter tail.

This phenomenon can also be caused by an increase in the default probability of each firm during recession. In these cases, credit VaR (unexpected loss under a given confidence level) would be higher, and the capital requirements should be higher as well.

Moreover, notice that the portfolio loss distribution depends on the level of macroeconomic conditions and deviations from the long-run mean. This implies that the default correlation and the probability of default of each firm will change as macroeconomic conditions change. As a result, the shape of the loss distribution given a specific time horizon will change dynamically. This would necessitate dynamic adjustments in measuring credit VaR.

V. CONCLUSION

We have developed an extended Merton model incorporating systematic risk factors that affect asset value movement in order to illustrate the relation between default probabilities and default correlations, credit grades, and business cycle fluctuations. We also extend the framework into a first-passage-time approach. The model represents an easily implementable approach to default probabilities and default correlations, using both firm-specific and macroeconomic information. Our results are consistent with empirical evidence concerning both time series and cross-sectional default correlation dynamics.

First, default probabilities and default correlations are higher in economic downturns. Second, default correlations among firms in a high-grade class with a higher level of systematic risk can be higher than default correlations among firms in a low-grade class with a lower level of systematic risk. Further, when all firms in various credit classes are assumed to have the same systematic risk exposure, default correlations increase as the credit quality of firms declines.

This approach is useful in pricing credit derivatives that are exposed to market risk and multiple defaults, such as credit default swaps, basket credit default swaps, and collateralized bond obligations. The approach also has critical implications for managing the credit risk of bond portfolios, since default correlations have a greater impact on credit portfolio risk during recessionary periods.

APPENDIX A

Extended First-Passage-Time Default Correlation Model with Multiple Firms and Multiple Factors

The asset value dynamics generally depend on multiple macroeconomic factors, such as interest rate, production growth rate, market portfolio index, and unemployment rate. Determining the joint default probability of multiple firms while incorporating these macro-dependencies is necessary for managing the credit risk of bond portfolios. We extend the first-passage-time default correlation model to a case with multiple firms and multiple common factors.

Let $\mathbf{Y(t)} = \mathbf{Y_t} = (Y_t^1, Y_t^2, ..., Y_t^N)^T$ and $\mathbf{l(t)} = \mathbf{l_t} = (l_t^1, l_t^2, ..., l_t^M)^T$ define the systematic risk vector and the logarithmic debt-to-asset value ratio vector, respectively.

Define $g[\mathbf{l_s}, \mathbf{Y_s}, s|\mathbf{l_0}, \mathbf{Y_0}, 0]$ as the probability that the first time all firms have defaulted is at time s, and that multiple macro factor $\mathbf{Y}$ assumes the value $\mathbf{Y_s}$ at that time. Let A be a set including all the subsets of $\{1, 2, 3, ..., M\}$ except the null set. For example, when $a = \{1, 3\}$ $(a \in A)$, $g_a[l_s^1 = 0, l_s^2 > 0, l_s^3 = 0, ..., l_s^M > 0, \mathbf{Y_s}, s|\mathbf{l_0}, \mathbf{Y_0}, 0]$ is the probability density that firm 1 and firm 3 have defaulted exactly at time s and other firms have defaulted prior to time s. Then Equation (A-1) holds:

$$g[\mathbf{l_s}, \mathbf{Y_s}, s|\mathbf{l_0}, \mathbf{Y_0}, 0] = \sum_{a\in A} g_a[\mathbf{l_s}, \mathbf{Y_s}, s|\mathbf{l_0}, \mathbf{Y_0}, 0] \tag{A-1}$$

Then the (N + 1)-dimensional generalization of the Fortet [1943] model is:[16]

$$\begin{aligned}&\pi(\mathbf{l_t}, \mathbf{Y_t}, t|\mathbf{l_0}, \mathbf{Y_0}, 0) = \\ &\int_0^t ds \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dY_s^1 \cdots dY_s^N g \times \\ &g[\mathbf{l_s}, \mathbf{Y_s}, s|\mathbf{l_0}, \mathbf{Y_0}, 0]\pi(\mathbf{l_t}, \mathbf{Y_t}, t|\mathbf{l_s} = \mathbf{0}, \mathbf{Y_s}, s)\end{aligned} \tag{A-2}$$

Integrating both sides of Equation (A-2) by

$$\int_0^\infty \cdots \int_0^\infty dl_t^1 \cdots dl_t^M$$

produces the equation:

$$\Psi(\mathbf{Y_t}, t) = \int_0^t ds \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dY_s^1 \cdots dY_s^N \; g(\mathbf{Y_s}, s)\psi(\mathbf{Y_t}, t, \mathbf{Y_s}, s) \tag{A-3}$$

where

$$\Psi(\mathbf{Y_t}, t) = \int_0^\infty \cdots \int_0^\infty dl_t^1 \cdots dl_t^M \pi(\mathbf{l_t}, \mathbf{Y_t}, t|\mathbf{l_0}, \mathbf{Y_0}, 0) \tag{A-4}$$

$$\psi(\mathbf{Y_t}, t, \mathbf{Y_s}, s) = \int_0^\infty \cdots \int_0^\infty dl_t^1 \cdots dl_t^M \times \pi(\mathbf{l_t}, \mathbf{Y_t}, t|\mathbf{l_s} = \mathbf{0}, \mathbf{Y_s}, s) \tag{A-5}$$

$$g(\mathbf{Y_s}, s) = g[\mathbf{l_s}, \mathbf{Y_s}, s|\mathbf{l_0}, \mathbf{Y_0}, 0] \tag{A-6}$$

Discretize time into n_t equal intervals, and define date $t_j = jt/n_t = \Delta t$ for $j \in (1, 2, \ldots, n_t)$. For $1 \le n \le N$, similarly discretize Y^n-space into n_{Y^n} equal intervals between some chosen minimum $\underline{Y^n}$ and maximum $\overline{Y^n}$, and define $Y_i^n = \underline{Y^n} + i\Delta Y^n$ for $i \in (1, 2, \ldots, n_{Y^n})$, where $\Delta Y^n = (\overline{Y^n} - \underline{Y^n})/n_{Y^n}$.

The discretized version of Equation (A-3) is:

$$\Psi(\mathbf{Y_i}, t_j) = \sum_{v=1}^{j} \sum_{u^1=1}^{n_{Y^1}} \cdots \sum_{u^N=1}^{n_{Y^N}} q(\mathbf{Y_u}, t_v)\psi(\mathbf{Y_i}, t_j|\mathbf{Y_u}, t_v)$$

where

$$q(\mathbf{Y_u}, t_v) = \Delta t \prod_{n=1}^{N} \Delta Y^n g(\mathbf{Y_u}, t_v)$$

The joint default probability of multiple firms influenced by market risk is:

$$P(D_1(t) = 1, ..., D_M(t) = 1) = \sum_{j=1}^{n_t} \sum_{i^1=1}^{n_{Y^1}} \cdots \sum_{i^N=1}^{n_{Y^N}} q(\mathbf{Y}_i, t_j) \tag{A-7}$$

where

$$\begin{aligned} q(\mathbf{Y}_i, t_1) &= \prod_{n=1}^{N} \Delta Y^n \Psi(\mathbf{Y}_i, t_1) \\ q(\mathbf{Y}_i, t_j) &= \prod_{n=1}^{N} \Delta Y^n [\Psi(\mathbf{Y}_i, t_j) - \end{aligned}$$

$$\sum_{v=1}^{j-1}\sum_{u^1=1}^{n_{Y1}}\cdots\sum_{u^N=1}^{n_{YN}} q(\mathbf{Y}_u, t_v)\psi(\mathbf{Y}_i, t_j|\mathbf{Y}_u, t_v)]$$
$$\text{for } \forall j \in (2, ..., n_t)$$

Applying Bayes' rule to the multivariate density $\pi(\mathbf{l_t}, \mathbf{Y_t}, t|\mathbf{l_v}, \mathbf{Y_v}, \upsilon)$ gives:

$$\pi(\mathbf{l_t}, \mathbf{Y_t}, t|\mathbf{l_v}, \mathbf{Y_v}, v) =$$
$$\pi(\mathbf{Y_t}, t|\mathbf{l_v}, \mathbf{Y_v}, v)\pi(\mathbf{l_t}, t|\mathbf{Y_t}, \mathbf{l_v}, \mathbf{Y_v}, v) \quad \forall\{v\} \tag{A-8}$$

Using this in Equations (A-4) and (A-5), we obtain:

$$\Psi(\mathbf{Y_t}, t) = \pi(\mathbf{Y_t}, t|\mathbf{l_0}, \mathbf{Y_0}, 0)\int_0^\infty \cdots \int_0^\infty dl_t^1 \cdots dl_t^M \pi(\mathbf{l_t}, t|\mathbf{Y_t}, \mathbf{l_0}, \mathbf{Y_0}, 0) \tag{A-9}$$

$$\psi(\mathbf{Y_t}, t|\mathbf{Y_s}, s) = \pi(\mathbf{Y_t}, t|\mathbf{l_s} = \mathbf{0}, \mathbf{Y_s}, s) \int_0^\infty \cdots \int_0^\infty dl_t^1 \cdots dl_t^M \times \pi(\mathbf{l_t}, t|\mathbf{Y_t}, \mathbf{l_s} = \mathbf{0}, \mathbf{Y_s}, s) \tag{A-10}$$

Because $(\mathbf{Y}, \mathbf{l})$ follow a joint Gaussian process, we can apply the projection theorem to obtain the conditional mean and covariance of $\mathbf{l_t}$ with respect to $\mathbf{Y_t}$ as follows:

$$\mu(\mathbf{Y_t}, t|\mathbf{l_s}, \mathbf{Y_s}, s) = E_s[\mathbf{l_t}|\mathbf{Y_t}] = E_s[\mathbf{l_t}] + COV_s[\mathbf{l_t}, \mathbf{Y_t}]\times COV_s^{-1}[\mathbf{Y_t}]\,(\mathbf{Y_t} - E_s[\mathbf{Y_t}])$$

$$\Sigma(\mathbf{Y_t}, t|\mathbf{l_s}, \mathbf{Y_s}, s) = COV_s[\mathbf{l_t}|\mathbf{Y_t}] = COV_s[\mathbf{l_t}] - COV_s[\mathbf{l_t}, \mathbf{Y_t}]COV_s^{-1}[\mathbf{Y_t}]COV_s[\mathbf{Y_t}, \mathbf{l_t}]$$

$$\Theta(\mathbf{Y_t}, t|\mathbf{l_s}, \mathbf{Y_s}, s) = CORR_s[\mathbf{l_t}|\mathbf{Y_t}]$$

Since multiple common factor processes are Markov, the equation:

$$\pi(\mathbf{Y_t}, t|\mathbf{l_s}, \mathbf{Y_s}, s) = \pi(\mathbf{Y_t}, t|\mathbf{Y_s}, s) \tag{A-11}$$

holds, and the final expressions are:

$$\Psi(\mathbf{Y_t}, t) = \pi(\mathbf{Y_t}, t|\mathbf{Y_0}, 0)MVN_M\,(Z(\mathbf{Y_t}, t|\mathbf{l_0}, \mathbf{Y_0}, 0), \Theta(\mathbf{Y_t}, t|\mathbf{l_0}, \mathbf{Y_0}, 0))$$

$$\psi(\mathbf{Y_t}, t|\mathbf{Y_s}, s) = \pi(\mathbf{Y_t}, t|\mathbf{Y_s}, s)MVN_M\,(Z(\mathbf{Y_t}, t|\mathbf{l_s} = \mathbf{0}, \mathbf{Y_s}, s), \Theta(\mathbf{Y_t}, t|\mathbf{l_s} = \mathbf{0}, \mathbf{Y_s}, s))$$

$$Z(\mathbf{Y_t}, t|\mathbf{l_s}, \mathbf{Y_s}, s) = \mu(\mathbf{Y_t}, t|\mathbf{l_s}, \mathbf{Y_s}, s)^T \circ \Lambda(\mathbf{Y_t}, t|\mathbf{l_s}, \mathbf{Y_s}, s)$$

where

$$\Lambda(\mathbf{Y_t}, t|\mathbf{l_s}, \mathbf{Y_s}, s) = (\Sigma_1^{-1}, \Sigma_2^{-1}, ..., \Sigma_M^{-1}),\ \Sigma_m = \sqrt{\Sigma_{mm}}$$

for all $m \in (1, 2, ..., M)$; $MVN_M(\cdot)$ means an M-dimensional standard multivariate normal distribution; $A \circ B$ means the multiplication of element by element in the matrices A and B; and Σ_{mk} defines the $m \times k$ element of the $M \times M$ matrix Σ.

APPENDIX B

Proof of Proposition 2

In order for both firms to start out solvent and end up defaulting, it is necessary that there is a first time when both have defaulted. Call that time s. Note that at this time, firm 2 has defaulted previously (and now the value of l_s^2 is positive) and $l_s^1 = 0$ (i.e., it has just defaulted), or firm 1 has defaulted previously (and now the value of l_s^1 is positive) and $l_s^2 = 0$ (i.e., it has just defaulted), or $l_s^1 = l_s^2 = 0$ (i.e., both firms have just defaulted).

Define $g_1[l_s^1 = 0, l_s^2 > 0, Y_s, s\,|\,l_0^1, l_0^2, Y_0, 0]$ as the probability density that firm 1 has defaulted exactly at time s and firm 2 has defaulted prior to time s. Analogously, define $g_2[l_s^1 > 0, l_s^2 = 0, Y_s, s\,|\,l_0^1, l_0^2, Y_0, 0]$ as the probability density that firm 2 has defaulted exactly at time s and firm 1 has defaulted prior to time s.

Finally, define $g_{1,2}[l_s^1 = 0, l_s^2 = 0, Y_s, s\,|\,l_0^1, l_0^2, Y_0, 0]$ as the probability density that both firms have defaulted exactly at time s. Let $g[l_s^1, l_s^2, Y_s, s\,|\,l_0^1, l_0^2, Y_0, 0]$ be the probability density that the first time both firms have defaulted is at time s, and that the stochastic process Y takes on the value Y_s at that time. In other words, Equation (B-1) holds.

$$\begin{aligned} g[l_s^1, l_s^2, Y_s, s|l_0^1, l_0^2, Y_0, 0] &= g_1[l_s^1 = 0, l_s^2 > 0, Y_s, s|l_0^1, l_0^2, Y_0, 0] \\ &+ g_2[l_s^1 > 0, l_s^2 = 0, Y_s, s|l_0^1, l_0^2, Y_0, 0] \\ &+ g_{1,2}[l_s^1 = 0, l_s^2 = 0, Y_s, s|l_0^1, l_0^2, Y_0, 0] \end{aligned} \tag{B-1}$$

Then the two-dimensional generalization of the Fortet [1943] model is:

$$\pi(l_t^1, l_t^2, Y_t, t|l_0^1, l_0^2, Y_0, 0) =$$
$$\int_0^t ds \int_{-\infty}^{\infty} dY_s g_1[l_s^1 = 0, l_s^2 > 0, Y_s, s|l_0^1, l_0^2, Y_0, 0]$$

Exhibit 8

Difference in Joint Default Probabilities (%) for Log-Leverage Ratios All Zeros at Time s when Both Firms have Defaulted for the First Time

$\rho_{V_1V_2}$	Maturity (years)	1	2	3	4	5	10
0.5	Analytical solution	0.2	2.4	5.7	8.9	11.7	21.5
	Monte Carlo simulation	0.3	2.4	5.2	8.8	11.6	23.7
0.3	Analytical solution	0.1	1.4	3.8	6.4	8.7	17.1
	Monte Carlo simulation	0.1	1.4	3.8	6.5	9.3	20.3

$$\pi(l_t^1, l_t^2, Y_t, t | l_s^1 = 0, l_s^2 > 0, Y_s, s)$$
$$+ \int_0^t ds \int_{-\infty}^{\infty} dY_s g_2[l_s^1 > 0, l_s^2 = 0, Y_s, s | l_0^1, l_0^2, Y_0, 0]$$
$$\pi(l_t^1, l_t^2, Y_t, t | l_s^1 > 0, l_s^2 = 0, Y_s, s)$$
$$+ \int_0^t ds \int_{-\infty}^{\infty} dY_s g_{1,2}[l_s^1 = 0, l_s^2 = 0, Y_s, s | l_0^1, l_0^2, Y_0, 0]$$
$$\pi(l_t^1, l_t^2, Y_t, t | l_s^1 = 0, l_s^2 = 0, Y_s, s)$$

When these equations are discretized, we can get the aggregate probability density $g(\cdot)$, but not three unknowns in each equation, namely, $g_1(\cdot)$, $g_2(\cdot)$, and $g_{1,2}(\cdot)$. Furthermore, at time s when both firms have defaulted for the first time, one of three cases occurs, namely, $[l_s^1 = 0, l_s^2 > 0]$; $[l_s^1 > 0, l_s^2 = 0]$; or $[l_s^1 = 0, l_s^2 = 0]$. Because we cannot know which case happens, we assume that the value of l_s is zero at time s. This assumption may have some impact on joint default probability over a long horizon, but the impact is still relatively small (*see Exhibit 8*).

Then Equation (B-2) holds:

$$\pi(l_t^1, l_t^2, Y_t, t | l_0^1, l_0^2, Y_0, 0) =$$
$$\int_0^t ds \int_{-\infty}^{\infty} dY_s \quad g[l_s^1, l_s^2, Y_s, s | l_0^1, l_0^2, Y_0, 0]$$
$$\pi(l_t^1, l_t^2, Y_t, t | l_s^1 = 0, l_s^2 = 0, Y_s, s) \tag{B-2}$$

Integrating both sides of Equation (B-2) by $\int_0^\infty \int_0^\infty dl_t^1 dl_t^2$ produces the equation:

$$\Psi(Y_t, t) = \int_0^t ds \int_{-\infty}^{\infty} dY_s g(Y_s, s) \psi(Y_t, t, Y_s, s) \tag{B-3}$$

where

$$\Psi(Y_t, t) = \int_0^\infty \int_0^\infty dl_t^1 dl_t^2 \pi(l_t^1, l_t^2, Y_t, t | l_0^1, l_0^2, Y_0, 0) \tag{B-4}$$

$$\psi(Y_t, t, Y_s, s) =$$
$$\int_0^\infty \int_0^\infty dl_t^1 dl_t^2 \pi(l_t^1, l_t^2, Y_t, t | l_s^1 = 0, l_s^2 = 0, Y_s, s) \tag{B-5}$$

$$g(Y_s, s) = g[l_s^1, l_s^2, Y_s, s | l_0^1, l_0^2, Y_0, 0] \tag{B-6}$$

Discretize time into n_t equal intervals, and define date $t_j = jt/n_t = j\Delta t$ for $j \in (1, 2, \ldots, n_t)$. Similarly discretize Y-space into n_Y equal intervals, and define $Y_i = \underline{Y} + i\Delta Y$ for i $\in (1, 2, \ldots, n_Y)$, where $\Delta Y = (\bar{Y} - \underline{Y})/n_Y$.

The discretized version of Equation (B-3) is:

$$\Psi(Y_i, t_j) = \sum_{v=1}^{j} \sum_{u=1}^{n_Y} q(Y_u, t_v) \psi(Y_i, t_j | Y_u, t_v)$$

where

$$q(Y_u, t_v) = \Delta t \Delta Y g(Y_u, t_v)$$

The joint default probability of firm 1 and firm 2 influenced by market risk is obtained as follows:

$$P(D_1(t) = 1 \ and \ D_2(t) = 1) = \sum_{j=1}^{n_t} \sum_{i=1}^{n_Y} q(Y_i, t_j) \tag{B-7}$$

where

$$q(Y_i, t_1) = \Delta Y \Psi(Y_i, t_1), \ \forall i \in (1, 2, \ldots, n_Y)$$
$$q(Y_i, t_j) = \Delta Y \left[\Psi(Y_i, t_j) - \sum_{v=1}^{j-1} \sum_{u=1}^{n_Y} q(Y_u, t_v) \psi(Y_i, t_j | Y_u, t_v) \right]$$
$$\forall i \in (1, 2, \ldots, n_Y), \ \forall j \in (2, \ldots, n_t)$$

Applying Bayes' rule to the trivariate density $\pi(l_t^1, l_t^2, Y_t, t | l_v^1, l_v^2, Y_v, v)$ gives:

$$\pi(l_t^1, l_t^2, Y_t, t | l_v^1, l_v^2, Y_v, v) =$$
$$\pi(Y_t, t | l_v^1, l_v^2, Y_v, v) \pi(l_t^1, l_t^2, t | Y_t, l_v^1, l_v^2, Y_v, v) \ \forall \{v\} \tag{B-8}$$

Using this in Equations (B-4) and (B-5), we obtain:

$$\Psi(Y_t, t) = \pi(Y_t, t|l_0^1, l_0^2, Y_0, 0) \int_0^\infty \int_0^\infty dl_t^1 dl_t^2 \pi(l_t^1, l_t^2, t|Y_t, l_0^1, l_0^2, Y_0, 0) \quad \text{(B-9)}$$

$$\psi(Y_t, t|Y_s, s) = \pi(Y_t, t|l_s^1 = 0, l_s^2 = 0, Y_s, s) \int_0^\infty \int_0^\infty dl_t^1 dl_t^2 \pi(l_t^1, l_t^2, t|Y_t, l_s^1 = 0, l_s^2 = 0, Y_s, s) \quad \text{(B-10)}$$

Because (Y, l^1, l^2) follow a joint Gaussian process, we can apply the projection theorem to obtain the conditional mean and covariance of l_t^1, l_t^2 with respect to Y_t as follows:

$$\begin{aligned}
&\mu_m(Y_t, l_s^m, Y_s) \\
&= E_s[l_t^m|Y_t] = E_s[l_t^m] + \frac{Cov_s[l_t^m, Y_t]}{Var_s[Y_t]}(Y_t - E_s[Y_t])
\end{aligned}$$

$$\begin{aligned}
&\Sigma_{12}(Y_t, l_s^1, l_s^2, Y_s) \\
&= Cov_s[l_t^1, l_t^2|Y_t] = Cov_s[l_t^1, l_t^2] - \frac{Cov_s[l_t^1, Y_t]Cov_s[l_t^2, Y_t]}{Var_s[Y_t]}
\end{aligned}$$

$$\begin{aligned}
&\Theta(Y_t, l_s^1, l_s^2, Y_s) \\
&= Corr_s[l_t^1, l_t^2|Y_t] = \frac{Cov_s[l_t^1, l_t^2|Y_t]}{\sqrt{Var_s[l_t^1|Y_t]Var_s[l_t^2|Y_t]}}
\end{aligned}$$

where

$$\begin{aligned}
E_s[l_t^m] &= \ln\frac{K_m(t)}{V_m(s)} - \left(b_m - \frac{1}{2}\sigma_m^2\right)(t-s) - \\
&\quad a_m\left[B(t-s;\kappa)(Y_0 - \theta) + \theta(t-s)\right] \\
E_s[Y_t] &= Y_s e^{-\kappa(t-s)} + \theta(1 - e^{-\kappa(t-s)}) \\
Var_s[l_t^m] &= \sigma_m^2(t-s) + 2a_m\left(\frac{\rho_{V_mY}\sigma_m\delta}{\kappa}\right)\times \\
&\quad [(t-s) - B(t-s;\kappa)] + a_m^2\left(\frac{\delta}{\kappa}\right)^2\times \\
&\quad [(t-s) - 2B(t-s;\kappa) + B(t-s;2\kappa)] \\
Var_s[Y_t] &= \delta^2 B(t-s;2\kappa)
\end{aligned}$$

$$\begin{aligned}
Cov_s[l_t^m, Y_t] &= \left(\frac{\delta^2}{\kappa} + \rho_{V_mY}\sigma_m\delta\right)B(t-s;\kappa) - \\
&\quad \frac{\delta^2}{\kappa}B(t-s;2\kappa) \\
Cov_s[l_t^1, l_t^2] &= a_1a_2\left(\frac{\delta}{\kappa}\right)^2[(t-s) - 2B(t-s;\kappa) + \\
&\quad B(t-s;2\kappa)] + \sigma_1\sigma_2\rho_{V_1V_2}(t-s) + \\
&\quad \frac{a_1}{\kappa}(\rho_{V_2Y}\sigma_2\delta)[(t-s) - B(t-s;\kappa)] + \\
&\quad \frac{a_2}{\kappa}(\rho_{V_1Y}\sigma_1\delta)[(t-s) - B(t-s;\kappa)] \\
B(t;\kappa) &= (1 - e^{-\kappa t})/\kappa
\end{aligned}$$

Consequently, the final expressions are:

$$\begin{aligned}
&\Psi(Y, t) = \pi(Y_t, t|Y_0, 0)\times \\
&BVN\left(\frac{\mu_1(Y_t, t|l_0^1, Y_0, 0)}{\Sigma_1(Y_t, t|l_0^1, Y_0, 0)}, \frac{\mu_2(Y_t, t|l_0^2, Y_0, 0)}{\Sigma_2(Y_t, t|l_0^2, Y_0, 0)}, \Theta(Y_t, t|l_0^1, l_0^2, Y_0, 0)\right)
\end{aligned}$$

$$\begin{aligned}
\psi(Y_t, t|Y_s, s) &= \pi(Y_t, t|Y_s, s)\times \\
&BVN\left(\frac{\mu_1(Y_t, t|l_s^1 = 0, Y_s, s)}{\Sigma_1(Y_t, t|l_s^1 = 0, Y_s, s)}, \frac{\mu_2(Y_t, t|l_s^2 = 0, Y_s, s)}{\Sigma_2(Y_t, t|l_s^2 = 0, Y_s, s)},\right. \\
&\left.\Theta(Y_t, t|l_s^1 = 0, l_s^2 = 0, Y_s, s)\right)
\end{aligned}$$

$BVN(\cdot)$ is the standard bivariate normal distribution function, and $\Sigma_m = \sqrt{\Sigma_{mm}}$.

APPENDIX C

Proof of Proposition 3

The asset return variance of firm m is:

$$\begin{aligned}
Var_0[\ln V_m(t)] &= \sigma_m^2 t + 2a_m\left(\frac{\rho_{V_mY}\sigma_m\delta}{\kappa}\right)[t - B(t;\kappa)] \\
&\quad + a_m^2\left(\frac{\delta}{\kappa}\right)^2[t - 2B(t;\kappa) + B(t;2\kappa)]
\end{aligned} \quad \text{(C-1)}$$

and the covariance between asset returns is:

$$\begin{aligned}
Cov_0[\ln V_1(t), \ln V_2(t)] &= a_1a_2\left(\frac{\delta}{\kappa}\right)^2[t - 2B(t;\kappa) + \\
&\quad B(t;2\kappa)] + \sigma_1\sigma_2\rho_{V_1V_2}t\ +
\end{aligned}$$

$$\frac{a_1}{\kappa}(\rho_{V_2Y}\sigma_2\delta)\left[t - B(t;\kappa)\right] +$$
$$\frac{a_2}{\kappa}(\rho_{V_1Y}\sigma_1\delta)\left[t - B(t;\kappa)\right] \quad \text{(C-2)}$$

where ρ_{V_mY} is the correlation between the asset return of firm m and the macroeconomic variable Y.

The expected asset return correlation $\rho_{AR}(t)$ conditional on time-0 information is found to be:

$$\rho_{AR}(t) = \frac{Cov_0[\ln V_1(t), \ln V_2(t)]}{\sqrt{Var_0[\ln V_1(t)]Var_0[\ln V_2(t)]}} \quad \text{(C-3)}$$

Using Equations (C-1) and (C-2), the partial derivative of the asset return correlation with respect to the correlation between the asset value and macroeconomic shock can be easily obtained as Equation (16).

APPENDIX D

Proof of Proposition 4

The variance of the asset level is:

$$Var_0[V(t)] = e^{2M(t)+S(t)^2}(e^{S(t)^2} - 1) \quad \text{(D-1)}$$

where $M(t) = E_0[\ln V(t)]$, $S^2(t) = \text{Var}_0[\ln V(t)]$, and the covariance between asset levels is:

$$Cov_0[V_1(t), V_2(t)] = e^{-M_1(t)-M_2(t)+\frac{1}{2}S_1^2(t)+\frac{1}{2}S_2(t)^2}Cov_0[\ln V_1(t), \ln V_2(t)] \quad \text{(D-2)}$$

The asset level correlation $\rho_{AL}(t)$ is defined as:

$$\rho_{AL}(t) = \frac{Cov_0[V_1(t), V_2(t)]}{\sqrt{Var_0[V_1(t)]Var_0[V_2(t)]}} \quad \text{(D-3)}$$

Using Equations (D-1) and (D-2), the partial derivative of the asset level correlation with respect to market condition Y_0 can be easily obtained as Equation (18).

ENDNOTES

[1]See Barnhill and Maxwell [2002], Crouhy, Galai, and Mark [2000, 2001], Erlenmaier and Gersbach [2001], Gersbach and Lipponer [2000], and Zhou [2001].

[2]Although macroeconomic state variables may or may not take on negative values, this does not affect general results on the relationship between default probabilities and default correlations, credit grades, and business cycle fluctuations.

[3]Since all the other values except a_m in Equation (14) are always positive regardless of time, the sign of the partial derivative

$$\frac{\partial N(d_m(t))}{\partial Y_0}$$

is determined only by a_m.

[4]Default probabilities are obtained using combined information from the balance sheet and stock market data.

[5]Das, Fong, and Geng [2001] show the converse: that higher-grade firms have higher systematic risk exposures than lower-grade firms.

[6]This means the two firms have the same instantaneous expected asset returns and standard deviations of asset returns: i.e., $a_1 = a_2 = a$; $b_1 = b_2 = b$, and $\sigma_1 = \sigma_2 = \sigma$.

[7]The default correlation ρ_{DC} is positively related to the asset return correlation ρ_{AR} as shown in Gersbach and Lipponer [2000] and Zhou [2001], where the sign of

$$\frac{\partial \rho_{DC}}{\partial \rho_{VY}}$$

is determined by the sign of

$$\frac{\partial \rho_{AR}}{\partial \rho_{VY}}$$

[8]Of course, when a is negative (i.e., when there is a negative macro shock on asset values), the value

$$\frac{\partial \rho_{AR}}{\partial \rho_{VY}}$$

is negative, and this means asset return correlations increase as there is greater systematic risk exposure, and the ρ_{DC} increases with absolute systematic risk exposure.

[9]Since the signs of

$$\frac{\partial \rho_{DC}}{\partial \rho_{AR}}$$

and

$$\frac{\partial \rho_{AR}}{\partial \rho_{AL}}$$

are positive, the sign of

$$\frac{\partial \rho_{DC}}{\partial \rho_{Y_0}}$$

is determined by the sign of

$$\frac{\partial \rho_{AL}}{\partial Y_0}$$

[10]The asset level correlation ρ_{AL} may be negative, but ρ_{AL} is assumed to be generally positive in our discussion of the relation between ρ_{AL} and Y_0.

[11]For parameter values of $a_1 = a_2 = -0.3$; $V_1/K_1 = V_2/K_2 = 2$; $\sigma_1 = \sigma_2 = 0.3$; $\kappa = 0.1$; $\theta = 0.08$; $\delta = 0.03$; $\rho_{V_1Y} = \rho_{V_2Y} = -0.3$; and $\rho_{V_1V_2} = 0.5$.

[12]*Local* asset return correlation means $\rho_{V_iY_j}$ such that $E[dW_{V_i}dW_{V_j}] = \rho_{V_iV_j}dt$.

[13]*Average* defines the mean of default correlations across various investment horizons.

[14]The effect related to the horizon occurs because macroeconomic state variables are assumed to follow a mean-reverting diffusion process.

[15]Zhou assumes that the time-dependent default boundary $K(t)$ takes an exponential form $e^{\lambda t}K_0$ where λ is the growth rate of the firm's liabilities.

[16]All the elements of $\mathbf{I}_s$ are assumed to be zero at time s when all firms have defaulted for the first time.

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