Modeling Default Dependence with Threshold Models

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In threshold models, for each credit the underlying ability-to-pay process can be modeled as a transformation of a Wiener process. The model for dependent defaults that is proposed here is based on correlated Wiener processes with suitably transformed time scales in order to calibrate the model to given marginal default distributions for each underlying credit.

This model allows for a straightforward analytic calibration of dependence information in the form of joint default probabilities.

Sophisticated default modeling has become indispensable, given explosive development of the credit derivatives market, scarce capital resources, and the rules of regulatory authorities concerning credit risk. Increasing securitization of credit risk via collateralized debt obligations and the growing popularity of multi-credit derivatives like basket default swaps calls not just for modeling default but also for modeling the dependence across defaults of different obligors.

Default models can be divided into structural models and reduced-form models. In structural models, as pioneered by Merton [1974], the default event is triggered when the value of the firm falls below a certain trigger level. Structural models, also called threshold models, are appealing because of their simplicity and the linking of the credit event to the firm's economic fundamentals. Structural models are relatively straightforward to extend to multiple credits with correlated defaults, although calibration to a given term structure of default probabilities is often not possible or computationally demanding.

Reduced-form models follow a completely different approach. Rather than model the event of default by some economic primitives, reduced-form models instead focus on modeling the (infinitesimal) likelihood of default (see Jarrow and Turnbull [1995], Jarrow, Lando, and Turnbull [1997], and Duffie and Singleton [1999]). Reduced-form models are in certain aspects similar to interest rate term structure models. They can easily be calibrated to a given term structure of default probabilities. Schönbucher and Schubert [2001] and Ward and Schmidt [2002], for example, have investigated default dependence using reduced-form models.

Dependencies between defaults for multi-credit products are often modeled using copulas. In the simplest case, the joint distribution of default times is generated from the marginal distributions, applying a normal copula; see, e.g., Li [2000]. In this approach, credit spreads are static until the time of the first default (see Ward and Schmidt [2002]).

We adopt the structural approach to model dependent defaults. It is straightforward to calibrate our model to any given term structure of defaults. We use a result in Zhou [1997] to derive an analytic expression for the probability of joint default, which then allows an efficient calibration of the model to given dependence information. As we follow a structural approach, credit spreads have a certain stochastic dynamic, contrary to the copula approach for dependent defaults.

Hull and White [2001] propose a model for dependent defaults that achieves the same goals but is much more involved computationally.

I. FORMULATION OF THE PROBLEM

Our goal is to develop a model for the random times $\tau_1, \ldots, \tau_n$ of default of credits $i = 1, \ldots, n$, where τ_i denotes the time of default of credit number i, $i = 1, \ldots, n$. The given, either derived from market prices of traded defaultable instruments like default swaps or from historical data, is the distribution function F_i of the random variable τ_i:[1]

$$\mathbf{P}(\tau_i < t) = F_i(t), \; t \geq 0, \; i = 1, \ldots, n \tag{1}$$

We assume F_i is continuous and strictly increasing, and $F_i(0) = 0$.

Information on the dependencies of the random default times involves, in the most general case, some conditions on the joint distribution of $\tau_1, \ldots, \tau_n$. As this is unrealistic in practice, we restrict ourselves to given pairwise joint default probabilities for a given fixed time horizon t_0:

$$p_{ij} = \mathbf{P}(\tau_i < t_0, \tau_j < t_0) \tag{2}$$

The event correlation ρ_{ij}^E (for time t_0) is defined as

$$\rho_{ij}^E = \frac{p_{ij} - F_i(t_0)F_j(t_0)}{\sqrt{F_i(t_0)(1 - F_i(t_0))F_j(t_0)(1 - F_j(t_0))}} \tag{3}$$

so specifying the joint default probability is equivalent to specifying the event correlation.

Note that, in view of $0 \leq \mathbf{P}(\tau_i < t_0, \tau_j < t_0) \leq \min[F_i(t_0), F_j(t_0)]$, the event correlation admits the natural bounds:

$$\begin{aligned} & \frac{-F_i(t_0)F_j(t_0)}{\sqrt{F_i(t_0)(1 - F_i(t_0))F_j(t_0)(1 - F_j(t_0))}} \leq \rho_{ij}^E \\ & \leq \sqrt{\frac{u(1-v)}{v(1-u)}} \end{aligned} \tag{4}$$

where $u = \min[F_i(t_0), F_j(t_0)]$, and $v = \max[F_i(t_0), F_j(t_0)]$.

Given distribution functions F_i and joint default probabilities p_{ij} (or, equivalently, event correlations), we are looking for stochastic processes (Y_t^i) called *ability-to-pay processes* and barriers $K_i(t)$ such that the default time τ_i for credit i can be modeled as the first hitting time of the barrier $K_i(t)$ by the process (Y_t^i):

$$\tau_i = \inf\{t \geq 0 : Y_t^i \leq K_i(t)\} \tag{5}$$

In other words, the hitting time τ_i defined by Equation (5) satisfies Equations (1) and (2).

Given the analytical tractability of the Wiener process and the fact that many other stochastic processes can be transformed into a Wiener process, a reasonable way to find appropriate processes (Y_t^i) would be to start with correlated Wiener processes (W_t^i) and then apply suitable transformations G^i:

$$Y_t^i = G^i(W_t^i, t)$$

II. HULL AND WHITE APPROACH REVISITED

Hull and White [2001] propose a model for dependent defaults where the default time takes values in an arbitrary fine discrete time grid $0 = t_0 < t_1 < t_2 \ldots \delta_k = t_k - t_{k-1}$. The default time for credit i is defined as

$$\tau_i = \inf\{t_k : W_{t_k}^i < K_i(t_k), \; k = 1, \ldots\} \tag{6}$$

where (W_t^i) is a Wiener process. The default barriers $K_i(t_k)$ are calibrated successively to match the default distribution function F_i for the time points $t_1, t_2, \ldots$

$$\begin{aligned} K_i(t_1) &= \sqrt{\delta_1}\mathrm{N}^{(-1)}(F_i(t_1)) \\ F_i(t_k) - F_i(t_{k-1}) &= \int_{K_i(t_{k-1})}^{\infty} f_i(t_{k-1}, u) \times \\ & \quad \mathrm{N}\left(\frac{K_i(t_k) - u}{\sqrt{\delta_k}}\right) du \end{aligned}$$

where $f_i(t_k, x)$ is the density of $W_{t_k}^i$ given that $W_{t_j}^i > K_i(t_j)$ for all $j < k$, and:

$$\begin{aligned} f_i(t_1, x) &= \frac{1}{\sqrt{2\pi\delta_1}} \exp\left(-\frac{x^2}{2\delta_1}\right) \\ f_i(t_k, x) &= \int_{K_i(t_{k-1})}^{\infty} f_i(t_{k-1}, u) \frac{1}{\sqrt{2\pi\delta_k}} \exp\left(-\frac{(x-u)^2}{2\delta_k}\right) du \end{aligned}$$

N denotes the standard normal distribution function, $N^{(-1)}$ its inverse. Numerical procedures must be used to evaluate the integrals recursively.

To match given default dependencies as in Equation (2), we have to calibrate the correlations ρ_{ij} of the Wiener processes W^i, W^j. To this end, one has to simulate the default times according to Equation (6), estimate the joint default probabilities from the samples, and calibrate over the results.

While the Hull and White approach provides an elegant solution to the problem, at least on a discrete time grid, a major shortcoming is that its calibration is computationally involved.

At the continuous-time limit, the Hull and White model also allows for interpretation as follows. Define:

$$\tau_i = \inf\{t \geq 0 : W_t^i < K_i(t)\}$$

If $K_i(t)$ is absolutely continuous:

$$K_i(t) = K_i(0) + \int_0^t \mu_s^i \mathrm{d}s$$

the default time τ_i is the first hitting time of the constant barrier $K_i(0)$ for a Wiener process with drift:

$$\begin{aligned} Y_t^i &= W_t^i - \int_0^t \mu_s^i \mathrm{d}s \\ \tau_i &= \inf\{t \geq 0 : Y_t^i < K_i(0)\} \end{aligned}$$

The drift can be interpreted as a *default trend*; the higher μ_s^i, the higher the increase in the likelihood of default.

In general, time-dependent default barriers in threshold models can be replaced by constant barriers at the price of some drift in the underlying firm value or ability-to-pay process.

III. MODELING DEFAULT WITH TIME-CHANGED WIENER PROCESSES

Our alternative solution to the problem of modeling dependent defaults is intuitive and straightforward to calibrate to both Equations (1) and (2). The idea is to use Wiener processes with suitably transformed time scales. As we shall see below, this is in principle equivalent to using Wiener processes with time-varying but deterministic volatilities.

For every credit, the default is triggered when an individual Wiener process (ability-to-pay process) reaches a certain default threshold. Because of differential creditworthiness of the underlyings, the individual Wiener processes run at different speeds—the intrinsic clock of the particular credit. The faster the intrinsic clock runs, the higher the chances of reaching the threshold; i.e., the higher the likelihood of default.

The intrinsic clock can also be interpreted as a firm-specific time-dependent but deterministic volatility. As time passes, volatility accumulates. Default is then triggered if the amount of accumulated volatility causes the ability-to-pay process to hit the default threshold.

As we shall see below, when we apply this idea, it is straightforward to calibrate the intrinsic clock (the explicit volatility function) explicitly in order to reproduce any given term structure of default probabilities.

Given Wiener processes (W_t^i) and strictly increasing time transformations (T_t^i):

$$T^i | [0, \infty) \to [0, \infty), \ T_0^i = 0$$

we define:[2]

$$Y_s^i = W_{T_s^i}^i \tag{7}$$

$$\tau_i = \inf\{s \geq 0 : Y_s^i < K_i\} \tag{8}$$

Calibrating the Term Structure of Defaults

The first result shows that it is straightforward to chose the time transformation appropriately to match a given term structure of default probabilities (1).

Proposition 1. *Let F_i be a continuous distribution function, strictly increasing in* $[0, \infty)$ *with* $F_i(0) = 0$. *If the time transformation* (T_t^i) *is given by:*

$$T_t^i = \left[\frac{K_i}{\mathrm{N}^{(-1)}\left(\frac{F_i(t)}{2}\right)} \right]^2 \text{ for } t \geq 0 \tag{9}$$

the default time τ_i defined by Equation (8) *admits the distribution function F_i; i.e., condition* (1) *is satisfied.*

Proof. The distribution of the hitting time of a Brownian motion W is well-known to be (see Karatzas and Shreve [1988]):

$$\mathbf{P}(\min_{s \leq t} W_s < K_i) = 2\,\mathrm{N}(K_i/\sqrt{t})$$

This yields for τ_i defined by Equation (8):

$$\begin{aligned}\mathbf{P}(\tau_i < t) &= \mathbf{P}(\min_{s \leq t} W^i_{T_s} < K_i) \\ &= \mathbf{P}(\min_{s \leq T^i_t} W^i_s < K_i) \\ &= 2\,\mathrm{N}\left(K_i/\sqrt{T^i_t}\right)\end{aligned}$$

and the assertion follows. ◇

The time transformation T^i_t admits an obvious interpretation: The greater the rate of increase of the function T^i_t, the faster the ability-to-pay process $Y^i_t = W^i_{T^i_t}$ passes along the Wiener path, thereby increasing the likelihood of default.

If the distribution function F_i admits a density $F_i' = f_i$, then the time transformation T^i_t is absolutely continuous:

$$T^i_t = \int_0^t (\sigma^i_s)^2 \mathrm{d}s \tag{10}$$

and by a well-known result the time-transformed Wiener process allows for representation as a stochastic integral with *volatility* σ^i_s:

$$W^i_{T^i_t} = \int_0^t \sigma^i_s \mathrm{d}\tilde{W}^i_s \tag{11}$$

with some new Wiener process $\tilde{W}^i$.

The relationship between the volatility σ^i_s and the distribution of τ_i is:

$$\sigma^i_s = \sqrt{-\left[\frac{K_i}{\mathrm{N}^{(-1)}\left(\frac{F_i(s)}{2}\right)}\right]^3 \frac{f_i(s)}{K_i \varphi\left(\mathrm{N}^{(-1)}\left(\frac{F_i(s)}{2}\right)\right)}}$$

where φ is the density of the standard normal distribution.

The volatility σ^i_s can be interpreted as *default speed*.[3] The higher the default speed, the greater the volatility of the ability-to-pay process and the higher the likelihood of crossing the default threshold K_i.

Calibrating Joint Default Probabilities

Our time-changed Wiener process model can be easily calibrated to a set of given pairwise joint default probabilities for a fixed time horizon. Specifying joint pairwise default probabilities is equivalent to specifying the default event correlation in Equation (3), an input in applications of the model. The model then needs to be calibrated to these default event correlations by finding the corresponding appropriate correlations of the underlying Wiener processes. This is quite similar to the approach in Hull and White [2001].

Using a result by Zhou [1997], we are able to derive an analytic expression for the joint default probability as a function of the correlation of the underlying Wiener processes. This is the key to an extremely efficient calibration of our model to given default event correlations.

Proposition 2. *Let (W^1_t) and (W^2_t) be Wiener processes with correlation ρ. For time changes (T^i_t), $i = 1, 2$, and default times τ_i, $i = 1, 2$, defined by Equation (8) we have an expression for the joint survival probability:*[4]

$$\mathbf{P}(\tau_1 > t_0, \tau_2 > t_0) = \tag{12}$$
$$\begin{cases} \frac{2}{\alpha T} \mathbf{e}^{\frac{-r_0^2}{2T}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \int_{\theta=0}^{\alpha} \int_{r=0}^{\infty} \sin\left(\frac{n\pi\theta}{\alpha}\right) r \mathbf{e}^{\frac{-r^2}{2T}} I_{\frac{n\pi}{\alpha}}\left(\frac{r r_0}{T}\right) \\ \qquad \left(-1 + 2\mathrm{N}\left(\frac{r\sin\theta}{\sqrt{\Delta}}\right)\right) \mathrm{d}\theta \mathrm{d}r, & \text{if} \Delta > 0 \\ \frac{2r_0}{\sqrt{2\pi T}} \mathbf{e}^{\frac{-r_0^2}{4T}} \sum_{n=1,3,5,\ldots} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \\ \qquad \left[I_{\frac{1}{2}(\frac{n\pi}{\alpha}+1)}\left(\frac{r_0^2}{4T}\right) + I_{\frac{1}{2}(\frac{n\pi}{\alpha}-1)}\left(\frac{r_0^2}{4T}\right)\right] & \text{if} \Delta = 0 \end{cases}$$

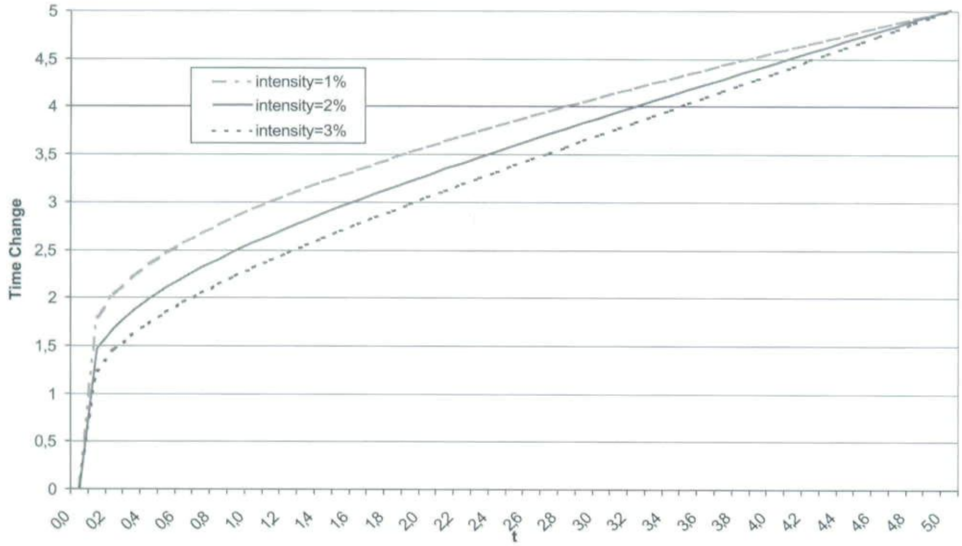
where

$$\begin{aligned} T &= \min(T^1_{t_0}, T^2_{t_0}) \\ \Delta &= \max(T^1_{t_0}, T^2_{t_0}) - \min(T^1_{t_0}, T^2_{t_0}) \\ \theta_0 &= \begin{cases} \tan^{(-1)}\left(\frac{K_2\sqrt{1-\rho^2}}{K_1 - \rho K_2}\right) & \textit{if } \left(\frac{K_2\sqrt{1-\rho^2}}{K_1-\rho K_2}\right) > 0 \\ \pi + \tan^{(-1)}\left(\frac{K_2\sqrt{1-\rho^2}}{K_1-\rho K_2}\right) & \textit{otherwise} \end{cases} \\ r_0 &= \frac{-K_2}{\sin\theta_0} \\ \alpha &= \begin{cases} \tan^{(-1)}\left(\frac{-\sqrt{1-\rho^2}}{\rho}\right) & \rho < 0 \\ \pi + \tan^{(-1)}\left(\frac{-\sqrt{1-\rho^2}}{\rho}\right) & \rho > 0 \end{cases} \end{aligned}$$

and I_k denotes the modified Bessel function with order k.

Proof. Assume $T^1_{t_0} \leq T^2_{t_0}$. Using the Markov property of the Wiener process (W^1_t, W^2_t) we obtain

EXHIBIT 1
Calibrated Time Changes T_t



$$
\begin{aligned}
&\mathbf{P}(\tau_1 > t_0, \tau_2 > t_0) \\
&= \mathbf{P}(W^1_s > K_1, s \le T^1_{t_0}, W^2_u > K_2, u \le T^2_{t_0}) \\
&= \mathbf{E}\left(\mathbf{1}_{\{W^1_s > K_1, W^2_s > K_2, s \le T\}} \mathbf{P}(W^2_u > K_2, u \in [T, T+\Delta] | W^2_T)\right) \\
&= \mathbf{E}\left(\mathbf{1}_{\{W^1_s > K_1, W^2_s > K_2, s \le T\}} \left(1 - 2\,\mathrm{N}\left(\frac{K_2 - W^2_T}{\sqrt{\Delta}}\right)\right)\right) \\
&= \int_{-\infty}^{-K_1}\int_{-\infty}^{-K_2} f(x_1, x_2, T, \rho)\left(1 - 2\mathrm{N}\left(\frac{K_2 + x_2}{\sqrt{\Delta}}\right)\right) \mathrm{d}x_1 \mathrm{d}x_2
\end{aligned}
$$

where $f(x_1, x_2, T, \rho)$ is the density of $(-W^1_t, -W^2_t)$, given that the barriers $-\mathrm{K}_1$, $-\mathrm{K}_2$, respectively, have not been hit by time T. This density is explicitly calculated in Zhou [1997]. Using his result and transforming the integration variables appropriately yields the assertion.

The case of $\Delta = 0$, i.e., $T^1_{t_0} = T^2_{t_0}$, follows directly from Zhou [1997]. ◇

In general, the dependence structure of defaults between n credits is not fully determined by just specifying pairwise correlations. In our framework, the dependencies are fully determined as soon as the correlations of the underlying Wiener processes are fixed. For a given set of pairwise default event correlations for a certain time horizon, there are typically a universe of models consistent with those correlations.

It remains an open question whether it is possible to also obtain an analytical result if we consider the underlying ability-to-pay process as Wiener processes with different volatilities as in Equation (11). Observe that in Equation (11) the correlations of the Wiener processes $\tilde{W}^1, \ldots, \tilde{W}^n$ are not identical to those between $W^1, \ldots, W^n$!

Calibration Examples

It is easy to calibrate our model to given default probabilities and default event correlations.

Term Structure of Default Probabilities. In practice, for each credit the primary information on its creditworthiness consists of spreads (credit default swap spread, yield spreads) or prices of default-risky instruments. Using standard techniques one can extract a curve of cumulative default probabilities $F(t)$, i.e., the probability of a default up and until time (see, for example, Khuong-Huu [1999]):

$$F(t) = \mathbf{P}(\tau < t)$$

In modeling Equation (8), we now have to specify the default threshold K and the time change T_t so that the model is consistent with the given term structure of default probabilities $F(t)$.

The calibration in Equation (9) allows some degree of freedom in determining the default threshold K. It would seem natural to choose K so that the time change T_t is, in some sense, close to the original time scale. More precisely, for a fixed final time horizon $t_0 > 0$, we require:

$$T_{t_0} = t_0 \tag{13}$$

This implies:

$$K = \mathrm{N}^{(-1)}\left(\frac{F(t_0)}{2}\right)\sqrt{t_0} \tag{14}$$

Now let us assume that the term structure of default probabilities is of exponential form, $F(t) = 1 - \exp(-\lambda t)$. The quantity λ is the so-called default intensity or hazard rate. It is approximately linked to the CDS spread s by the rule of thumb formula $\lambda = s/(1 - \text{recovery rate})$.

Exhibit 1 shows calibrated time transformations for default times with default intensities (hazard rates) $\lambda = 1\%$, 2%, and 3%. We apply the additional condition in Equation (13) with $t_0 = 5$, and obtain the corresponding threshold barriers $K = -4.406$, $K = -3.731$, and $K = -3.306$.

The sharp increase of the time changes near time zero is due to the fact that for threshold models with continuous ability-to-pay processes there is basically little likelihood of default over short time periods near zero. In our example, we calibrate the model to a flat default rate λ_i per time unit. Due to the time change, the ability-to-pay process, although starting at zero, becomes random enough for very short times to allow for the given non-vanishing default rate.

An alternative way to overcome the general problem for threshold models with continuous ability-to-pay processes Y would be to allow for a random initial value Y_0 with some distribution $\mathbf{Q}$. Conditioning on the values of Y_0 and integrating with regard to $\mathbf{Q}$, it would be straightforward to extend our approach thus.

In practice, the model will be recalibrated every day. The time decay will in fact make this a slightly different model. This is a quite common feature of almost any modeling approach that requires recalibration and also inherent in the Hull and White [2001] model. One should be aware that this can lead to inconsistencies of the implied hedges.

Default Event Correlation. In our numerical examples on the calibration of the model to given dependence information, we express the joint default probability via its event correlation as defined in Equation (3). To simplify the exposition, we again assume that the default curves correspond to constant hazard rates λ:

$$F_i(t) = 1 - \exp(-\lambda_i t)$$

for each credit i. Our final time horizon is again $t_0 = 5$, and we calibrate the default event correlations to also refer to time t_0; i.e., we calibrate to given pairwise default probabilities for time t_0. The calibration results are the correlations ρ between the underlying Wiener processes in our model.

Exhibits 2 and 3 show the results. The event correlations ρ^E range from 0% to 90%. The columns show the calibrated correlations ρ of the driving Wiener processes for the given pair (τ_1, τ_2) of default times with default rates (λ_1, λ_2). Missing results indicate that the event correlation percentage is not valid for that combination.[5]

In the examples, we calibrate only to non-negative event correlations as in practice negative default correlations are rather artificial.

EXHIBIT 2

Wiener Processes Correlation Calibrated to Event Correlations I

$\rho^E/(\lambda_1,\lambda_2)$	(1%,1%)	(1%,2%)	(1%,3%)
0.00%	0.00%	0.00%	0.00%
5.00%	18.51%	16.27%	15.28%
10.00%	31.59%	28.82%	27.68%
15.00%	41.96%	39.23%	38.29%
20.00%	50.60%	48.16%	47.63%
25.00%	57.98%	55.99%	55.99%
30.00%	64.40%	62.92%	63.55%
35.00%	70.03%	69.11%	70.46%
40.00%	74.98%	74.66%	76.82%
45.00%	79.35%	79.64%	82.77%
50.00%	83.21%	84.12%	88.49%
55.00%	86.58%	88.15%	
60.00%	89.53%	91.79%	
65.00%	92.07%		
70.00%	94.23%		
75.00%	96.02%		
80.00%	97.47%		
85.00%	98.59%		
90.00%	99.37%		

EXHIBIT 3

Wiener Processes Correlation Calibrated to Event Correlations II

$\rho^E/(\lambda_1,\lambda_2)$	(2%,2%)	(2%,3%)	(3%,3%)
0.00%	0.00%	0.00%	0.00%
5.00%	13.98%	12.97%	11.94%
10.00%	25.52%	24.07%	22.48%
15.00%	35.43%	33.85%	31.94%
20.00%	44.13%	42.59%	40.52%
25.00%	51.87%	50.47%	48.32%
30.00%	58.78%	57.59%	55.44%
35.00%	64.99%	64.05%	61.92%
40.00%	70.56%	69.91%	67.82%
45.00%	75.55%	75.20%	73.16%
50.00%	80.01%	79.96%	77.97%
55.00%	83.96%	84.22%	82.27%
60.00%	87.43%	88.00%	86.07%
65.00%	90.45%	91.33%	89.40%
70.00%	93.03%	94.23%	92.25%
75.00%	95.19%	96.74%	94.65%
80.00%	96.94%		96.59%
85.00%	98.29%		98.09%
90.00%	99.24%		99.15%

IV. IMPLEMENTATION OF THE MODEL

To implement our model for practical applications, a Monte Carlo simulation seems to be appropriate and necessary for most applications. We provide step-by-step instructions on how to implement the model.

The model inputs are the default probability curves $F_i(t)$, $t \geq 0$ for each credit i and the default event correlations ρ^E_{ij} for each pair (i, j) of credits at a fixed time horizon t_0.

1. Choose a final time horizon t_0, naturally the maturity of the transaction to evaluate, and determine the corresponding threshold level K_i for each credit $i = 1, \ldots, n$ according to Equation (14).
2. Choose a time discretization $0 = s_0 < s_1 < \ldots < s_m = t_0$ for the time interval $[0, t_0]$, which determines the simulation time points and thus the time points for sampling the default times.
3. For each credit i, the time change T^i_t of the underlying Wiener process W^i is calibrated for each simulation point $s_1, \ldots, s_m$ according to Equation (9):

$$T^i_t = \left[\frac{K_i}{\mathrm{N}^{(-1)}\left(\frac{F_i(t)}{2}\right)} \right]^2, \; k = 1, \ldots, m$$

4. Translate the given default event correlation ρ^E_{ij} into probabilities of pairwise joint survivals [see Equation (3)]:

$$\begin{aligned}\mathbf{P}(\tau_i > t_0, \tau_j > t_0) = \\ &(1 - F_i(t_0))(1 - F_j(t_0)) + \\ &\rho^E_{ij}\sqrt{F_i(t_0)(1 - F_i(t_0))F_j(t_0)(1 - F_j(t_0))}\end{aligned}$$

   Use the result of Proposition 2, Equation (12), to back out by numerical search the corresponding correlation ρ_{ij} between the driving Wiener processes W^i, W^j that reproduces the given joint survival probability.
5. Now we are ready to start the Monte Carlo simulation. For the points s_j of the chosen time discretization simulate the values $W^i_{T^i_{sj}}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ of the correlated Wiener processes. This is achieved by sequentially simulating the increments $W^i_{T^i_{sj}} - W^i_{T^i_{sj-1}}$, which are normal with variance $T^i_{sj} - T^i_{sj-1}$. Of course, in simulating we have to take into account the calibrated correlations of the Wiener processes.
6. The first time s_j when $W^i_{T^i_{sj}} < K_i$, we set $\tau_i = s_j$. The first hitting time:[6]

$$\min\{s_j : W^i_{T^i_{s_j}} < K_i\}$$

   of the discrete path clearly systematically overestimates the true default time τ_i as defined by Equation (8) since we are missing the paths where the barrier is hit in between the simulation time points.
7. To overcome the bias in the simulation of the default time, we apply in addition a Brownian bridge technique to capture the probability of possible defaults in between the simulation points. This technique connects the discretely simulated points of the path by assuming a so-called Brownian bridge between these points (see, e.g., Karatzas and Shreve [1988]). For a Brownian bridge between two points α and β there is an explicit formula for the probability of hitting a certain barrier by the path in between these points that can be used to capture this missing piece of probability in our simulation. This is a standard technique in simulating barrier effects in many financial applications; see, e.g., Metwally and Atiya [2002].

   For simulation results $\alpha = W^i_{T^i_{sj}} > K_i$, and $\beta = W^i_{T^i_{sj+1}} > K_i$, the process W^i_u, $u \in [T^i_{sj}, T^i_{sj+1}]$ follows a Brownian bridge with starting point α and end point β. The probability of crossing the boundary K_i during the time interval $[T^i_{sj}, T^i_{sj+1}]$ is known to be:

$$\begin{aligned}\mathbf{P}(\min_{u\in[s_j,s_{j+1}]} W^i_{T^i_u} < K_i) &= \mathbf{P}(\min_{u\in[T^i_{s_j},T^i_{s_{j+1}}]} W^i_u < K_i) \\ &= \exp\left(\frac{-2(\beta - K_i)(\alpha - K_i)}{\Delta}\right)\end{aligned}$$

   where $\Delta = T^i_{sj+1} - T^i_{sj}$. For each time interval $[s_j, s_{j+1}]$, given that the threshold K_i has not been crossed before and that $W^i_{T^i_{sj+1}} > K_i$, we draw an additional uniform random variable U^i_j and set:

$$\tau_i = (s_j + s_{j+1})/2$$

$$\text{if } U^i_j < \exp\left(\frac{-2(\beta - K_i)(\alpha - K_i)}{\Delta}\right)$$

   and proceed with our simulations if:

$$U^i_j \geq \exp\left(\frac{-2(\beta - K_i)(\alpha - K_i)}{\Delta}\right)$$

EXHIBIT 4
Cumulative Default Probabilities from CDS Spreads

t	$F_1(t)$	$F_2(t)$	$F_3(t)$	$F_4(t)$	$F_5(t)$
1	0.950%	1.069%	1.187%	1.305%	1.422%
2	1.884%	2.117%	2.350%	2.582%	2.813%
3	2.809%	3.155%	3.499%	3.842%	4.184%
4	3.726%	4.181%	4.635%	5.087%	5.536%
5	4.636%	5.200%	5.761%	6.318%	6.872%

EXHIBIT 5
Calibrating Correlations

i	j	ρ_{ij}^{asset}	p_{ij}	ρ_{ij}^{E}	ρ_{ij}
1	2	30%	0.693%	9.68%	30.80%
1	3	40%	0.989%	14.73%	40.94%
1	4	35%	0.927%	12.40%	35.90%
1	5	25%	0.748%	8.07%	25.74%
2	3	35%	0.946%	12.49%	35.90%
2	4	40%	1.155%	15.30%	40.97%
2	5	30%	0.947%	10.49%	30.85%
3	4	33%	1.043%	11.98%	33.89%
3	5	30%	1.029%	10.74%	30.86%
4	5	25%	0.969%	8.69%	25.77%

This technique allows the simulation results to capture the true theoretical marginal default distribution $F_i(t)$ quite accurately.[7]

As a result of our simulation, we end up with the simulated default times $\tau_1, \ldots, \tau_n$ which are the primary input to a Monte Carlo evaluation of any multi-credit product.

V. APPLICATION EXAMPLE

Credit default swaps are the dominant plain vanilla credit derivative product and serve as a building block for many other credit products. A credit default swap offers protection against default of a certain underlying entity over a specified time horizon. A premium (spread) s is paid on a regular basis (e.g., quarterly) and on a certain notional amount N as an insurance fee against the losses from default of a risky position of notional N, e.g., a bond. Payment of the premium s stops at maturity or at default of the underlying credit, whichever comes first. In case of default before maturity of the credit, the protection buyer receives a payment $N(1 - R)$, where R is the recovery rate of the underlying credit risky instrument.

A basket credit default swap is an insurance contract that offers protection against the event of the k-th default on a basket of n ($n \geq k$) underlying names. It is quite similar to a plain credit default swap, but the credit event to be insured against is the event of the k-th default. Again a premium (spread) s is paid as insurance fee until maturity or the event of the k-th default in return for compensation for the loss. We denote by $s^{k\text{-th}}$ the fair spread in a k-th-to-default swap. This is the spread making the value of this swap today equal to 0.

Most popular are first-to-default swaps, i.e., $k = 1$, as they offer highly attractive spreads to a credit investor (protection seller).

Pricing a Basket CDS

We apply the model to a basket with $n = 5$ names trading in the credit default swap market with fair CDS spreads $s_1 = 0.80\%$, $s_2 = 0.90\%$, $s_3 = 1.00\%$, $s_4 = 1.10\%$, and $s_5 = 1.20\%$ for all maturities. We suppose a recovery rate of $R = 15\%$ for all credits. The riskless interest rate curve is assumed to be flat at 5.00%. As common in practice, we extract from the given CDS spreads and the recovery assumption the (risk-neutral) distribution function $F_i(t)$ of the default time τ_i, $i = 1, \ldots, 5$ as shown in Exhibit 4.

The cumulative default probabilities in Exhibit 4 are the inputs to the calibration of the time changes T^i using Equation (9).

Since the market increasingly quotes default dependence in terms of (implied) asset correlations, the input correlations to our example are given as asset return correlations ρ_{ij}^{asset}. From these we calculate the joint default probabilities p_{ij}.[8]

Finally, we calibrate the correlation ρ_{ij} of the time-changed Wiener processes in our model to the given joint default probabilities p_{ij}. Since the calibration of the joint probabilities is based on the explicit formula in Proposition 2, the computation time to calibrate the whole basket is a fraction of a second.

Exhibit 5 shows the detailed results. For convenience we also show the implicit default event correlations ρ_{ij}^{E} [see Equation (3)].

We price five-year to maturity k-th-to-default basket default swaps for $k = 1, \ldots, 5$ and determine their fair spreads. The model uses a Monte Carlo simulation on a monthly time grid with 10,000 simulations.

There is only 1% to 2% relative error for each name between the true cumulative five-year default probability $F_i(5)$ and the in-sample cumulative default frequency,

EXHIBIT 6

Fair Basket CDS Spreads

	run 1	run 2	run 3	run 4	run 5
s^{first}	4.1611%	4.148%	4.150%	4.1750%	4.163%
s^{2nd}	0.9504%	0.969%	0.973%	0.9565%	0.977%
s^{3rd}	0.2350%	0.215%	0.219%	0.2340%	0.219%
s^{4th}	0.0441%	0.062%	0.060%	0.0421%	0.058%
s^{5th}	0.0049%	0.003%	0.005%	0.0067%	0.003%

which is quite good, given that we use only 10,000 simulations and defaults are a rare event.

The final pricing results are quite stable, even with that number of simulations. The seed variance of the fair basket spreads is about 0.01% to 0.03%. Exhibit 6 shows the results for repeated Monte Carlo pricing runs of the model.

Comparison to Normal Copula Approach

The normal copula model, as described in detail in Li [2000] and Ward and Schmidt [2002], is currently the market standard for the valuation of basket credit derivatives. To compare the models and to focus the exposition, we restrict ourselves to a constant correlation $\rho_{ij} = \rho$ between the driving Wiener processes in our model.

The correlations in the normal copula model are calibrated to produce the same pairwise default probabilities for the time horizon $t_0 = 5$ as in our time change model. A flat correlation of $\rho_{ij} = \rho = 30\%$ between our underlying time-changed Wiener processes translates into an asset correlation matrix (ρ_{ij}^{asset}) as follows for the normal copula model:

$$(\rho_{ij}^{\text{asset}}) = \begin{pmatrix} 100.00\% & 29.22\% & 29.20\% & 29.19\% & 29.18\% \\ 29.22\% & 100.00\% & 29.19\% & 29.18\% & 29.17\% \\ 29.20\% & 29.19\% & 100.00\% & 29.17\% & 29.16\% \\ 29.19\% & 29.18\% & 29.17\% & 100.00\% & 29.15\% \\ 29.18\% & 29.17\% & 29.16\% & 29.15\% & 100.00\% \end{pmatrix}$$

The two models produce the numerical results for the basket default swap valuations shown in Exhibits 7 and 8. Overall the results are relatively close. For the first- and second-to-default basket, the time change model yields slightly higher spreads. For the fourth to default, the normal copula model produces slightly higher spreads. This indicates that the normal copula model yields a slightly higher dependence, even though the pairwise joint default probabilities for maturity are the same in both models.

This is supported by repeated simulations with different seeds for the random number generator. For the normal copula model, we use a variance reduction technique based on stratified sampling, and for the time change model we apply an in-sample orthogonalization of the increments of the Wiener processes.

Finger [2000] compares the results of the pricing of multi-credit products for various stochastic default rate models. Models were calibrated against each other in a single-period time horizon but then applied to a multi-period valuation problem. Finger's comparisons show much higher discrepancies among the different models.

To some extent this can be attributed to the fact

EXHIBIT 7

Fair Basket Default Swap Spreads in the Time Change Model

ρ	10%	20%	30%	40%	50%	60%	70%
s^{first}	4.791%	4.563%	4.296%	3.953%	3.620%	3.252%	2.845%
s^{2nd}	0.625%	0.799%	0.941%	1.055%	1.131%	1.201%	1.259%
s^{3rd}	0.058%	0.130%	0.201%	0.296%	0.398%	0.523%	0.635%
s^{4th}	0.003%	0.015%	0.040%	0.076%	0.126%	0.202%	0.306%
s^{5th}	0.000%	0.003%	0.006%	0.013%	0.030%	0.057%	0.118%

EXHIBIT 8

Fair Basket Default Swap Spreads in the Normal Copula Model

ρ	10%	20%	30%	40%	50%	60%	70%
s^{first}	4.788%	4.507%	4.209%	3.896%	3.575%	3.229%	2.851%
s^{2nd}	0.604%	0.744%	0.875%	0.988%	1.108%	1.185%	1.228%
s^{3rd}	0.070%	0.140%	0.220%	0.301%	0.386%	0.500%	0.630%
s^{4th}	0.006%	0.021%	0.045%	0.092%	0.139%	0.209%	0.312%
s^{5th}	0.001%	0.002%	0.007%	0.023%	0.037%	0.069%	0.119%

that the models were not calibrated for the final time horizon. This might explain why our comparison yields much closer results.

VI. CONCLUSIONS

We have investigated models for dependent defaults based on so-called structural models, where default is triggered by an underlying ability-to-pay process reaching a certain threshold barrier. Our new model is based on correlated and suitably time-changed Wiener processes. The model is flexible and easy to calibrate to any given term structure of default probabilities backed out from market information, as, for example, CDS spreads or bond prices. An analytic expression for pairwise joint default probabilities also allows us to calibrate the model efficiently to given dependence information.

Potential applications of the model are the pricing of multi-credit derivative products or the valuation of counter-party default risks. Our sample application is pricing a basket credit default swap. A comparison of the results of our model and the normal copula model, both calibrated to the same dependence information, shows that both models give quite similar results, but our model produces slightly higher fair first-to-default premiums.

ENDNOTES

[1]See, e.g., Khuong-Huu [1999].

[2]Because of the space-time scaling properties of the Wiener process, we could set the default barrier as $K_i = -1$. But as we shall see later it is useful to keep this degree of freedom.

[3]Compare this with interpretation of the drift as default trend in our presentation of the Hull and White model.

[4]The joint survival probability and the joint default probability are linked by

$$\mathbf{P}(\tau_1 > t, \tau_2 > t) = 1 - \mathbf{P}(\tau_1 \leq t) - \mathbf{P}(\tau_2 \leq t) + \mathbf{P}(\tau_1 \leq t, \tau_2 \leq t)$$

[5]Compare Equation (4) for necessary bounds of the default event correlation.

[6]As usual, we follow the convention min $(\emptyset) = \infty$.

[7]In the case of non-vanishing correlations between the underlying Wiener processes, we may lose some degree of correlation in our implementation since, theoretically, the uniform U_i would have to be somehow dependent.

[8]For a given asset return correlation ρ, the joint default probability for time horizon T is $p_{ij} = \mathrm{N}_2(\mathrm{N}^{(-1)}[F_i(T)], \mathrm{N}^{(-1)}[F_j(T)], \rho)$, where N_2 denotes the two-dimensional cumulative normal distribution function.

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