Static Hedging of Asian Options under Lévy Models

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A simple static superhedging strategy for the payoff of an arithmetic Asian option in terms of a portfolio of European options based on stop-loss transforms is applicable under general stock price models. The hedge obtained is optimal in some sense.

Numerical illustrations of the hedging performance are given for various Lévy models calibrated to market data of the S&P 500.

Even in the Black-Scholes world, pricing of arithmetic Asian options is not straightforward. There is generally no explicit analytical expression for the average, so we have to use Monte Carlo simulation techniques to obtain numerical estimates of the price (see Broadie and Glasserman [1996] and Broadie, Glasserman, and Kou [1999]). Alternatives are to follow a partial differential equation approach (Dewynne and Wilmott [1993], Večeř [2001, 2002]) (or a partial integro-differential equation approach in more general market models; Večeř and Xu [2004]). Typically these approaches do not lead to practicable hedging strategies.

Approximations of the distribution of the average that produce closed-form expressions have also been studied (see, e.g., Turnbull and Wakeman [1991], Vorst [1992]), but in general it is difficult to assess the approximation error, and this method is often unsatisfactory for hedging purposes. For an approach based on fast Fourier transforms, see Benhamou [2002].

Another route is to try to derive upper and lower bounds for the option price. This can be done by use of comonotonic theory as described in Dhaene et al. [2002a, 2002b] and Vanmaele et al. [2002]. We follow this path and derive a static (super)hedge for fixed-strike Asian call options based on a buy-and-hold strategy consisting of European call options maturing with and before the Asian option.

This is useful because European call options are typically available on the market and quite liquid. Moreover, only when the contract is struck does one have to take a position in these calls, and no dynamic trading is needed.

Static hedging has several advantages over dynamic hedging. It is less sensitive to the assumption of zero transaction costs (both commissions and the cost of paying to monitor the positions). Furthermore, dynamic hedging tends to fail when liquidity dries up or in the face of large market moves, but it is particularly in these situations that effective hedging is needed (see, e.g., Carr, Ellis, and Gupta [1998] and Carr and Picron [1999]).

The hedging error of our simple superhedging strategy is very small if the option is in the money. For options at and out of the money, the strategy can be quite conservative, but it is still much cheaper than the trivial superhedge of the Asian option by a European option with identical strike and maturity (when dividend yield q is lower than the continuously compound interest rate r).

The procedure we develop may be used

with general stock price models. We focus on models that describe the asset price by an exponential of a general Lévy process. It has come to be realized that the stock price dynamics are much better described by a Lévy model than by the classic Black-Scholes model (see, for instance, Schoutens [2003]). In a Lévy model, the Brownian motion is replaced by a more general Lévy process, taking into account the typical non-normality of asset returns.

Among the Lévy models that are analytically tractable and at the same time fit empirical data well enough are the NIG-, the VG-, and the Meixner-Lévy models. For NIG- and VG-Lévy models, Albrecher and Predota [2002, 2004] observe that Asian option prices in these more realistic models differ significantly from corresponding Black-Scholes prices.

Lévy market models are incomplete, except in the Brownian and the unrealistic Poisson cases. There are many equivalent martingale measures for risk-neutral valuation of derivative securities. Our approach is based on the risk-neutral densities of the distribution of the asset price, and thus works for all equivalent martingale measures that produce tractable numerical estimates of these density functions.

I. STATIC HEDGING STRATEGY FOR ARITHMETIC AVERAGE OPTIONS

We assume throughout an arbitrage-free frictionless market model, which consists of a riskless bond (bank account) and one risky financial asset, a stock or an index. The market dictates that there is a fixed interest rate $r \geq 0$, and that the bond price process behaves (deterministically) as $B = \{B_t = \exp(rt), t \geq 0\}$. The stock price process follows a stochastic process and is denoted by $S = \{S_t, t \geq 0\}$. We assume that the stock pays a continuous compound dividend yield at a rate q per year.

We always work with the natural filtration $\mathbb{F} = \mathbb{F}^S = \{\mathcal{F}_t, 0 \leq t \leq T\}$ of S. Later on, we choose an exponential of a Lévy process for the stock price process, after we develop the theory for a general model.

In an arbitrary arbitrage-free incomplete market model with an equivalent martingale measure Q, the price of a European-style arithmetic average call option with strike price K, maturity T, and n averaging days $0 \leq t_1 < \ldots < t_n \leq T$ at time t is given by:

$$
\begin{aligned}
AA_t &= \exp(-r(T-t))E_Q\left[\left(\frac{\sum_{k=1}^n S_{t_k}}{n} - K\right)^+ \Big| \mathcal{F}_t\right] \\
&= \frac{\exp(-r(T-t))}{n} E_Q\left[\left(\sum_{k=1}^n S_{t_k} - nK\right)^+ \Big| \mathcal{F}_t\right]
\end{aligned}
$$

where r is the risk-free interest rate, S_t is the asset price at time t, and $(x - K)^+$ means max $(x - K, 0)$.

The main difficulty in evaluating this expression is that typically the distribution of the average $\sum_{k=1}^n S_{t_k}/n$, which is a sum of dependent random variables, is not available. We focus on upper bounds based on a portfolio of European options. Assume for simplicity that we are at time $t = 0$ and that the averaging has not yet started.

First, note that for any $K_1, \ldots, K_n \geq 0$ with $K = \sum_{k=1}^n K_k$ we have almost surely

$$
\begin{aligned}
\left(\sum_{k=1}^n S_{t_k} - nK\right)^+ &= \left((S_{t_1} - nK_1) + \cdots + (S_{t_n} - nK_n)\right)^+ \\
&\leq \sum_{k=1}^n (S_{t_k} - nK_k)^+
\end{aligned}
$$

Hence:

$$
\begin{aligned}
AA_0(K,T) &= \frac{\exp(-rT)}{n} E_Q\left[\left(\sum_{k=1}^n S_{t_k} - nK\right)^+ \Big| \mathcal{F}_0\right] \\
&\leq \frac{\exp(-rT)}{n} \sum_{k=1}^n E_Q\left[(S_{t_k} - nK_k)^+ \Big| \mathcal{F}_0\right] \\
&= \frac{\exp(-rT)}{n} \sum_{k=1}^n \exp(rt_k) EC_0(\kappa_k, t_k) \qquad (1)
\end{aligned}
$$

where $EC_0(\kappa_k, t_k)$ denotes the price of a European call option at time 0 with strike $\kappa_k = nK_k$ and maturity t_k.

In terms of hedging this means we have a static superhedging strategy: For each k, buy $\exp(-r(T - t_k))/n$ European call options at time $t = 0$ with strike κ_k and maturity t_k, and hold these until their expiration. Then put their payoff in the bank.

Since relation (1) holds for all combinations of $\kappa_k \geq 0$ that satisfy $\sum_{k=1}^n \kappa_k = nK$, we have a variety of portfolios of n European options whose payoff dominates the Asian option. The simplest choice is $\kappa_k = K$ ($k = 1, \ldots, n$). If $q \leq r$, we have $EC_0(K,t) \leq EC_0(K,T)$ for every $K \geq 0$ and $0 \leq t \leq T$. This trivial choice shows that the Asian option price is dominated by the price of a European option with the same strike and maturity:

$$AA_0(K,T) \leq EC_0(K,T)$$

For superhedging purposes, we naturally look for the combination of κ_k that minimizes the right-hand side of (1). As Dhaene et al. [2002a] show, this optimal combination can be determined by using stop-loss transforms and the theory of comonotonic risks.

We briefly summarize these techniques and adapt them to our setting of general market models. Let $F(x)$ be a distribution function of a non-negative random variable X. Then (in accordance with actuarial practice) its stop-loss transform $\Phi_F(m)$ is defined by:

$$\Phi_F(m) = \int_m^{+\infty} (x-m)\mathrm{d}F(x) = E[(X-m)^+]$$

for $m \geq 0$.

A convex ordering of distribution functions $F(x)$ and $G(x)$ (or equivalently of the corresponding random variables) on the non-negative real line can be defined as follows. $F(x)$ is said to precede $G(x)$ in convex order ($F \leq_{cx} G$), if the corresponding means of the distribution functions (random variables) are equal and:

$$\Phi_F(m) \leq \Phi_G(m) \text{ for all } m \geq 0$$

If we write

$$A_n = \sum_{k=1}^{n} S_{t_k}$$

and $F^t_{A_n}(x) = \mathbb{P}_Q(A_n \leq x \mid \mathcal{F}_t)$ for the distribution function under Q of A_n, given the information $\mathcal{F}_t$, we have

$$AA_t = \frac{\exp(-r(T-t))}{n} \Phi_{F^t_{A_n}}(nK) \tag{2}$$

In this way we transform the problem of pricing an arithmetic average option into calculation of the stop-loss transform of a sum of dependent risks. Concretely, we look at bounds for stop-loss transforms based on comonotonic risks. A positive random vector $(X_1, \ldots, X_n)$ with marginal distribution functions $F_1(x_1), \ldots, F_n(x_n)$ is called *comonotonic* if the joint distribution function $F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \min\{F_1(x_1), \ldots, F_n(x_n)\}$ holds for every $x_1, \ldots, x_n \geq 0$.

It immediately follows that the distribution of a comonotonic random vector $(X_1, \ldots, X_n)$ with given marginal distributions $F_1(x_1), \ldots, F_n(x_n)$ is uniquely determined. A comonotonic random vector $(X_1, \ldots, X_n)$ has the same joint distribution as $(F_1^{-1}(U), \ldots, F_n^{-1}(U))$ where U is a uniform (0, 1) random variable. For a detailed account of comonotonicity and its connection to stochastic orderings, see Dhaene et al. [2002b].

Note that comonotonicity represents the strongest possible positive dependence among the marginals. That is, all components of $(X_1, \ldots, X_n)$ are non-decreasing functions of the same random variable U.

Dhaene et al. [2002b] show that an upper bound for the stop-loss transform of the sum of arbitrary dependent positive random variables $\sum_{k=1}^n X_k$ with marginal distributions $F_1(x_1), \ldots, F_n(x_n)$ is given by the stop-loss transform of the sum $S^c = \sum_{k=1}^n Y_k$, where $(Y_1, \ldots, Y_k)$ is the comonotonic random vector with marginal distributions $F_1(x_1), \ldots, F_n(x_n)$:

$$\sum_{k=1}^{n} X_k \leq_{\mathrm{cx}} \sum_{k=1}^{n} Y_k$$

Let $F_{S^c}(x)$ denote the distribution function of $\sum_{k=1}^n Y_k$. Then we have the relation for its inverse:

$$F_{S^c}^{-1}(x) = \sum_{k=1}^{n} F_{X_k}^{-1}(x) \tag{3}$$

for $x \geq 0$.

From Theorem 6 in Simon, Goovaerts, and Dhaene [2000], it follows that the stop-loss transform of a sum of comonotonic random variables can be obtained as a sum of the stop-loss transforms of the marginals evaluated at specified points:

$$\Phi_{F_{S^c}}(m) = \sum_{k=1}^{n} \Phi_{F_{X_k}}\left(F_{X_k}^{-1}(F_{S^c}(m))\right), \quad m \geq 0 \tag{4}$$

given that the marginal distribution functions involved are strictly increasing (which will always be the case in our applications).

At the same time, we have:

$$\Phi_{F_{S^c}}(m) = E\Big(\Big(\sum_{k=1}^{n} Y_k - m\Big)^+\Big) \leq \sum_{k=1}^{n} E\left((Y_k - m_k)^+\right)$$

$$= \sum_{k=1}^{n} \Phi_{F_{X_k}}(m_k) \tag{5}$$

whenever $\sum_{k=1}^n m_k = m$.

Thus the stop-loss transform of the comonotonic sum given by (4) represents at the same time the lowest possible bound in terms of a sum of stop-loss transforms of the marginal distributions.

We apply this result to our setting of an arithmetic Asian option. Let $F(x_k; t_k)$ ($k = 1, \dots, n$) denote the conditional distribution of S_{t_k} under the risk-neutral measure Q (given the information available at time $t = 0$). That is, for x_k, $t_k > 0$:

$$F(x_k; t_k) \quad = \quad \mathbb{P}_Q\left(S_{t_k} \leq x_k | \mathcal{F}_0\right) \tag{6}$$

Combining (1), (2), (4), and (5), we have the optimal combination of strike prices κ_k:

$$\kappa_k = F^{-1}\left(F_{S^c}(n\,K); t_k\right), \quad k = 1, \dots, n \tag{7}$$

We have obtained the optimal static superhedge in terms of European call options with maturity dates equal to the averaging dates.

For practical determination of the strike prices κ_k, the distribution function of the comonotonic sum $F_{S^c}(x)$ as given by (3) must be calculated and evaluated at nK (note that the marginal distribution functions are strictly increasing and continuous). If the risk-neutral density (or an approximation of it) is available, this can be done numerically in a straightforward way. The κ_k are then obtained by evaluating the inverse distribution function of $F(x; t_k)$.

II. LÉVY MARKET MODEL

Suppose $\phi(u)$ is the characteristic function of a distribution. If for every positive integer n, $\phi(u)$ is also the n-th power of a characteristic function, we say that the distribution is infinitely divisible.

One can define for every such infinitely divisible distribution a stochastic process, $X = \{X_t, t \geq 0\}$, called a Lévy process, which starts at zero and has independent and stationary increments so that the distribution of an increment over $[s, s + t]$, $s, t \geq 0$, i.e., $X_{t+s} - X_s$, has $(\phi(u))^t$ as its characteristic function.

Every Lévy process has a modification with sample paths that are almost everywhere continuous from the right and limited from the left (*càdlàg*) and this modification is itself a Lévy process. We work with this càdlàg version of the process. The cumulant characteristic function $\psi(u) = \log\phi(u)$ is often called the *characteristic exponent* (see, e.g., Bertoin [1996]).

We assume the market consists of one riskless asset (the bond) with the price process given by $B_t = \exp(rt)$ and one risky asset (the stock or index). The risk-neutral model for the risky asset is given by

$$S_t = S_0 \frac{\exp((r-q)t)}{E[\exp(X_t)]} \exp(X_t)$$

The factor $\exp((r - q)t / E[\exp(X_t)]$ puts us immediately in a risk-neutral setting by a mean correcting argument.

Note that the argument underlying the choice of a risk-neutral measure is consistent with the classic risk-neutral mean-correcting technique used in a Black-Scholes setting. We should stress, however, that our proposed hedging strategies are not restricted to this particular choice of a risk-neutral density.

In this case, we have for (6):

$$\begin{aligned} F(x_k; t_k) &= \mathbb{P}_Q\left(S_{t_k} \leq x_k | \mathcal{F}_0\right) && (8) \\ &= \mathbb{P}_Q\left(S_0 \frac{\exp((r-q)t)}{E[\exp(X_t)]} \exp(X_{t_k}) \leq x_k | \mathcal{F}_0\right) && (9) \\ &= \mathbb{P}_Q\left(X_{t_k} \leq \log(x_k/S_0) + \psi(-\mathrm{i}) - (r-q)t \middle| \mathcal{F}_0\right) && (10) \end{aligned}$$

There are three popular Lévy processes often used in the modeling of financial assets: the variance gamma (VG) process, the normal inverse Gaussian (NIG) process, and the Meixner process. To obtain the price $EC(K,T)$ of a European call option with strike K and time to maturity T under these models, one can use the Carr and Madan [1998] formula, which is formulated in terms of the characteristic function of the underlying Lévy process. Let α be a positive constant such that the α-th moment of the stock price exists (typically a value of $\alpha = 0.75$ will suffice). Then:

$$EC(K,T) = \frac{\exp(-\alpha \log(K))}{\pi} \int_0^{+\infty} \exp(-\mathrm{i}v \log(K)) \varrho(v) \mathrm{d}v \tag{11}$$

where

$$\varrho(v) \quad = \quad \frac{\exp(-rT) E[\exp(\mathrm{i}(v - (\alpha+1)\mathrm{i}) \log(S_T))]}{\alpha^2 + \alpha - v^2 + \mathrm{i}(2\alpha+1)v} \tag{12}$$

$$= \frac{\exp(-rT)\phi(v-(\alpha+1)\mathrm{i})}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v} \tag{13}$$

The fast Fourier transform can be used to invert the generalized Fourier transform of the call price. Using Equation (11), one can typically calculate the complete option surface over all strikes and maturities in a fraction of a second.

Variance Gamma Process

The VG(C, G, M) law has a characteristic function of the form:

$$\phi_{VG}(u; C, G, M) = \left(\frac{GM}{GM+(M-G)\mathrm{i}u+u^2}\right)^C$$

Its density function is given by:

$$f_{\mathrm{VG}}(x; C, G, M)(x) = \frac{(GM)^C}{\sqrt{\pi}\Gamma(C)} \exp\left(\frac{(G-M)x}{2}\right) \times \left(\frac{|x|}{G+M}\right)^{C-1/2} \times K_{C-1/2}((G+M)|x|/2)$$

where C, G, $M > 0$, $\Gamma(x)$ denotes the gamma function, and $K_\nu(x)$ denotes the modified Bessel function of the third kind with index ν. This distribution is infinitely divisible and has the convolution property: $\phi_{VG}(u; C, G, M) = \phi_{VG}(u; C/n, G, M))^n$.

Thus one can define the VG process $X^{(VG)} = \{X_t^{(\mathrm{VG})}, t \geq 0\}$ as a process that starts at zero and has independent and stationary increments and where the increment $X_{s+t}^{(VG)} - X_s^{(VG)}$ over the time interval $[s, t+s]$ follows a VG(Ct, G, M) law. Sometimes another parameterization of the VG distribution is used (see, e.g., Schoutens [2003]).

The class of variance gamma distributions as a model for stock returns was developed by Madan and Seneta [1990] (where the symmetric case $G = M$ is considered); see also Madan and Milne [1991]. Madan, Carr, and Chang [1998], discuss the general case with skewness.

Normal Inverse Gaussian Process

The normal inverse Gaussian (NIG) distribution with parameters $\alpha > 0, |\beta| < \alpha$, and $\delta > 0$ has a characteristic function given by:

$$\phi_{NIG}(u; \alpha, \beta, \delta) = \exp\left(-\delta\left(\sqrt{\alpha^2-(\beta+\mathrm{i}u)^2} - \sqrt{\alpha^2-\beta^2}\right)\right)$$

One can see that this is an infinitely divisible characteristic function with $\phi_{NIG}(u; \alpha, \beta, \delta) = (\phi_{NIG}(u; \alpha, \beta, \delta/n))^n$. Hence we can define the NIG process $X^{(NIG)} = \{X_t^{(NIG)}, t \geq 0\}$, with $X_0^{(NIG)} = 0$, stationary and independent NIG distributed increments:

To be precise, $X_t^{(NIG)}$ has a NIG(α, β, $t\delta$) law. The density of the NIG(α, β, δ) distribution is given by:

$$f_{NIG}(x; \alpha, \beta, \delta) = \frac{\alpha\delta}{\pi} \exp\left(\delta\sqrt{\alpha^2-\beta^2}+\beta x\right) \times \frac{K_1(\alpha\sqrt{\delta^2+x^2})}{\sqrt{\delta^2+x^2}}$$

The NIG distribution was developed by Barndorff-Nielsen [1995]. See also Rydberg [1997].

Meixner Process

The density of the Meixner distribution is given by

$$f_{Meixner}(x; \alpha, \beta, \delta) = \frac{(2\cos(\beta/2))^{2\delta}}{2\alpha\pi\Gamma(2d)} \exp\left(\frac{bx}{a}\right)\left|\Gamma\left(\delta+\frac{\mathrm{i}x}{\alpha}\right)\right|^2$$

where $\alpha > 0$, $-\pi < \beta < \pi$, and $\delta > 0$.

The characteristic function of the Meixner(α, β, δ) distribution is given by:

$$\phi_{Meixner}(u; \alpha, \beta, \delta) = \left(\frac{\cos(\beta/2)}{\cosh\frac{\alpha u - \mathrm{i}\beta}{2}}\right)^{2\delta}$$

The Meixner(α, β, δ) distribution is infinitely divisible: $\phi_{Meixner}(u; \alpha, \beta, \delta) = (\phi_{Meixner}(u; \alpha, \beta, \delta/n))^n$. It thus generates a Lévy process that we call the Meixner process. More precisely, a Meixner process $X^{(Meixner)} = \{X_t^{(Meixner)}, t \geq 0\}$ is a stochastic process that starts at zero, i.e., $X_0^{(Meixner)} = 0$, and has independent and stationary increments, and where the distribution of $X_t^{(Meixner)}$ is given by the Meixner distribution Meixner(α, β, δt).

The Meixner process was developed in Schoutens and Teugels [1998] and extended to fitting stock returns in Grigelionis [1999]. This application in finance is worked out in Schoutens [2002].

EXHIBIT 1
Lévy Models (mean correcting): Parameter Estimation (S&P 500)

Model	Parameters		
VG	C	G	M
	1.3574	5.8704	14.2699
NIG	α	β	δ
	6.1882	-3.8941	0.1622
Meixner	α	β	δ
	0.3977	-1.4940	0.3462
Black-Scholes	σ		
	0.1812		

EXHIBIT 2
Strike Prices κ_k for Hedge Portfolio (monthly averaging)

$100K/S_0$	t_k	VG model	NIG model	Meixner model	BS model
80	0.083	92.82	94.87	95.14	91.26
	0.167	89.23	90.66	90.65	87.84
	0.250	86.39	87.23	87.03	85.29
	0.333	83.59	84.34	84.07	83.18
	0.417	81.42	81.84	81.57	81.36
	0.500	79.54	79.62	79.41	79.75
	0.583	77.87	77.64	77.51	78.28
	0.667	76.37	75.83	75.80	76.94
	0.750	75.00	74.18	74.24	75.70
	0.833	73.74	72.65	72.82	74.54
	0.917	72.56	71.23	71.50	73.45
	1.000	71.46	69.90	70.27	72.42
90	0.083	98.36	98.23	98.50	95.86
	0.167	96.56	96.40	96.71	94.16
	0.250	94.86	94.67	94.90	92.86
	0.333	92.88	93.06	93.19	91.77
	0.417	91.30	91.56	91.59	90.81
	0.500	89.91	90.17	90.12	89.94
	0.583	88.64	88.86	88.75	89.14
	0.667	87.49	87.62	87.48	88.41
	0.750	86.43	86.46	86.29	87.71
	0.833	85.43	85.36	85.18	87.06
	0.917	84.50	84.31	84.13	86.44
	1.000	83.62	83.31	83.14	85.85
100	0.083	100.84	100.33	100.38	100.12
	0.167	101.33	100.48	100.57	100.13
	0.250	101.49	100.51	100.63	100.12
	0.333	101.11	100.47	100.59	100.10
	0.417	100.75	100.38	100.48	100.07
	0.500	100.36	100.25	100.32	100.04
	0.583	99.96	100.09	100.12	100.01
	0.667	99.57	99.91	99.90	99.97
	0.750	99.19	99.72	99.65	99.93
	0.833	98.82	99.51	99.39	99.88
	0.917	98.46	99.29	99.13	99.84
	1.000	98.11	99.06	98.85	99.79

III. NUMERICAL RESULTS

We illustrate the performance of the static hedge portfolio for the Lévy market models applied to a liquid market. For completeness, we also include the Black-Scholes model. We calibrate our model parameters to the set of vanilla options on the S&P 500 as given in Schoutens [2003, Appendix C]. The yearly risk-free interest rate and the dividend rate are given by $r = 0.019$ and $q = 0.012$, respectively.

Exhibit 1 provides results of the calibration in the least squared sense, that is, with the minimal value of:

$$\text{lse} = \sum_{options} (\text{Market price} - \text{Model price})^2$$

We examine an arithmetic Asian call option with a maturity of one year and averaging every month (i.e., 12 averaging days) and every week (i.e., 52 averaging days). To set up our hedge portfolio, we have to determine the inverse distribution function of the asset price at these 12 and 52 days.

We do this by discretizing the real line in an appropriate range and numerically building up the distribution function from the density function. The inverse is then found by a bisection method from the table using linear interpolation between grid points. It turns out that 40,000 points in the grid are sufficient (in the sense that a further increase does not change the significant digits of the results).

Next, the inverse of the distribution of the comonotonic sum is built up according to (3) and then itself inverted. Finally, the strike prices κ_k of the European options are obtained by evaluating the inverse distribution functions of the marginals according to (7).

For the models discussed, this numerical procedure to obtain the strike prices for our hedging strategy is both accurate and very quick. For monthly averaging, it takes less than a minute on a standard PC to determine the entire hedge portfolio.

Exhibit 2 shows the strike prices as a percentage of the spot price for this example with monthly averaging under various models calibrated to the S&P 500 (all numbers are rounded to their last digit). Note that the optimal strike prices are little dif-

Exhibit 2 (continued)
Strike Prices κ_k for Hedge Portfolio (monthly averaging)

$100K/S_0$	t_k	VG model	NIG model	Meixner model	BS model
110	0.083	101.87	102.35	102.24	104.09
	0.167	103.61	104.21	104.09	105.80
	0.250	105.24	105.83	105.72	107.11
	0.333	106.93	107.28	107.19	108.21
	0.417	108.42	108.62	108.56	109.18
	0.500	109.81	109.86	109.84	110.05
	0.583	111.12	111.03	111.04	110.86
	0.667	112.36	112.14	112.18	111.61
	0.750	113.53	113.20	113.27	112.31
	0.833	114.65	114.20	114.31	112.97
	0.917	115.72	115.17	115.30	113.60
	1.000	116.74	116.10	116.26	114.21
120	0.083	106.63	106.05	105.86	107.83
	0.167	109.81	109.68	109.55	111.21
	0.250	112.52	112.61	112.54	113.85
	0.333	115.20	115.18	115.16	116.11
	0.417	117.54	117.53	117.54	118.13
	0.500	119.71	119.73	119.75	119.98
	0.583	121.76	121.80	121.84	121.70
	0.667	123.72	123.78	123.84	123.31
	0.750	125.60	125.69	125.75	124.84
	0.833	127.42	127.54	127.61	126.30
	0.917	129.19	129.33	129.40	127.70
	1.000	130.91	131.07	131.15	129.04

ferent for the various Lévy models, but the Black-Scholes model calibrated to the same data produces strike prices that are significantly closer to the strike of the Asian option.

The price of the hedging strategy is then easily determined using the Carr and Madan European call option pricing formula and (1). Exhibits 3-6 compare the Monte Carlo simulated price of the Asian option AA_{MC} and the comonotonic superhedge price AA_c with the prices of two simple superhedging strategies: the trivial superhedge using the European option price EC with identical strike and maturity (note $q \leq r$), and the superhedge (1) with all $\kappa_i = K$ with price AA_{tr}.

For the Monte Carlo price, we use 1 million simulated paths. The VG process is simulated as a difference of two gamma processes (see Schoutens [2003, Section 8.4.2]). NIG paths are obtained as described in Schoutens [2003, Section 8.4.5], and Meixner paths are obtained by a compound Poisson approximation as described in Schoutens [2003, Section 8.2.1].

In Exhibits 3-6, we can see that the more in the money the Asian option is, the less the difference between the option price and the comonotonic hedge. For an option with moneyness of 80%, the difference is typically around 1.5%. For the classic hedge with the European call the difference is almost 10%.

For options out of the money, the difference increases, but it is still substantially smaller than the differences for the other two simple hedges. In view of its easy and cost-effective implementation, this comonotonic approach seems to be competitive in these cases also.

Exhibits 7-10 give the corresponding results for the same Asian options with weekly averaging. This lets us investigate the sensitivity of the hedging performance to the number of sampling points.

In principle, we would expect the relative difference between the price of the option and the price of the comonotonic hedge to increase for an increasing number of sampling points (as we limit each additional component by a comonotonic dependence structure). Yet, as Exhibits 7-10 show, this effect is not very pronounced for the values in this numerical example, indicating that the strategy is quite robust with respect to number of sampling points.

Note that the Black-Scholes model underestimates the prices of Asian call options for in-the-money options and overestimates them for out-of-the-money options. The comonotonic hedge, however, turns out to perform in a similar way with regard to the moneyness of the option.

IV. CONCLUSION

The pricing of exotic derivatives rests in general on rather weak foundations. Schoutens, Simons, and Tistaert [2004] note that calibration of a variety of market models may produce widely differing prices of exotic options, which indicates that obtaining concrete superhedging strategies is of utmost importance.

We have shown that static hedging of an Asian option in terms of a portfolio of European options is a simple and quick alternative to current approaches. Unlike most current techniques, this approach is applicable in general market models whenever we have the risk-neutral density of the asset price distribution or an approximation of it. A static hedging strategy is much less sensitive to the assumption of zero transaction costs and to hedging performance in the presence of large market movements; no dynamic rebalancing is required. These advantages may sometimes offset the difference between the hedging price and the option price, even for OTM Asian options.

Our method is not restricted to Asian options. It works whenever one is faced with a payoff defined as a sum of dependent asset prices (such as basket options). From the structure of the technique, one can see that the stronger the dependence of the involved asset prices, the better the performance.

ENDNOTE

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EXHIBIT 3

Black-Scholes Option Prices as Percentage of Spot (monthly averaging)

$100K/S_0$	AA_{MC}	AA_c	AA_{tr}	EC
80	20.0727	20.1523	20.2756	21.0819
90	11.0662	11.4577	11.5952	13.2550
100	4.5532	5.1833	5.1835	7.4552
110	1.3491	1.8359	1.9573	3.7642
120	0.2944	0.5174	0.6952	1.7243

EXHIBIT 4

VG Option Prices as Percentage of Spot (monthly averaging)

$100K/S_0$	AA_{MC}	AA_c	AA_{tr}	EC
80	20.5233	20.7895	20.9331	22.0739
90	11.7384	12.1649	12.3462	14.2015
100	4.5979	5.0555	5.0764	7.7732
110	0.9585	1.2261	1.5090	3.3712
120	0.2108	0.3364	0.4824	1.2554

EXHIBIT 5

NIG Option Prices as Percentage of Spot (monthly averaging)

$100K/S_0$	AA_{MC}	AA_c	AA_{tr}	EC
80	20.6067	20.9335	21.0906	22.3345
90	11.7500	12.2184	12.3885	14.3309
100	4.4899	5.0184	5.0223	7.7433
110	0.9208	1.2477	1.5039	3.3441
120	0.1865	0.3149	0.4660	1.2381

EXHIBIT 6

Meixner Option Prices as Percentage of Spot (monthly averaging)

$100K/S_0$	AA_{MC}	AA_c	AA_{tr}	EC
80	20.7128	20.8870	21.0459	22.2530
90	11.8590	12.2050	12.3861	14.3029
100	4.5133	5.0147	5.0204	7.7499
110	0.8768	1.2471	1.5085	3.3476
120	0.1961	0.3382	0.4862	1.2601

EXHIBIT 7

Black-Scholes Option Prices as Percentage of Spot (weekly averaging)

$100K/S_0$	AA_{MC}	AA_c	AA_{tr}	EC
80	20.0198	20.0964	20.2280	21.0819
90	10.9226	11.3088	11.4833	13.2550
100	4.3373	4.9738	4.9741	7.4552
110	1.1970	1.6747	1.8345	3.7642
120	0.2342	0.4390	0.6384	1.7243

EXHIBIT 8

VG Option Prices as Percentage of Spot (weekly averaging)

$100K/S_0$	AA_{MC}	AA_c	AA_{tr}	EC
80	20.3915	20.6882	20.8543	22.0739
90	11.5263	11.9870	12.2064	14.2015
100	4.3138	4.8178	4.8365	7.7732
110	0.8012	1.0884	1.3991	3.3712
120	0.1651	0.2905	0.4409	1.2554

EXHIBIT 9

NIG Option Prices as Percentage of Spot (weekly averaging)

$100K/S_0$	AA_{MC}	AA_c	AA_{tr}	EC
80	20.5274	20.8217	21.0036	22.3345
90	11.6063	12.0419	12.2443	14.3309
100	4.2758	4.7829	4.7861	7.7433
110	0.8010	1.1060	1.3948	3.3441
120	0.1543	0.2696	0.4250	1.2381

EXHIBIT 10

Meixner Option Prices as Percentage of Spot (weekly averaging)

$100K/S_0$	AA_{MC}	AA_c	AA_{tr}	EC
80	20.3915	20.7760	20.9615	22.2530
90	11.5263	12.0285	12.2430	14.3029
100	4.3138	4.7785	4.7834	7.7499
110	0.8012	1.1071	1.3993	3.3476
120	0.1651	0.2914	0.4446	1.2601

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