Optimal Calibration of LIBOR Market Models to Correlations

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Here is a simple, fast, and non-parametric method for calibrating LIBOR market models to historical or implied correlation matrices. For a given symmetric matrix, the method uses alternating projections to find the nearest correlation matrix of a lower rank.

This short article addresses the problem of calibrating the market models developed by Brace, Gatarek, and Musiela [1997] and Miltersen, Sandmann, and Sondermann [1997] to correlation matrices. Correlation matrices can be either historical or implied by the option markets, but they usually exhibit a high-factor structure that hinders efficient use of market models.

I describe a simple algorithm that finds the closest correlation matrix for any number of factors. The algorithm provides an efficient calibration of low-factor market models to the observed correlation matrices.

The correlation matrices in LIBOR market models appear in two ways: first, as the historical correlation of instantaneous changes in the LIBOR, and second, as the correlation of instantaneous changes of the LIBOR implied by derivatives data, usually swaption prices.[1] This implied correlation matrix can be obtained by first fitting the model using some parametric correlation function, and then evaluating it for a given set of forward rates.

For a given set of n LIBOR, the associated instantaneous correlation matrix usually has rank n. To reproduce this correlation matrix, the implementation of the LIBOR market model will require Brownian motions. To reduce the dimensionality of the problem, one could implement the model using only k Brownian motions, with $k < n$. In this case, however, one needs to choose a rank k, $n \times n$, correlation matrix that is closest in some sense to the original rank, n, $n \times n$ correlation matrix.

We define the sets:

$$S = \{Y = Y^T \in \mathbb{R}^{n\times n} : Y \geq 0\}$$

$$U = \{Y = Y^T \in \mathbb{R}^{n\times n} : y_{ii} = 1, i = 1 : n\}$$

$$S_k = \{Y = Y^T \in \mathbb{R}^{n\times n} : \mathrm{Rank}(Y) \leq k, k \in \mathbb{N}, k \leq n\}$$

For a symmetric Y, the notation $Y \geq 0$ means Y is positive semidefinite.

The problem we consider here can be stated as follows. For arbitrary symmetric $A \in \mathbb{R}^{n\times n}$, find a matrix $X \in S \cap U \cap S_k$. That is, find a correlation matrix X of rank less or equal k minimizing the distance

$$\gamma(A) = \min\{\|A - X\|_F\} \quad (1)$$

The norm is the Frobenius norm, $\|A\|_F^2 := \Sigma_{i,j} a_{ij}^2$.[2]

Rebonato [1999] suggests a procedure for constructing a correlation matrix for the calibration of LIBOR market models. He approximates the implied correlation matrix $\hat{C}$ by a matrix of the form BB^T, where the matrix B is constructed via the algorithm:

EXHIBIT 1
Alternating Projection on Closed Subsets of $\mathbb{R}^2$

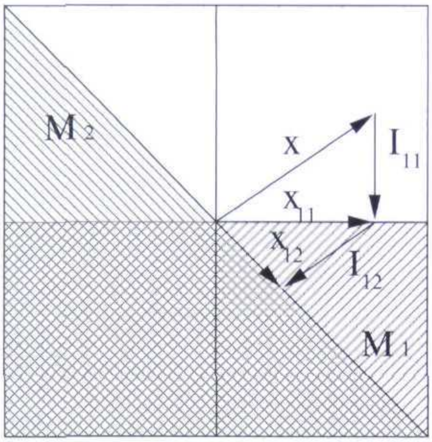
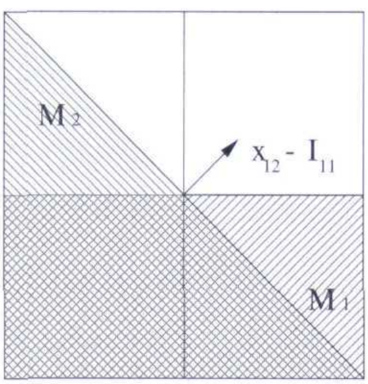


EXHIBIT 2
Effect of Dykstra's Correction



$$b_{ij} = \cos\theta_{ij} \prod_{k=1}^{j-1} \sin\theta_{ik} \quad \text{for } j = 1, \ldots, k-1$$
$$b_{ij} = \prod_{k=1}^{j-1} \sin\theta_{ik} \quad \text{for } j = k$$

for an arbitrary set of angles θ_{ij}. The matrix BB^T is a correlation matrix of rank k by construction. To find the closest correlation matrix of this form, one needs to minimize the distance function $\gamma(A)$ in (1) over the set of $n \times k$ angles θ_{ij}. The computation time to calibrate a low number n of LIBOR may be acceptable, but for a large number of LIBOR this non-linear minimization can be difficult to handle.

Zhang and Wu [2003] approach the problem using a Lagrange multiplier approach.[3] They have rigorously justified the convergence of the Lagrange multiplier method for the low-rank approximation of correlation matrices.

Our approach is more simple to understand and to implement. The iterative algorithm involves only projections on the sets S, K, and S_k, with the optimal correlation matrix of rank as its limit. Implementation of the algorithm takes only a few lines of code.

I. ALTERNATING PROJECTIONS

Our method is based on a procedure generally known as the *method of alternating projections*. This is an iterative scheme for finding the best approximation to any given point in a Hilbert space from the intersection of a finite collection of closed subspaces. This result was originally proved by von Neumann [1950], and then independently by Wiener [1955] for the case of two subspaces. It has been generalized to the case of several subspaces by Halperin [1962]. In our case, the sets under consideration are not subspaces, so there is no guarantee that the alternating projection method will converge.

To illustrate this point, we borrow an example from Han [1988]. Consider the space $\mathbb{R}^2$ equipped with the Euclidean inner product. Let $M_1 := \{(\zeta_1, \zeta_2) \mid \zeta_2 \leq 0\}$ and $M_1 := \{(\zeta_1, \zeta_2) \mid \zeta_1 + \zeta_2 \leq 0\}$. The standard projection method does not work for any vector x outside M_1 and M_2. This is demonstrated in Exhibit 1.

The graph represents application of the standard projection method to the vector x. The vector $x_{12} = P_{M_2} P_{M_1}$ is clearly suboptimal. The iterative projection terminates at x_{12} since the vector is in the intersection of M_1 and M_2.

A generalization of the method to convex closed sets has been given by Dykstra [1983] and Boyle and Dykstra [1985].[4] Their algorithm converges strongly to the nearest point in the intersection. Dykstra's idea is to introduce a correction procedure for pathological cases such as the Exhibit 1 example. It consists of subtracting a correction term before the projection. One obtains this term from the previous run as a difference between the starting value and the projection on the first set.

Exhibit 2 represents the effect of correction by I_{11}. The vector x_{12} is effectively taken out of the intersection $M_1 \cap M_2$, and the iterative procedure will proceed with the limit as the projection x of onto $M_1 \cap M_2$.

Next, we identify the projections of a symmetric matrix A, $P_S(A)$, $P_{S_k}(A)$, and $P_U(A)$, on the sets S, S_k, and U. The projection of a symmetric matrix A on the set S of symmetric positive semidefinite matrices is

$$P_S(A) = \Pi \text{diag}(\max(\lambda_i, 0))\Pi^T \tag{2}$$

where $A = \Pi\Lambda\Pi^T$ is a spectral decomposition, with Π an orthogonal matrix of eigenvectors and $\Lambda = \text{diag}(\lambda_i)$ a diag-

EXHIBIT 3
Model Correlation Matrix Resulting from Function $\alpha + (1-\alpha)\exp(\beta|t_i - t_j|)$

	1	2	3	4	5	6	7	8	9	10	11
2	0.958										
3	0.916	0.957									
4	0.876	0.915	0.956								
5	0.837	0.874	0.913	0.955							
6	0.799	0.834	0.872	0.912	0.954						
7	0.763	0.796	0.832	0.870	0.910	0.953					
8	0.729	0.760	0.793	0.829	0.867	0.909	0.953				
9	0.696	0.725	0.756	0.790	0.826	0.865	0.907	0.952			
10	0.665	0.692	0.721	0.753	0.787	0.824	0.863	0.906	0.951		
11	0.635	0.660	0.688	0.718	0.750	0.784	0.821	0.861	0.904	0.950	
12	0.607	0.631	0.656	0.684	0.714	0.746	0.781	0.819	0.859	0.902	0.949

onal matrix with eigenvalues on the diagonal. This result is derived by Higham [1988].[5]

The projection of a general matrix A on the set of matrices with a lower rank can be found using singular value decomposition.[6] When the matrix A is symmetric, the projection P_{S_k} is achieved by setting the eigenvalues in the spectral decomposition of A to zero, with the exception of the first k:

$$P_{S_k}(A) = \Pi \mathrm{diag}(\max(1_{k<i}\lambda_i, 0))\Pi^T \tag{3}$$

where $A = \Pi\Lambda\Pi^T$ is again a spectral decomposition, and $1_{k<i}$ is the standard indicator function.

The projection $P_U(A)$ of a symmetric matrix A on the set of symmetric matrices with unit diagonal U is simply

$$P_U(A) = \begin{cases} a_{ij} & i \neq j \\ 1, & i = j \end{cases}$$

That is, we set the diagonal entries of A to 1.

The algorithm is:

$\Delta I_0 = 0$, $Z_0 = A$ for $n = 1, 2, \ldots$
$R_n = Z_{n-1} - \Delta I_{n-1}$, w
where ΔI_{n-1} is Dykstra's correction

$$X_n = P_S(R_n)$$

$$\Delta I_n = X_n - R_n$$

$$Y_n = P_{S_k}(X_n)$$

$$Z_n = P_U(Y_n)$$

end

One could use the relative changes in one of the projections as a stopping criterion.

II. NUMERICAL RESULTS

For numerical analysis of the method, we use the examples in Zhang and Wu [2003]. The first example is based on the correlation matrix C, generated by the correlation function as suggested by Rebonato [1999]. The entries of the matrix C are defined as:

$$c_{ij} = \alpha + (1-\alpha)\exp(\beta|t_i - t_j|)$$

$$\beta = d_1 - d_2 \max(t_i, t_j)$$

with parameters $\alpha = 0.3$, $d_1 = -0.12$, and $d_2 = 0.005$.[7]

Exhibit 3 presents the values of the correlation matrix C resulting from this choice of parameters. We plot the corresponding distance function $\gamma(C)$ in Exhibit 4. The eigenvectors of the rank-three approximation together with the first three eigenvectors of the model correlation are plotted in Exhibit 5. The approximating correlation matrices for several ranks together with the model correlation matrix C are plotted in Exhibits 6 through 11.

We see that approximations take on the shape of the model correlation quite quickly. The approximation at the diagonal, however, becomes reasonable only for a high number of factors, which is unavoidable because of smoothness of the first principal components.

We take the correlation matrix for the second example in Exhibit 12 from Brace, Gatarek, and Musiela [1997], who give the market forward rate correlations for the British pound in 1995. As in the first example, we plot

EXHIBIT 4
Convergence to Model Correlation Matrix Measured in Frobenius Norm

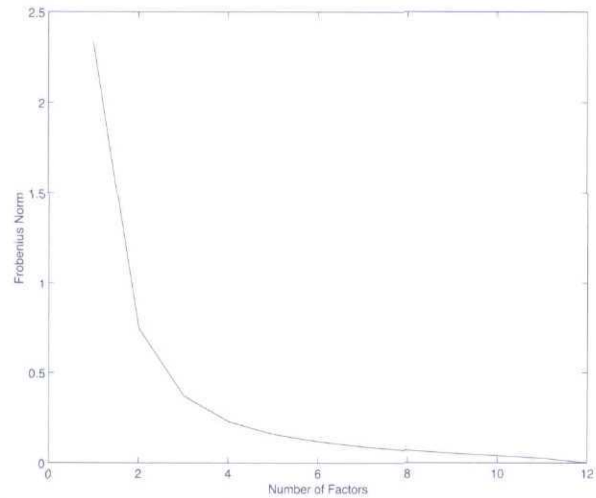
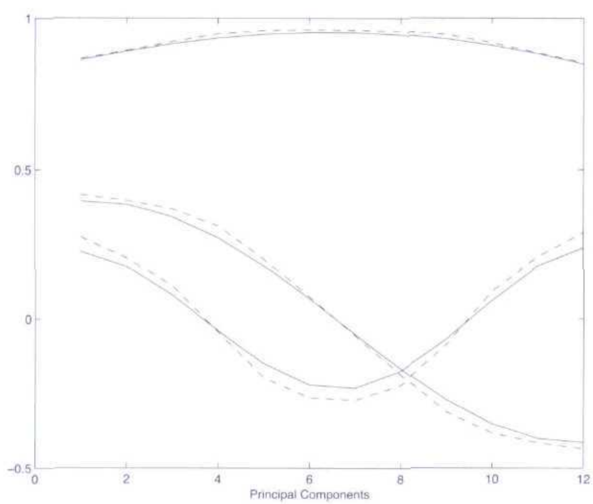


EXHIBIT 5
First Three Principal Components



the distance function γ for an increasing number of factors in the approximating correlation matrix in Exhibit 13. The eigenvectors of the rank-three approximation matrix together with the first three eigenvectors of the market correlation matrix are plotted in Exhibit 14. We plot the approximating correlation matrices for several ranks together with the market correlation matrix in Exhibits 15 through 20.

The shape of the market correlation matrix is not as well-behaved as in the first example. To capture its basic shape, one needs an approximation of at least rank six. Even

EXHIBIT 6
Rank-One Approximation

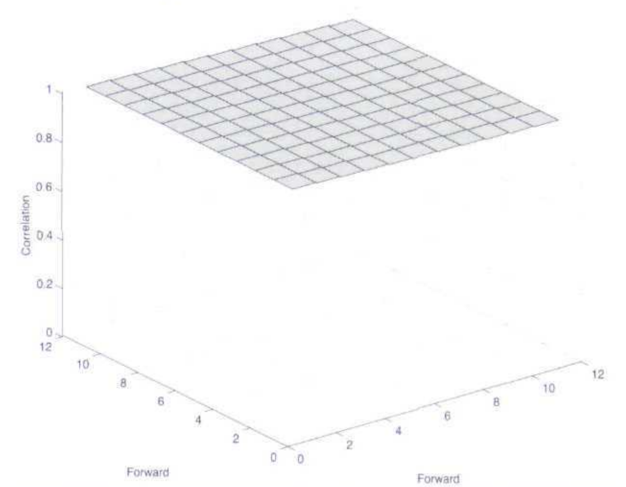


EXHIBIT 7
Rank-Three Approximation

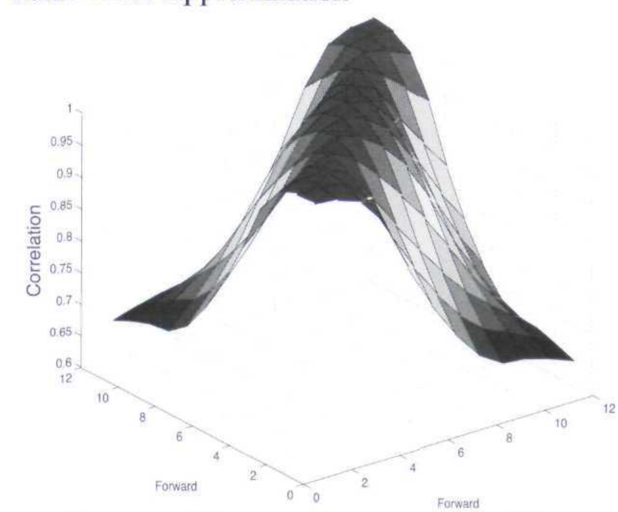


EXHIBIT 8
Rank-Five Approximation

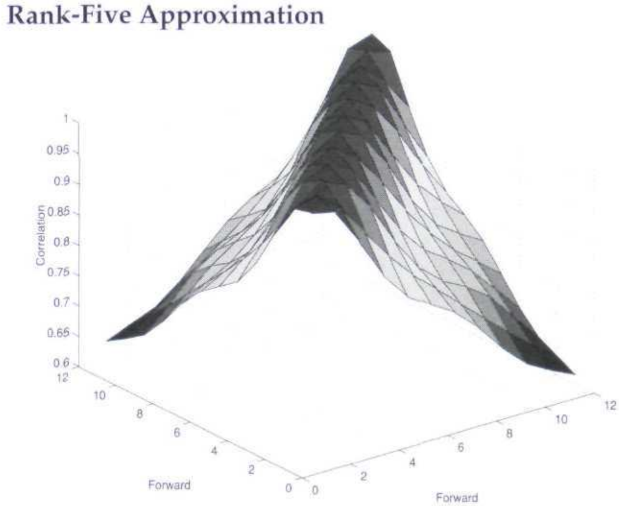


EXHIBIT 9
Rank-Seven Approximation

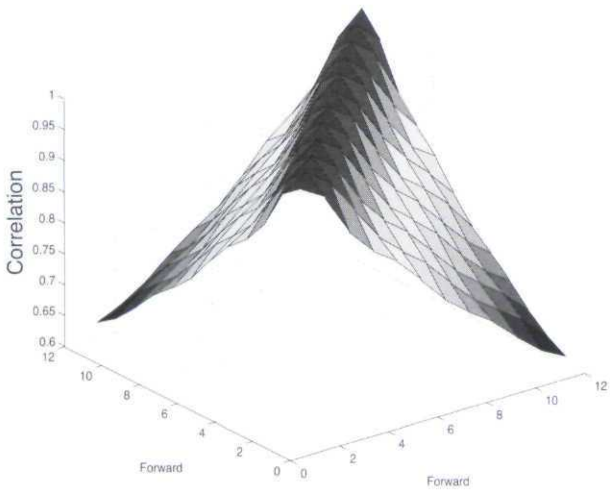
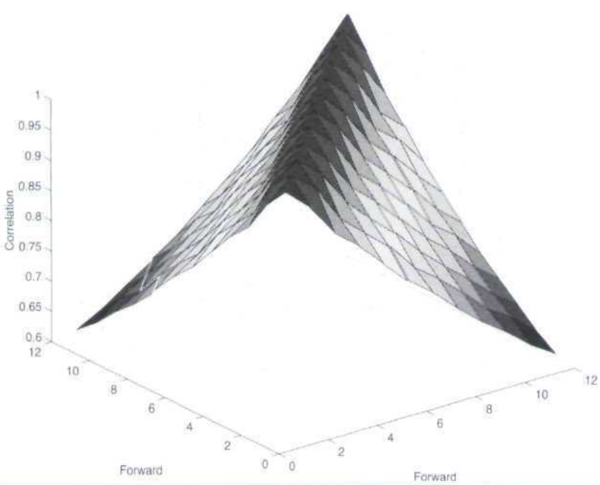


EXHIBIT 10
Rank-Nine Approximation

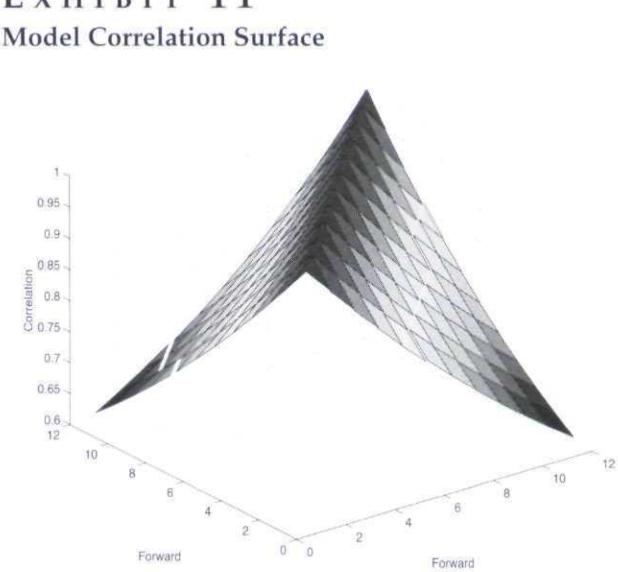


for a non-optimal implementation, the computation time for the correlation approximations is a fraction of a second.

III. CONCLUSIONS

We have demonstrated a simple, fast, and non-parametric method for calibrating LIBOR market models to historical or implied correlation matrices. The method is based on alternating projections of the solution on the set of all positive semidefinite matrices, the set of matrices of rank k, and the set of matrices with diagonal one. This iterative algorithm converges to a correlation matrix of rank k in the intersection of these three sets.

EXHIBIT 11
Model Correlation Surface



Implementation of the algorithm is simple and requires only a few lines of code. Tests of the method on a model correlation matrix and a historical market correlation matrix indicate that convergence is achieved in a fraction of a second in both cases.

ENDNOTES

The author thanks Riccardo Rebonato for comments on an earlier version of this article.

[1]Brigo and Mercurio [2001] and Rebonato [2002] describe LIBOR modeling in detail.

[2]Other norms can be considered as well, such as the two-norm, $\|A\| = \rho\sqrt{A^TA}$, where the spectral radius $\rho(B) = \max\{|\lambda|: \det(B - \lambda I) = 0\}$. Halmos [1972] shows that positive approximants of the Frobenius and 2-norms are the same when A is symmetric.

[3]Wu [2003] applies this method to calibration of the LIBOR market model.

[4]Bregman [1965] also considers such a generalization. His algorithm, however, converges only weakly to a point in the intersection, not necessarily to the best approximation.

[5]Our method is closely related to the algorithm developed by Higham [2002], who considers the problem of finding the closest correlation matrix to any given matrix.

[6]See, for example, Horn and Johnson [1985], Chapter 7.4.

[7]For some choices of the parameters, this function fails to be positive-definite, but the correlation approximant will always be positive-semidefinite.

EXHIBIT 12
Market Forward Rate Correlations for GBP

	0.25	0.5	1	1.5	2	2.5	3	4	5	7
0.5	0.842									
1	0.625	0.790								
1.5	0.623	0.784	0.997							
2	0.533	0.732	0.811	0.815						
2.5	0.428	0.635	0.724	0.729	0.976					
3	0.327	0.452	0.543	0.538	0.568	0.546				
4	0.446	0.581	0.612	0.617	0.686	0.658	0.594			
5	0.244	0.344	0.443	0.446	0.497	0.492	0.608	0.485		
7	0.333	0.453	0.519	0.523	0.573	0.551	0.675	0.645	0.602	
9	0.263	0.366	0.425	0.430	0.477	0.458	0.602	0.567	0.520	0.989

EXHIBIT 13
Convergence to Market Correlation Matrix Measured in Frobenius Norm

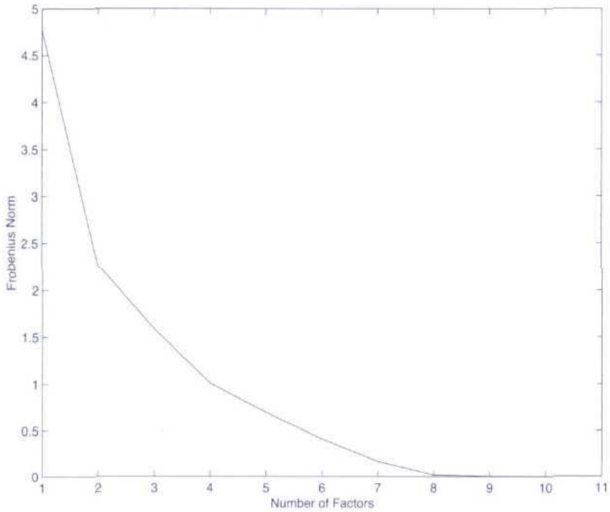


EXHIBIT 14
First Three Principal Components

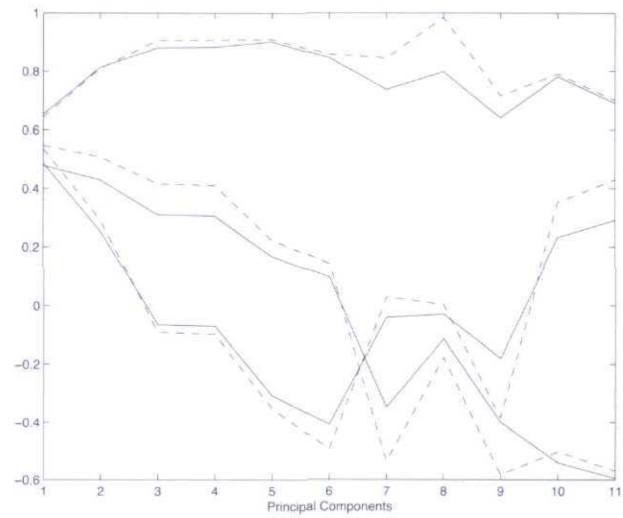


Exhibit 15
Rank-One Approximation

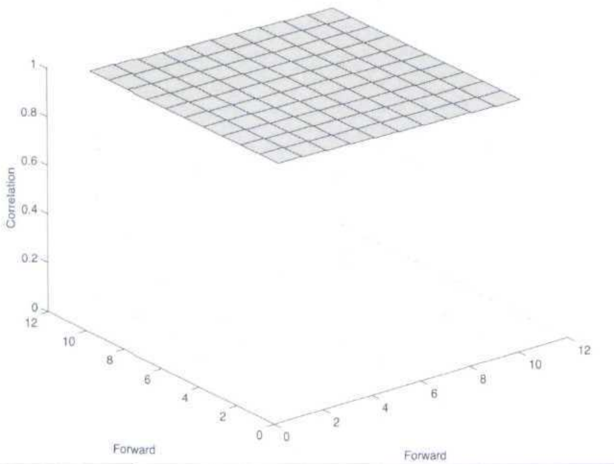


Exhibit 16
Rank-Two Approximation

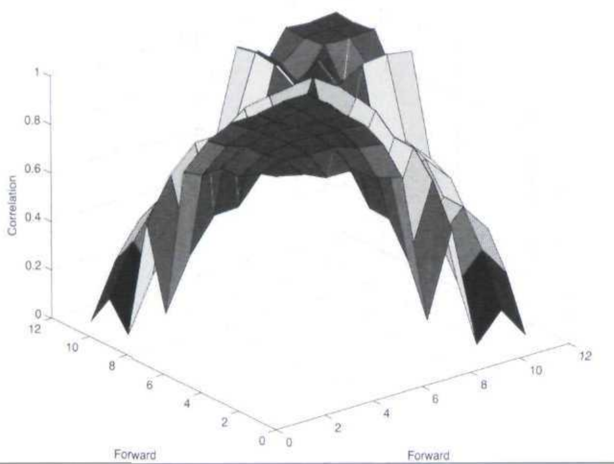


Exhibit 17
Rank-Four Approximation

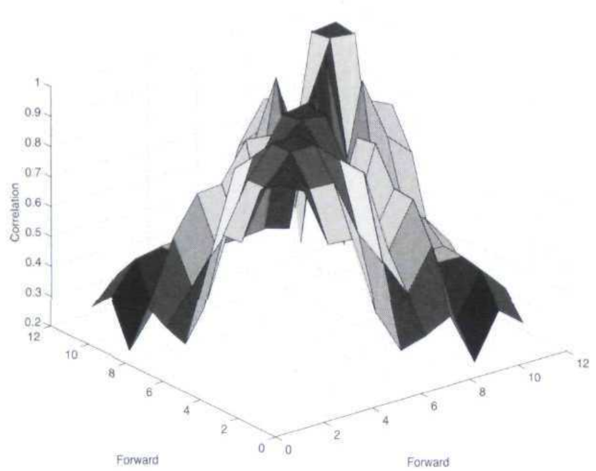


Exhibit 18
Rank-Six Approximation

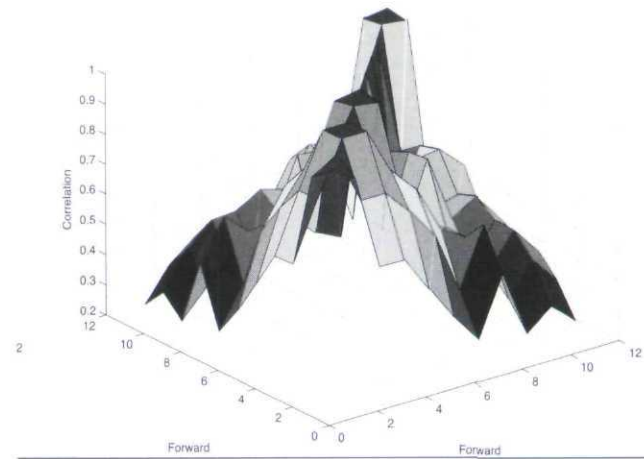


Exhibit 19
Rank-Nine Approximation

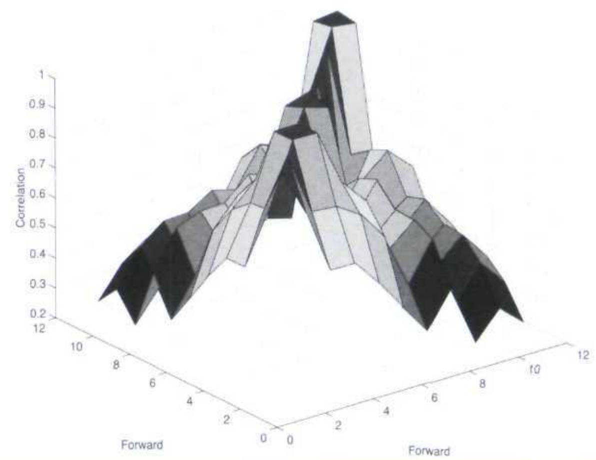
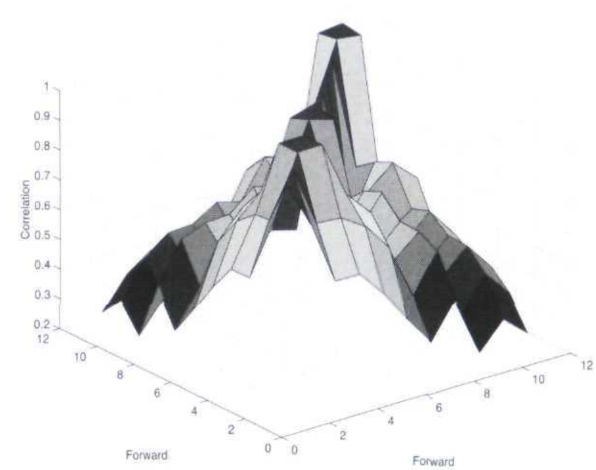


Exhibit 20
Market Correlation Surface



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