Pricing of Electricity Swing Options

JUSSI KEPPO

JUSSI KEPPO
is an assistant professor of industrial and operations engineering at the University of Michigan, in Ann Arbor, MI.
keppo@umich.edu

Electricity swing options hedge electricity price risk and also partly hedge the risks in the option owner's electricity consumption. The swing derivative sets instantaneous and cumulative boundaries for electricity consumption, and it specifies the price at which the option owner can buy energy. The name swing option *comes from these consumption boundaries, as consumption swings between lower and upper boundaries.*

Swing options can be priced and hedged with regular electricity forwards and call options. A numerical example illustrates how swing options can be used to model and hedge power plant operation.

One major difference between the stock market and many commodity markets, especially the electricity market, is that commodity usage is usually given by an exogenous consumption process, while a stock investor can choose asset holdings. Therefore, in commodity derivative markets there is a need to produce derivative instruments that take into account uncertain commodity usage. One example of this kind of instrument is a swing option.

The aim of this article is to model electricity swing options in terms of regular electricity derivatives. Our pricing model helps demonstrate how different variables affect swing option prices and also gives a practical hedging strategy for the options. The model can be used in the optimization and hedging of power plant operations.

Swing options have been traded in electricity over-the-counter (OTC) markets for many years, but their embedded optionality is sometimes mispriced. A swing option, also called a *variable base-load factor,* is an agreement to purchase energy at a certain fixed price during a fixed time period (see, e.g., Barbieri and Garman [1996], Garman and Barbieri [1997], and Pilipovic and Wengler [1998]).

The option gives some flexibility on both the purchased energy amount and the purchase timing. In practice, swing option owners do not always use this flexibility to maximize the option value because often they are unable to control their electricity usage. Purchase constraints are therefore selected so that the consumption process remains within the constraints with a high probability.

There are two types of swing options. The first type is called a one-swing option, and the other is a full-swing option. Under the one-swing option, a single consumption swing right is acquired; under the full-swing option, the number of swing rights is equal to the number of delivery dates in the delivery period.

These option types have similar instantaneous and cumulative consumption constraints (power and energy boundaries), and thus can be analyzed following our framework. Using this model, we can price and hedge more general swing options that may have constantly changing instantaneous consumption constraints. We examine an option seller that to protect itself assumes the option owner maximizes the value of the swing option by optimizing the consumption process.

A swing option is a portfolio of electricity forwards and options. These derivatives are traded on OTC markets and electricity exchanges (e.g., Nord Pool (Scandinavia), APX (U.S. and the Netherlands), NYMEX (New York), and VicPool of Victoria, Australia). Results obtained from our model show that the higher the instantaneous and cumulative consumption upper boundaries, the higher the swing option price, because then there is more optionality in the swing option, so the electricity consumer can better hedge consumption uncertainties. In this high optionality case, the swing option price depends on uncertainties in the electricity forward curve, as the value of the flexibility is an increasing function of the forward volatility.

In a numerical example, we illustrate application of the model in a power plant's production optimization and hedging. For instance, hydropower plants can be modeled under our framework because water reservoir constraints are similar to swing options' consumption constraints. The optimization of hydropower plant operations is considered in Pereira [1989], Gjelsvik, Røtting, and Røynstrand [1992], and Keppo [2002].

Many other authors also consider commodity and electricity derivative pricing. The basic financial asset pricing methods can be found in Harrison and Kreps [1979], Harrison and Pliska [1981], Duffie [1992], and Heath, Jarrow, and Morton [1992]. Black [1976] and Wolf [1982] derive pricing models for commodity contracts. Deng, Johnson, and Sogomonian [2001] study electricity spark and location spread options.

Keppo and Räsänen [1999] derive a pricing model for an electricity contract that does not have explicit constraints on electricity consumption. An extension is illustrated in Keppo and Räsänen [2000], where fixed budget instruments are introduced. These instruments allow customers to fix the amount they spend on electricity consumption. The difference between fixed budget instruments and swing options is that swing options assume the option owner optimizes electricity consumption, while in the models of Keppo and Räsänen the electricity consumption process is an exogenous stochastic variable.

Swing option pricing is considered in Thompson [1995] and Jaillet, Ronn, and Tompaidis [2001]. In both cases, swing option prices are solved using numerical techniques, e.g., a lattice-based method. We instead first characterize swing option prices in terms of electricity forwards and options that are traded in electricity financial markets.

Because of the swing option's path-dependence and Bermudan exercise style, we are not able to obtain an analytical pricing function for swing options (for the pricing of Bermudan options, see Carr and Yang [2001], Carrière [2001], Broadie and Yamamoto (2002), and Schweizer [2002]). The swing option's optimal consumption process depends not only on the current electricity spot price (and the cumulative energy usage) but also on the whole electricity forward curve, which is a function of several risk factors, so optimal exercise of the Bermudan option feature depends on multidimensional uncertainty (for the modeling of electricity forward curves, see, e.g., Koekebakker and Ollmar [2001] and Audet et al. [2004]). This means that the numerical pricing algorithms are hard to implement and the swing option price depends to a great extent on the forward curve model.

By assuming Markov electricity consumption processes, however, we solve for the swing option's analytical lower boundary in a linear optimization model. This lower boundary and the corresponding hedging strategy are easily implemented in everyday industry practice.

I. MODEL

We consider an electricity market where financial instruments are traded continuously within a time horizon $[0, \tau]$. We denote the swing option's delivery period by $[T_0, T_1]$, which is a subset of $[0, \tau]$. The swing option's owner has to purchase at least a minimum amount of energy at a certain price during $T_0 - T_1$. The owner also has an option to purchase more energy at the same fixed price during the same delivery period.

We assume the option seller expects the option's owner to select a consumption process, p, that maximizes the option's value. In the probabilistic structure of the market, a filtered complete probability space $(\Omega, F, P, (F_t)_{0 \le t \le \tau})$ satisfies the usual hypotheses (see, e.g., Protter [1992]). Here Ω is a set, F is a σ algebra, P is a probability measure on F, and $(F_t)_{0 \le t \le \tau}$ is an increasing family of σ algebras. Two assumptions characterize our electricity market.

Assumption 1. There are European call options and forward contracts on the electricity price. The electricity derivative market is complete and there is no arbitrage.

In a competitive electricity market, there are options and forward contracts continuously traded on exchanges and in the OTC market. This implies that the electricity derivative market has priced the risks associated with the electricity price process.

In deriving our swing option pricing model, we use different-maturity forwards and European call options that have the same strike price as the swing option under consideration. Given the no-arbitrage condition, all the electricity derivative portfolios with the same future payoffs have the same current value; i.e., the law of one price holds in our electricity market. If this were not the case, traders would sell the overpriced portfolio and buy the lower-priced portfolio and collect arbitrage profits from the market.

Market completeness means that tradable instruments are priced according to the same unique linear pricing function (see, e.g., Duffie [1992]). Therefore, in the electricity market there must be at least as many derivative contracts as there are sources of uncertainty. For instance, if the market is driven by n independent Brownian motions, then we assume there are at least n electricity derivative instruments traded in the market. Note that without the derivative instruments the electricity market is incomplete, because electricity is a non-tradable instrument.

We denote by $f(t, T)$ the T-maturity electricity forward price at time t. By allowing T to vary from t to τ, we get the whole forward curve $f(t,\cdot)$: $[t, \tau] \rightarrow \mathbf{R}_+$. There are huge cycles in the forward curve because of cycles in electricity demand. The maturity $T = t$ corresponds to the starting point of the curve $f(t, t) = S(t)$, which is the electricity spot price at time t. We assume that at each time $t \in [0, \tau]$ there is a right continuous electricity forward curve.

The T-maturity European call option price at time t is denoted by $C(t, T, K)$, where K is the strike price of the option, which equals the swing option's strike price (defined later). We assume there are call options for all maturities, and for all t the curve $C(t, \cdot, K)$: $[t, \tau] \rightarrow \mathbf{R}_+$ is right continuous.

Assumption 2 characterizes the derivative price processes and other stochastic processes followed.

Assumption 2. The stochastic variables follow right continuous semimartingale processes that are driven by continuous and jump uncertainties in the probability space (Ω, F, P) along with the standard filtration $\{F_t: t \in [0, \tau]\}$. These variables are of the class of functions $g(t, \omega)$: $[0, \tau] \times \Omega \rightarrow \mathbf{R}$ that satisfy:

1. $(t, \omega) \rightarrow g(t, \omega)$ is $\boldsymbol{B} \times F$-measurable, where $\boldsymbol{B}$ denotes the Borel σ algebra on $[0, \tau]$.
2. $g(t, \omega)$ is F_t-adapted.
3. $E[g^2(t, \omega)] < \infty$ for all $t \in [0, \tau]$.

Assumption 2 implies that we have both continuous and jump uncertainties in the electricity market. For simplicity, we work in Hilbert space $L^2(\Omega, F, P)$. Semimartingale processes in financial markets are examined in Delbaen and Schachermayer [1994, 1998], Goll and Kallsen [2000], and Carr and Wu [2004].

Under Assumptions 1 and 2, the price of a T-maturity electricity derivative instrument at time t is given by the risk-neutral pricing formula:

$$\pi(t, T) = e^{-r(T-t)} E^Q\left[\Phi(S(T)) \middle| F_t\right] \quad (1)$$

$$\text{for all } t \in [0,T],\ T \in [0,\tau]$$

where r is the risk-free interest rate that is for simplicity assumed to be constant, E^Q is the expectations operator under the risk-neutral pricing measure Q, and $\Phi(\cdot)$ is the payoff function.

If π is the value of a T-maturity forward contract we get that the forward price at time t

$$f(t, T) = E^Q\left[S(T) \middle| F_t\right] \quad (2)$$

because now $\Phi[S(T)] = S(T) - f(t, T)$, and the value of the forward contract when initiated is by definition zero, i.e., $\pi(t, T) = 0$. The pricing measure Q can be estimated from the electricity forward prices, because, given Assumptions 1 and 2, the only unknown term in Equation (2) is the Radon-Nikodym derivative dQ/dP.

By the definition of a swing option, there are constraints on the swing option's consumption process. The objective of the swing option holder is to maximize the value of the contract by selecting the optimal consumption process among those processes that satisfy the cumulative and instantaneous constraints, namely, the energy and power constraints:

$$e_{low} \le \int_{T_0}^{T_1} p(y)dy \le e_{up}$$

$$p_{low}(t) \le p(t) \le p_{up}(t) \text{ for all } t \in \left[T_0, T_1\right] \quad (3)$$

where $p(\cdot)$: $[T_0, T_1] \rightarrow \mathbf{R}_+$ is the swing option's stochastic consumption process; e_{low} and e_{up} are constant lower and upper boundaries for the total energy purchased; and $p_{low}(\cdot)$: $[T_0, T_1] \rightarrow \mathbf{R}_+$ and $p_{up}(\cdot)$: $[T_0, T_1] \rightarrow \mathbf{R}_+$ are deterministic right continuous lower and upper boundary functions for power. Hence, p_{low} and p_{up} are the constraints on instantaneous power consumption, and e_{low} and e_{up} are the constraints on cumulative electricity purchased.

The instantaneous constraints imply that the option owner must buy at least $p_{low}(t)$ and cannot buy more than $p_{up}(t)$ at each time $t \in [T_0, T_1]$. Due to the cumulative constraints, total purchases during $T_0 - T_1$ must be at least e_{low} and cannot be more than e_{up}. Because of the instantaneous constraints, the cumulative constraints must satisfy:

$$\int_{T_0}^{T_1} p_{low}(y)dy \leq e_{low}, e_{up} \leq \int_{T_0}^{T_1} p_{up}(y)dy \tag{4}$$

We assume that the swing option seller hedges the option by trading electricity derivative instruments. Because the value of the hedging portfolio equals the swing option price, we have

$$\int_{t}^{T_1} e^{-ry} p^*(y)[S(y) - K]dy = e^{-rt} z(t) + \int_{t}^{T_1} dM^Q(y) \tag{5}$$

where $p^*(\cdot)$ is the optimal consumption process of the swing option; which satisfies Assumption 2 and Equation (3); K is the electricity price of the swing option (the strike price); $z(t)$ is the value of the hedging portfolio at time t; and $M^Q(\cdot)$ is a martingale under Q corresponding to the discounted gains and losses from the swing option's hedging strategy.

Thus, the left-hand side of (5) is the discounted cash flows from the swing option, and the right-hand side is the discounted gains and losses from the hedging strategy. Each time $t \in [T_0, T_1]$ the swing option's cash flow equals $p^*(t)[S(t) - K]$. If $p^*(t) > 0$, which is the case for all states of the world if $p_{low}(t) > 0$, and if $S(t) < K$, the cash flow is positive. If $p^*(t) > 0$ and $S(t) < K$, the cash flow is negative; i.e., if the strike price K is high enough, the swing option holder might incur losses, because the consumption process has to satisfy the instantaneous and cumulative lower boundaries in (3).

As we noted earlier, the objective of the swing option's owner is to maximize the value of the contract by controlling electricity consumption. Therefore, by taking the conditional expectation under Q, Equation (5) can be represented as follows:

$$z(t, T_0, T_1) = \sup_{p(\cdot) \in A} E^Q \left[\int_{t \vee T_0}^{T_1} e^{-r(y-t)} p(y)[S(y) - K]dy \middle| F_t \right]$$
$$\text{for all } t \in [0, T_1] \tag{6}$$

where $z(t, T_0, T_1)$ $(=z(t))$ is the arbitrage-free swing option price at time t for delivery period $[t \vee T_0, T_1]$; $t \vee T_0 = \max[t, T_0]$; and A is a non-empty set of consumption processes that satisfy the conditions of Assumption 2 and the constraints in Equations (3) and (4).

Thus, according to (6), the swing option is defined by the set of parameters $(p_{low}(\cdot), p_{up}(\cdot), e_{low}, e_{up}, T_0, T_1, K)$, and the price is given by the usual risk-neutral pricing equation.

As indicated in (5) and (6), we will see that swing options can be replicated by a basket of regular electricity derivatives, and the amounts of these derivatives depend on the instantaneous and cumulative consumption constraints. In practice, the constraints are set so that the realized consumption process of the option's owner is between the boundaries with a high probability. Because the realized consumption is unknown when the swing option is sold, however, and because in many cases the option owner is not able to control the consumption process, there is always a risk that the realized consumption is outside the given region. Therefore, the option owner always bears some consumption risk.

The swing option owner buys or sells the excess or the shortage of energy on the spot market. Thus, with swing options, planning the power and energy constraints is very important in serving consumer needs. An example of a different contract that also hedges the stochastic consumption process is given in Keppo and Räsänen [2000].

II. PRICING

We can price swing options using different electricity derivative instruments. First, we divide the swing option's consumption into three terms as described in Proposition 1:

Proposition 1. The swing option's electricity consumption can be represented as:

$$p(t) = p_{low}(t) + p_S(t) + p_C(t)\mathbf{1}\{S(t) \geq K\} \text{ for all } t \in [T_0, T_1] \tag{7}$$

where $p_S(\cdot) \in A_S$, $p_C(\cdot) \in A_C$ (p_S), A_S is the class of stochastic processes that satisfy the conditions of Assumption 2, and:

$$\int_{T_0}^{T_1} p_S(y)dy = e_{low} - \int_{T_0}^{T_1} p_{low}(y)dy$$

$$0 \le p_S(t) \le p_{up}(t) - p_{low}(t) \text{ for all } t \in [T_0, T_1]$$

$A_C(p_S)$ is the class of stochastic processes that satisfy the conditions of Assumption 2, and

$$0 \le \int_{T_0}^{T_1} p_C(y)\mathbf{1}\{S(y) \ge K\}dy \le e_{up} - e_{low}$$

$$0 \le p_C(t)\mathbf{1}\{S(t) \ge K\} \le p_{up}(t) - p_{low}(t) - p_S(t)$$
$$\text{for all } t \in [T_0, T_1]$$

and the indicator $\mathbf{1}\{S(t) \ge K\} = \begin{cases} 1, \text{if } S(t) \ge K \\ 0, \text{otherwise} \end{cases}$

Proof. Because with p_C the lower instantaneous and cumulative constraints (power and energy constraints) are zero, and because the owner of the swing option maximizes the value of the swing option, the holder selects $p_C(t) = 0$ if $S(t) < K$. Therefore, the third term of (7) is $p_C(t)\mathbf{1}\{S(t) \ge K\}$. From Equation (7) we get:

$$e_{low} \le \int_{T_0}^{T_1} \begin{pmatrix} p_{low}(y) + p_S(y) + \\ p_C(y)\mathbf{1}\{S(y) \ge K\} \end{pmatrix} dy \le e_{up}$$

and

$$p_{low}(t) \le p_{low}(t) + p_S(t) + p_C(t)\mathbf{1}\{S(t) \ge K\} \le p_{up}(t)$$
$$\text{for all } t \in [T_0, T_1]$$

These are the same as Equations (3) and (4). *Q.E.D.*

All the consumption terms in Equation (7) are non-negative, and the two last terms are stochastic. The first term, p_{low}, is the lower power boundary in (3). Thus, each time $t \in [T_0, T_1]$, the swing option owner consumes at least this minimum power. The set A_S for the second term, p_S, corresponds to additional purchases, whose cumulative value equals the difference between the lower energy boundary, e_{low}, and the cumulative lower power boundary:

$$\int_{T_0}^{T_1} p_{low}(y)dy$$

Therefore, $p_{low} + p_s$ is the minimum consumption required to satisfy both of the swing option constraints in Equation (3).

Note that there is some optionality in the timing of p_S because the option holder can select the times when $p_S(t) > 0$, but the cumulative value must satisfy

$$\int_{T_0}^{T_1} p_S(y)dy = e_{low} - \int_{T_0}^{T_1} p_{low}(y)dy$$

If the swing option's holder selects $p_S(t) > 0$, the cash flow at time t is $[p_{low}(t) + p_s(t)][S(t) - K]$, and the owner loses the opportunity to use this consumption option in the future and the possibility to collect even higher cash flows. This assumes $p_C(t) = 0$, which it will be, due to the constraints of p_C, as is explained below. In some cases, however, the cumulative constraint on p_S forces $p_S(t)$ to be strictly positive, even though we might have negative cash flow at time t.

The set A_C for the third term, p_C, is the set of purchase patterns that have timing and quantity optionality. Thus, the holder of the swing option can set $p_C(t) = 0$ for all $t \in [T_0, T_1]$. Because the holder maximizes the swing option value, the third term is modeled as $p_C(t)\mathbf{1}\{S(t) \ge K\}$; i.e., the third term is non-zero only if $S(t) \ge K$.

The maximum cumulative value of p_C is $e_{up} - e_{low}$. If the swing option's holder selects $p_C(t) > 0$, then the cash flow at time t is $[p_{low}(t) + p_C(t)][S(t) - K] \ge 0$ (now setting $p_S(t) = 0$). That is, due to the quantity optionality, if $p_C(t) > 0$, the cash flow at time t is positive. By using the consumption option at time t, however, the option owner loses the opportunity to use this option in the future, i.e., forgoes the possibility of even higher profits in the future.

To sum this up, the first term in Equation (7) is the deterministic lower instantaneous consumption constraint; the second term corresponds to the lower cumulative consumption constraint; and the last term corresponds to the upper cumulative constraints.

Now, by using Proposition 1 and Equation (6), we get Proposition 2, which characterizes the swing option price in terms of electricity forwards:

Proposition 2. The swing option price is given as follows:

$$z(t,T_0,T_1)=\int_{t\vee T_0}^{T_1} e^{-r(y-t)}p_{low}(y)(f(t,y)-K)dy+$$

$$\int_{t\vee T_0}^{T_1}\left[p_{up}(y)-p_{low}(y)\right]e^{-r(y-t)}|f(t,y)-K|\begin{bmatrix} Q_{f(t,y)}\left(p_S^*(y)>0,S(y)\geq K\right)- \\ Q_{f(t,y)}\left(p_S^*(y)>0,S(y)<K\right)+ \\ Q_{f(t,y)}\left(p_C^*(y)>0,p_S^*(y)=0\right)\end{bmatrix}dy$$

$$\text{for all } t\in[0,T_1],\ T_0\in[0,T_1),\ T_1\in(0,\tau] \tag{8}$$

where the optimal consumption terms $p^*_S(\cdot)$ and $p^*_C(\cdot)$ satisfy

$$\int_{t\vee T_0}^{T_1}\left[p_{up}(y)-p_{low}(y)\right]\mathbf{1}\left\{p_S^*(y)>0\right\}dy=e_S(t)$$

$$\int_{t\vee T_0}^{T_1}\left[p_{up}(y)-p_{low}(y)\right]\mathbf{1}\left\{p_C^*(y)>0\right\}\mathbf{1}\left\{p_S^*(y)=0\right\}dy\leq e_C(t)$$

and $f(t, T) = E^Q[S(T)\mid F_t]$ is the T maturity forward price at time t; $Q_{f(t,T)}$ is the electricity forward martingale measure corresponding to $f(t, T)$ (see, e.g., Geman [1989] and Björk [1998]); $Q_{f(t,T)}(0 < x, 0 < y) = E^{Q_{f(t,T)}}[\mathbf{1}\{0 < x\}\mathbf{1}\{0 < y\}\mid F_t]$ is the probability under $Q_{f(t,T)}$ that x and y are strictly positive;

$$e_S(t)=e_{low}-\int_{T_0}^{T_1}p_{low}(y)dy-\int_{T_0}^{t\vee T_0}p_S^*(y)dy$$

is the cumulative constraint of p_S at time t; and

$$e_C(t)=e_{up}-e_{low}-\int_{T_0}^{t\vee T_0}p_C^*(y)dy$$

is the cumulative upper constraint of p_C at time t.

The proof is in the appendix.

Proposition 2 implies that swing options can be priced and hedged in terms of regular forwards. These different-maturity forward contracts have forward prices equal to the swing option's strike price, and these forwards can be replicated using the current forward contracts and risk-free instruments. For instance, $e^{-r(T-t)}(f(t, T) - K)$ in Proposition 2 is equal to the value of one T-maturity forward contract plus $e^{-r(T-t)}(f(t, T) - K)$ in the risk-free asset at time t. Therefore, the presence of a risk-free asset and a forward curve implies the presence of the forwards used in Proposition 2. From now on when we refer to forward contracts, we mean forwards with forward price equal to the swing option's strike price.

We use the absolute values of forward contracts in Equation (8) because then the numeraires are always positive (see, e.g., Björk [1998]). This is done in order to represent swing option prices in terms of forward contracts in Proposition 2. Note that if $f(t, T) \geq K$, then $e^{-r(T-t)}|f(t, T) - K| = e^{-r(T-t)}(f(t, T) - K)$, which is the value of one long forward contract at time t; and if $(f(t, T) < K$, then $e^{-r(T-t)}|f(t, T) - K| = e^{-r(T-t)}(K - f(t, T))$, and this is the value of the short position.

Proposition 2 is quite obvious, as it says that, given the optimal exercise times of swing rights that satisfy the consumption constraints, i.e., given that $p^*_S(t) > 0$ and $p^*_C(t) > 0$ for all $t \in [T_0, T_1]$, the swing option price is equal to the portfolio of forward contracts, and the forward holdings are given by the probabilities $Q_{f(t,T)}(p^*_S(T) > 0, S(T) \geq K)$; $Q_{f(t,T)}(p^*_S(T) > 0,\ S(T) < K)$; and $Q_{f(t,T)}(p^*_C(T) > 0,\ p^*_S(T) = 0)$, where $T \in [T_0, T_1]$. Note that these probabilities are not under the objective measure P but under the forward measures.

The exercise events when $p^*_S(t) > 0$ or $p^*_C(T) > 0$ depend on the energy used during $[T_0, t]$ as well as the forward curve and its uncertainties. Therefore, solving for these exercise times is difficult even using numerical methods.

The first integral in Equation (8) is the value of the lower instantaneous consumption boundary. The term $Q_{f(t,T)}(p^*_S(T) > 0,\ S(T) \geq K)$ is the probability under the forward measure $Q_{f(t,T)}$ that $p^*_S(T) > 0$ and $S(T) \geq$ K, and $Q_{f(t,T)}(p^*_S(T) > 0,\ S(T) <$ K) is the probability that $p^*_S(T) > 0$ and $S(T) < K$. According to the appendix, we have the expectation under the risk-neutral probability measure:

$$E^Q\left[p_S^*(T)(S(T)-K)\middle|F_t\right]=|f(t,T)-K|\left[p_{up}(T)-p_{low}(T)\right]\times$$
$$\left[Q_{f(t,T)}\left(p_S^*(T)>0,S(T)\geq K\right)-Q_{f(t,T)}\left(p_S^*(T)>0,S(T)<K\right)\right]$$

where $t \in [0, T]$ and $T \in [T_0, T_1]$. Therefore, in (8) the term:

$$\left[p_{up}(T)-p_{low}(T)\right]e^{-r(T-t)}|f(t,T)-K|\begin{bmatrix} Q_{f(t,T)}\left(p_S^*(T)>0,S(T)\geq K\right)- \\ Q_{f(t,T)}\left(p_S^*(T)>0,S(T)<K\right)\end{bmatrix}$$

is the present value of $p^*_S(T)$ units of T-maturity forwards. Note that $p^*_S(T) \in \{p_{up}(T) - p_{low}(T), 0\}$ and it is an F_T-measurable random variable; i.e., its value is unknown before time T. Similarly, the expectation:

$$E^Q\left[p_C^*(T)(S(T) - K)|F_t\right] = |f(t, T) - K|\left[p_{up}(T) - p_{low}(T)\right] \times Q_{f(t,T)}\left(p_C^*(T) > 0, p_S^*(T) = 0\right)$$

and this gives that in (8) the term

$$\left[p_{up}(T) - p_{low}(T)\right] e^{-r(T-t)}|f(t, T) - K| \, Q_{f(t,T)}\left(p_C^*(T) > 0, p_S^*(T) = 0\right)$$

is the present value of $p_C^*(T)$ units of T-maturity forwards. Combining terms, we see that Equation (8) represents the net present value corresponding to the consumption terms of Proposition 1 under optimal policy.

These equations imply several facts about the swing option's optimal consumption strategy. First, the second and third consumption terms satisfy $p_S^*(T)p_C^*(T) = 0$ for all $T \in [T_0, T_1]$, due to the representation of Proposition 1. Second, the optimal consumption process is either p_{low} or p_{up}; i.e., the consumption process swings between the instantaneous lower and upper boundaries. As illustrated in the appendix, this is due to the fact that the swing option value is linear with respect to p_S and p_C, and the instantaneous consumption at time t never violates the cumulative upper constraint if

$$\int_0^{t-} p(y)dy < e_{up}$$

According to Equation (8), if the swing option seller holds T-maturity forward contracts at time t as follows

$$p_{low}(T) + \left[p_{up}(T) - p_{low}(T)\right] \times \begin{bmatrix} Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) \geq K\right) - \\ Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) < K\right) + \\ Q_{f(t,T)}\left(p_C^*(T) > 0, p_S^*(T) = 0\right) \end{bmatrix} \text{if } f(t, T) \geq K$$

$$p_{low}(T) - \left[p_{up}(T) - p_{low}(T)\right] \times \begin{bmatrix} Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) \geq K\right) - \\ Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) < K\right) + \\ Q_{f(t,T)}\left(p_C^*(T) > 0, p_S^*(T) = 0\right) \end{bmatrix} \text{if } f(t, T) < K$$

for all $T \in [T_0 \vee t, T_1]$, the swing option is hedged.

Note that the seller needs to hold forwards with maturities between T_0 and T_1, and if this holding of T-maturity forwards is negative, the seller has a short position. Because the forward holdings depend on probabilities that are calculated with respect to information at time t, these probabilities change over time, so the hedging strategy of a swing option is dynamic.

To illustrate this, let us assume that $f(0, T) > K$ for all $T \in [T_0, T_1]$. In this case, if the forward prices on $[T_0, \tilde{T}]$ increase and the forward prices on $[\tilde{T}, T_1]$ decline, then the forward holdings corresponding to maturities between T_0 and $\tilde{T}$ increase and the forward holdings corresponding to maturities between $\tilde{T}$ and T_1 decline, where $\tilde{T} \in [T_0, T_1]$. This is because in the last equation above the term

$$Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) \geq K\right) - Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) < K\right) + Q_{f(t,T)}\left(p_C^*(T) > 0, p_S^*(T) = 0\right)$$

increases for all $T \in [T_0, \tilde{T}]$ and declines for all $T \in [\tilde{T}, T_1]$. Note that when the forward prices increase, there is a higher probability that the corresponding spot price $S(T) \geq K$, so it is more likely that the optimal consumption at time T is $p_{up}(T)$.

In Proposition 2 we use only forward contracts to hedge the swing options. We can also solve for the corresponding swing option's replication strategy by using the forwards and call options. This is given in Corollary 1:

Corollary 1. The swing option price is given by:

$$\begin{aligned} z(t, T_0, T_1) = &\int_{t \vee T_0}^{T_1} e^{-r(y-t)} p_{low}(y)(f(t, y) - K)dy + \\ &\int_{t \vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right] e^{-r(y-t)} |f(t, y) - K| \begin{bmatrix} Q_{f(t,y)}\left(p_S^*(y) > 0, S(y) \geq K\right) - \\ Q_{f(t,y)}\left(p_S^*(y) > 0, S(y) < K\right) \end{bmatrix} dy + \\ &\int_{t \vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right] \times C(t, y, K)\, Q_{C(t,y,K)}\left(p_C^*(y) > 0, p_S^*(y) = 0\right) dy \\ &\text{for all } t \in [0, T_1], T_0 \in [0, T_1), T_1 \in (0, \tau] \end{aligned} \tag{9}$$

where

$$\int_{t\vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right]\mathbf{1}\{p_S^*(y) > 0\}dy = e_S(t)$$

$$\int_{t\vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right]\mathbf{1}\{p_C^*(y) > 0\}\mathbf{1}\{p_S^*(y) = 0\}dy = e_C(t)$$

the T-maturity European call option price with strike price K at time t is $C(t, T, K) = e^{-r(T-t)} E^Q[\mathbf{1}\{S(T) \geq K\}(S(T) - K) \mid F_t]$; $Q_{C(t, T, K)}$ is the martingale measure corresponding to $C(t, T, K)$; and $Q_{C(t, T, K)}(x > 0, y > 0) = E^{Q_{C(t,T,K)}}[\mathbf{1}\{x > 0\}\mathbf{1}\{y > 0\} \mid F_t]$ is the probability under $Q_{C(t,T,K)}$ that x and y are strictly positive.

The proof is in the appendix.

In Corollary 1 we have written Proposition 2 in a different form by using the call option measures, $Q_{C(t, T, K)}$. Thus, we get from the last integrals of (8) and (9):

$$\frac{Q_{C(t,T,K)}\left(p_C^*(T) > 0, p_S^*(T) = 0\right)}{Q_{f(t,T)}\left(p_C^*(T) > 0, p_S^*(T) = 0\right)} = \frac{e^{-r(T-t)}|f(t,T) - K|}{C(t,T,K)} \text{ for all } t \in [0, T],\ T \in [T_0, T_1]$$

That is, the ratio of the probabilities is given by the ratio of the absolute value of the forward and the call option price. The representation of Corollary 1 is more convenient than Proposition 2 in the sense that the indicator in p_C is modeled using call options, so it is easier to understand the consumption terms of the swing option. $p_{low} + p_S^*$ is the minimum consumption that satisfies both swing option constraints in Equation (3), so it is modeled using forward holdings. Because p_C^* has quantity optionality, this consumption term is modeled with call options.

From Corollary 1, we get that if $p_{low} = p_{up}$ or if $e_{low} = e_{up}$, the portfolio consists only of forwards. Further, if e_{low}, $p_{low} = 0$, and e_{up}, $p_{up} > 0$, then there are only options. In this case, the swing option price depends more on the forward curve volatility because the forward curve uncertainty affects the call option prices. Since option prices are always positive, an increase in e_{up} and p_{up} never reduces the swing option value.

Proposition 2 and Corollary 1 imply that the swing option is equivalent to a basket of regular electricity derivatives. That is, swing options can be represented in terms of established electricity derivatives. Note that because the consumption process depends on the market forward and call option prices and because these prices are stochastic, the swing option's replication strategy is dynamic.

The purchase times, where $p_S^*(t) > 0$ or $p_C^*(t) > 0$, are stopping times of stochastic analysis, which are solved similarly to exercise strategies of Bermudan options. Bermudan derivatives fall in between American and European derivatives because they have a set of permitted exercise dates. Bermudan options are examined in Carr and Yang [2001], Carrière [2001], Broadie and Yamamoto [2002], and Schweizer [2002]. The optimal purchase times can be solved using stochastic dynamic programming. This implies that since there are at most four consumption alternatives we get the optimal decision at time t by assuming an optimal consumption policy on $(t \vee T_0, T_1]$ and by comparing the swing option value under the alternatives:

$$p_S(t) = \begin{cases} p_{up}(t) - p_{low}(t) \\ 0 \end{cases}$$

and

$$p_C(t) = \begin{cases} p_{up}(t) - p_{low}(t) \\ 0 \end{cases}$$

where we assume that $p_S(t) = 0$, $p_S(t) = p_{up}(t) - p_{low}(t)$, and $p_C(t) = p_{up}(t) - p_{low}(t)$ are admissible. That is, they satisfy the conditions of Proposition 1 (and if one or more is not admissible, then we just have fewer alternatives to compare).

This dynamic programming problem is hard to solve because the swing option's optimal consumption strategy depends on the whole forward curve and its uncertainties as well as on the cumulative consumption process. In other words, the problem is a multidimensional stochastic control problem.

According to Koekebakker and Ollmar [2001] and Audet et al. [2004], there are more uncertainties in the electricity forward curve than there typically are in the yield curve. Further, the stochastic processes of the electricity forward curve's risk factors are more complicated than risk factor processes in regular financial markets.

Because of these problems in applying stochastic control techniques to modeling swing options, we develop a practical pricing and hedging approach that gives a lower boundary for the swing option. We model $p_S^*(\cdot)$ and $p_C^*(\cdot)$ using Markov processes that depend on the electricity spot price. This implies that the probabilities of Equation (9) are indicator functions of time because otherwise there is no guarantee that the power and energy constraints are satisfied. Because the class of Markov consumption processes is smaller than the initial space A, we get the result:

Corollary 2. The swing option's lower boundary is given by

$$z_{low}(t, T_0, T_1) = \int_{T_0 \vee t}^{T_1} e^{-r(y-t)}(f(t,y) - K)p_{low}(y)dy +$$
$$\int_{T_0 \vee t}^{T_1} e^{-r(y-t)}(f(t,y) - K)\left[p_{up}(y) - p_{low}(y)\right]\mathbf{1}\left\{y \in \Gamma_S^*(t)\right\}dy +$$
$$\int_{T_0 \vee t}^{T_1} C(t,y,K)\left[p_{up}(y) - p_{low}(y)\right]\mathbf{1}\left\{y \in \Gamma_C^*(t)\right\}dy$$
$$\text{for all } t \in [0, T_1], T_0 \in [0, T_1), T_1 \in (0, \tau] \tag{10}$$

where

$$p_{low}(\cdot) + \left[p_{up}(\cdot) - p_{low}(\cdot)\right]\left[\mathbf{1}\left\{\cdot \in \Gamma_S^*(t)\right\} + \mathbf{1}\left\{\cdot \in \Gamma_C^*(t)\right\}\right]: \left[t \vee T_0, T_1\right] \to \mathbf{R}_+$$

is the optimal Markov consumption process at time t for the future time period $[t \vee T_0, T_1]$; and $\Gamma_S^*(t)$ and $\Gamma_C^*(t)$ are disjoint sets. These sets are obtained from the linear optimization problem at time t:

$$\max_{\Gamma_S(t) \subset [t,\tau]} \left\{ \int_{T_0 \vee t}^{T_1} \begin{array}{l} e^{-r(y-t)}(f(t,y) - K) \times \\ \left[p_{up}(y) - p_{low}(y)\right]\mathbf{1}\left\{y \in \Gamma_S(t)\right\}dy \end{array} + \max_{\Gamma_C(t) \subset [t,\tau] - \Gamma_S(t)} \int_{T_0 \vee t}^{T_1} \begin{array}{l} C(t,y,K) \times \\ \left[p_{up}(y) - p_{low}(y)\right]\mathbf{1}\left\{y \in \Gamma_C(t)\right\}dy \end{array} \right\}$$

subject to

$$\int_{t \vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right]\mathbf{1}\left\{y \in \Gamma_S(t)\right\}dy = e_S(t)$$
$$\int_{t \vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right]\mathbf{1}\left\{y \in \Gamma_C(t)\right\}\mathbf{1}\left\{y \notin \Gamma_S(t)\right\}dy = e_C(t)$$

where

$$e_S(t) = e_{low} - \int_{T_0}^{T_1} p_{low}(y)dy - \int_{T_0}^{t \vee T_0} \left[p_{up}(y) - p_{low}(y)\right]\mathbf{1}\left\{y \in \Gamma_S^*(y)\right\}dy$$

and

$$e_C(t) = e_{up} - e_{low} - \int_{T_0}^{t \vee T_0} \left[p_{up}(y) - p_{low}(y)\right]\mathbf{1}\left\{y \in \Gamma_C^*(y)\right\}\mathbf{1}\left\{y \notin \Gamma_S^*(y)\right\}dy$$

Proof. Under the Markov consumption strategies, we get from Corollary 1:

$$z(t, T_0, T_1) = \int_{t \vee T_0}^{T_1} e^{-r(y-t)}p_{low}(y)(f(t,y) - K)dy +$$
$$\int_{t \vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right]e^{-r(y-t)}|f(t,y) - K| \begin{bmatrix} Q_{f(t,y)}(S(y) \geq K) \\ - Q_{f(t,y)}(S(y) < K) \end{bmatrix} \mathbf{1}\left\{y \in \Gamma_S^*(t)\right\}dy +$$
$$\int_{t \vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right]C(t,y,K)\mathbf{1}\left\{y \in \Gamma_C^*(t)\right\}dy$$

where events $p_S^*(t) > 0$ and $p_C^*(t) > 0$ are modeled with the indicator functions. Indicator functions are used because otherwise there is no guarantee with spot price-dependent Markov strategies that the consumption constraints are satisfied.

According to Equations (A-2) and (A-5) in the appendix, we have:

$$E^Q\left[p_S^*(T)(S(T) - K)|F_t\right] = |f(t,T) - K|\left[p_{up}(T) - p_{low}(T)\right] \times \left[Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) \geq K\right) - Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) < K\right)\right]$$

By using Equation (A-5) again and the Markov assumption, we get:

$$E^Q\left[(S(T) - K)|F_t\right] = |f(t,T) - K|\begin{bmatrix} Q_{f(t,T)}(S(T) \geq K) - \\ Q_{f(t,T)}(S(T) < K) \end{bmatrix}$$

Using this with (2) and the second integral above gives Equation (10). *Q.E.D.*

Because of the Markov assumption, Corollary 2 solves the swing option's lower boundary. We select the sets Γ_S and Γ_C (times where $p_S(t) > 0$ and $p_C(t) > 0$), given the current forward curve and European call option values. Since the forward curve and option prices change constantly, these sets and therefore also the asset holdings are dynamic.

By the definition of a swing option, there is some freedom on the purchase timing of p_S and p_C, but the Markov assumption implicitly partly ignores the swing option's Bermudan option characteristics. Because a Bermudan derivative is always more valuable than the corresponding European derivative, Corollary 2 gives the lower boundary of the swing option. This is similar to approximating a Bermudan option with the highest European option as follows:

$$\tilde{X}(t) = \max_{\tau \in D} X(t, \tau)$$

where $\tilde{X}(t)$ is the price of the Bermudan option at time t, D is the set of its possible exercise dates, and $X(t, T)$ is the price of a T-maturity European option at time t. Because the Bermudan option is more valuable than any European option on the right-hand side, this equation gives a lower boundary for the Bermudan option price. The difference between the lower boundary and the true value depends on the stochastic process for the underlying asset as well as the option parameters. Broadie and Yamamoto [2002] report in their numerical example about a 3.5% difference between a European option and the corresponding Bermudan option.

The optimization problem of the additional electricity purchase

$$\left[p_{up}(\cdot) - p_{low}(\cdot)\right]\left[\mathbf{1}\{\cdot \in \Gamma_S(t)\} + \mathbf{1}\{\cdot \in \Gamma_C(t)\}\right]$$

in Corollary 2 is a linear optimization problem that can be solved using standard techniques (for linear optimization problems, see Taha [1997]). Note that if $f(t, T) \leq K$ then $e^{-r(T-t)}f(t, T) - K) \leq 0$ and $C(t, T, K) \geq 0$. In the optimization, we thus try to select $\Gamma_S(t)$ and $\Gamma_C(t)$ so that $\mathbf{1}\{T \in \Gamma_S(t)\} = 0$ and $\mathbf{1}\{T \in \Gamma_C(t)\} = 1$. That is, at points where the forward curve is above the swing option's strike price K, we prefer to use forwards, and at points where the curve is below the strike price, we prefer options. This is illustrated in our numerical example.

From Corollary 2 and Equation (A-1) in the appendix, we get the lower and upper boundaries of a swing option. That is, the swing option price z satisfies

$$z_{low}(t, T_0, T_1) \leq z(t, T_0, T_1) \leq z_{up}(t, T_0, T_1) < \infty \tag{11}$$

where

$$z_{up}(t, T_0, T_1) = \int_{T_0 \vee t}^{T_1} \left\{ \begin{aligned} & e^{-r(y-t)}\left[f(t, y) - K\right]p_{low}(y) + \\ & C(t, y, K)\left[p_{up}(y) - p_{low}(y)\right] \end{aligned} \right\} dy$$

Thus, Corollary 2 gives the correct swing option price if

$$e_{up} \geq \int_{T_0}^{T_1} p_{up}(y)\,dy$$

and

$$e_{low} \leq \int_{T_0}^{T_1} p_{low}(y)\,dy$$

(also if $e_{up} = e_{low}$). This is a special case, however, that is ruled out in (4). In general, $z_{low}(t, T_0, T_1)$ should be closer to $z(t, T_0, T_1)$, the closer e_{up} and e_{low} are to

$$\int_{T_0}^{T_1} p_{up}(y)\,dy$$

and

$$\int_{T_0}^{T_1} p_{low}(y)\,dy$$

The difference between the true value and the lower boundary depends on the forward curve and its uncertainties.

Because of the additional consumption jumps between the power constraints, i.e., $p_S^*(t) + p_C^*(t) = 0$ or $p_S^*(t) + p_C^*(t) = p_{up}(t) - p_{low}(t)$, we can model different swing options in our framework. If we use discrete time intervals $T_0 = \tau_1 < \ldots < \tau_{n+1} = T_1$, where $\Delta\tau = \tau_{i+1} - \tau_i$ for all $i \in \{1, \ldots, n\}$, Corollary 3 models the one-swing option whose swing period is equal to $\Delta\tau$.

Corollary 3. If A in Equation (6) is a set of consumption processes satisfying

$$\begin{aligned} &(T_1 - T_0)p_{low} \leq \Delta\tau \sum_{i \in \{1,\ldots,n\}} p(\tau_i) \leq (T_1 - T_0)p_{low} + \\ &\qquad\qquad (p_{up} - p_{low})\Delta\tau \\ &p(\tau_1) = p_{low};\, p_{low} \leq p(\tau_i) \leq p_{up} \text{ for all } i \in \{2, \ldots, n\} \end{aligned} \tag{12}$$

where electricity consumption is constant over a discrete time interval and the lower and upper boundaries are constant, then Equation (12), Proposition 1, Proposition 2, and Corollaries 1 and 2 solve the one-swing option price at time t.

Proof. From Proposition 2 we get $p(\tau_i) = p_{low}$ or $p(\tau_i) = p_{up}$, and at the beginning of the swing option delivery period we have $p(T_0) = p_{low}$ because $p(\tau_1) = p_{low}$. If the option holder chooses $p(\tau_i) = p_{up}$ at some point $\tau_i > T_0$, then due to the cumulative constraints we must have again $p(\tau_j) = p_{low}$ for all $j > i$. Thus, there is only one swing from p_{low} to p_{up} and the time spent in p_{up} is $\Delta\tau$. *Q.E.D.*

By using Proposition 2 and Corollaries 1-3 we get the value of the one-swing option at time t. Note that Proposition 2 and Corollaries 1 and 2 directly give the full-swing option price. By using the cumulative boundaries e_{low} and e_{up}, we can give constraints for the time that the consumption process p equals p_{low} and p_{up}. That is, if we use the same discrete intervals as in Corollary 3, and if the consumption process satisfies:

$$\begin{aligned}
&(T_1 - T_0)p_{low} + (p_{up} - p_{low})k_{low}\Delta\tau \le \\
&\quad \Delta\tau \sum_{i\in\{1,\dots,n\}} p(\tau_i) \le (T_1 - T_0)p_{low} + (p_{up} - p_{low})k_{up}\Delta\tau \\
&p_{low} \le p(\tau_i) \le p_{up} \text{ for all } i \in \{1, \dots, n\}
\end{aligned} \tag{13}$$

where $0 \le k_{low} \le k_{up} \le n$, then consumption is equal to p_{up} at least $\frac{k_{low}\Delta\tau}{T_1-T_0}(100\%)$ of the time and at most $\frac{k_{up}\Delta\tau}{T_1-T_0}(100\%)$ of the time.

III. EXAMPLE

We illustrate our swing option pricing model using two examples. For simplicity, we compute the option's lower boundary; i.e., we use Corollary 2. The first example solves a swing option price's lower boundary with different energy constraints, and the second illustrates how the pricing model can be used in optimizing and hedging the electricity production process. We use forward price data from the Scandinavian Exchange Nord Pool (the Nordic Power Exchange) for the year 2001.

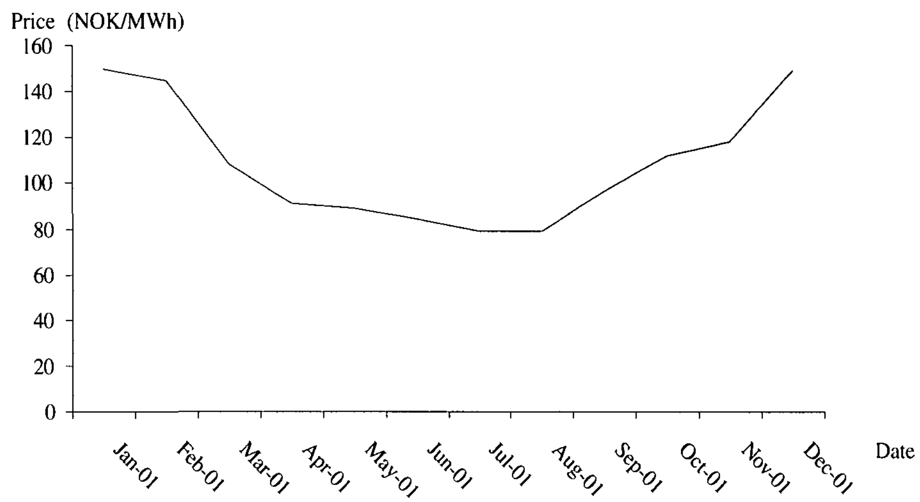
The Scandinavian electricity market is hydropower dominated; roughly 50% of the electricity supply is hydro-based. The winters are cold, and much of the precipitation comes as snow. In the spring the snow melts, causing floods whose timing varies considerably from year to year as a function of the temperature. There is a significant electricity heating load, but mild summers do not require much air conditioning, so electricity demand is concentrated in the winter season.

The time-dependent variation in demand results in seasonal, weekly, and daily patterns in electricity spot prices and the electricity forward curve. These price variations are smoothed to some extent in the Scandinavian market because of hydropower production.

Exhibit 1 illustrates the one-year forward curve during 2001 in terms of block electricity forward products whose underlying asset is the average electricity price over a four-week period. Because in the pricing of swing options we need a forward curve on the electricity spot price that has instantaneous duration, we have interpo-

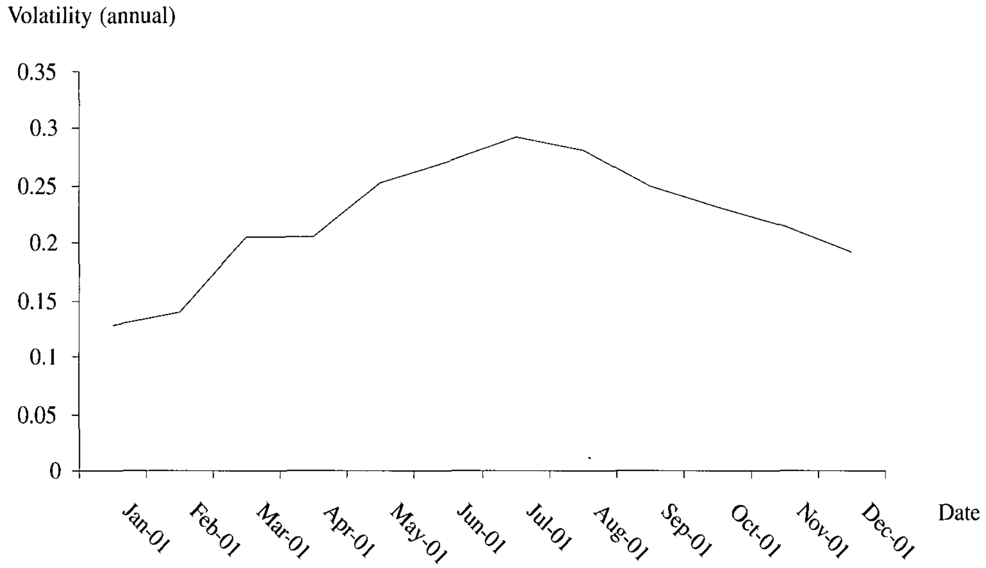
EXHIBIT 1

Forward Curve Based on Nord Pool's Block Products in 2001



NOK is Norwegian krone.

EXHIBIT 2
Historical Volatility Curve Based on Nord Pool's Block Products in 2001



lated the block data points to better model the forward curve on the spot price.

As would be expected, according to Exhibit 1, the forward price is higher during the winter and lower during the summer. Usually the price variation is much greater in Scandinavia, but 2001 was unusual in that the price was always quite close to the yearly average price. Hydrological conditions changed from relatively wet conditions to dry at the beginning of 2001. The change was due to cold and relatively dry weather in the first months of 2001, reflected as a sharp rise in the spot electricity price. Hydrological conditions then improved during the year, causing the spot price to fall toward autumn.

Exhibit 2 illustrates the corresponding historical volatility curve. Usually in Scandinavia, volatility is high during the spring because the melting snow causes floods whose timing depends on temperature. As we have noted, in winter 2000-2001 there was less snow than normal. Therefore, there was little uncertainty about the spring flood, and the volatility in Exhibit 2 is significantly lower than usual.

We assume the current date is December 1, 2000, and the swing option's delivery period is one year, from January 1, 2001, to December 31, 2001. The strike price is 100 NOK/MWh (average forward price 112 NOK/MWh), and the annual continuously compounded risk-free rate is 5%, where NOK is the Norwegian krone. Now from Exhibits 1 and 2, we can solve the European call option prices of Corollary 2 by using the Black [1976] model.

This is illustrated in Exhibit 3. The option prices for the summer months are low and for the winter period high because of the forward curve in Exhibit 1.

Swing Option Example

We analyze the swing option price's lower boundary under different energy (cumulative) constraints. The results are illustrated in Exhibit 4.

Comparing the first row and the second row, we see that if the energy lower boundary is raised, the price's lower boundary drops, because some call options are replaced with forwards. The third row implies that if the upper energy boundary is dropped, the swing option's lower boundary drops even more, because, according to Corollary 2, the call option holding is reduced.

Production Example

Now we show how the swing option model can be used in electricity production optimization and hedging. We assume production costs are constant and equal to 100 NOK/MWh. For reason of production constraints, the minimum production power p_{low} = 500 MW and the maximum production power p_{up} = 1,000 MW.

There are also boundaries on cumulative production, so the energy constraints for production between January

EXHIBIT 3

Call Option Prices

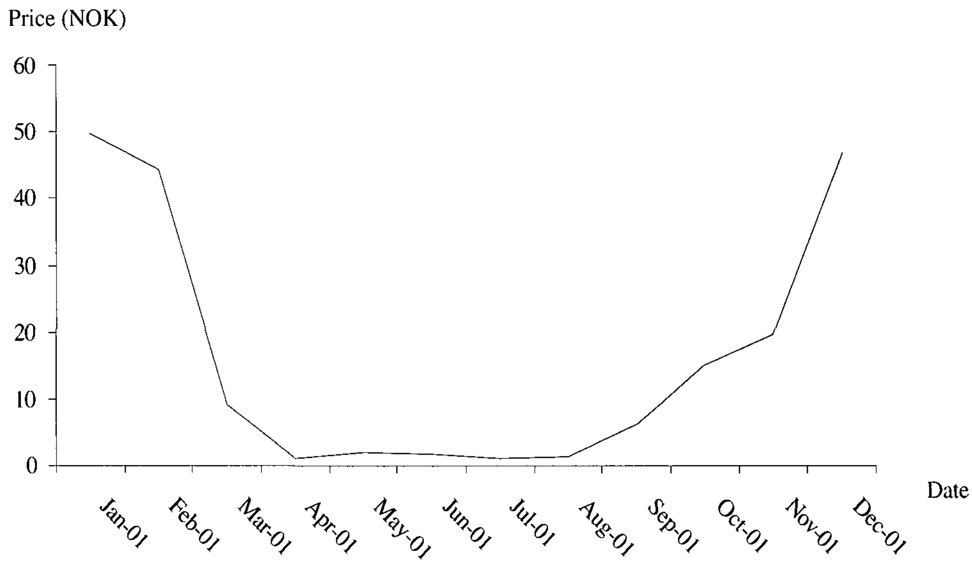


EXHIBIT 4

Lower Boundary of Swing Option Price Under Different Consumption Constraints

e_{low} (MWh)	e_{up} (MWh)	p_{low} (MW)	p_{up} (MW)	**Swing Option Price (NOK)**
400	800	0.05	0.1	10456
600	800	0.05	0.1	10196
600	600	0.05	0.1	9693

1 and December 31, 2001, are $e_{low} = 4\ (10^6)$ MWh and $e_{up} = 8\ (10^6)$ MWh. In the case of a hydroelectric plant, these upper and lower energy constraints correspond to the size of the water reservoir (see Pereira [1989], Gjelsvik, Røtting, and Røynstrand [1992], and Keppo [2002]).

From the swing option's instantaneous and cumulative constraints, we see that owning a power plant is similar to holding a swing option with strike price equal to production cost, so we model the production strategy using Corollary 2. Exhibit 5 illustrates the production strategy that corresponds to the forward holding p_S and that can be hedged by selling the electricity forwards.

Note in Exhibit 5 that production is high during the winter and low during the summer. This is due to the shape of the forward curve in Exhibit 1.

Exhibit 6 illustrates the optional production strategy p_C. Optional electricity is produced during the summer if the electricity spot price is higher than production cost. Because this optional production strategy corresponds to the call option holding, this part of the production can be hedged by selling call options.

According to Exhibits 5 and 6, the energy production strategy of a hydropower plant, for example, can be modeled as $p_{low} + p_S + p_C$, where $p_{low} + p_S$ is a production strategy that depends only on time and p_C is an optional production strategy that depends on time and electricity spot price. Therefore, the production component $p_{low} + p_S$ can be hedged by selling forwards and p_C by selling call options.

IV. CONCLUSIONS

We show in the pricing and hedging of electricity swing options that the swing options can be replicated using regular electricity forwards and call options. Therefore, the pricing of swing options can be based on electricity market data available from electricity exchanges.

The hedging strategy for a swing option may be solved following a constrained optimization problem. Because this optimization problem requires numerical methods, we introduce a practical hedging strategy that replicates the swing option's lower boundary. In this

EXHIBIT 5
Production Strategy

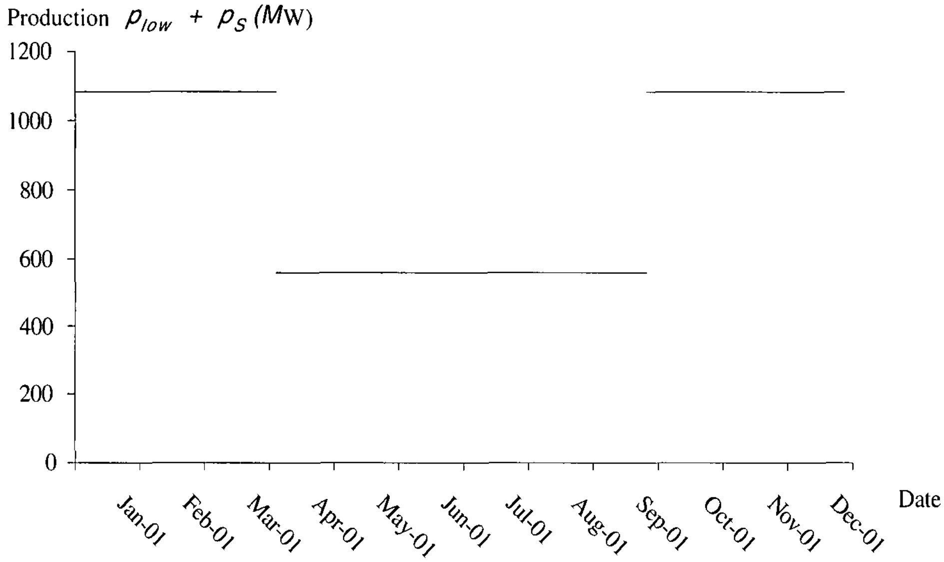
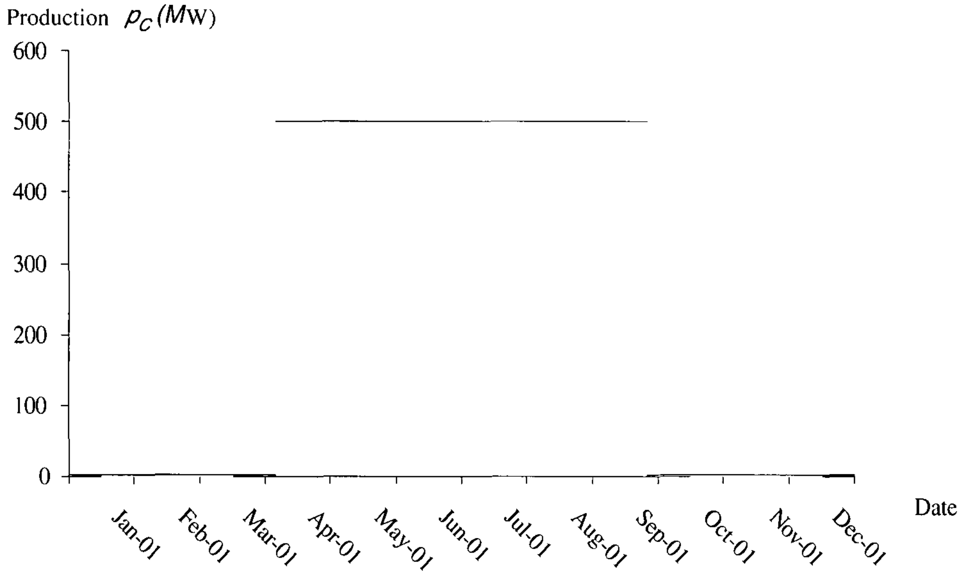


EXHIBIT 6
Optional Production Strategy



hedging strategy, the minimum energy amount is modeled using electricity forward holdings and the optional energy amount using call option holdings. Because electricity options are functions of the forward volatility, electricity forward curve uncertainty affects the swing option price more if there is flexibility in the purchased energy amount.

The pricing model helps us understand how different variables affect swing option prices. By providing a hedging strategy, the model also makes the production of these instruments easier.

As a numerical example illustrates, the swing option pricing framework helps in the optimization and hedging of power plant operation, because many power plants have production and consumption constraints that follow patterns of swing options.

APPENDIX A

Proof of Proposition 2

From Proposition 1 and Equation (6), we get by using dynamic programming, since p_C is a function of p_S (see Hillier and Lieberman [2001]):

$$z(t,T_0,T_1)=\int_{T_0}^{T_1} e^{-r(y-t)}(f(t,y)-K)p_{low}(y)dy+$$
$$\sup_{p_S\in A_S}\left\{E^Q\left[\int_{T_0}^{T_1} e^{-r(y-t)}(S(y)-K)p_S(y)dy\middle|F_t\right]+\right.$$
$$\left.\sup_{p_C\in A_C(p_S)} E^Q\left[\int_{T_0}^{T_1} e^{-r(y-t)}\mathbf{1}\{S(y)\ge K\}(S(y)-K)p_C(y)dy\middle|F_t\right]\right\} \tag{A-1}$$

where $t < T_0$,

$$z(t,T_0,T_1)\le\int_{T_0}^{T_1}\left\{\begin{array}{l} e^{-r(y-t)}\left[f(t,y)-K\right]p_{low}(y)+ \\ \mathbf{1}\left\{e_{up}>e_{low}\right\}C(t,y,K)\times \\ \left[p_{up}(y)-p_{low}(y)\right]\end{array}\right\}dy<\infty$$

and the T-maturity European call option price with strike price K at time t is $C(t, T, K) = e^{-r(T-t)}E^Q[\mathbf{1}\{S(T) \ge K\}(S(T) - K) \mid F_t]$. Note that the optimal consumption $p_{low} + p^*_S(t) + p^*_C(t)$ depends on power and energy constraints as well as on the forward curve and its dynamics.

Next we represent Equation (A-1) under the electricity forward measures. That is, by using the absolute value of a T-maturity forward contract that has forward price equal to K as a numeraire with the second term of (A-1) we get (see, e.g., Geman [1989] and Björk [1998]):

$$\frac{\pi_S(t\,T)}{e^{-r(T-t)}|f(t,T)-K|}=E^{Q_{f(t,T)}}\left[\frac{(S(T)-K)p_S(T)}{|S(T)-K|}\right]$$
$$=E^{Q_{f(t,T)}}\left[\begin{pmatrix}\mathbf{1}\{S(T)\ge K\}- \\ \mathbf{1}\{S(T)<K\}\end{pmatrix}p_S(T)\right]=p_S(t,T,K)$$
$$\text{for all } t\in[0,T],\ T\in[T_0,T_1] \tag{A-2}$$

where $\pi_S(t, T) = e^{-r(T-t)}E^Q[p_S(T)(S(T) - K) \mid F_t]$; $f(t, T) = E^Q[S(T) \mid F_t]$, $P(|S(T) - K| > 0) = 1$, and the numeraire forward contract (forward price K) is equal at time t to a portfolio of one long forward contract with forward price $f(t, T)$ and $e^{-r(T-t)}(f(t, T) - K)$ in the risk-free asset.

Note first that if $f(t, T) = K$ in (A-2), we set $\pi_S(t, T) = 0$. Second, we can use the absolute value of the forward contract as a numeraire since $|f(y, T) - K| > 0$, with probability one, for each $y \in (t, T]$. In the same way, the third term of (A-1):

$$\frac{\pi_C(t,T)}{e^{-r(T-t)}|f(t,T)-K|}$$
$$=E^{Q_{f(t,T)}}\left[\frac{\mathbf{1}\{S(y)\ge K\}(S(T)-K)p_C(T)}{|S(T)-K|}\middle|F_t\right]$$
$$=E^{Q_{f(t,T)}}\left[\mathbf{1}\{S(T)\ge K\}p_C(T)\middle|F_t\right]$$
$$=p_C(t,T,K)\text{ for all } t\in[0,T],\ T\in[T_0,T_1] \tag{A-3}$$

where $\pi_C(t, T) = e^{-r(T-t)}E^Q[p_C(T)\mathbf{1}\{S(T) \ge K\}S(T) - K)F_t]$.

That is: $p_S(t, t, K) = p_S(t)(\mathbf{1}\{S(t) \ge K\} - \mathbf{1}\{S(t) < K\})$ and $p_C(t, t, K) = p_C(t)(\mathbf{1}\{S(t) \ge K\}$.

From (A-1), we now get by changing the order of the integrations (we assume technical conditions for this; see Ikeda and Watanabe [1981]):

$$z(t,T_0,T_1)=\int_{t\vee T_0}^{T_1} e^{-r(y-t)}p_{low}(y)\left[f(t,y)-K\right]dy+$$
$$\sup_{p_S\in A_S,\,p_C\in A_C(p_S)}\int_{t\vee T_0}^{T_1}\left[p_S(t,y,K)+p_C(t,y,K)\right]e^{-r(y-t)}|f(t,y)-K|dy \tag{A-4}$$

where $t \vee T = max\ [t, T]$ and $t < T_1$.

This optimization problem is hard to solve because it is path-dependent and also depends on the Radon-Nikodym derivative $\frac{dQ_{f(t,T)}}{dP}$ as well as on the forward curve dynamics. Because the objective function is linear with respect to p_S and p_C, $p^*_S(t)$, $p^*_C(t) \in [0, p_{up}(t) - p_{low}(t)]$, and because if

$$\int_0^{t-} p(y)dy<e_{up}$$

instantaneous consumption $p(t)$ does not violate this cumulative constraint ($p_{up}(t)$ is bounded), we can characterize the optimal terms $p^*_S(t) = p^*_S(t, \omega)$ and $p^*_C(t) = p^*_C(t, \omega)$ by using indicator functions as follows:

$$p^*_S(t,\omega)=\left[p_{up}(t)-p_{low}(t)\right]\mathbf{1}\left\{p^*_S(t,\omega)>0\right\}$$
$$p^*_C(t,\omega)=\left[p_{up}(t)-p_{low}(t)\right]\mathbf{1}\left\{p^*_C(t,\omega)>0\right\}$$
$$\text{for all } t\in[T_0,T_1] \tag{A-5}$$

where we assume that we know the sets $\{\omega \mid p^*_S(t, \omega) > 0\}$ and $\{\omega \mid p^*_C(t, \omega) > 0\}$ for all $t \in [T_0, T_1]$, and where we have written the consumption terms as functions of event $\omega \in \Omega$ to emphasize the fact that electricity consumption is a stochastic process. As we will see, the main problem in the swing option pricing is to solve these sets. They depend on the forward curve and its uncertainties at time t as well as the energy used up to time t.

By using Equations (A-4) and (A-5) we can characterize (A-1) as follows:

$$z(t, T_0, T_1) = \int_{t\vee T_0}^{T_1} e^{-r(y-t)} p_{low}(y)(f(t,y) - K)dy +$$

$$\int_{t\vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right] \times$$

$$e^{-r(y-t)}|f(t,y) - K| \begin{bmatrix} Q_{f(t,y)}\left(p_S^*(y) > 0, S(y) \geq K\right) - \\ Q_{f(t,y)}\left(p_S^*(y) > 0, S(y) < K\right) + \\ Q_{f(t,y)}\left(p_C^*(y) > 0, p_S^*(y) = 0\right) \end{bmatrix} dy \tag{A-6}$$

where $Q_{f(t,T)}(x > 0, y > 0) = E^{Q_{f(t,T)}}[\mathbf{1}\{x > 0\}\mathbf{1}\{y > 0\} \mid F_t]$ is the probability under $Q_{f(t,T)}$ that x and y are strictly positive;

$$\int_{t\vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right]\mathbf{1}\left\{p_S^*(y) > 0\right\}dy = e_S(t)$$

$$\int_{T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right]\mathbf{1}\left\{p_C^*(y) > 0\right\}dy \leq e_C(t)$$

$$e_S(t) = e_{low} - \int_{T_0}^{T_1} p_{low}(y)dy - \int_{T_0}^{t\vee T_0} p_S(y)dy$$

and

$$e_C(t) = e_{up} - e_{low} - \int_{T_0}^{t\vee T_0} p_C(y)dy$$

This gives Proposition 2. *Q.E.D.*

APPENDIX B

Proof of Corollary 1

To prove Corollary 1, we proceed in the same way as for Proposition 2.

By using the electricity call option as a numeraire with the third term of Equation (A-1) we get:

$$\frac{\pi_C(t, T)}{C(t, T, K)} = E^{Q_{C(t,T,K)}}\left[p_C(T)|F_t\right] = p_C(t, T)$$
$$\text{for all } t \in [0, T],\ T \in \left[T_0, T_1\right] \tag{B-1}$$

where $\pi_C(t, T) = e^{-r(T-t)}E^Q[p_C(T)\mathbf{1}\{S(T) \geq K\}S(T) - K)F_t]$ and European call option $C(t, T, K) = e^{-r(T-t)}E^Q[\mathbf{1}\{S(T) \geq K\}S(T) - K)F_t]$.

That is, $p_C(t, t) = p_C(t)$. Note that in (A-3) the indicator $\mathbf{1}\{S(T) \geq K\}$ is inside the expectation, because the numeraires of (A-3) and (B-1) are different. Because $C(t, T, K) > 0$ for all $T > t$, we have $\pi_C(t, T) \geq 0$. As in the proof of Proposition 2, we can use the value of the call option as a numeraire since it is always positive.

From (A-1), (A-2), and (B-1), we get by changing the order of the integrations:

$$z(t, T_0, T_1) = \int_{t\vee T_0}^{T_1} e^{-r(y-t)} p_{low}(y)(f(t,y) - K)dy +$$

$$\sup_{p_S \in A_S}\left\{\int_{t\vee T_0}^{T_1} p_S(t, y, K)e^{-r(y-t)}|f(t,y) - K|dy + \sup_{p_C \in A_C(p_S)} \int_{t\vee T_0}^{T_1} p_C(t, y)C(t, y, K)dy\right\} \tag{B-2}$$

where $t < T_1$. By using (A-5) and (B-2) we get

$$z(t, T_0, T_1) = \int_{t\vee T_0}^{T_1} e^{-r(y-t)} p_{low}(y)(f(t,y) - K)dy +$$

$$\int_{t\vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right]e^{-r(y-t)}|f(t,y) - K| \times$$

$$\left[Q_{f(t,y)}\left(p_S^*(y) > 0, S(y) \geq K\right) - Q_{f(t,y)}\begin{pmatrix} p_S^*(y) > 0, \\ S(y) < K \end{pmatrix}\right]dy +$$

$$\int_{t\vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right]C(t, y, K)Q_{C(t,y,K)}\left(p_C^*(y) > 0, p_S^*(y) = 0\right)dy \tag{B-3}$$

where

$$\int_{t\vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right]\mathbf{1}\left\{p_S^*(y) > 0\right\}dy = e_S(t)$$

and, because the value of an option is always non-negative, we have now the equality constraint for p_C^*:

$$\int_{t\vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right]\mathbf{1}\left\{p_C^*(y) > 0\right\}dy = e_C(t)$$

This gives Corollary 1. *Q.E.D.*

ENDNOTE

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