Calculation of Volatility in a Jump-Diffusion Model

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A common way to incorporate discontinuities in a model of asset returns is to add a Poisson process to a Brownian motion. The jump-diffusion process provides probability distributions that typically fit market data better than distributions of a simple diffusion process. To compare the performance of models in pricing options, the total volatility of the jump-diffusion process must be used in the Black-Scholes formula.

Several authors miscalculate this volatility because they do not fully include the effect of uncertainty on the jump size. We calculate the volatility correctly and show how this affects option prices.

The diffusion process is widely used to describe the evolution of asset returns over time. In option pricing, this enables the use of Black and Scholes [1973] formulas to value European options on stocks, foreign currencies, interest rates, and commodities and futures. Under this process, instantaneous asset returns are normally distributed. The distributions observed in the market, however, exhibit non-zero skewness and higher kurtosis than the normal distribution, which produces pricing errors when the Black-Scholes formula is used.[1]

One way to obtain distributions consistent with market data is to assume that the stock price follows a jump-diffusion process. Merton [1976b] develops an approach that models the arrival of normal information as a diffusion process and the arrival of abnormal information as a Poisson process. The jump-diffusion process can potentially describe stock prices more accurately at the cost of making the market incomplete, since jumps in the stock price cannot be hedged using traded securities.

If the market is incomplete, the payoffs of the option cannot be replicated, and the option cannot be priced. To overcome this problem, Merton assumes that the jump risk is diversifiable and consequently not priced in equilibrium. He then derives a closed-form expression for the price of a call option.

To study the performance of the jump-diffusion model, Merton [1976a] compares option prices computed with his model with those obtained with the Black-Scholes formula. For this comparison to be meaningful, the total volatility of the jump-diffusion process (v) must be used in the Black-Scholes formula.

Merton [1976a, 1976b] miscalculates this volatility because he overlooks the full effect of jump size uncertainty on return volatility, so his measure is smaller than v. We refer to this measure as "Merton's volatility rate" (v_M). Merton finds there can be significant differences in option prices for deep out-of-the-money options and short-maturity options when there are large and infrequent jumps.

Ball and Torous [1985] analyze a sample of New York Stock Exchange listed common stocks, and find lognormal jumps in most of the daily returns considered. They then study Merton and Black-Scholes call option prices and find small differences. They too miscalculate the total volatility of the jump-diffusion process, so that their results should be interpreted with care.

Other examples of incorrect computation of the volatility of the jump-diffusion process are Jorion [1988], who adapts Merton's model to price foreign currency options, and Amin [1993], who develops a discrete-time option pricing model under a jump-diffusion process.

In this note we show how to compute the total volatility of the jump-diffusion process correctly, and examine the effect of the miscalculation on option prices. We find that that pricing errors can be higher or lower than those previously reported, and that some options undervalued with υ_M are in fact overvalued using the correct volatility measure.

I. MERTON'S JUMP-DIFFUSION FORMULA

We consider a continuous trading economy with trading interval $[0, \tau]$ for a fixed $\tau > 0$, in which there are three sources of uncertainty, represented by a standard Brownian motion $\{Z(t): t \in [0, \tau]\}$ and an independent Poisson process $\{N(t): t \in [0, \tau]\}$ with constant intensity λ and random jump size Y on the filtered probability space $\{\Omega, \mathcal{F}, Q, \{\mathcal{F}_t: t \in [0, \tau]\}\}$.

We define the broader filter $\mathcal{F}_t \equiv \mathcal{F}^Z_t \vee \mathcal{F}^N_t$ and $\mathcal{F} \equiv \mathcal{F}_\tau$, where $\mathcal{F}^Z_t$ and $\mathcal{F}^N_t$ are the smallest right-continuous complete σ-algebras generated by $\{Z(s): s \leq t\}$ and $\{N(s): s \leq t\}$, respectively. (The symbol $\vee$ denotes "the greater of.")

Merton [1976b] assumes that the stock price dynamics are described by the stochastic differential equation:

$$\frac{dS(t)}{S(t-)} = (\alpha - \lambda\kappa)\, dt + \sigma dZ(t) + (Y(t) - 1)\, dN(t) \tag{1}$$

where α is the instantaneous expected stock return conditional on no jumps; $\kappa \equiv E\{Y(t) - 1\}$ is the expected relative jump of $S(t)$; and σ is the instantaneous volatility of returns conditional on no jumps.

Assuming that jump sizes are lognormally distributed with parameters μ and δ, and that the jump risk is diversifiable, Merton shows that the option price is given by:

$$F(S(t), \tau, K, \sigma^2, r; \mu, \delta^2, \lambda) = \sum_{n=0}^{\infty} \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!} \times W(S(t), \tau, K, \sigma_n^2, r_n) \tag{2}$$

where $W(S(t), \tau, K, \sigma_n^2, r_n)$ is the Black-Scholes option price for a European option with exercise price K and maturity τ on a non-dividend-paying stock. The risk-free interest rate is r, and:

$$\begin{aligned}
\lambda' &\equiv \lambda(1+\kappa) = \lambda \exp\left(\mu + \frac{\delta^2}{2}\right) \\
\sigma_n^2 &\equiv \sigma^2 + \frac{n}{\tau}\delta^2 \\
r_n &\equiv r + \frac{n}{\tau}\left(\mu + \frac{\delta^2}{2}\right) - \lambda\,\kappa
\end{aligned}$$

When changes in stock prices are given by Equation (1), the instantaneous stock returns are not normally distributed. The distribution will have non-zero skewness and will exhibit leptokurtosis compared to a Gaussian distribution, which is consistent with the empirical evidence.

There will not be, in general, a closed-form expression for the density function of the distribution, but we can easily compute the moments of stock returns. Applying Itô's formula to (1), we have that, under Q, the expected natural logarithm of the stock price is given by:

$$\begin{aligned}
E\left\{\ln\frac{S(t)}{S(0)}\right\} &= \left(\alpha - \lambda\kappa - \frac{\sigma^2}{2}\right)t + E\left\{E\left\{\ln Y(n(t))|\mathcal{F}^N_T\right\}\right\} \\
&= \left(\alpha - \lambda\kappa - \frac{\sigma^2}{2}\right)t + E\{n(t)\mu\} \\
&= \left(\alpha - \lambda\kappa - \frac{\sigma^2}{2}\right)t + \lambda t \mu
\end{aligned}$$

where $Y(n) \equiv \Pi^n_{i=1} Y_i$, $\{Y_i, i = 1, 2, \ldots, n\}$ are the random jump amplitudes, and $n(t)$ is a Poisson random variable with parameter λt.

Since the Brownian motion and the Poisson process are independent by construction (see Protter [1995]), to calculate the variance, we can write:

$$Var\left\{\ln\frac{S(t)}{S(0)}\right\} = Var\{\sigma Z(t)\} + Var\{\ln Y(n(t))\} \tag{3}$$

Notice that $\Sigma^n_{i=1} \ln Y_i$ is conditionally distributed as a Gaussian distribution with mean $n\mu$ and variance $n\delta^2$. This does not imply that Var $\{\ln Y(n(t)\} = \lambda t\delta^2$ since now $n(t)$ is random. Using the fact that for any random variable u, Var$\{u\} = -E\{u\}^2 + E\{u^2\}$, we have that:

$$\begin{aligned}
Var\{\ln \mathrm{Y}(\mathrm{n(t)})\} &= -E\{\ln \mathrm{Y}(\mathrm{n(t)})\}^2 + E\left\{[\ln \mathrm{Y}(\mathrm{n(t)})]^2\right\} \\
&= -(\lambda t\mu)^2 + E\left\{E\left\{[\ln \mathrm{Y}(\mathrm{n(t)})]^2 \,|\mathcal{F}_{\mathrm{T}}^{\mathrm{N}}\right\}\right\} \\
&= -(\lambda t\mu)^2 + E\left\{n(t)\delta^2 + n(t)^2\mu^2\right\} \\
&= -(\lambda t\mu)^2 + \delta^2\lambda t + \mu^2\left(\lambda t + \lambda^2 t^2\right) \\
&= \lambda\left(\mu^2 + \delta^2\right)t \qquad (4)
\end{aligned}$$

Hence, the total variance of the natural logarithm of the stock price under a jump-diffusion process is given by

$$Var\left\{\ln\frac{\mathrm{S(t)}}{\mathrm{S(0)}}\right\} = \left(\sigma^2 + \lambda\left(\mu^2+\delta^2\right)\right)t \qquad (5)$$

A similar result can be found in Press [1967], Navas [1994], and Das and Sundaram [1999]. Merton [1976a, 1976b], Ball and Torous [1985], Jorion [1988], and Amin [1993] miscalculate this variance as $(\sigma^2 + \lambda\delta^2)t$, which will be correct only when $\mu = 0$. They assume instead that the expected jump size is zero, i.e., $E\{Y-1\} = 0$, which is equivalent to taking $\mu = -(\delta^2/2)$. Obviously, when δ^2 increases, the error in the variance will increase.

II. IMPLICATIONS FOR OPTION PRICING

If an investor prices options according to the Black-Scholes model, but the market prices options according to Merton's jump-diffusion process, she will underprice out-of-the-money and in-the-money options, and she will overprice at-the-money (ATM) options, since the true stock return distribution will have fatter tails than the Gaussian distribution. Hence, if investors estimate the volatility parameter of the diffusion process implicitly from market prices, they will observe the well-known volatility smile.

In Exhibit 1 we price a call option with different exercise prices using Merton's [1976b] model with the parameters $\tau = 0.5$, $\sigma^2 = 0.05$, $r = 0.10$, $\mu = -0.025$, $\delta^2 = 0.05$, and $\lambda = 1$. Then, we take those values as market prices and compute implied volatilities using the Black-Scholes formula.

We see that in order to fit market prices correctly, the volatility of the diffusion process in the Black-Scholes model must be set higher for out-of-the-money (OTM) and in-the-money (ITM) options than for ATM options. The curve is not symmetric because of the presence of skewness in the distribution of stock returns.

We examine the pricing error that our investor will face when she estimates the volatility of stock returns from the time series generated by the jump-diffusion process, and then uses the Black-Scholes formula to price options. In this case, her estimate of the volatility of stock returns will be an unbiased estimate of the total volatility of the jump-diffusion process (see Merton [1976a] for further details).

Exhibit 2 compares Merton and Black-Scholes prices for different parameters of the jump-diffusion process. The exhibit draws upon Ball and Torous [1985, Table IV]. They value a six-month stock call option with exercise price 35 when the stock is trading at 38. The variance of the diffusion part is $\sigma^2 = 0.05$, and the interest rate is $r = 0.10$.

In the first and second rows, we follow Merton [1976a, 1976b] and Ball and Torous [1985] where the expected relative jump size of the stock (κ) is 0; i.e., we take $\mu = -(\delta^2/2)$. To be consistent with their notation, we represent Merton prices by F; that is, $F \equiv F(S(t), \tau, K, \sigma^2, r; \mu, \delta^2, \lambda)$. We denote the investor appraised option value using the Black-Scholes formula by $F_c(\upsilon^2)$, and $F_c(\upsilon_{\mathrm{M}}^2)$, depending on the variance rate used; that is, $F_c(\upsilon^2), \equiv W(S(t), \tau, K, \upsilon^2, r)$ and $F_c(\upsilon_{\mathrm{M}}^2) \equiv W(S(t), \tau, K, \upsilon_{\mathrm{M}}^2, r)$.[2]

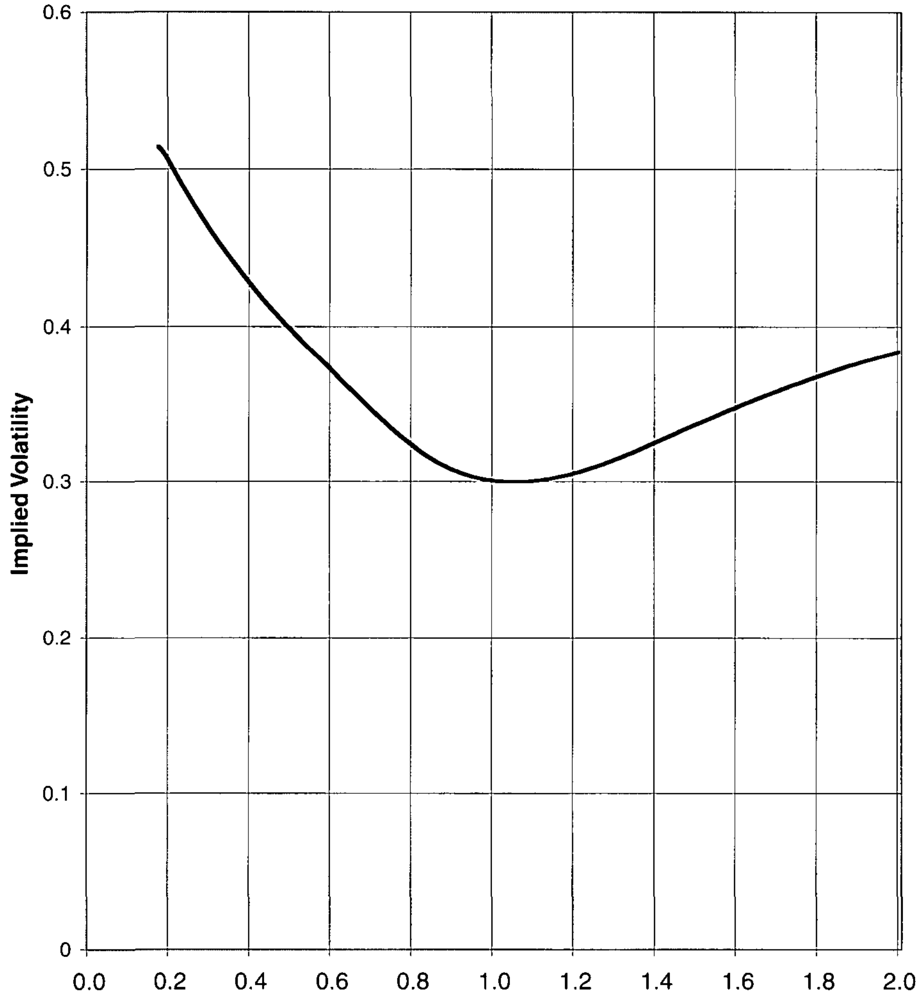
In the first row, we assume the variance of the log of the jump size is equal to the variance of the diffusion ($\delta^2 = 0.05$), and there is only one jump per year on average ($\lambda = 1$). This implies that the variance of the diffusion process is practically half of the total variance of the process. With these values, the Merton option price is $F = 5.9713$.

As a reference, the Black-Scholes price *if* there were no jumps would be $F_c(\sigma^2) = 5.3396$. This will not be the appraised option value of an investor who incorrectly believes that the stock price follows a diffusion process and estimates the volatility from historical data, however. As stated before, her estimate of the variance will be $\upsilon^2 = 0.10062$ and her appraised option value $F_c(\upsilon^2) = 6.0711$. In this case the investor will overprice the option by only 1.7%.

The pricing error increases, however, as there are larger and fewer jumps per year. For instance, when $\delta^2 = 0.50$ and $\lambda = 0.1$ (second row), the Merton option price is $F = 5.6979$, while the investor-estimated value is $F_c(\upsilon^2) = 6.1447$ (7.8% higher). Exhibit 2 also shows the appraised option values incorrectly computed by Ball and Torous [1985] using υ_M, $F_c(\upsilon_M^2)$.

An interesting question is whether the lack of significant difference between the Black-Scholes and Merton model prices found by Ball and Torous [1985] could be

EXHIBIT 1
Implied Black-Scholes Volatility from Merton [1976b] Call Option Prices



Parameters of the model are $\tau = 0.5$, $\sigma^2 = 0.05$, $r = 0.10$, $\mu = -0.025$, $\delta^2 = 0.05$, and $\lambda = 1$.

due to their miscalculation of the total volatility of the process. Using their parameter estimates, we find that the difference between υ and υ_M is insignificant for most stocks. Thus, using the correct volatility measure apparently will not change their results.

Rows 3-10 of Exhibit 2, however, show that if we relax the Ball and Torous assumption that the mean relative jump size is zero, the difference in option prices can be great. For example, when investors expect one jump per year that will produce a mean relative jump in stock price of −20% (row 9), $F = 6.6872$, $F_c(\upsilon_M^2) = 6.0628$, and $F_c(\upsilon^2) = 6.8066$.

That is, an investor incorrectly using the Black-Scholes formula and miscalculating the volatility of the process will underprice the option by 9.3%, while using the correct volatility she will overprice it by 1.8%. Hence, the restriction imposed on the jump size together with the miscalculation of the volatility could partially explain the results of Ball and Torous [1985].

We can further analyze the pricing errors standardizing the variables as in Merton [1976a]. In this way, we reduce from eight to four the number of parameters needed to compute option prices in the jump-diffusion model. Like other authors, Merton considers the special case in which $\mu = -(\delta^2/2)$. Substituting $\kappa = 0$ into Equation (2), the option pricing formula simplifies to

EXHIBIT 2
Call Option Prices for Different Parameters of the Jump-Diffusion Process

	δ^2	λ	μ	ν_M^2	ν^2	F	$F_e(\sigma^2)$	$F_e(\nu_M^2)$	$F_e(\nu^2)$
$\kappa = 0$	0.05	1.00	-0.025	0.10	0.10062	5.9713	5.3396	6.0628	6.0711
	0.50	0.10	-0.250	0.10	0.10625	5.6979	5.3396	6.0628	6.1447
$\kappa = 0.1$	0.05	1.00	0.070	0.10	0.10494	5.9647	5.3396	6.0628	6.1277
	0.50	0.10	-0.155	0.10	0.10239	5.6826	5.3396	6.0628	6.0944
$\kappa = 0.2$	0.05	1.00	0.157	0.10	0.12475	6.1554	5.3396	6.0628	6.3778
	0.50	0.10	-0.068	0.10	0.10046	5.6758	5.3396	6.0628	6.0689
$\kappa = -0.1$	0.05	1.00	-0.130	0.10	0.11699	6.2055	5.3396	6.0628	6.2817
	0.50	0.10	-0.355	0.10	0.11263	5.7234	5.3396	6.0628	6.2266
$\kappa = -0.2$	0.05	1.00	-0.248	0.10	0.16158	6.6872	5.3396	6.0628	6.8066
	0.50	0.10	-0.473	0.10	0.12239	5.7603	5.3396	6.0628	6.3488

Source: Draws on Ball and Torous [1985, Table IV] who assume that the mean relative jump size of the stock (κ) is zero (rows one and two). The other parameters of the model are $S(0) = 38$, $\tau = 0.5$, $K = 35$, $\sigma^2 = 0.05$, and $r = 0.10$, where σ is the volatility of the diffusion process. F represents the Merton model price, and F_e is the investor-appraised option value using the Black-Scholes formula. The total volatility per unit of time of the jump-diffusion process (needed to compute F_e) is denoted by υ, and υ_M is the volatility Ball and Torous use to compute Black-Scholes prices.

$$F = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} W(S(t), \tau, K, \sigma_n^2, r) \qquad (6)$$

where $\sigma_n^2 = \sigma^2 + (n/\tau)\delta^2$.

If we define $W'(S, \tau) \equiv W(S, \tau, 1, 1, 0)$, it is easy to show that

$$W'(X, \tau_n) = \frac{W(S, \tau, K, \sigma_n^2, r)}{Ke^{-r\tau}}$$

where $X \equiv S/Ke^{-r\tau}$ and $\tau_n = \sigma_n^2 \tau$.

With this notation, we rewrite Equation (6) as

$$F = Ke^{-r\tau} \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} W'(X, \tau_n) \qquad (7)$$

Similarly, we can express the investor's value of the option using υ_M as

$$F_e(\nu_M^2) \equiv W(S, \tau, K, \nu_M^2, r) = Ke^{-r\tau} W'(X, \bar{\tau}) \qquad (8)$$

where $\bar{\tau} \equiv \upsilon_M^2 \tau$, which can be considered a maturity measure of the option.

Merton [1976a] defines two other standardized variables: Γ, a variance measure, and Λ, the expected number of jumps during the life of the option divided by the maturity measure $\bar{\tau}$.[3] Their expressions are given by:

$$\Gamma \equiv \frac{\lambda\delta^2}{\nu_M^2},$$
$$\Lambda \equiv \lambda\frac{\tau}{\bar{\tau}}$$

From these definitions it is easy to show that $\tau_n = (1 - \Gamma)\,\bar{\tau} + n(\Gamma/\Lambda)$ and $\lambda\tau = \Lambda\bar{\tau}$.

Finally, from Equations (7) and (8), we can express the option prices normalized by the present value of the exercise price as:

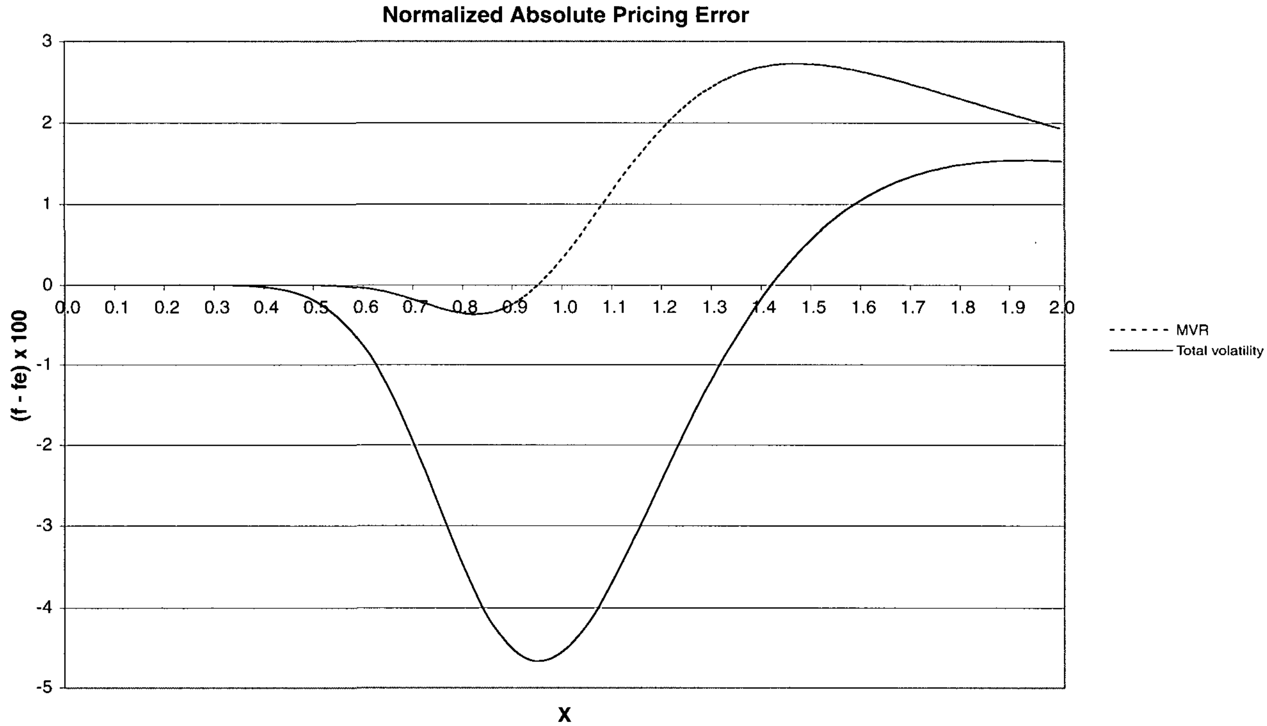
$$f = \sum_{n=0}^{\infty} \frac{e^{-\Lambda\bar{\tau}}(\Lambda\bar{\tau})^n}{n!} W'\left(X, (1-\Gamma)\bar{\tau} + n\frac{\Gamma}{\Lambda}\right) \qquad (9)$$

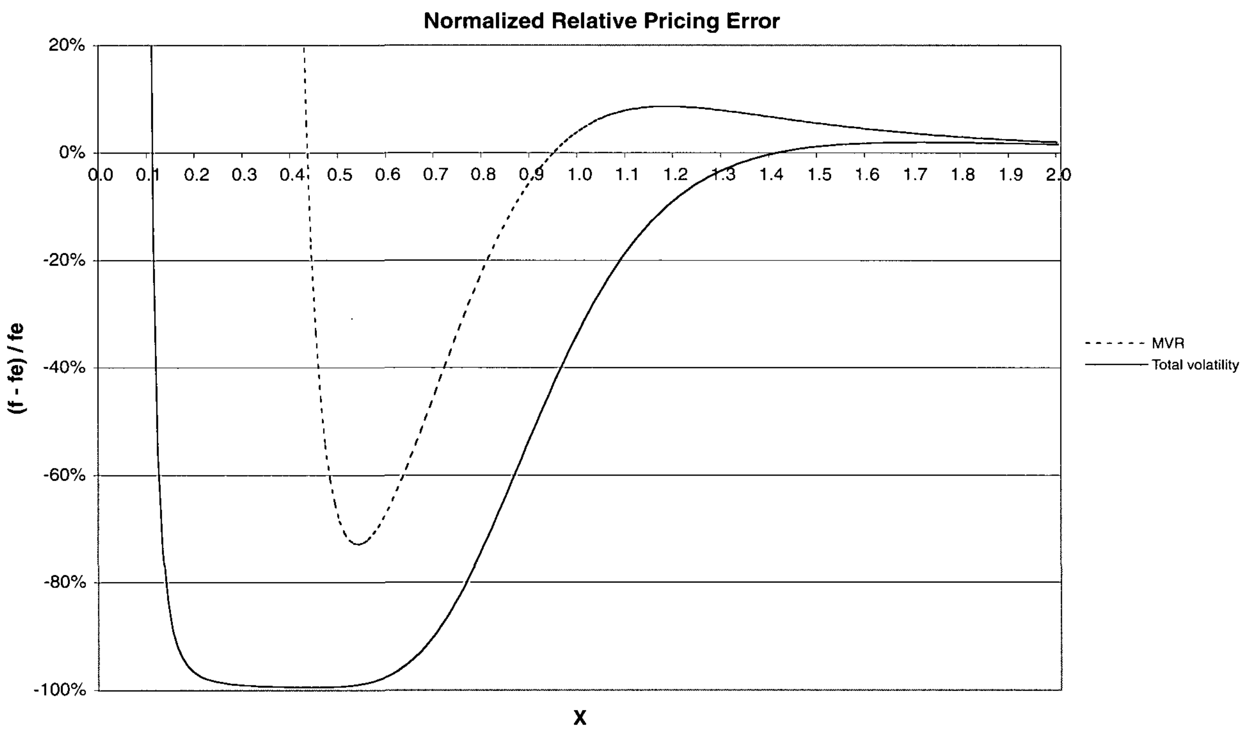
and

$$f_{eM} \equiv \frac{F_e(\nu_M^2)}{Ke^{-r\tau}} = W'(X, \bar{\tau}) \qquad (10)$$

Exhibit 3 shows the absolute and relative normalized pricing errors, defined as $f - f_e$ and $(f - f_e)/f_e$, respectively, for different stock prices. We take $\bar{\tau} = 0.0492$, $\Gamma = 0.3670$, and $\Lambda = 1.8084$. This represents, for example, a one-year call option with an annual Merton volatility rate (υ_M) of

EXHIBIT 3
Option Price Error for Actual Parameter Values





MVR stands for Merton's volatility rate. Stock and option prices are expressed in units of the present value of the exercise price. X is the normalized stock price, f is the normalized Merton [1976a] call option price, and f_e is the normalized investor estimate of the option value using the Black-Scholes formula. The parameters of the model are $\tau = 0.0492$, $\Gamma = 0.367$, and $\Lambda = 1.808$, and are obtained from Andersen and Andreasen [1999].

0.2218, when the number of jumps per year (λ) is 0.0890 and the volatility of the diffusion part (σ) is 0.1765.

These values are obtained from Andersen and Andreasen [1999], who estimate the jump-diffusion parameters for a sample of S&P 500 index options, without assuming that $\mu = -(\delta^2/2)$. Notice that, in this case, the total volatility of the process (υ) is 0.3459, considerably higher than Merton's volatility rate. An investor using υ_M in the Black-Scholes formula will observe a pricing error that is larger (in absolute value) for deep in-the-money options and smaller for most of the other options than what she would actually observe in the market (using υ).

Moreover, if the normalized stock price (X) is higher than 0.94, the Black-Scholes model with υ_M underprices the option, while with υ the model overprices the option. For example, when $S = 100$ and $K = 100$ ($X = 1.10$), an investor using υ_M in the Black-Scholes formula will underprice the option by 7.85%, when in fact the model overprices it by 18.33%. Thus, the practical relevance of the miscalculation of the total variance of the jump-diffusion process can be significant.

III. CONCLUSIONS

If the true process describing the dynamics of stock returns is a jump-diffusion process, but investors incorrectly believe that stock returns follow a diffusion process, they will use the Black-Scholes formula to price options. If they estimate the volatility of stock returns from time series data, they will actually estimate the total volatility of the jump-diffusion process.

Using this volatility in the Black-Scholes formula produces pricing errors. Many authors thus miscalculate, since they compute the total volatility of the jump-diffusion process incorrectly.

We compute the total volatility of the process correctly and recalculate the pricing error. For reasonable parameter values, the pricing error can behave very differently from what was previously reported.

ENDNOTES

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[1]See, for example, Fama [1965], Hsieh [1988], Jorion [1988], Bates [1996], and Campa, Chang, and Rieder [1998].

[2]Recall that the total variance of the process per unit of time, υ^2, is $\sigma^2 + \lambda(\mu^2 + \delta^2)$, while υ_M^2 is defined as $\sigma^2 + \lambda\delta^2$.

[3]Notice that Γ does not represent the variance of the jump process divided by the total variance of the jump-diffusion process, as stated by Merton [1976a].

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