Lookback Option Valuation: A Simplified Approach

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This simple method to value lookback options handles path-dependence without introducing an additional dimension or requiring a transformation of underlying state variables. We show that lookback options can be valued as if they are path-independent standard options. This proposed method would be very useful in valuation of a lookback option with multiple sources of uncertainty.

Lookback options are options whose payoff depends on the maximum or minimum underlying asset price reached during the life of the option. They are in widespread use in derivatives markets, particularly over-the-counter currency options markets.

The academic literature proposes several numerical valuation methods to accommodate the path-dependence of these options (e.g., Hull and White [1993], Wilmott, Howison, and Dewynne [1995], and Cheuk and Vorst [1997]). The numerical approaches deal with the path-dependence feature of lookback options either by explicitly including the historical minimum or maximum underlying asset values as an additional dimension or by tracing the possible minimum or maximum values in the underlying asset price process. These approaches have the drawback that they may become computationally burdensome or difficult to extend to more general cases when there are multiple sources of uncertainty.

We propose an alternative methodology that accommodates the path-dependence features of lookback options simply and naturally. It requires neither the additional dimension that may arise from the path-dependence nor a transformation of underlying state variables as in Cheuk and Vorst [1997]. Indeed, we show that well-known backward numerical procedures used to value standard options, such as lattice, finite-difference, or finite-element methods, can be applied to lookback options despite their path-dependence.

This turns out to be possible because, at each time step in the backward computation, the option values at some nodes can be replaced by values readily available in the computation process that capture the effects of path-dependence. Thus, the proposed technique is easily implemented by adding one simple step to any numerical approach, such as lattice or finite-difference methods, that is used for standard option valuation. An advantage of the technique is that it may also be applied to numerical approaches for multiple sources of uncertainty without introducing additional complexity.

I. LOOKBACK OPTIONS AND VALUATION METHODS

There are two classes of lookback options: floating-strike and fixed-strike options. A European lookback floating-strike call option gives the holder the right to buy the underlying asset at expiration at the historical minimum price reached during the option's life.

A European lookback fixed-strike call option pays either zero or the difference between the historical maximum price and the strike price, whichever is greater. While there are closed-form solutions for lookback options of the European type, lookback options commonly exhibit other features that complicate their valuation.[1]

Two features that seem to be particularly important in practice are discrete monitoring (monitoring to determine historical extreme prices only at discrete, contractually specified intervals) and multiple sources of uncertainty that affect lookback option value. Examples of the latter include the currency option modeled by Amin and Jarrow [1991] and options written on multiple assets as considered by Stulz [1982].[2] Closed-form solutions do not allow for either of these possible complicating features. Instead, they assume the underlying asset price is monitored continuously, and is driven by a single source of uncertainty following a geometric Brownian motion.

Suggestions have been raised that address, to one degree or another, each of these problems individually. When both occur simultaneously, however, as is usually the case in practice, these approaches tend to conflict.

For example, Broadie, Glasserman, and Kou [1997, 1999] introduce correction terms that allow continuous-time valuation formulas to be applied to discretely monitored options driven by a single source of uncertainty. It is not clear, however, how these continuity correction factors can be extended to a multidimensional context.

Alternatively, Cheuk and Vorst [1997] incorporate discrete monitoring in a numerical approach based on a binomial lattice.[3] They show that the added state variable due to path-dependence of lookback options can be avoided if valuation proceeds after a transformation of the state space.[4]

There are two difficulties associated with this method. First, it applies only to lattice numerical techniques. It is not clear the method can be used with other numerical techniques, such as finite-difference methods, that may be more suitable for certain option valuation problems. Second, it is unclear whether this transformation technique can be extended to cases involving multiple sources of uncertainty.

II. VALUATION OF LOOKBACK OPTIONS

For the sake of simplicity, we consider a currency lookback option valuation problem including only one source of uncertainty and with constant domestic and foreign interest rates. In the risk-neutral world, the exchange rate, S, denoted in terms of domestic currency per unit of foreign currency is assumed to follow:

$$\frac{dS}{S} = (r^d - r^f)dt + \sigma dW \tag{1}$$

where r^d and r^f are constants representing the domestic and foreign risk-free interest rate, respectively, σ is the volatility of the exchange rate, and dW is a Wiener process.

The computational domain for the underlying exchange rates consists of a set of uniform mesh points $[s_1, s_2, \ldots, s_N]$. Let k^* denote the index such that s_{k^*} is the historical minimum or maximum exchange rate at the time the option is valued. The time index runs from 0 at origination to T at expiration, and appears in parentheses following the variable when this information is useful. Thus the value of the underlying exchange rate is $S(T)$ when the option expires and $S(0) = s_{k^*}$ at the moment it is valued.

It is often useful to index variables by the time step at which their value is calculated. Given an appropriate number of time steps, M, such that the uniform time step size is $\Delta\tau = \frac{T}{M}$, we index the time steps by $m = 0, 1, 2, \ldots, M$. We use the superscript in parenthesis, (m), to mean the number of time steps updated backward from the option expiration date. Thus the typical element, $S^{(m)}$, implies the underlying exchange rate at time $T - m\Delta\tau$, so $S^{(m)} = S(T - m\Delta\tau)$. The value at expiration is $S^{(0)} = S(T)$ and at option valuation it is $S^{(M)} = S(0)$.

Floating-Strike Lookback Call

The payoff for a floating-strike lookback call depends on the historical minimum exchange rate reached during the option's life. If we define

$$\underline{J}(t) = \min_{0 \le t' \le t} S(t')$$

a continuously monitored floating-strike lookback call has a payoff at expiration of:

$$\max[0, (S(T) - \underline{J}(T))]$$

It turns out that this particular form of path-dependence has two characteristics that permit us to deal with the time-dependence problem. First, by definition the historical minimum can only drop, never rise, over the life of the option. There is no need to consider the possibility that the historical minimum may exceed the level it had at the time of origination. That is, the possibility that $\underline{J}^{(m)} > s_{k*}$ can be disregarded for all m.

Second, because a continuously monitored floating-strike lookback, by definition, uses the historical minimum as the strike price, it follows that the option is at the money every time a new historical minimum is established. We exploit this at-the-money property to simplify valuation for mesh points where new historic minima are set.

Given these properties, it proves sufficient to consider the case $\underline{J}^{(m)} = \underline{J}(T - m\Delta\tau) = s_{k*}$; that is, where the historical minimum at each backward time step still equals the initial exchange rate. In effect this is to assume that no new minimum has been set prior to the time step under consideration. Because the calculations for each time step are carried out for only one value of the historical minimum, this eliminates the additional computational burden of the path-dependence.

As will become clear below, this does not mean we ignore the possibility of setting a new historical minimum in the future. We incorporate this possibility at each step through the at-the-money property.

We can offer an inductive argument to show that this approach yields correct values for the option. We begin with the values at expiration and work backward to origination. Consider first the terminal values of the option:

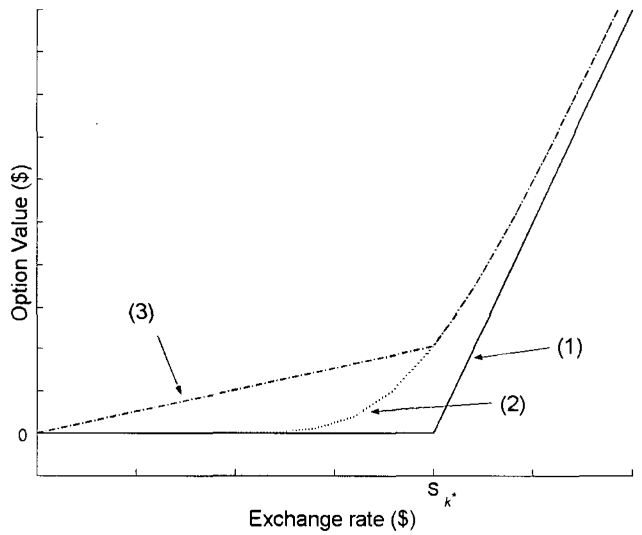
$$V(T) = V^{(0)} = max\left[0, \left(S(T) - \underline{J}^{(0)}\right)\right] \qquad (2)$$

The assumption that no new minimum has been set prior to this time means $\underline{J}^{(0)} = \min(s_{k*}, S(T))$. In this case, the payoff is represented by line (1) in Exhibit 1. If $S(T) < s_{k*}$, a new minimum is set at expiration, and the option expires at the money with no payoff. For $S(T) < s_{k*}$, the option pays like a call with strike price s_{k*} as indicated in the exhibit. In either case, the correct terminal values result.

Consider now the option values one step before expiration, i.e., $m = 1$. Again we assume no new minimum has been set prior to this time so that $\underline{J}(T - 2\Delta\tau) = s_{k*}$ and $\underline{J}(T - 1\Delta\tau) = \min(s_{k*}, S(T - 1\Delta\tau))$. If no new minimum is set at step $m = 1$ either, the previously calculated terminal ($m = 0$) payoffs as depicted in line (1) in Exhibit 1 can be used as inputs for a recursive numerical approach such as the partial differential equation (PDE) or lattice technique to update the option values for $m = 1$.

Line (2) in Exhibit 1 depicts the results of this exer-

Exhibit 1
Floating-Strike Lookback Call Option



cise. When $S(T - 1\Delta\tau) \geq s_{k*}$, the no-new-minimum condition is satisfied, so the calculated values at those mesh points are the values of the lookback call at that step.

When $S(T - 1\Delta\tau) < s_{k*}$, a new minimum is set, so this approach cannot be used to value a lookback option. In this case, the option is in effect an at-the-money call option. Merton [1973, theorem 6] shows that if the strike price and the underlying asset price decline proportionally, the arbitrage-free option value must also be diminished proportionally. Therefore, the numerical value calculated at s_{k*}, when the lookback call is at the money but no new minimum is set, provides a usable value for an at-the-money option.

The proportionality rule combined with the option value at s_{k*} gives an option value in these cases as follows:

$$V_i^{(m)} = \frac{s_i}{s_{k^*}} V_{k^*}^{(m)} \text{ for all } i < k^* \quad (\text{that is, for } s_i < s_{k^*}) \qquad (3)$$

The option values derived by the proportionality rule are represented by line (3) in Exhibit 1.

To summarize this, when no new minimum is set (that is, $S = s_i \geq s_{k*}$), the option values, $V_i^{(m)}$, for the step are calculated numerically based on the values from step $m = 0$. When a new minimum is set (that is, $S = s_i < s_{k*}$) the value is calculated using Equation (3).

In practice, unless the updating procedure generates the option values for all mesh points simultaneously, there is no reason to calculate numerical values when $S < s_{k*}$. Instead, these are calculated by the proportionality rule.

Although the new minimum invalidates the assumption under which step $m = 0$ was calculated, this does not affect the accuracy of the step $m = 1$ results. This is because Equation (3), not a direct numerical updating procedure, is used in all such cases.

The proportionality rule means that if the terminal values of the option are recalculated to reflect the newly set historical minimum, and these recalculated values are then used to compute the values for time step $m = 1$ numerically, the result would be the same as that obtained using Equation (3). The proportionality property of at-the-money options makes this recalculation unnecessary.

The same argument applies for each time step. We continue to assume that no new minimum is set before the step at which $m = 2$. Then at all mesh points where no new minimum is set, the option values for $m = 2$ are updated using the chosen numerical techniques as in step $m = 1$ with the option values for $m = 1$ (line (3) in Exhibit 1). When a new minimum is set, the proportionality rule in Equation (3) applies. Again, the fact that, for example, setting a new minimum at $m = 2$ invalidates the assumption for $m = 1$ does not matter, because the results at $m = 1$ are not directly used when this occurs. Repeating this process for each time step leads ultimately to the option value at valuation, $m = M$.

Note that this procedure applies equally to the PDE formulation or the lattice version. For a trinomial approach, as shown in Exhibit 2, all option values above $V_{k*}^{(1)}$ at time step $m = 1$ are obtained by the standard risk-neutral valuation technique, and the remaining option values are obtained by Equation (3).[5]

A slight modification of this approach is required to accommodate an option for which the underlying exchange rate is monitored and the strike price is updated only at discrete intervals. With continuous monitoring, a floating-strike lookback call can never be out of the money. Discrete monitoring means the call can be out of the money during the interval between monitoring dates.

For time steps when no monitoring occurs, updating simply follows the numerical procedure for all values of the underlying exchange rate. In terms of Exhibit 1, that means values from line (2) are used. This does allow the option to be out of the money at those steps, and that possibility is reflected in the option valuations calculated for previous dates.

For time steps when a monitoring does occur, the procedure described above still applies. Regular numerical updating [line (2)] is used when no new minimum is set, and the proportionality rule in Equation (3) applies when there is a new minimum.

Fixed-Strike Lookback Call

The procedure for a fixed-strike lookback call (or lookback rate call) is analogous. Here the payoff depends on the historical maximum price reached during the option's life:

$$\bar{J}(t) = \max_{0 \le t' \le t} S(t')$$

Specifically, the payoff at expiration is:

$$\max\left[0, \left(\bar{J}(T) - K\right)\right]$$

where K is a fixed strike price.

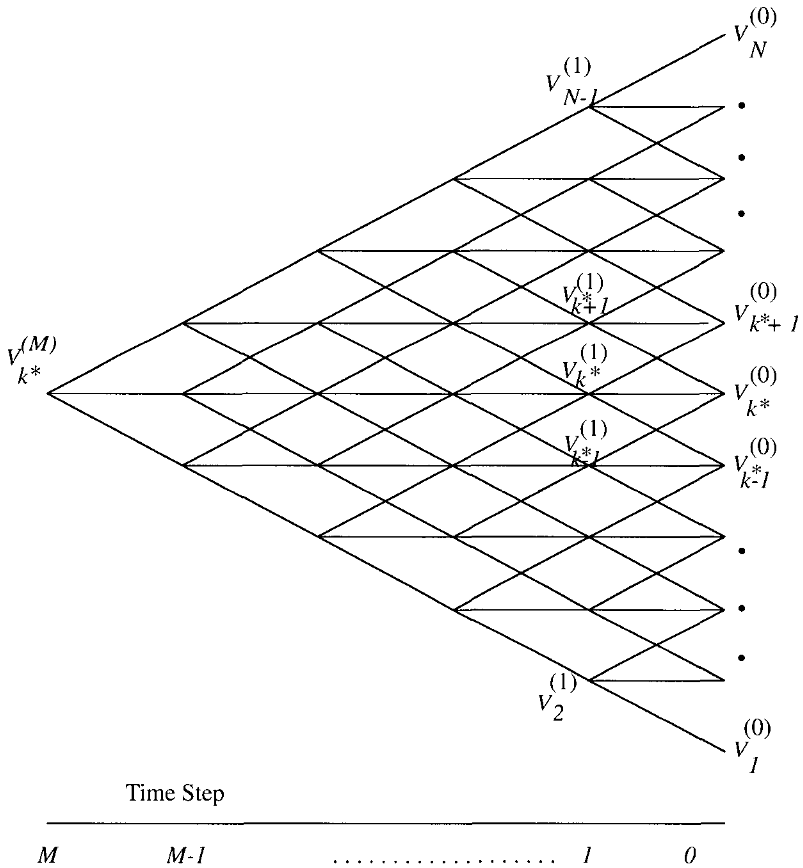
We consider a fixed-strike lookback option whose strike price, K, is set at s_{k*}, which is the historical maximum price at option valuation. Again, it is sufficient to consider the case of $\bar{J}^{(m)} = \bar{J}(T - m\Delta\tau) = s_{k*}$; that is, when no new maximum has been set since origination.

We begin by considering terminal values of the option $V^{(0)} = V(T) = \max[0, (\bar{J}^{(0)} - K)]$ for the chosen mesh points on the solid line in Exhibit 3. When no new maximum has been set prior to this time, $\bar{J}(T - 1\Delta\tau) = s_{k*}$ and $\bar{J}(T) = \max(s_{k*}, S(T))$ In this case, the payoff is represented by line (1) in Exhibit 3.

If the exchange rate at expiration falls below s_{k*}, s_{k*} remains the historical maximum, and the payoff diagram is correct. Of course, if the exchange rate at expiration exceeds s_{k*}, then $\bar{J}(T)$ is the new exchange rate, which is now the historical maximum. Accordingly, the payoff at these exchange rates becomes $S(T) - K$ as indicated in Exhibit 3.

Consider the option values one step before expiration, i.e., $m = 1$. Again we assume no new maximum has been set prior to this time so that $\bar{J}(T - 2\Delta\tau) = s_{k*}$ and $\bar{J}(T - 1\Delta\tau) = \max[s_{k*}, S(T - 1\Delta\tau)]$. If no new maximum is set at step $m = 1$ either, the previously calculated terminal ($m = 0$) payoffs as depicted in line (1) of Exhibit 3 can be used as inputs to update the option values for $m = 1$. Line (2) in Exhibit 3 depicts the results of this exercise. When $S(T - 1\Delta\tau) \le s_{k*}$, the no-new-maximum condition is satisfied, so the calculated values at those mesh points are the values of the lookback call at that step.

EXHIBIT 2
Trinomial Tree



Note: $Vi^{(m)}$ means the updated option value at a time step m when $S = S_i$.

When $S(T - 1\Delta\tau) > s_{k^*}$, a new maximum is set, and this approach cannot be used. Instead, the values of the remaining nodes are calculated as a portion of the value of the node s_{k^*} that was at the money. Setting a new maximum exchange rate means that holding this option is equivalent to holding an at-the-money call option with the new strike price $\overline{J}^{(1)}$ and the certain promised payment equal to the amount of $(\overline{J}^{(1)} - K)$, the excess of the maximum to date over the fixed strike. Thus the value of the option at $m = 1$ is the sum of the value of the at-the-money call (computed as a proportion of the value of the at-the-money call with strike $\overline{J}^{(0)}$ used originally—this is analogous to the previous case) and the present value of this fixed sum.

That is, the option values for $S = s_i \geq s_{k^*}$ are:

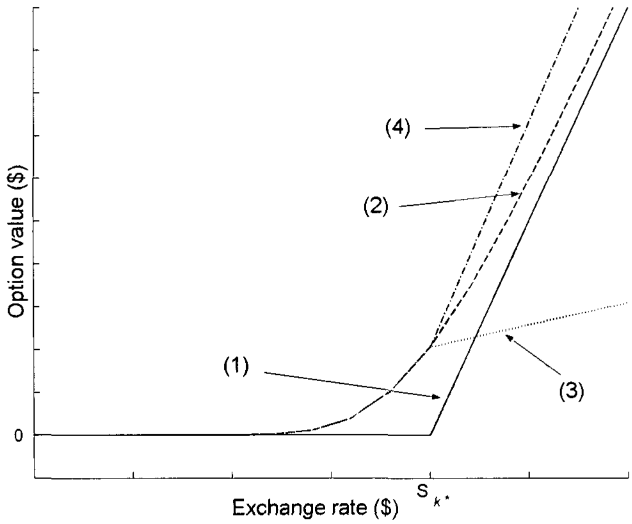
$$V_i^{(m)} = \frac{s_i}{s_{k^*}} V_{k^*}^{(m)} + PV^{(m)}[s_i - s_{k^*}] \text{ for all } i > k^* \qquad (4)$$

where $PV^{(m)}[s_i - s_{k^*}] = e^{-r^{d,m}\Delta\tau}(s_i - s_{k^*})$ is the present value of $s_i - s_{k^*}$. The first term in Equation (4) is depicted by line (3) in Exhibit 3, and the value including both terms is depicted by line (4). Thus the option values of line (4) are now used to update the option values for $m = 2$. The same updating procedure is followed iteratively for $m = 2, 3, \ldots, M$.

III. NUMERICAL EXAMPLES

For the numerical examples, we use two numerical option valuation approaches: the trinomial and the radial basis function (RBF) approach. We apply our proposed methodology to these two approaches. Because the RBF approach is newer and not as widely known as some other numerical techniques, we explain this method first.

Exhibit 3
Fixed-Strike Lookback Call Option



RBF and Trinomial Approaches

To demonstrate the RBF approach, we describe its application to a currency option with a single source of uncertainty, the exchange rate itself. Choi and Marcozzi [2001] describe its application to a plain vanilla currency option when the domestic and foreign interest rate are also stochastic (three sources of uncertainty). Our method works perfectly well to value lookback options when there are multiple sources of uncertainty, because the proportional substitution rule [Equations (3) or (4)] still applies under these circumstances. As long as there is a numerical method (such as the RBF) that can value a plain vanilla option with multiple sources of uncertainty, the adjustment we describe to value a lookback option can be applied.

As Choi and Marcozzi [2001] show, the RBF approach, similar to finite-difference methods, approximates the true option value by discretizing the partial differential equation (PDE), but using radial basis functions as the basis of the approximation. It requires only that the form of the radial basis function be selected and a mesh generated, independent of the dimensionality or geometry of the problem. The option value is then approximated by a linear combination of the basis functions with time-varying coefficients.[6]

Suppose we choose N mesh points, $\boldsymbol{x} = [x_1, x_2, \ldots, x_N]$ covering a broad enough interval for the log exchange rates, $x_j = \ln(s_j)$. For a given mesh, $\boldsymbol{x}$, suppose the unknown option value, $V_k(t)$, at x_k and time t can be approximated by the RBF, φ:

$$V_k(t) \approx \sum_{j=1}^{N} \alpha_j(t)\varphi(|x_k - x_j|) \tag{5}$$

where the α_j are unknown coefficients depending only on time, t, and $\varphi(|x_k - x_j|)$ is an RBF so that:

$$\varphi(|x_k - x_j|) = \sqrt{(x_k - x_j)^2 + c^2}$$

where c is a constant known as the shape parameter. This RBF is known as *Hardy's multiquadratic*, introduced in Hardy [1971]. With a uniform mesh with spacing $h = |x_j - x_{j-1}|$, we choose $c = 4h$ in the numerical example.

The N-dimensional column vector of option values, $\boldsymbol{V}(t)$, at time t, evaluated at all $\boldsymbol{x}$, can now be expressed in matrix form:

$$\boldsymbol{V}(t) = \boldsymbol{\Phi}\alpha(t) \tag{6}$$

where $\boldsymbol{\Phi}$ is the $N \times N$ matrix such that the (i, j)-th element is $\varphi(|x_i - x_j|)$, and $\alpha(t)$ is the N-dimensional column vector whose j-th element is $\alpha_j(t)$. As $\boldsymbol{\Phi}$ is symmetric and positive-definite (and so invertible), $\alpha(t)$ is uniquely determined as follows:

$$\alpha(t) = \boldsymbol{\Phi}^{-1}\boldsymbol{V}(t) \tag{7}$$

Following standard assumptions under the Black-Scholes option framework, the arbitrage-free option values, V, must satisfy the partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + (r^d - r^f - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial x} - r^d V = 0 \tag{8}$$

with a boundary condition given by the terminal option values. When (5) is substituted into (8), we obtain a system of N linear equations, where the partial derivatives in (8) are, for $i = 1, 2, \ldots, N$:

$$\frac{\partial V_i(t)}{\partial t} = \sum_{j=1}^{N}\frac{d\alpha_j(t)}{dt}\varphi(|x_i - x_j|)$$

$$\frac{\partial V_i(t)}{\partial x_i} = \sum_{j=1}^{N} \alpha_j(t) \frac{\partial \varphi(|x_i - x_j|)}{\partial x_i} = \sum_{j=1}^{N} \alpha_j(t) \frac{(x_i - x_j)}{\varphi(|x_i - x_j|)}$$

and

$$\frac{\partial^2 V_i(t)}{\partial x_i^2} = \sum_{j=1}^{N} \alpha_j(t) \frac{\partial^2 \varphi(|x_i - x_j|)}{\partial x_i^2}$$
$$= \sum_{j=1}^{N} \alpha_j(t) \left[\frac{1}{\varphi(|x_i - x_j|)} - \frac{(x_i - x_j)^2}{\left[\varphi(|x_i - x_j|)\right]^3} \right]$$

The resulting system of N linear equations can also be written succinctly in a matrix form as follows:

$$\mathbf{\Phi}\dot{\alpha} = \mathbf{B}\alpha \tag{9}$$

such that

$$\mathbf{B} \equiv -0.5\sigma^2 \mathbf{\Phi}_{\mathbf{xx}} - (r^d - r^f - 0.5\sigma^2)\, \mathbf{\Phi}_{\mathbf{x}} + r^d \mathbf{\Phi}$$

where the dot denotes the time derivatives, and $\mathbf{\Phi}_{\mathbf{x}}$ and $\mathbf{\Phi}_{\mathbf{xx}}$ are the N × N matrix of $\partial\varphi/\partial x$ and $\partial^2\varphi/\partial x^2$, respectively. Given a fixed collocation point, x_j, (9) constitutes a system of first-order homogeneous ordinary differential equations in $\alpha(t)$. The matrix $\mathbf{B}$ is a known constant.

Given the terminal condition at time T, $\alpha(T)$ is uniquely determined by (7). With an appropriate time discretization, the system (9) may be recursively solved using various backward time integration methods. We use an implicit backward time integration scheme such that, for $m = 1, 2, \ldots, M$:

$$\frac{d\alpha_j(t)}{dt} \approx \frac{\alpha_j^{(m)} - \alpha_j^{(m-1)}}{\Delta\tau} \tag{10}$$

where $\Delta\tau > 0$, $\alpha_j^{(m)} \equiv \alpha_j(T - m\Delta\tau)$, and $M\Delta\tau \equiv T$.

Substituting (10) into (9) and rearranging the terms, we obtain

$$\mathbf{V}^{(m)} = \mathbf{\Phi}(\mathbf{\Phi} - \Delta\tau\mathbf{B})^{-1}\mathbf{V}^{(m-1)} \tag{11}$$

Thus, the European currency lookback call solution, $\mathbf{V}^{(M)} = \mathbf{\Phi}\alpha^{(M)}$, may be obtained by successive iteration of (11) with the substitution procedures discussed in the last section. This iteration procedure requires inverting the matrix, $(\mathbf{\Phi} - \Delta\tau\mathbf{B})$, once. In our computation, we use Gaussian elimination to invert the matrix.

Let us summarize the procedure for floating-strike lookback option valuation with the RBF approach. First, choose N mesh points of log exchange rates, $\boldsymbol{x}$, such that one of the mesh points corresponds to the historical minimum, x_{k*}. Then choose M, the number of time steps and construct matrices, $\mathbf{\Phi}$ and $\mathbf{B}$.

Second, given the terminal values of Equation (2), find $\boldsymbol{V}^{(1)}$ using Equation (11). Third, if the option is monitored at time step 1, then substitute all option values at $x < x_{k*}$ by using Equation (3). Otherwise, skip this step. Fourth, given $\boldsymbol{V}^{(1)}$, find $\boldsymbol{V}^{(2)}$ using Equation (11) again, and then go to the third step.

We repeat this procedure until the final numerical solution $\boldsymbol{V}^{(M)}$ is obtained.

For a large enough number of time steps including the third step, the resulting option values will be close to the continuously monitored case. Omitting the third step when the option values are not monitored, we obtain a discretely monitored lookback option solution. For the fixed-strike lookback option, the same procedure is applied, except the substitutions in step 3 take place for all $x > x_j$ using Equation (4).

The trinomial approach in the single source of uncertainty case assumes that state S at time t moves to state uS, S, or dS at time $t + \Delta\tau$ with probabilities p_u, p, and p_d, respectively, where:

$$u = e^{\lambda\sigma\sqrt{\Delta\tau}}, \quad d = \frac{1}{u}$$

and

$$p_u = \frac{1}{2\lambda^2} + \frac{(r^d - r^f - \sigma^2/2)\sqrt{\Delta\tau}}{2\lambda\sigma}$$
$$p = 1 - \frac{1}{\lambda^2}$$
$$p_d = 1 - p_u - p$$

For $\lambda = 1$, $p = 0$, this becomes the binomial approach of Cox, Ross, and Rubinstein [1979]. There is one important difference between implementing the binomial tree in the usual way, and implementing it as a trinomial with $\lambda = 1$, $p = 0$. Although the middle of the three trinomial nodes receives no weight when calculating the value at the next step, the algorithm does calculate a value for each node. When the binomial is implemented in the usual way, however, half the time steps do not have a node at exchange rate s_{k*}. Instead,

EXHIBIT 4
European Floating-Strike Currency Lookback Call

N	(1) Binomial Model	(2) Trinomial Model ($\lambda = 1.22474$)	RBF (3) Time Step = N	RBF (4) Time Step = $6 \times N$
50	8.969	8.783	8.752	9.063
100	9.201	9.067	9.045	9.272
500	9.522	9.461	9.450	9.551
1000	9.600	9.556	9.595	9.661
5000	9.705	9.686	9.700*	9.765*

Parameters are $S = 100$, $r_d = 0.04$, $r_f = 0.07$, $\sigma = 0.2$, $T = 0.5$. The analytic solution for a continuously monitored option is 9.792. Number of mesh points is 2,001.*

EXHIBIT 5
European Fixed-Strike Currency Lookback Call

N	(1) Binomial Model	(2) Trinomial Model	RBF (3) Time Step = N – 1	RBF (4) Time Step = $6 \times N$
50	9.740	9.514	9.478	9.861
100	10.027	9.863	9.836	10.117
500	10.427	10.351	10.337	10.464
1000	10.524	10.471	10.460	10.530
5000	10.656	10.632	10.630*	10.661*

Parameters are $S = 100$, $r_d = 0.04$, $r_f = 0.07$, $\sigma = 0.2$, $T = 0.5$. The analytic solution for a continuously monitored option is 10.977. Number of mesh points is 2,001.*

nodes occur one step above and one step below this value. This means that at these time steps no value is calculated for an option that is exactly at the money.

Ordinarily this would not affect the final option value reported. For the method used here, however, where we substitute the computed at-the-money value into formulas for value at many nodes, it is important for numerical accuracy to have values computed exactly at the money. For this reason, we implement the binomial as a special case of the trinomial.

Numerical Results

Exhibits 4 and 5 report the numerical examples obtained applying the three methods to currency lookback call options featuring floating and fixed strike prices. In each case it is assumed the calls are European, monitoring is continuous, and interest rates are non-stochastic. The specific parameter values in each exhibit are $S = 100$, $r_d = 0.04$, $r_f = 0.07$, $\sigma = 0.2$, and $T = 0.5$. These are the same values as in Cheuk and Vorst [1997]. An analytic solution is available for continuously monitored European lookback options; the values in this case are 9.792 and 10.977.

The stubs in the exhibits indicate the number of time steps used to implement the binomial and trinomial calculations. Column (1) reports the binomial results. (As we note above, these were actually implemented as a trinomial with $\lambda = 1$.) These results are identical to those reported by Cheuk and Vorst [1997], verifying the computational equivalence of the two methods in this special case.

The second column reports results for the trinomial model where $\lambda = 1.22474$. This gives approximately equal weight to all three nodes in each calculation. Note that in these particular numerical examples, the trinomial consistently underestimates the true value by more than the binomial ($\lambda = 0$). This may be because the binomial puts more weight on the in-the-money outcome than does the trinomial in the very first updating procedure. This tends to amplify the calculated result because the at-the-money and out-of-the-money outcomes are worth zero at expiration. This effect assumes magnified importance when the at-the-money value is substituted in the computation of the out-of-the-money values. It is diminished when the discrete monitoring case is considered, because the substitution is carried out only at monitoring times.

The last two columns report the results obtained using the radial basis function method. A potential advantage of this or any other PDE approach is that it permits separate determination of the computational domain—the number and location of mesh points—and the number and location of the time steps.

For the binomial and trinomial methods, the location and number of mesh points is automatically linked to the number of time steps. If N is the number of time steps, the traditional binomial approach generates $N + 1$ mesh points at expiration, the trinomial $2N + 1$. The resulting computational domain at expiration depends on this number and on the parameter λ through its influence on the step size. In both applications of the RBF method, the computational domain and mesh points are set to coincide with that produced by the binomial method.

In column (3), the number of time steps used in the calculation also corresponds to the number used in the binomial case. In the final column, the number of time steps is set at six times the number in the binomial case, while maintaining the same computational domain as the binomial.

When the numbers of time steps are the same, the RBF and the binomial performance are very similar. When the number of time steps is increased, the RBF outperforms the other methods. Nevertheless, as already noted by Cheuk and Vorst [1997], convergence to the continuous monitoring value is very slow. In fact, if we did not already know the exact value by analytical methods, it would be extremely difficult to recognize the progression of values as numerical convergence to some constant.

Having shown that the RBF performs as well as or better than the alternatives when comparison is readily possible, we now consider its application in cases that are computationally quite burdensome for lattice methods. Specifically, we increase the sources of uncertainty from one to three, adding stochastic domestic and foreign interest rates to the stochastic exchange rate. In addition, we assume that monitoring occurs only at discrete intervals.

The results are reported for a floating-strike lookback currency option with parameter values used in Choi and Marcozzi [2001].[7] Here we know the exact solution for the lookback option in only one case, a standard European call option, since the standard option is a special case of the lookback option when the underlying asset value is monitored only once, at expiration. The exact value for a standard call option following this process can be found in Amin and Jarrow [1991] to be 0.0835.

Exhibit 6 describes the result of applying the RBF approach to value this option using 125 time steps. The computational domain is designed to conform as nearly as possible to those in the previous exhibits. First, we calculate the computational domain that would be generated by a binomial model when the only source of uncertainty is an exchange rate process with the indicated variance. The range of values for both interest rates is then set to an equal size, with the result that the computational domain at each time step is a cube.

The stub in Exhibit 6 describes how finely the mesh points are set in this cube. For example, $11 \times 5 \times 5$ means that 11, 5, and 5 mesh points are used for exchange rates, domestic interest rates, and foreign interest rates, respectively. Thus the number of nodes at which values are calculated at each time step ranges from 125 to slightly over 4,000. Experimentation suggests that devoting additional mesh points to the interest rates does not improve accuracy much. For that reason, most of the variation reported in the exhibit is with respect to the exchange rate.

Note that for the option monitored only at maturity, reported in column (1), the results are quite accurate with the mesh set at $21 \times 5 \times 5$. The remaining columns report the results for options that are monitored more frequently.

EXHIBIT 6

European Floating-Strike Currency Lookback Call With Stochastic Interest Rates (time step = 125)

Mesh Points $X \times R^d \times R^f$	Monitoring Frequency				
	(1) 1	(2) 2	(3) 5	(4) 25	(5) 125
$5 \times 5 \times 5$	0.0774	0.0954	0.1094	0.1193	0.1215
$11 \times 5 \times 5$	0.0828	0.0987	0.1140	0.1297	0.1341
$21 \times 5 \times 5$	0.8300	0.0994	0.1150	0.1326	0.1384
$41 \times 7 \times 7$	0.0831	0.0998	0.1154	0.1342	0.1412
$51 \times 9 \times 9$	0.0831	0.1000	0.1156	0.1345	0.1418

Option values evaluated for at-the-money option when $S = 1$ ($x = \log(S) = 0$). Parameter values are: $T = 0.5$, $\theta_t^d = \theta_t^f = 0.005$, $\sigma_{xx} = 0.0918$, $\sigma_{xd} = \sigma_{xf} = 0.0032$, $\sigma_{dd} = \sigma_{ff} = 0.0026$, and $\sigma_{df} = 0.0005$. The exact European standard call option value is 0.0835, in which monitoring occurs only once [column (1)].

EXHIBIT 7

European Floating-Strike Currency Lookback Call With Stochastic Interest Rates (time step = 5,000)

Mesh Points $X \times R^d \times R^f$	Monitoring Frequency				
	(1) 1	(2) 2	(3) 5	(4) 25	(5) 125
$5 \times 5 \times 5$	0.0775	0.0951	0.1098	0.1200	0.1223
$11 \times 5 \times 5$	0.0829	0.0979	0.1143	0.1306	0.1354
$21 \times 5 \times 5$	0.0831	0.0984	0.1153	0.1336	0.1400
$41 \times 7 \times 7$	0.0831	0.0988	0.1160	0.1353	0.1433
$51 \times 9 \times 9$	0.0831	0.0989	0.1162	0.1361	0.1447

Option values evaluated for at-the-money option when $S = 1$ ($x = \log(S) = 0$). Parameter values are: $T = 0.5$, $\theta_t^d = \theta_t^f = 0.005$, $\sigma_{xx} = 0.0918$, $\sigma_{xd} = \sigma_{xf} = 0.0032$, $\sigma_{dd} = \sigma_{ff} = 0.0026$, and $\sigma_{df} = 0.0005$. The exact European standard call option value is 0.0835, in which monitoring occurs only once [column (1)].

In the final column, results are reported for monitoring the option at every one of the 125 steps, corresponding to an option that is continuously monitored. Although true values are not known for these options, we note that the estimates are well behaved in the sense that, as the number of mesh points increases, values show convergence behavior similar to that observed in Exhibits 4 and 5. In particular, the increments of value are smaller as the mesh points increase.

As a final check, Exhibit 7 reports the results for the same options using a large number of time steps; 5,000. The results are reassuring in that they generally show that the extra time steps do not much change the computed values, suggesting that convergence is substantially complete with 125 time steps. The continuous monitoring case is slightly more sensitive than the other cases, probably due to more substitutions, resulting from the frequent monitoring.

IV. SUMMARY

We have introduced a simple numerical methodology for valuing lookback options. The method handles the path-dependence of lookback options without increasing the dimension of the computational problem. It also permits the mesh structure and location of the time steps in the calculation to be determined independently. This flexibility and relative parsimony permit convenient application when alternative approaches may become unwieldy, such as for options with values dependent on multiple sources of uncertainty or subject to monitoring at discrete times.

The numerical results presented show that the method performs at least as well as other methods when applied to relatively simple cases with known analytic solutions (such as European lookback calls with continuous monitoring). The method also handles more complicated cases (such as discretely monitored currency lookback options when the option value depends on multiple sources of uncertainty).

ENDNOTES

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[1]See Goldman, Sosin, and Gatto [1979], Garman [1989], and Conze and Viswanathan [1991] for the closed-form solution of lookback options.

[2]The practical importance of multidimensional path-dependent options is underlined by Heynen and Kat [1995] and Ritchken [1995].

[3]In principle, analytic values for discretely monitored path-dependent options can be derived according to the multivariate normal distribution (Heynen and Kat [1995]). Because the dimensionality of the multivariate normal distribution required equals the number of monitoring periods, this approach is tractable only for options subject to a very limited number of monitoring times.

[4]A numerical approach to path-dependence, based on Hull and White [1993], compounds the multidimensionality problem by treating the past extreme as yet another state variable.

[5]The binomial approach presented by Cheuk and Vorst [1997] is a special case of the trinomial approach, just as the standard binomial approach is a special case of the trinomial approach.

[6]The performance of RBF approaches is examined numerically in Marcozzi, Choi, and Chen [2001] with application to the standard Black-Scholes option pricing model and options on multiple assets.

[7]We replicate Choi and Marcozzi [2001] with values $T = 0.5$, $\theta_t^d = \theta_t^f = 0.005$, $\sigma_{xx} = 0.0918$, $\sigma_{xd} = \sigma_{xf} = 0.0032$, $\sigma_{dd} = \sigma_{ff} = 0.0026$, and $\sigma_{df} = 0.0005$. The results shown in Exhibits 6 and 7 are at $S = 1$ (or $x = \log(S) = 0$).

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