

including measures of recent volatility such as the most recent five- or ten-day standard deviations of NDX returns does not affect the results.

¹³It is important to keep in mind the extent of the effects. For example, if the NDX is equal to its five-day moving average, a 1% NDX rally leads to a one-half percentage point drop in the VXN, and the corresponding deviation from the five-day moving average leads to roughly a one-fifth percentage point increase in the VXN. In this case, the effect of the NDX rally on lowering the VXN would be about two-thirds of what it otherwise would be.

¹⁴Although the GJR GARCH model allows equal-magnitude positive and negative return innovations to have different impacts on conditional volatility, the quadratic specification for both positive and negative return innovations imposes constraints on the way larger shocks have greater per unit effects on conditional volatility. Other specifications were also estimated, such as GARCH with leverage models, where conditional volatility is a function of a constant lagged conditional volatility, the lagged squared return innovation, and separate terms for positive and negative (unsquared) lagged return innovations. GJR GARCH models outperform the GARCH with leverage models according to the news impact curve bias tests suggested by Engle and Ng [1993]. Also, estimates of the Engle and Bollerslev [1986] non-linear GARCH specification, where the lagged absolute return innovation is raised to a power that is estimated, result in a parameter estimate close to 2.0, consistent with the quadratic news impact curve of the GJR GARCH model.

¹⁵If the asymmetry indicator variable is set equal to one because the most recent return innovation is negative, forecasting more than one step ahead using $\alpha_0 + (\beta + \delta_1 + \delta_2) E_t(h^2_{t+s-1})$ would be incorrect, because it would assume a zero probability that future innovations will be positive. By the same reasoning, forecasting more than one step ahead following a positive innovation with the same formula absent the δ_2 term would be incorrect because of the possibility that future innovations are negative. As a result, the asymmetry term is allowed to affect the forecast of the next day's volatility, but it is assumed that positive and negative innovations have the same mean-reversion patterns and revert to the same unconditional mean. See Potter [1999] for a discussion of the complexities of forecasting multiple periods ahead with asymmetric GARCH models.

¹⁶These regressions are not run for subsamples because with monthly data the post-bubble period has too few observations.

¹⁷The coefficient estimates indicate that the GARCH forecasts tend to underpredict actual volatility only when the GARCH forecast is lower than 0.21, which, as shown in Exhibit 6, is not far from its lowest decile.

¹⁸The results are similar for an analysis using puts rather than calls.

¹⁹The intuition is that higher implied volatility makes it more likely both that options that move in the money will go out of the money before expiration and options that move out of the money will go in the money before expiration. As a result, higher implied volatilities cause the absolute delta of at-the-money options that are going in the money to rise by less and the absolute delta of at-the-money options that are going out of the money to fall by less.

²⁰In this exercise, we abstract from the impact of changing implied volatility and focus on the impacts of actual volatility through gamma. Because of the option skew, VXN changes do not necessarily correspond to implied volatility changes for specific close-to-the-money options.

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Valuation of Stock Option Grants Under Multiple Severance Risks

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Executive stock option (ESO) grants have a number of important features that do not conform to assumptions of the Black-Scholes model. This article develops a risk-neutral model for valuing such options where the holder is exposed to multiple severance risks like termination with cause or without cause, or mortality, with varying cause-contingent ex-severance payoffs and stock holding restrictions. The Black-Scholes model significantly overestimates the cost of ESOs to the firm by as much as 28% to 39% for a severance rate of 5%, and the bias is inversely related to volatility.

The severance event is modeled using a flexible doubly stochastic Poisson process that permits the rich information structures of the state variables and the severance event to be endogenously captured in the valuation. Valuation is accomplished using a multi-severance binomial ESO model and a multi-severance partial differential equation.

Stock option grants have become an increasingly dominant portion of the total compensation of CEOs and senior executives. Such a great transfer of value from shareholders to corporate executives has raised calls for corporate expensing of stock option grants. Advocates argue that option grants represent a cost of acquiring managerial services whose market value should be an adjustment to earnings in the same way that interest and depreciation reflect a firm's cost of debt and expenditure of physical capital.

Financial Accounting Standards Board (FASB) guidelines suggest the use of Black-

Scholes (BS) [1973] pricing to measure the cost of stock options granted by a corporation. In fact, the Black-Scholes method significantly overestimates the value of stock options by as much as 28% to 39%, at a cumulative severance rate of 5%, because it ignores the severance risks employees face and the cause-dependent ex-severance values of option plans.

Executive stock options (ESOs) have a number of important features that make Black-Scholes model assumptions inappropriate. First, the holder of the stock options faces substantial severance risk; dismissal prior to option expiration (typically, ten years at issue) may lead to loss of the option award, or immediate exercise, or reduction in maturity, depending on the type of employment exit.

Second, stock option plans may impose a stock holding restriction, such as 12 months, on options exercised early (this may be circumvented by registering the exercised stock to a financial intermediary at a discount to the market price, of about 10%-30%). The holding restriction may be plan-specific, or may arise from regulations such as Rule 144 of the U.S. Securities and Exchange Commission (SEC), which restricts the ability of corporate insiders and affiliates to sell shares in their firm. Third, there are restrictions on vesting before which stock options may not be exercised.

We thus need a risk-neutral method to value stock options that prices the multiple severance risks, the cause-contingent severance payoffs, and the stock holding restrictions

embedded in option grants. We develop for this purpose a *multi-severance binomial ESO* model, and also identify an ESO multi-severance partial differential equation (PDE) that offers an alternative method of valuation.

The implementation of the multi-severance ESO model is illustrated by valuing a stock option grant exposed to three sources of severance: with cause, without cause, and mortality. The example also investigates the effect of volatility and exercise price on option valuation and the early exercise rates. As benchmarks, results are compared with the Black-Scholes (BS) model and an alternative way of accounting for option loss risk by exogenously adjusting the BS value by the probability of non-severance.

I. VALUATION APPROACHES AND MODELS

Two general approaches have been adopted to model executive exercise decisions and valuation of the cost of ESOs to a firm. In the first approach, executives select a policy of option exercise that maximizes their expected utility, subject to restrictions on hedging and option sale (Lambert, Larcker, and Verrecchia [1991], Huddart [1994], Marcus and Kulatilaka [1994], and Huddart and Lang [1996]). The utility framework, however, requires information and modeling of unobservable variables such as the executive's risk aversion, wealth holding, and the impact of employment change.

In an alternative risk-neutral approach, early option exercise is modeled as a random stopping time with a Poisson exercise trigger as in Jennergren and Naslund [1993]. Carpenter [1998] shows that a reduced-form American call option model performs as well as or better than the more complicated structural model in predicting exercise patterns for a sample of 40 firms. Cuny and Jorion [1995] and Rubinstein [1995] allow exercise probabilities to depend functionally on the value of the stock, and Brenner, Sundaram, and Yermack [2000] consider the valuation of stock options whose exercise price may be reset. Carr and Linetsky [2001] use a stock price barrier trigger in the intensity process to model the executive's diversification and liquidity considerations.

This article makes three contributions. First, the proposed ESO valuation model incorporates multiple severance risks with ex-severance values that depend on the executive's mode of exit from the firm. Other models assume a single severance payoff for the ESO, and this restriction fails to price the cause-dependent severance structure of actual stock option grants.

Second, the severance event is modeled as a doubly

stochastic Poisson probability process in the multi-severance binomial tree and the PDE gain process. This specification allows the stock price and other state variables to affect the severance probabilities in a very flexible and stochastic fashion, allowing complex and rich information structures to be endogenously captured in the valuation.

Finally, the risk-neutral valuation approach finds near-arbitrage-free prices that are independent of the option holder's personal risk and wealth attributes. This is an attractive feature of the model from the perspective of a corporation attempting to find the cost of its employee stock option grants.

II. CREDIT RISK MODELING

There are some parallels in our mathematical formulation of the severance event and the default event in *reduced-form* credit risk models for fixed-income securities (see Artzner and Delbaen [1995], Jarrow and Turnbull [1995], Duffie and Singleton [1995, 1997], Duffie, Schroder, and Skiadas [1996], Jarrow, Lando, and Turnbull [1997], Lando [1998], Madan and Unal [1998], and others). Despite the similarities, there are important differences between the two settings (credit risk versus severance risk) arising from the type of underlying asset, the source of the risk, the option plan's severance-contingent payoffs, and holding restrictions.

For stock options, the underlying asset is company stock as opposed to the spot rate of interest or forward rates in credit risk models. Second, the credit risk in fixed-income claims (e.g., corporate bonds, swaps) pertains to the discontinuation risk of the corporation or the counterparty, while severance risk affects an individual employee and is essentially internal to the firm. Third, the stock option grants are exposed to multiple sources of severance and exhibit a wide array of severance-dependent values (e.g., forfeiture of options, immediate exercise, reduction in maturity).

III. ESO PRICING UNDER MULTIPLE SEVERANCE RISKS

We first describe the doubly stochastic Poisson probability representation for the severance event under multiple causes, and then extend the standard binomial option pricing model in Cox, Ross, and Rubinstein [1979] (CRR) to incorporate the ESO's multi-severance and cause-contingent ex-severance features. The corresponding PDE is identified, and the effect of stock holding constraints is also discussed.

Severance Risks and the Doubly Stochastic Poisson Representation

Stock option awards typically have very long maturities; ten years is a common maturity at issue. During this time, a stock option holder may exit the firm for a number of reasons: dismissal with cause, dismissal without cause, mortality, and so on. If termination occurs due to cause, the options may be forfeited completely, while in the case of dismissal without cause or mortality the options must be exercised immediately or within a short period of time (e.g., three months). Therefore, the ESO holder is exposed to substantial *severance risk*.

Let τ be the random future time at which severance occurs. Then, the severance event is conveniently represented by the survival indicator $I_{[\tau > u]}$, which takes the value of 1 prior to a severance event at time u and 0 thereafter. There are $j \in \{1, 2, \dots, J\}$ possible causes of severance, and τ_j represents the future random time at which severance results from cause j . The future time of severance (from any cause) is given by $\tau = \min(\tau_1, \tau_2, \dots, \tau_J)$.

The severance event is modeled as the first jump-time of a doubly stochastic Poisson process where severance hazard rate functions $h_{ij} \equiv h_j(s_i, t): \mathcal{R} \times [0, \infty] \rightarrow [0, \infty)$ depend on the random stock price s_i . The information set under which probabilities of future events are computed is represented by $G_t = H_t \vee D_t$ which comprises information on both the stock price evolution (H_t) and the severance event (D_t).

Under multiple severance risks, the probability of non-severance by time s computed at time $t < s$ is given by:

$$\begin{aligned} \Pr(\tau > s | G_t) &= \prod_{j=1}^J \Pr(\tau_j > s | G_t) = E\left(\prod_{j=1}^J I_{[\tau_j > s]} | G_t\right) \\ &= E\left(E\left(\prod_{j=1}^J I_{[\tau_j > s]} | H_s \vee D_s\right) | G_t\right) \\ &= E\left(e^{-\int_t^s (h_{u1} + \dots + h_{uJ}) du} | G_t\right) \end{aligned} \quad (1)$$

where the law of iterated expectations is applied at the third equality using the fact that $(H_s \vee D_s) \subset (H_t \vee D_t) = G_t$. The conditioning argument allows the severance probability at future times to evolve with the state variables revealed at that time (e.g., stock price). The outer expectation then averages over the uncertainty in the future value of these state variables.

Multi-Severance Binomial ESO Model

Risk-neutral arguments are applied to construct the *multiple-severance binomial ESO* model (MSB-ESO). Let s_0 be the initial stock price, and let, $s_t \in [(s_0 u^t d^0), (s_0 u^{t-1}, d^1), \dots, (s_0 u^1 d^{t-1}), (s_0 u^0 d^t)]$, $t = 1, \dots, n$, represent the stock prices in the binomial tree where n is the number of time steps up to maturity T (in years) of length $\Delta = T/n$. At each node, the stock price either moves up by the factor $u = \exp(\sigma\sqrt{\Delta})$ or down by the factor $d = 1/u$, where σ is the stock return volatility. Further, given the continuously compounded risk-free rate r and dividend yield δ , the risk-neutral probabilities for the up and down movements of the stock price at each node are given by $p = \exp[(r - \delta)\Delta] - d/(u - d)$ and $1 - p$, respectively.

In the context of the binomial model, the probability of survival over the next time step under Equation (1) becomes:

$$\begin{aligned} \Pr(\tau > \Delta(t+1) | \tau > \Delta t, s_t) \\ = \left(p e^{-(h_1(s_t, u, \Delta(t+1)) + \dots + h_J(s_t, u, \Delta(t+1)))\Delta} \right. \\ \left. + (1-p) e^{-(h_1(s_t, d, \Delta(t+1)) + \dots + h_J(s_t, d, \Delta(t+1)))\Delta} \right) \end{aligned} \quad (2)$$

Let $W_t(s_i, I_{[\tau > \Delta t]}) \equiv W_t(s_i, I_{[\tau > \Delta t]}; \sigma, r, K, T)$ represent the ESO call option value when exposed to multiple severance risks with exercise price K . $W_t(s_i, I_{[\tau > \Delta t]} = 1)$ represents the survived value of the ESO at time Δt , and $W_t(s_i, I_{[\tau > \Delta t]} = 0, j)$ is the ex-severance value of the ESO when severance occurs due to cause $j \in \{1, 2, \dots, J\}$ (we assume that the severance event occurs at the end of the period $[\Delta(t-1), \Delta t]$). The corresponding hazard rate of severance at time Δt is $h_{ij} \equiv h_j(s_i, \Delta t): \mathcal{R} \times [0, \infty) \rightarrow [0, \infty)$.

ESO plans stipulate differing terms for options under various modes of severance. We note two examples.

Severance with cause: Loss of options. If the ESO holder is terminated due to *cause*, the stock option award usually becomes null and void. Then, the ex-severance payoff at the end of period $[\Delta t, \Delta(t+1)]$ may be written as:

$$W_{t+1}(s_{t+1}, I_{[\tau > \Delta(t+1)]} = 0, j) = 0 \quad (3)$$

Severance without cause or mortality: Reduction in ESO maturity. If the employee leaves voluntarily or is terminated without cause, the holder retains the vested options but may be required to exercise them within a specified period T_s (e.g., three months), thereby reducing

the maturity of the ESO. Let $V(s_{t+1}, T_s)$ represent the value of an American call option with time to maturity T_s . Then, the value of the stock option upon severance at the end of period $[\Delta t, \Delta(t+1)]$ is:

$$W_{t+1}(s_{t+1}, I_{[\tau > \Delta(t+1)]} = 0, j) = V(s_{t+1}, T_s) \quad (4)$$

Alternatively, if the option holder must exercise *immediately* upon severance, the continuation value of the stock option at each node of the binomial tree changes to

$$W_{t+1}(s_{t+1}, I_{[\tau > \Delta(t+1)]} = 0, j) = \max(s_{t+1} - K, 0) \quad (5)$$

Risk-neutral valuation of contingent claims is based on the principle that if the price risk of the derivative security can be dynamically eliminated until expiration by holding positions in the underlying tradable asset, then to rule out arbitrage the hedged position must earn a return equal to the risk-free rate r .

Equivalently, the derivative's arbitrage-free value may be determined by taking an expectation of the contingent claim under a special risk-neutral probability measure. We can apply risk-neutral arguments to our setting because although the option holder cannot freely sell the stock or the ESO, the corporation is free to hedge the option using its stock directly or through a financial intermediary.

Working backward through the binomial tree, the ESO value at time t in state s_t is given by:

$$\begin{aligned} W_t(s_t, I_{[\tau > \Delta t]} = 1) \\ = \max([s_t - K], W_t^c(s_t, I_{[\tau > \Delta t]} = 1)) \end{aligned} \quad (6)$$

where the continuation value $W_t^c(s_t, I_{[\tau > \Delta t]} = 1)$ of the stock option is given by:

$$\begin{aligned} W_t^c(s_t, I_{[\tau > \Delta t]} = 1) &= E\left(e^{-r\Delta} W_{t+1}(s_{t+1}, I_{[\tau > \Delta(t+1)]}) \mid G_{\Delta t}\right) \\ &= E\left(e^{-r\Delta} E\left(W_{t+1}(s_{t+1}, I_{[\tau > \Delta(t+1)]}) \mid H_{\Delta t+1} \vee D_{\Delta t}\right) \mid G_{\Delta t}\right) \\ &= e^{-r\Delta} \left\{ p \left[e^{-h(s_t^u, \Delta(t+1))\Delta} W_{t+1}(s_t^u, I_{[\tau > \Delta(t+1)]} = 1) \right. \right. \\ &\quad \left. \left. + (1 - e^{-h(s_t^u, \Delta(t+1))\Delta}) W_{t+1}(s_t^u, I_{[\tau > \Delta(t+1)]} = 0) \right] \right. \\ &\quad \left. + (1 - p) \left[e^{-h(s_t^d, \Delta(t+1))\Delta} W_{t+1}(s_t^d, I_{[\tau > \Delta(t+1)]} = 1) + \right. \right. \\ &\quad \left. \left. (1 - e^{-h(s_t^d, \Delta(t+1))\Delta}) W_{t+1}(s_t^d, I_{[\tau > \Delta(t+1)]} = 0) \right] \right\} \end{aligned} \quad (7)$$

The expectation in Equation (7) is taken under the doubly stochastic Poisson probability process, and the conditioning arguments of Equations (1) and (2) are used in the second and third equalities. The new quantity $W_{t+1}(s_{t+1}, I_{[\tau > \Delta(t+1)]} = 0)$ represents the expected value of the ESO over the severance states, and is given by:

$$W_{t+1}(s_{t+1}, I_{[\tau > \Delta(t+1)]} = 0) = \sum_{j=1}^J W_{t+1}(s_{t+1}, I_{[\tau > \Delta(t+1)]} = 0, j) \frac{h_{\Delta(t+1),j}}{h_{\Delta(t+1)}} \quad (8)$$

for $s_{t+1} \in (s_t^u, s_t^d)$. This follows from calculating the option's conditional expected value under severance at the end of period $[\Delta t, \Delta(t+1)]$. This is given by

$$\sum_{j=1}^J W_{t+1}(s_{t+1}, I_{[\tau > \Delta(t+1)]} = 0, j) \Pr(\tau = \tau_j \mid \tau = \Delta(t+1))$$

where

$$\begin{aligned} \Pr(\tau = \tau_j \mid \tau = \Delta(t+1)) \\ = \frac{\Pr(\tau > \Delta(t+1)) h_{\Delta(t+1),j}}{\Pr(\tau > \Delta(t+1)) h_{\Delta(t+1)}} = \frac{h_{\Delta(t+1),j}}{h_{\Delta(t+1)}} \end{aligned}$$

Severance Probabilities and Estimation

Can we use severance probabilities (or hazard rates) based on the company's historical employment data in ESO valuation?

The company's human resource department has historical employment turnover data to allow estimation of annual severance probabilities of its managerial staff, and these can be disaggregated by cause. An important issue here is whether such empirical severance probabilities may be used in the multi-severance binomial ESO model when the risk-neutral option pricing framework requires risk-neutral probabilities of severance. We consider this from the viewpoint of both the option writer (the corporation) and the ESO holder (the executive).

From the perspective of the corporation, it may be argued that employee severance risk is relatively well diversified (Jennergren and Naslund [1993], Cuny and Jorion [1995], and Carr and Linetsky [2001]). A corporation with many senior executives, constant managerial turnover, and a ready pool of potential replacements has relatively low exposure to the severance jump risk arising from any single employee. Therefore, on an aggregate level, the company's employee severance risk is not sys-

tematic and is relatively diversified (this is similar to the CAPM argument that the market should not compensate diversifiable security risk in equilibrium).

This may also be a reasonable practical assumption on empirical grounds. For example, Marquardt [1999] examines 58 Fortune 500 firms for over 21 years, and finds an average of 17 ESO grants per firm. This line of reasoning implies that, from the company's perspective, one may treat the firm's empirical severance probabilities as risk-neutral severance probabilities for the purpose of valuing its stock option grants.

These arguments for the firm do not apply to the option holder, and a utility framework is necessary for complete analysis. This approach, however, requires information and assumptions on unobservables such as the executive's risk aversion and wealth endowment that pose substantial problems in implementation. Here, we can assert that the risk-neutral MSB-ESO model offers an easy-to-compute upper bound (much lower than Black-Scholes) on the value of a stock option grant to the executive.

A company can readily estimate from its employment turnover data the annual probabilities q_j that a manager will involuntarily leave the firm from each cause j . These probabilities can be converted to annualized hazard rates of severance (e.g., for $q_j = 0.05$, $h_j = -\ln(1 - q_j) = 0.051293$), or, alternatively, one can use more refined statistical techniques such as Cox proportional hazard modeling to estimate hazard functions in terms of covariates, including the stock price. For severance related to mortality, age- and sex-specific actuarial life tables published by the Society of Actuaries may be used to quantify this probability distribution.

Stock Holding Restrictions and Discounted Exercise

If the employee exercises an award prior to expiration, some option plans prevent the option holder from selling the stock for a certain period (e.g., 12 months). This restriction imposes an additional risk, as the stock price can change over the holding period. In the risk-neutral world, however, the held stock is expected to grow at the risk-free rate over the holding period, and its present value is just the current value of the stock. Therefore, the holding restriction does not affect the ESO.

Rather than wait one year to sell the stock, the option holder may exercise the option and register the stock in the name of a financial intermediary at a $(1 - \lambda)\%$ discount (typical discounts fall in the 10%-30% range).

The effects of the discount sale alternative on option valuation are incorporated by substituting for Equation (6):

$$W_t(s_t, I_{[\tau > t]} = 1) = \max([\lambda s_t - K], W_t^c(s_t, I_{[\tau > t]} = 1)) \quad (9)$$

ESO Multi-Severance Risk PDE

An alternative method of valuation is to solve the *multi-severance risk-adjusted partial differential equation* (PDE) representing the ESO. Merton [1976] considers option pricing with discontinuous jumps in the stock return process. This framework is not exactly applicable here because severance risk is independent of jumps in the stock price, and severance affects the ESO value directly (not the stock price). Further, the ESO's ex-severance values depend on the mode of severance.

Variants of a single-discontinuity risk PDE are considered by Jennergren and Naslund [1993] and Rubinstein [1995] under a constant hazard specification and by Carr and Linetsky [2001] who allow the constant hazard rate to jump when the stock price hits a predefined barrier. This work, however, considers only a single aggregate cause of severance and restricts the ESO to one ex-severance value. We now obtain a multi-severance ESO PDE.

We take t to represent continuous time, and X_t represents the corresponding stock price (as opposed to s_t in the binomial model). We make the standard BS assumption that the stock price obeys a diffusion process as follows under the risk-neutral probability measure Q :

$$dX_t = (r - q)X_t dt + \sigma X_t dB_t^Q \quad (10)$$

where r is the risk-free rate (drift under Q), q is the dividend payout rate, σ is the return volatility, and B_t^Q is standard Brownian motion. To avoid a dramatic change from the earlier notation, we understand $W_t(X_t, I_{[\tau > t]} = 1) \equiv W(t, X_t, I_{[\tau > t]} = 1): [0, \infty) \times \{0, 1\} \rightarrow [0, \infty)$ to be twice-differentiable with respect to the first two arguments.

Mathematically, the risk-neutral pricing condition is that the discounted value of the derivative claim must be a martingale under the risk-neutral probability measure Q :

$$E^Q\{d[W_s(X_s, I_{[\tau > s]} = 1)e^{-rs}]|G_t\} = 0, 0 < t < s < T \quad (11)$$

The terms of $d[W_s(X_s, I_{[\tau > s]} = 1)e^{-rs}]$ are obtained by applying the generalized form of Itô's formula (see Protter [1990, p. 71]), and the expectation in Equation (11) is made using the law of iterated expectations:

$$E(\cdot|G_t) = E(E(\cdot|H_s \vee D_t)|G_t)$$

The resulting multi-severance risk ESO PDE is given by:

$$\begin{aligned} \frac{\partial W}{\partial s} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 W}{\partial X^2} + (r - q)X_s \frac{\partial W}{\partial X} - rW_s(X_s, I_{[\tau > s]} = 1) \\ + \left[\sum_{j=1}^J h_{sj} W_s(X_s, I_{[\tau > s]} = 0, j) - (h_{s1} + \dots + h_{sJ}) W_s(X_s, I_{[\tau > s]} = 1) \right] = 0 \end{aligned} \quad (12)$$

for $0 \leq s < T$ where $W_T(X_T, I_{[\tau > T]} = 1) = (X_T - K)^+$ is the stock option's terminal value; $W_s(X_s, I_{[\tau > s]} = 0, j)$ is the ESO's ex-severance value when triggered by cause $j = 1, \dots, J$; and $h_{sj} \equiv h_j(X_s, t): \mathfrak{R}[0, \infty] \rightarrow [0, \infty]$ is the corresponding *hazard rate function of severance*. (Details are in the appendix.)

The ESO may be valued by solving the multi-severance risk ESO PDE (12) using numerical methods (e.g., explicit finite differences, implicit finite differences, Crank-Nicholson). The early exercise and discounted exercise features can be easily applied at each iteration in the finite-difference procedure.

IV. APPLICATION AND NUMERICAL STUDY

We use the multi-severance binomial ESO (MSB-ESO) model to price a stock option grant under three potential causes of severance. The numerical study also investigates the valuation bias from using alternative models (Black-Scholes, exogenous adjustment to BS) and the effect of volatility on the bias and early exercise.

ESO Valuation Under Three Severance Risks

The ESO grant consists of 500,000 shares with a maturity of ten years and a vesting period of three years. The initial market price of the stock is $X_0 = \$50$; the annualized return volatility is $\sigma = 50\%$; and the stock is non-dividend-paying. We consider three causes of job severance ($J = 3$): with cause ($j = 1$), without cause ($j = 2$), and mortality ($j = 3$). The executive's annual probabilities of severance are taken to be 2% for severance with cause ($q_1 = 0.02$) and 3% for severance without cause ($q_2 = 0.03$). The probabilities of mortality ($q_3, t = 1, \dots, 10$) as reported in the actuarial life tables published by the Society of Actuaries for an insured man aged 50 are used: $q_3 = (0.00317, 0.00343, 0.00379, 0.00420, 0.00472, 0.00534,$

$0.00599, 0.00668, 0.00724, 0.00789)$. In the implementation, the annual probabilities are converted to the relevant severance hazard rate by cause.

The valuation of the ESO is considered according to several models:

1. Black-Scholes call option value (no severance risk adjustment).
2. MSB-ESO model with the cause-dependent severance payoffs:
 - A. Loss of stock option plan upon severance for cause $j = 1$.
 - B. Immediate exercise upon severance for cause $j = 2$.
 - C. Three-month option expiration upon severance for cause $j = 3$.
3. Severance probability-adjusted Black-Scholes.

Valuation is considered under both the 12-month stock holding restrictions as well as the 90% discount sale alternative. Scenario 3 refers to a simple approach of adjusting the Black-Scholes option value by the probability of non-severance:

$$\Pr(\tau > T | \tau > t) = \exp(-\sum_{i=1}^n h_i \Delta)$$

As reported in Exhibit 1, valuation under severance risk is substantially lower than under the Black-Scholes (BS) model. A comparison of columns (1) and (2) shows that BS overestimates the option value by between 28% and 39%, leading to significant overvaluation of the stock option grant. Black-Scholes gives a valuation of \$16,801,026 while the multi-severance binomial ESO model (MSB-ESO) yields a value of \$12,527,992 with a vesting period of three years (second panel). Meanwhile, the probability-adjusted BS method undervalues the option grants by 19% to 25%.

Exhibits 2 and 3 graph the results. It is clear that ignoring severance risk, as occurs in Black-Scholes pricing, leads to dramatic overvaluation of the ESO value (top curve in Exhibits 2 and 3), while the simple adjustment to the Black-Scholes value by the probability of survival (along the lines of Jennergren and Naslund [1993]) leads to significant undervaluation. This adjustment ignores the additional value from early exercise under severance risk exposure as well as the severance-contingent payoffs of the stock option grant.

EXHIBIT 1

Comparison of Executive Stock Option Valuation under Multiple Severance Risks and Exercise/Vesting Restrictions

Type of Stock Holding Restriction	Time to Vesting/ Shares/ BS Bias	(1) Black-Scholes Valuation	(2) MSB-ESO (Severance Risk Valuation)	(3) Black-Scholes Probability- Adjusted
No Restriction / Stock Holding	0 Yrs Vesting	33.6021	26.2847	19.6135
	500,000 shrs	(\$16,801,026)	(\$13,142,349)	(\$9,806,745)
	BS Bias	28%		-25%
	3 Yrs Vesting	33.6021	25.0560	19.6135
90% Discount Sale	500,000 shrs	(\$16,801,026)	(\$12,527,992)	(\$9,806,745)
	BS Bias	34%		-22%
	0 Yrs Vesting	33.6021	25.3250	19.6135
	500,000 shrs	(\$16,801,026)	(\$12,662,487)	(\$9,806,745)
	BS Bias	33%		-23%
	3 Yrs Vesting	33.6021	24.1067	19.6135
	500,000 shrs	(\$16,801,026)	(\$12,053,342)	(\$9,806,745)
	BS Bias	39%		-19%

BS Bias is the Black-Scholes overvaluation from ignoring severance risk and is computed as the percentage difference between the BS and MSB-ESO value.

Effects of Volatility and Exercise Price

To consider the effect of stock volatility and strike price on ESO valuation, we maintain the same parameter settings as in Exhibit 1, but use two levels of volatility ($\sigma = 20\%$ and 50%) and three levels of exercise prices ($0.8X_0$, X_0 , and $1.2X_0$). Numerical results for vested options with no stock holding restriction (same as 12-month holding) are reported in Exhibit 4.

Again, Black-Scholes (BS) valuation and the probability adjustment for severance lead to significant over- and undervaluation, respectively, but the extent of the bias depends on the volatility and moneyness of the option. An increase in volatility from $\sigma = 20\%$ to $\sigma = 50\%$ reduces the BS pricing bias from the range of 30%–41% to 25%–30%. Conversely, the undervaluation bias in the probability-adjusted Black-Scholes increases from 18%–24% to 24%–27% as volatility rises. Therefore, the use of Black-Scholes for option expensing will lead to a greater overvaluation bias for lower-volatility blue-chip stocks than the more volatile Nasdaq technology stocks. We also observe that the BS bias increases uniformly with the option's exercise price.

Column (2) also reports the early exercise rate for the MSB-ESO model. This is the fraction of the total states in the binomial tree where early exercise occurred. An increase in volatility increases the early exercise rate. As stock volatility increases from $\sigma = 20\%$ to $\sigma = 50\%$, the

early exercise rates increase from 28%–33% to 38%–40%.

While volatility increases the continuation value of the option under severance, it also has the offsetting effect of raising the immediate payoff from early exercise. The empirical results suggest that the latter effect dominates, yielding a positive relationship between volatility and early exercise. Further, early exercise rates fall as the exercise price rises.

Effect of Ex-Severance Payoffs

To examine the impact on ESO valuation of specific ex-severance option values independently, we maintain the same parametric settings as before. Now there is only one source of severance ($J = 1$), and the annual probability of severance is set at $q = 5\%$ (severance hazard rate of $h = 0.051293$).

The stock options are valued using three models:

1. Black-Scholes call option value (no severance risk adjustment).
2. MSB-ESO valuation with ex-severance payoffs:
 - A. Loss of stock option plan.
 - B. Immediate exercise.
 - C. Three-month option expiration.
3. Severance probability-adjusted Black-Scholes.

EXHIBIT 2

Stock Option Valuation to Maturity by Valuation Model—No Restriction on Stock Holding

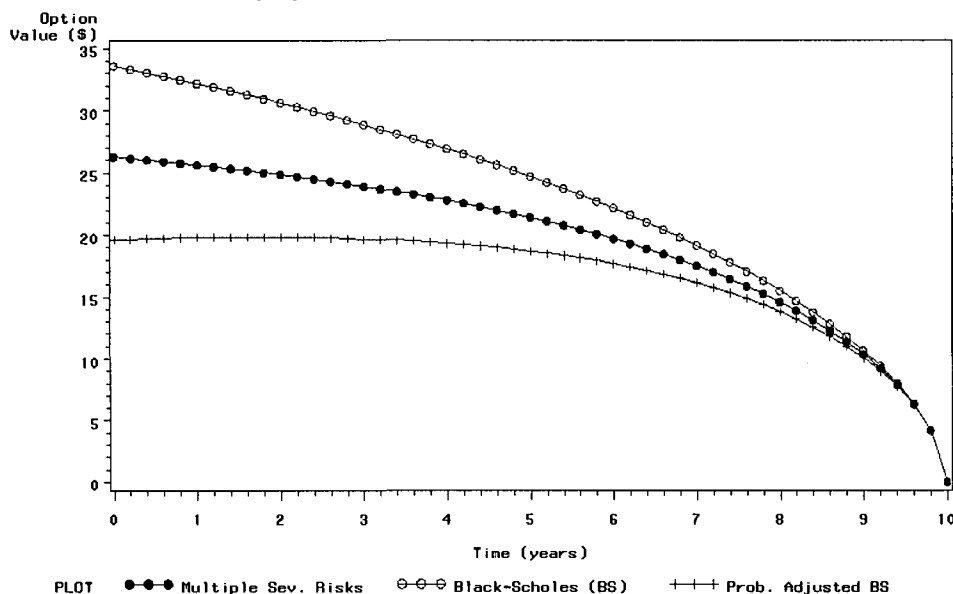


EXHIBIT 3

Stock Option Valuation to Maturity by Valuation Model—90% Discount Sale

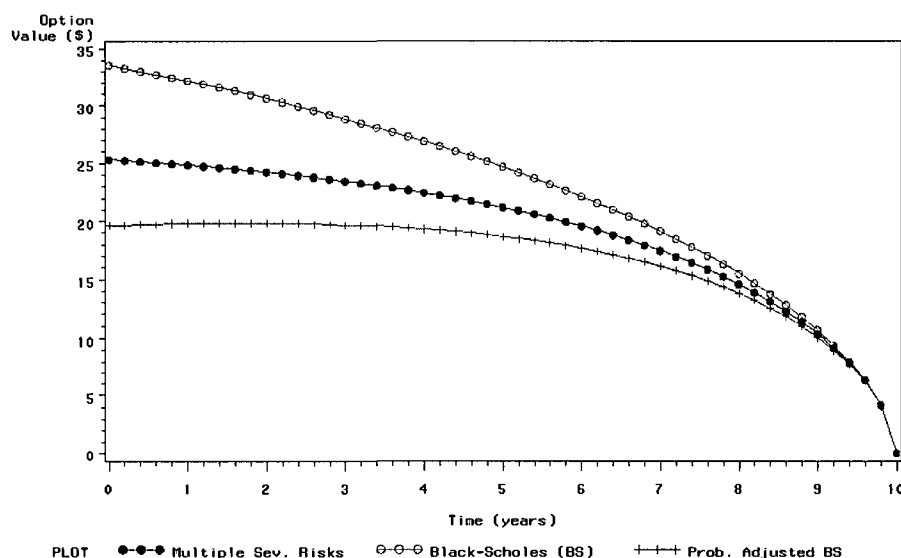


Exhibit 5 shows that the greatest impact on option valuation occurs when options are lost upon severance (2.A). In this case, the Black-Scholes overvaluation bias increases from 36%–41% to 51%–59% when volatility falls from $\sigma = 50\%$ to $\sigma = 20\%$. When options must be exercised within three months of severance (2.C), the bias in pricing drops to 13%–19% for $\sigma = 50\%$ and 14%–26% for $\sigma = 20\%$.

The early exercise rates show that early exercise is optimal only when options are lost upon severance (2.A), while there is no early exercise for immediate exercise (2.B) and three-month expiration (2.C). For scenario 2.A, a detailed breakdown of early exercise rates at different times in the life of the stock option (strike at the money) is given in Exhibits 6 and 7.

EXHIBIT 4

Effect of Volatility and Exercise Price on ESO Valuation and Early Exercise

Stock Volatility σ	Strike Moneyiness (m) $K = mX_0$		(1) Black-Scholes Valuation	(2) MSB-ESO (Severance Risk Valuation)	(3) Black-Scholes Probability- Adjusted
$\sigma = 50\%$	0.8	Option Value	35.7487	28.6995	20.8665
		BS Bias	24.6%		-27.3%
		Early Ex. Rate		39.7%	
	1.0	Option Value	33.6021	26.2847	19.6135
		BS Bias	27.8%		-25.4%
		Early Ex. Rate		38.6%	
	1.2	Option Value	31.8869	24.4089	18.6124
		BS Bias	30.6%		-23.7%
		Early Ex. Rate		37.6%	
$\sigma = 20\%$	0.8	Option Value	27.0798	20.8436	15.8064
		BS Bias	29.9%		-24.2%
		Early Ex. Rate		32.6%	
	1.0	Option Value	22.5682	16.6001	13.1730
		BS Bias	36.0%		-20.6%
		Early Ex. Rate		30.0%	
	1.2	Option Value	18.7949	13.3260	10.9706
		BS Bias	41.0%		-17.7%
		Early Ex. Rate		27.8%	

Early Exercise Rate is the percentage of nodes in the binomial tree where early exercise occurred. BS Bias gives the bias in Black-Scholes valuation due to ignoring severance risk.

Exhibit 6 shows that under the no-stock holding restriction (and 12-month stock holding), as the volatility drops from 50% to 20%, the interval over which early exercise occurs shortens from 0.9–10.0 years to 1.4–10.0 years. The same shortening for the 90% discount sale feature is from 1.0–8.0 years to 1.6–8.0 years (Exhibit 7). It is clear from the two graphs that volatility increases both the rates of early exercise and the span of the early exercise time zone.

V. CONCLUSION

We have developed a risk-neutral model for valuing executive stock options exposed to severance risk from multiple sources with varying cause-contingent severance payoffs and stock holding restrictions. These features of executive stock option grants cannot be captured under the assumptions of the Black-Scholes model. In the case of option expensing, the model offers the further advantage that valuation does not depend on the risk aversion and endowment of the option holder. This is an attractive feature of the model for a corporation attempting to

find the total severance-adjusted value of the options it has granted.

Valuation may be performed by implementing either the multi-severance binomial ESO model (MSB-ESO) or by numerically solving a multi-severance PDE. Both representations accommodate the option's varying cause-contingent payoffs. Modeling the severance event using a flexible doubly stochastic Poisson process with stochastic hazard parameters enables incorporation of complex interaction between state variables and the severance event during valuation. The model also endogenously values the executive's early exercise decision, as severance risk diminishes the ESO's continuation value.

The numerical study shows that valuation by Black-Scholes—as suggested in FASB guidelines—results in significant overvaluation, as this method ignores the severance risk faced by the option holder. The Black-Scholes approach inflates the option expense by the range of 28% to 39% for at-the-money options with a cumulative severance rate of 5%, while an exogenous probability adjustment for option loss to the Black-Scholes formula leads to undervaluation of 19% to 25%. Further, volatility reduces

EXHIBIT 5

Relative Effect of Severance Payoffs on ESO Valuation and Early Exercise

Stock Volatility σ	Strike Moneyness (m) $K = mX_0$		(1) Black-Scholes (No Severance Adjustment)	(2) MSB-ESO (Severance Risk Valuation)			(3) Prob.-Adjusted Black-Scholes
			(1) BS Call Option	(2.A) Option Loss	(2.B) Immediate Exercise	(2.C) Three-month Expiry	(3) Prob. Adj. To A
$\sigma = 50\%$	0.8	Option Value	35.7487	26.3086	31.7063	31.7167	21.4041
		Early Ex.		40.3%	0	0	
		Exogenous		35.9%	12.7%	12.7%	67.0%
	1.0	Option Value	33.6021	24.1902	28.9779	28.9895	20.1188
		Early Ex.		41.4%	0	0	
		Exogenous		38.9%	16.0%	15.9%	67.0%
$\sigma = 20\%$	0.8	Option Value	31.8869	22.5533	26.8613	26.8762	19.0919
		Early Ex.		40.3%	0	0	
		Exogenous		41.4%	18.7%	18.6%	67.0%
	1.0	Option Value	27.0798	17.9606	23.8241	23.8268	16.2137
		Early Ex.		40.0%	0	0	
		Exogenous		50.8%	13.7%	13.7%	67.0%
$\sigma = 20\%$	1.0	Option Value	22.5682	14.4890	18.7638	18.7695	13.5124
		Early Ex.		37.0%	0	0	
		Exogenous		55.8%	20.3%	20.2%	67.0%
	1.2	Option Value	18.7949	11.8290	14.9109	14.9120	11.2532
		Early Ex.		34.7%	0	0	
		Exogenous		58.9%	26.0%	26.0%	67.0%

EXHIBIT 6

Early Exercise Rates over Time under Option Forfeiture upon Severance—No Restriction on Stock Holding

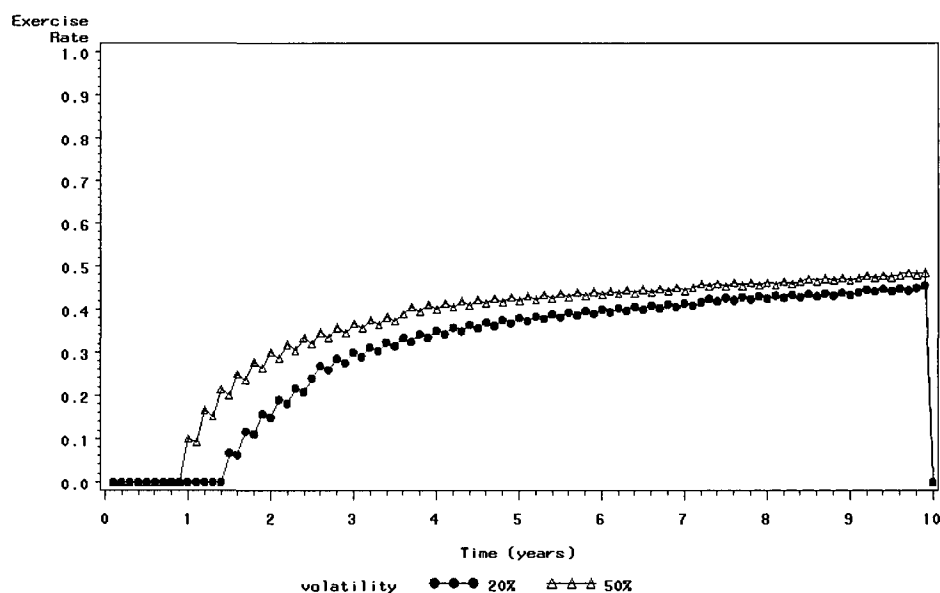
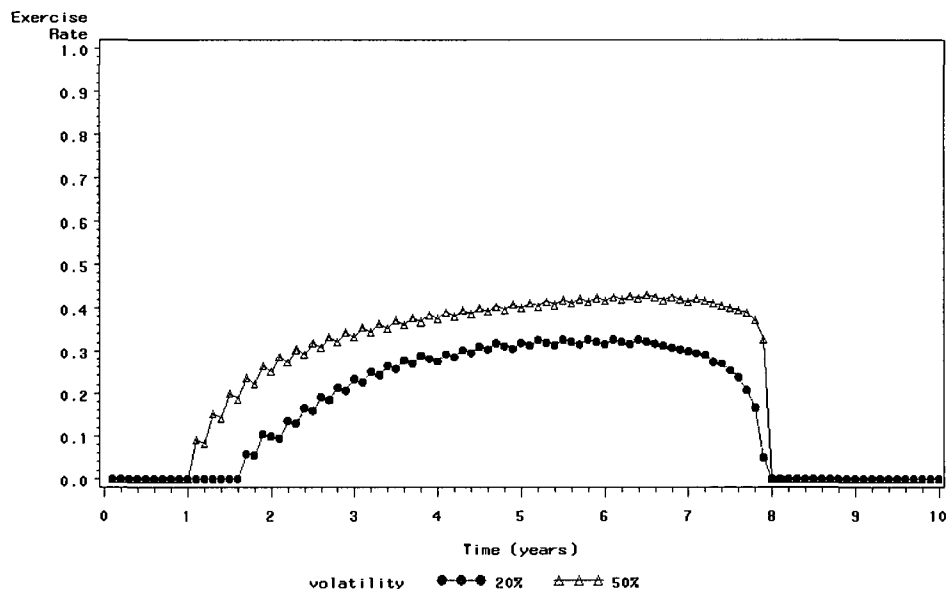


EXHIBIT 7

Early Exercise Rates over Time under Option Forfeiture upon Severance—90% Discount Sale



the BS overvaluation bias and increases both the rate of early exercise and the span of the exercise time horizon. Therefore, Black-Scholes valuation will overvalue employee stock option grants on blue-chip stocks to a greater extent than more volatile Nasdaq technology companies.

APPENDIX

Multi-Severance ESO PDE

The components of $d\{W_s(X_s, I_{[\tau > s]} = 1)e^{-rs}\}$ are obtained by applying the generalized form of Itô's formula (Protter [1990, p. 71]). This gives:

$$\begin{aligned} & d\{W_s(X_s, I_{[\tau > s]} = 1)e^{-rs}\} \\ &= dW_s(X_s, I_{[\tau > s]} = 1)e^{-rs} - rW_s(X_s, I_{[\tau > s]} = 1)e^{-rs} \\ &= \left\{ \left[\frac{\partial W}{\partial s} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 W}{\partial X^2} + ((r-q)X + \sigma X dB_s^Q) \frac{\partial W}{\partial X} \right] \right. \\ & \quad \left. + \left[W_s(X_s, I_{[\tau > s]} = 0, j) - W_s(X_s, I_{[\tau > s]} = 1) \right] - rW_s(X_s, I_{[\tau > s]} = 1) \right\} e^{-rs} \end{aligned} \quad (A-1)$$

where $\Delta W_s \equiv W_s(X_s, I_{[\tau > s]} = 0, j) - W_s(X_s, I_{[\tau > s]} = 1)$ is the jump in ESO value due to severance at the moment $[s, s + ds]$, and the remaining terms represent the diffusive change. From the doubly stochastic Poisson representation of $\Pr(\tau > s | G_t)$, $\tau < s$, given by Equation (1), the instantaneous severance measure is:

$$\begin{aligned} \frac{\partial \Pr(\tau < s | G_t)}{\partial s} &= \frac{\partial [1 - \Pr(\tau > s | G_t)]}{\partial s} \\ &= E \left((h_{s1} + \dots + h_{sj}) e^{-\int_t^s (h_{u1} + \dots + h_{uj}) du} | G_t \right) \end{aligned} \quad (A-2)$$

Similarly, the instantaneous severance measure due to cause $j = 1, \dots, J$ is given by

$$\frac{\partial \Pr(\tau < s, j | G_t)}{\partial s} = E \left(h_{sj} e^{-\int_t^s (h_{u1} + \dots + h_{uj}) du} | G_t \right) \quad (A-3)$$

Taking the conditional expectation of Equation (11) under the law of iterated expectations $E(\cdot | G_t) = E(E(\cdot | H_s \vee D_t) | G_t)$ with $t < s < T$ where $(H_s \vee D_t) \subset (H_t \vee D_t) = G_t$ yields:

$$\begin{aligned} & E^Q \{ d\{W_s(X_s, I_{[\tau > s]} = 1)e^{-rs}\} | G_t \} \\ &= E^Q \left\{ \left(\left[\frac{\partial W}{\partial s} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 W}{\partial X^2} + ((r-q)X + \sigma X dB_s^Q) \frac{\partial W}{\partial X} \right] \right. \right. \\ & \quad \left. \left. + \left[W_s(X_s, I_{[\tau > s]} = 0, j) - W_s(X_s, I_{[\tau > s]} = 1) \right] - rW_s(X_s, I_{[\tau > s]} = 1) \right] \right\} e^{-rs} H_s \vee D_t | G_t \end{aligned}$$

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