Computing Option Price Sensitivities Using Homogeneity and Other Tricks

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No practitioner software can survive without providing derivatives of option prices with respect to underlying market or model parameters, the so-called Greeks. We present a list of common Greeks and exploit homogeneity properties of financial markets to derive many model-independent relationships among Greeks.

We apply the results to European style options and multi-asset options, and avoid time-consuming computations of derivatives.

The computation of sensitivities of option prices, the so-called Greeks, is often cumbersome—both for the mathematician and for symbolic calculators. We provide methods to avoid differentiation as much as possible.

Many Greeks are related. The relationships are based on both model-independent homogeneity of time and price level of a financial product and model-dependent relationships such as the partial differential equation that the value function must satisfy and the assumed distribution of the underlying.

The basic market model we use is the Black-Scholes model. This model supports the homogeneity properties that are valid in general, but its structure is so simple that we can concentrate on the essential elements.

For special cases, we look at the Greeks of European options in the Black-Scholes model in one dimension. It turns out that one needs to know only two Greeks in order to calculate all the other Greeks without differentiating.

Another interesting example is a European derivative security depending on two assets. For such rainbow options, the analysis of risk due to changing correlation of the two assets is very important. We will show how this risk is related to simultaneous changes of the two underlying securities.

There are several advantages of these homogeneity relations:

1. Time saving in computing derivatives.
2. Robust implementation compared to Greeks via difference quotients.
3. Checks on the quality and consistency of Greeks produced by finite-difference, tree, or Monte Carlo methods.
4. Computation of Greeks for Monte Carlo-based values.
5. Evidence of relationships among Greeks that wouldn't be seen merely by looking at difference quotients.

Related work about the computation of Greeks includes Pelsser and Vorst [1994], who show how to compute Greeks in a binomial tree, and Carr [2001], who views Greeks as values of suitable derivative securities and then computes them using risk-neutral valuation. The computation of Greeks using Monte Carlo simulation has been discussed by Broadie and Glasserman [1996] and Glasserman and Zhao [1999].

I. FUNDAMENTAL PROPERTIES

We list notation and the commonly used Greeks and their symbols in Exhibits 1 and 2. We do not claim this list is complete, because one can always define more derivatives of the option price function.

Homogeneity of Time

In most cases, the price of the option is not a function of both the current time t and the maturity time T, but rather *only* a function of the time to maturity $\tau = T - t$ implying the relations

$$\Theta = v_t = -v_\tau = -v_T = -\mathrm{Dual}\Theta \tag{1}$$

Scale-Invariance of Time

The principle of the scale-invariance of time holds in general. In a market model, parameters may be quoted on an annual basis. We illustrate this idea in a Black-Scholes framework, in which the volatility is a model parameter. The same idea can easily be applied to other market models.

We may want to measure time in units other than years, in which case interest rates and volatilities, which are normally quoted on an annual basis, must be changed according to rules as follows for all $a > 0$:

$$\begin{aligned} \tau &\rightarrow \frac{\tau}{a} \\ r &\rightarrow ar \\ q &\rightarrow aq \\ \sigma &\rightarrow \sqrt{a}\sigma \end{aligned} \tag{2}$$

The option's value must be invariant under this rescaling:

$$v(x, \tau, r, q, \sigma, \dots) = v(x, \frac{\tau}{a}, ar, aq, \sqrt{a}\sigma, \dots) \tag{3}$$

We differentiate this equation with respect to α and obtain for $a = 1$

$$0 = \tau\Theta + r\rho + q\rho_q + \frac{1}{2}\sigma\Phi \tag{4}$$

a general relationship among the Greeks *theta, rho, rhoq,* and *vega*. Applying the same principle to a derivative based on multiple underlyings $(1, \dots, n)$:

$$\begin{aligned} &v(x_1, \dots, x_n, \tau, r, q_1, \dots, q_n, \sigma_{11}, \dots, \sigma_{nn}) = \\ &v(x_1, \dots, x_n, \frac{\tau}{a}, ar, aq_1, \dots, aq_n, \sqrt{a}\sigma_{11}, \dots, \sqrt{a}\sigma_{nn}) \end{aligned} \tag{5}$$

we obtain Theorem 1 (*scale invariance of time*):

$$0 = \tau\Theta + r\rho + \sum_{i=1}^{n} q_i \rho_{q_i} + \frac{1}{2}\sum_{i,j=1}^{n} \Phi_{ij}\sigma_{ij}. \tag{6}$$

where Φ_{ij} denotes the differentiation of v with respect to σ_{ij}.

Scale-Invariance of Prices

The general idea is that values of securities may be measured in a different unit, just like values of European stocks are now measured in Euro instead of local currencies. Option contracts usually depend on strikes and barriers. Rescaling can have different effects on the value of the option.

Essentially we consider several types of homogeneity classes. Let $v(x, k)$ be the value function of an option, where x is the spot (or a vector of spots) and k the strike or barrier or a vector of strikes or barriers.

Definition (*homogeneity classes*): We call a value function k-homogeneous of degree n if for all $a > 0$:

$$v(ax, ak) = a^n v(x, k) \tag{7}$$

We call an option whose value function is strike-homogeneous of degree 1 a *strike-defined option* and an option whose value function is barrier-homogeneous of degree 0 a *barrier-defined option*.

The value function of a European call or put option with strike K is then K-homogeneous of degree 1; a digital option that pays a fixed amount if the stock price is higher than a barrier B is B-homogeneous of degree 0. The path-independent barrier call option paying $(S - k)^+ I_{\{S > K\}}$ is (k, K)-homogeneous of degree 1. A power call with cap paying $\min(C, [(S - k)^+]^2)$ has a homogeneity structure of $v(aS, aK, a^2C) = a^2 v(S, K, C)$.

We show how such a scale-invariance can be used to determine some relations among the Greeks. We explain this using two examples. In the first example, we analyze a strike-defined option, and in the second one we concentrate on a barrier-defined option. Generalization to options with some more parameters like a capped call or power call is easily done.

Strike Delta and Strike Gamma. For a strike-defined value function we have for all $a, b > 0$:

$$abv(x, k) = v(abx, abk) \tag{8}$$

We differentiate with respect to a and get for $a = 1$:

$$bv(x, k) = bxv_x(bx, bk) + bkv_k(bx, bk) \tag{9}$$

We now differentiate with respect to b get for $b = 1$:

$$\begin{aligned} v(x, k) &= xv_x + xv_{xx}x + xv_{xk}k + \\ &\quad kv_k + kv_{kx}x + kv_{kk}k \qquad (10) \\ &= x\Delta + x^2\Gamma + 2xkv_{xk} + k\Delta^k + k^2\Gamma^k. \qquad (11) \end{aligned}$$

If we evaluate Equation (9) at $b = 1$ we get:

$$v = x\Delta + k\Delta^k. \tag{12}$$

We differentiate this equation with respect to k and obtain

$$\Delta^k = xv_{kx} + \Delta^k + k\Gamma^k, \tag{13}$$

$$kxv_{kx} = -k^2\Gamma^k. \tag{14}$$

Together with Equation (11) we conclude

$$x^2\Gamma = k^2\Gamma^k \tag{15}$$

Barrier Delta and Barrier Gamma. For a barrier-defined value function we have for all $a, b > 0$:

$$v(x, B) = v(abx, abB) \tag{16}$$

We differentiate with respect to a and get at $a = 1$

$$0 = v_x(bx, bB)bx + v_B(bx, bB)bB \tag{17}$$

If we set $b = 1$ we get the relation

$$\Delta x + \Delta^B B = 0 \tag{18}$$

Now we differentiate Equation (17) with respect to b and get at $b = 1$:

$$0 = v_{xx}x^2 + 2v_{xB}xB + v_{BB}B^2. \tag{19}$$

We can also differentiate the relation between delta and barrier delta with respect to B and get

$$v_{xB}x + B\Gamma^B + \Delta^B = 0 \tag{20}$$

With Equation (19) we conclude:

$$x^2\Gamma + x\Delta = B^2\Gamma^B + B\Delta^B \tag{21}$$

In general we obtain Theorem 2 (*price homogeneity*):

$$v = \sum_{i=1}^{n} x_i\Delta_i + \sum_{j=1}^{m} k_j\Delta_j^k \tag{22}$$

$$\sum_{i,j=1}^{n} x_i x_j\Gamma_{ij} = \sum_{i,j=1}^{m} k_i k_j\Gamma_{ij}^k \tag{23}$$

for strike-defined options and

$$0 = \sum_{i=1}^{n} x_i\Delta_i + \sum_{j=1}^{m} B_j\Delta_j^B \tag{24}$$

$$\begin{aligned} \sum_{i,j=1}^{n} x_i x_j\Gamma_{ij} + \sum_{i=1}^{n} x_i\Delta_i &= \\ \sum_{i,j=1}^{m} B_i B_j\Gamma_{ij}^B + \sum_{i=1}^{m} B_i\Delta_i^B \end{aligned} \tag{25}$$

for barrier-defined options.

II. EUROPEAN OPTIONS IN THE BLACK-SCHOLES MODEL

We start with relations among Greeks for European claims in the n-dimensional Black-Scholes model:

$$dS_i(t) = S_i(t)[(r - q_i)\,dt + \sigma_i\,dW_i(t)] \quad \text{for } i = 1, \ldots, n \tag{26}$$

$$\mathbf{Cov}(W_i(t), W_j(t)) = \varsigma_{ij} t, \tag{27}$$

where r is the risk-free rate, q_i the dividend rate of asset i or foreign interest rate of exchange rate i, σ_i the volatility of asset i, and $(W_1, \ldots, W_n)$ a standard Brownian motion (under the risk-neutral measure) with correlation matrix ς.

Let v denote today's value of the payoff $f[S_1(T), \ldots, S_n(T)]$ at maturity T. Then it is known that v satisfies the Black-Scholes partial differential equation:

$$0 = -v_\tau - rv + \sum_{i=1}^{n} x_i (r - q_i) v_{x_i} + \frac{1}{2}\sum_{i,j=1}^{n} (\sigma \circ \sigma^T)_{ij}\, \varsigma_{ij}\, x_i x_j v_{x_i x_j} \tag{28}$$

Relationships Among Greeks Based on the Lognormal Distribution

The value function v has a representation given by the n-fold integral:

$$v = e^{-r\tau} \int f\left(\ldots, S_i(0)e^{\sigma_i\sqrt{\tau}x_i + \mu_i \tau}, \ldots\right) g(\vec{x}, \varsigma)\, d\vec{x}. \tag{29}$$

where $\mu_i = r - q_i - \frac{1}{2}\sigma_i^2$ and $g(\vec{x}, \varsigma)$ is the n-variate standard normal density with correlation matrix ς.

Since we do not want to assume differentiability of the payoff f, but we know that the transition density g is differentiable, we define a change in the variables $y_i \triangleq S_i(0)e^{\sigma_i\sqrt{\tau}x_i + \mu_i\tau}$, which leads to

$$v = e^{-r\tau} \int f(\ldots, y_i, \ldots) \times g\left(\frac{\ln\frac{y_i}{S_i(0)} - \mu_i\tau}{\sigma_i\sqrt{\tau}}, \varsigma\right) \frac{d\vec{y}}{\prod y_i \sigma_i \sqrt{\tau}} \tag{30}$$

Properties of the Normal Distribution

We collect some properties of the multivariate normal density function g. We suppose that the vector X of n random variables with means zero and unit variances has a non-singular normal multivariate distribution with probability density function:

$$g(x_1, \ldots, x_n; c_{11}, \ldots, c_{nn}) = (2\pi)^{-\frac{1}{2}n}|\mathbf{C}|^{\frac{1}{2}} \exp\left(-\frac{1}{2}\mathbf{x}^T\mathbf{C}\mathbf{x}\right) \tag{31}$$

Here $\mathbf{C}$ is the inverse of the covariance matrix of X, which is denoted by ς.

Then the identity developed by Plackett [1954] can be proved easily by writing the density in terms of its characteristic function: Theorem 3 (*Plackett's identity*):

$$\frac{\partial g}{\partial \varsigma_{ij}} = \frac{\partial^2 g}{\partial x_i \partial x_j} \tag{32}$$

In the two-dimensional case, this reads as:

$$\frac{\partial n_2(x, y; \varsigma)}{\partial \varsigma} = \frac{\partial^2 n_2(x, y; \varsigma)}{\partial x \partial y} \tag{33}$$

which can be extended readily to the corresponding cumulative distribution function:

$$\frac{\partial N_2(x, y; \varsigma)}{\partial \varsigma} = \frac{\partial^2 N_2(x, y; \varsigma)}{\partial x \partial y} = n_2(x, y; \varsigma) \tag{34}$$

Correlation Risk and Cross-Gamma. Using the abbreviation $g_{jk} \triangleq \frac{\partial^2 g}{\partial x_j \partial x_k}$, the cross-gamma and correlation risk are

$$\frac{\partial^2 v}{\partial S_j(0)\partial S_k(0)} = e^{-r\tau}\frac{1}{S_j(0)S_k(0)\sigma_j\sigma_k\tau}\times$$

$$\int f(\ldots,y_i,\ldots)g_{jk}\frac{d\vec{y}}{\prod y_i\sigma_i\sqrt{\tau}} \quad (35)$$

$$\frac{\partial v}{\partial \varsigma_{jk}} = e^{-r\tau}\int f(\ldots,y_i,\ldots)g_{\varsigma_{jk}}\frac{d\vec{y}}{\prod y_i\sigma_i\sqrt{\tau}} \quad (36)$$

Invoking Plackett's identity (32) that $g_{\varsigma_{jk}} = g_{jk}$ leads to: Theorem 4 (*cross-gamma-correlation risk relationship*):

$$\frac{\partial v}{\partial \varsigma_{jk}} = S_j(0)S_k(0)\sigma_j\sigma_k\tau\frac{\partial^2 v}{\partial S_j(0)\partial S_k(0)} \quad (37)$$

Interest Rate Risk and Delta. A similar computation yields: Theorem 5 (*delta-rho relationship*):

$$\frac{\partial v}{\partial q_j} = -S_j(0)\tau\frac{\partial v}{\partial S_j(0)} \quad (38)$$

$$\frac{\partial v}{\partial r} = -\tau\left(v - \sum_{j=1}^{n} S_j(0)\frac{\partial v}{\partial S_j(0)}\right) \quad (39)$$

Volatility Risk and Gamma. The first and second derivative of the density g satisfy:

$$g_j = -g\sum_{i=1}^{n} x_i C_{ij} \quad (40)$$

$$g_{jk} = g\sum_{i=1}^{n} x_i C_{ij}\sum_{i=1}^{n} x_i C_{ik} - gC_{kj} \quad (41)$$

For the j-th vega we find thus

$$\sigma_j\frac{\partial v}{\partial \sigma_j} = e^{-r\tau}\int f\cdot g\cdot\left(\sum_{i=1}^{n} x_i C_{ij}x_j^- - 1\right)\times$$

$$\frac{d\vec{y}}{\prod y_i\sigma_i\sqrt{\tau}} \quad (42)$$

$$x_j^- \triangleq \frac{\ln\frac{y_i}{S_i(0)} - (r - q_i + \frac{1}{2}\sigma_i^2)\tau}{\sigma_i\sqrt{\tau}}$$

$$= x_j - \sigma_j\sqrt{\tau} \quad (43)$$

where we omit the arguments of f and g to simplify the notation.

For the cross-gammas, we derive:

$$\sigma_j\sigma_k S_j(0)S_k(0)\tau\frac{\partial^2 v}{\partial S_j(0)\partial S_k(0)} =$$

$$e^{-r\tau}\int f\cdot g\cdot B_{jk}\frac{d\vec{y}}{\prod y_i\sigma_i\sqrt{\tau}} \quad (44)$$

$$B_{jk} \triangleq \sum_{i=1}^{n} x_i C_{ij}\sum_{i=1}^{n} x_i C_{ik} - C_{kj} -$$

$$\sum_{i=1}^{n} x_i C_{ij}\sigma_k\sqrt{\tau}\delta_{jk} \quad (45)$$

We now multiply by ς_{jk} and sum over k, remembering that ς is the inverse matrix of C, and obtain:

$$\sum_{k=1}^{n}\varsigma_{jk}\sigma_j\sigma_k S_j(0)S_k(0)\tau\frac{\partial^2 v}{\partial S_j(0)\partial S_k(0)} =$$

$$e^{-r\tau}\int f\cdot g\cdot D_j\frac{d\vec{y}}{\prod y_i\sigma_i\sqrt{\tau}} \quad (46)$$

$$D_j \triangleq \sum_{i=1}^{n} x_i C_{ij}x_j - 1 - \sum_{i=1}^{n} x_i C_{ij}x_j +$$

$$\sum_{i=1}^{n} x_i C_{ij}x_j^- \quad (47)$$

In summary, we obtain Theorem 6 (*gamma-vega relationship*):

$$\sigma_j \frac{\partial v}{\partial \sigma_j} = \sum_{k=1}^{n} s_{jk}\sigma_j\sigma_k S_j(0)S_k(0)\tau \times \frac{\partial^2 v}{\partial S_j(0)\partial S_k(0)} \quad (48)$$

In one dimension, the gamma-vega and delta-rho relationships are also mentioned by Shaw [1998]. Shaw shows there that $v_\sigma - \sigma\tau S^2(t)v_{S(t)S(t)}$ satisfies the Black-Scholes partial differential equation and is hence identically zero for path-independent options. We note that the gamma-vega relationship does not hold for barrier options, simply because gamma and vega are not equal at the barrier.

III. THE ONE-DIMENSIONAL CASE

Results for European Options in the Black-Scholes Model

The relationships among Greeks for European options in the one-dimensional Black–Scholes model are listed in Exhibit 3. An interpretation of the gamma-vega relationship in Equation (111) has been given by Taleb [1996]. There are surely more relations one can prove, but the next theorem will give a deeper insight into the relations of the Greeks.

Theorem 7. If the price and two Greeks g_1, g_2 of a European option are given by:

$$g_1 \in G_1 = \{\Delta, \Delta^k, \Delta^B, \rho, \rho_q\} \quad (49)$$

$$g_2 \in G_2 = \{\Gamma, \Gamma^k, \Gamma^B, \Phi, \Theta\} \quad (50)$$

then all the other Greeks ($\in G_1 \cup G_2$) can be calculated. Furthermore, if Θ and another Greek from G_2 is given, it is also possible, to determine all other Greeks.

Proof. Equations (99)-(106) in Exhibit 3 are independent of each other. Equations (107)-(109) are conclusions. Exhibit 4 provides an overview of all these relations.

Let us now assume the option price and one Greek from the set G_1 are given. A look at Exhibit 4 shows that all Greeks of the set G_1 can be evaluated. If all Greeks of the set G_1 are known and additionally one Greek of the set G_2 is given, all other Greeks can be determined. Yet only eight equations are independent, so the knowledge of two Greeks is also the minimum knowledge one needs to determine all ten Greeks. This is the proof of the first statement.

If Θ and another Greek from G_2 are given, then it is always possible to determine one Greek of the set G_1, and one applies the part of this theorem already proved. If Γ, Γ^k, or Γ^B is given, one can use one of the three Black-Scholes Equations (106)-(108). If vega Φ is given, one can use (111) to get Γ.

Numerous formulas for Greeks can be found in Wystup's formula catalogue [1999]. In the case of plain vanilla calls and puts in a foreign exchange market, all relations for the Greeks presented above are valid.

Options Depending on Final Time Value and Barrier

Some relationships remain valid for claims depending on the final time value and period extremes of the underlying asset. This applies to barrier options, one-touch options, and look-back options.

Pricing Formula. We denote the running extremum by

$$M_T \stackrel{\Delta}{=} \eta \min_{0\le t\le T}[\eta S_t] \quad (51)$$

where $\eta = +1$ for minimum, $\eta = -1$ for maximum (52)

The payoff is a function $f(M_T, S_T)$, and its value at time zero is computed via

$$v = e^{-r\tau}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(S_0e^{\sigma y}, S_0e^{\sigma x})g(y,x)\,dy\,dx \quad (53)$$

where the joint density of the random vector

$$\left(W(T) + \mu T, \eta \min_{0\le t\le T}[\eta(W(t)+\mu t)]\right) \quad (54)$$

is given by (see Shreve [1996]):

$$g(x,y) = -\eta e^{\mu x - \frac{1}{2}\mu^2 T} \frac{2(2y-x)}{T\sqrt{2\pi T}} \times$$

$$\exp\left\{ -\frac{(2y-x)^2}{2T} \right\} \tag{55}$$

for $\eta y \leq \min(S_0, \eta x)$.

Using the density (55), the value function can be written as the integral:

$$v = e^{-rT} \int_{x=0}^{x=+\infty} \int_{\eta y \leq \min(\eta S_0, \eta x)}$$

$$\frac{f(y,x)}{\sigma^2 xy} g\left(\frac{1}{\sigma} \ln \frac{y}{S_0}, \frac{1}{\sigma} \ln \frac{x}{S_0} \right) dy\, dx \tag{56}$$

Foreign-Domestic Symmetry. Since $\frac{\partial g}{\partial r} = -\frac{\partial g}{\partial q}$, we obtain

$$v_r + v_q = -Tv \tag{57}$$

Remaining Relations. The rho-delta relationships as in Theorem 5 do not hold, and the vega-gamma relationship generalizes to take the form

$$(r-q)[Txv_x + v_q] + \frac{T}{2}\sigma^2 x^2 v_{xx} = \frac{1}{2}\sigma v_\sigma \tag{58}$$

which agrees with Theorem 6 if the cost of carry $r - q$ equals zero.

IV. EUROPEAN OPTIONS IN TWO-DIMENSIONAL BLACK-SCHOLES MODEL

Pricing of a European Option

Rainbow options are financial instruments that depend on several risky assets. Many of them are very sensitive to changes of correlation. We call kappa (κ) the derivative of the option value v with respect to the correlation ς.

Extensive computational effort is needed to compute kappa, even in a simple framework, but in the Black-Scholes model with two stocks and one cash bond, the cross-gamma-correlation risk relationship can be used easily to find kappa.

Let the stock price processes S_1 and S_2 be described by

$$\ln \frac{S_1(\tau)}{S_1(0)} = (r - q_1 - \frac{1}{2}\sigma_1^2)\tau + \sigma_1 W_\tau^1. \tag{59}$$

$$\ln \frac{S_2(\tau)}{S_2(0)} = (r - q_2 - \frac{1}{2}\sigma_2^2)\tau + \sigma_2 \varsigma W_\tau^1 +$$

$$\sigma_2\sqrt{1-\varsigma^2} W_\tau^2. \tag{60}$$

W^1 and W^2 are two independent Brownian motions under the risk-neutral measure. The probability density for the distribution of $S_1(\tau)$ is denoted by $h_1(x)$ and is given by the lognormal density:

$$h_1(x) = \frac{1}{\sqrt{2\pi\sigma_1^2\tau}} \frac{1}{x} \exp\left(-\frac{A^2}{2\sigma_1^2\tau} \right) \tag{61}$$

$$A \triangleq \ln\left(\frac{x}{S_1(0)} \right) - r\tau + q_1\tau + \frac{1}{2}\sigma_1^2\tau. \tag{62}$$

The equation for the second stock price process can be written as

$$\ln \frac{S_2(\tau)}{S_2(0)} = (r - q_2 - \frac{1}{2}\sigma_2^2)\tau + \frac{\sigma_2\varsigma}{\sigma_1} \times$$

$$\left(\ln\left(\frac{S_1(\tau)}{S_1(0)} \right) - (r - q_1 - \frac{1}{2}\sigma_1^2)\tau \right) +$$

$$\sigma_2\sqrt{1-\varsigma^2} W_\tau^2. \tag{63}$$

The conditional distribution of $S_2(\tau)$, given $S_1(\tau)$, is thus lognormal with density:

$$h_{2|1}(y|x) = \frac{1}{y\sqrt{2\pi\sigma_2^2(1-\varsigma^2)\tau}} \exp\left(-\frac{B^2}{2\sigma_2^2(1-\varsigma^2)\tau} \right) \tag{64}$$

$$B \triangleq \left[\ln\left(\frac{y}{S_2(0)}\right) - r\tau + q_2\tau + \frac{1}{2}\sigma_2^2\tau - \frac{\sigma_2\varsigma}{\sigma_1}A\right] \tag{65}$$

The joint distribution of $S_1(\tau)$ and $S_2(\tau)$ is given by the product of h_1 and h_2:

$$h(x,y) = h_1(x) \cdot h_{2|1}(y|x) \tag{66}$$

A European option with maturity τ and payoff $f[S_1(\tau), S_2(\tau)]$ will be priced by:

$$v = e^{-r\tau} \int_0^\infty \int_0^\infty h(x,y) \cdot f(x,y)\,dx\,dy \tag{67}$$

Using these results, one can establish several relationships for the Greeks in the two-dimensional case in Exhibit 5. The fundamental symmetric scale-invariance of time is valid too. Since we concentrate on European options, the two dimensional Black-Scholes partial differential equation also holds.

European Options on the Minimum/Maximum of Two Assets

We treat one example in full detail. Further examples such as outside barrier options and spread options are available in Wystup [1999]. We consider the payoff:

$$[\phi(\eta\min(\eta S_1(T), \eta S_2(T)) - K)]^+ \tag{68}$$

This is a European put or call on the minimum ($\eta = +1$) or maximum ($\eta = -1$) of the two assets $S_1(T)$ and $S_2(T)$ with strike K. As usual, the binary variable ϕ takes the value $+1$ for a call and -1 for a put. Its value function has been published in Stulz [1982] and can be written as:

$$\begin{aligned} &v(t, S_1(t), S_2(t), K, T, q_1, q_2, r, \sigma_1, \sigma_2, \varsigma, \phi, \eta) \\ &= \phi\Big[S_1(t)e^{-q_1\tau}\mathcal{N}_2(\phi d_1, \eta d_3; \phi\eta\varsigma_1) \\ &\quad + S_2(t)e^{-q_2\tau}\mathcal{N}_2(\phi d_2, \eta d_4; \phi\eta\varsigma_2) \\ &\quad - Ke^{-r\tau}\left(\frac{1-\phi\eta}{2} + \right. \\ &\quad \left.\left. \phi\mathcal{N}_2(\eta(d_1 - \sigma_1\sqrt{\tau}), \eta(d_2 - \sigma_2\sqrt{\tau})\ \kappa)\right)\right] \end{aligned} \tag{69}$$

$$\sigma^2 \triangleq \sigma_1^2 + \sigma_2^2 - 2\varsigma\sigma_1\sigma_2, \tag{70}$$

$$\varsigma_1 \triangleq \frac{\varsigma\sigma_2 - \sigma_1}{\sigma} \tag{71}$$

$$\varsigma_2 \triangleq \frac{\varsigma\sigma_1 - \sigma_2}{\sigma}, \tag{72}$$

$$d_1 \triangleq \frac{\ln(S_1(t)/K) + (r - q_1 + \frac{1}{2}\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}} \tag{73}$$

$$d_2 \triangleq \frac{\ln(S_2(t)/K) + (r - q_2 + \frac{1}{2}\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}} \tag{74}$$

$$d_3 \triangleq \frac{\ln(S_2(t)/S_1(t)) + (q_1 - q_2 - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \tag{75}$$

$$d_4 \triangleq \frac{\ln(S_1(t)/S_2(t)) + (q_2 - q_1 - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}. \tag{76}$$

Computing the Greeks proceeds as follows.
Delta. Space homogeneity implies that

$$v = S_1(t)\frac{\partial v}{\partial S_1(t)} + S_2(t)\frac{\partial v}{\partial S_2(t)} + K\frac{\partial v}{\partial K} \tag{77}$$

from which we can read off the deltas as*

$$\frac{\partial v}{\partial S_1(t)} = \phi e^{-q_1\tau}\mathcal{N}_2(\phi d_1, \eta d_3; \phi\eta\varsigma_1) \tag{78}$$

$$\frac{\partial v}{\partial S_2(t)} = \phi e^{-q_2\tau} \mathcal{N}_2(\phi d_2, \eta d_4; \phi\eta\varsigma_2) \tag{79}$$

$$\frac{\partial v}{\partial K} = -\phi e^{-r\tau}\left(\frac{1-\phi\eta}{2} + \phi\mathcal{N}_2(\eta(d_1 - \sigma_1\sqrt{\tau}),\right.$$

$$\left.\eta(d_2 - \sigma_2\sqrt{\tau}); \varsigma)\right) \tag{80}$$

Gamma. Computing the gammas is the last place differentiation is needed. We use the identities:

$$\frac{\partial}{\partial x}\mathcal{N}_2(x, y; \varsigma) = n(x)\mathcal{N}\left(\frac{y - \varsigma x}{\sqrt{1-\varsigma^2}}\right) \tag{81}$$

$$\frac{\partial}{\partial y}\mathcal{N}_2(x, y; \varsigma) = n(y)\mathcal{N}\left(\frac{x - \varsigma y}{\sqrt{1-\varsigma^2}}\right) \tag{82}$$

and obtain

$$\frac{\partial^2 v}{\partial(S_1(t))^2} = \frac{\phi e^{-q_1\tau}}{S_1(t)\sqrt{\tau}}\left[\frac{\phi}{\sigma_1} n(d_1)\mathcal{N}\left(\eta\sigma\frac{d_3 - d_1\varsigma_1}{\sigma_2\sqrt{1-\varsigma^2}}\right) - \frac{\eta}{\sigma} n(d_3)\mathcal{N}\left(\phi\sigma\frac{d_1 - d_3\varsigma_1}{\sigma_2\sqrt{1-\varsigma^2}}\right)\right] \tag{83}$$

$$\frac{\partial^2 v}{\partial(S_2(t))^2} = \frac{\phi e^{-q_2\tau}}{S_2(t)\sqrt{\tau}}\left[\frac{\phi}{\sigma_2} n(d_2)\mathcal{N}\left(\eta\sigma\frac{d_4 - d_2\varsigma_2}{\sigma_1\sqrt{1-\varsigma^2}}\right) - \frac{\eta}{\sigma} n(d_4)\mathcal{N}\left(\phi\sigma\frac{d_2 - d_4\varsigma_2}{\sigma_1\sqrt{1-\varsigma^2}}\right)\right] \tag{84}$$

$$\frac{\partial^2 v}{\partial S_1(t)\partial S_2(t)} = \frac{\phi\eta e^{-q_1\tau}}{S_2(t)\sigma\sqrt{\tau}} n(d_3)\mathcal{N}\left(\phi\sigma\frac{d_1 - d_3\varsigma_1}{\sigma_2\sqrt{1-\varsigma^2}}\right) \tag{85}$$

Kappa. The sensitivity of the value with respect to correlation is directly related to the cross-gamma:

$$\frac{\partial v}{\partial\varsigma} = \sigma_1\sigma_2\tau S_1(t)S_2(t)\frac{\partial^2 v}{\partial S_1(t)\partial S_2(t)} \tag{86}$$

Vega. We use Equations (117) and (118) in Exhibit 5 to get formulas for the vegas:

$$\frac{\partial v}{\partial\sigma_1} = \frac{\varsigma v_\varsigma + \sigma_1^2\tau(S_1(t))^2 v_{S_1(t)S_1(t)}}{\sigma_1}$$

$$= S_1(t)e^{-q_1\tau}\sqrt{\tau}\left[\varsigma_1\phi\eta n(d_3)\mathcal{N}\left(\phi\sigma\frac{d_1 - d_3\varsigma_1}{\sigma_2\sqrt{1-\varsigma^2}}\right)\right. \tag{87}$$

$$\left. + n(d_1)\mathcal{N}\left(\eta\sigma\frac{d_3 - d_1\varsigma_1}{\sigma_2\sqrt{1-\varsigma^2}}\right)\right] \tag{88}$$

$$\frac{\partial v}{\partial\sigma_2} = \frac{\varsigma v_\varsigma + \sigma_2^2\tau(S_2(t))^2 v_{S_2(t)S_2(t)}}{\sigma_2}$$

$$= S_2(t)e^{-q_2\tau}\sqrt{\tau}\left[\varsigma_2\phi\eta n(d_4)\mathcal{N}\left(\phi\sigma\frac{d_2 - d_4\varsigma_2}{\sigma_1\sqrt{1-\varsigma^2}}\right)\right. \tag{89}$$

$$\left. + n(d_2)\mathcal{N}\left(\eta\sigma\frac{d_4 - d_2\varsigma_2}{\sigma_1\sqrt{1-\varsigma^2}}\right)\right] \tag{90}$$

Rho. Looking at (112), (113), and (120), the rhos are given by:

$$\frac{\partial v}{\partial q_1} = -S_1(t)\tau\frac{\partial v}{\partial S_1(t)} \tag{91}$$

$$\frac{\partial v}{\partial q_2} = -S_2(t)\tau\frac{\partial v}{\partial S_2(t)} \tag{92}$$

$$\frac{\partial v}{\partial r} = -K\tau\frac{\partial v}{\partial K} \tag{93}$$

Theta. Among the various ways to compute theta one may use one based on (114).

$$\frac{\partial v}{\partial t} = -\frac{1}{\tau}\left[q_1 v_{q_1} + q_2 v_{q_2} + r v_r + \frac{\sigma_1}{2}v_{\sigma_1} + \frac{\sigma_2}{2}v_{\sigma_2}\right] \tag{94}$$

V. SUMMARY

We have shown how to employ homogeneity-based methods to compute formulas of Greeks for analytically known value functions of options in one- and higher-dimensional markets. Our strategy to compute Greeks can also be applied successfully in a stochastic volatility model such as the one by Heston [1993], where plain vanilla options allow the computation of delta and dual-delta just as in the Black-Scholes formula, and the delta-rho relationship holds as well. For Black-Scholes models alone, there are numerous further relationships among various Greeks that are of fundamental interest.

The method saves computation time for the mathematician who has to differentiate complicated formulas. It saves computer time as well, because analytical results for Greeks are usually faster to evaluate than finite differences involving at least twice the computation of the option's value. Knowing how the Greeks are related can speed up finite-difference, tree, or Monte Carlo-based computation of Greeks or at least provide a quality check. Many of the results are valid beyond the Black-Scholes model.

Most remarkably, some relationships among Greeks are based on properties of the normal distribution, confirming the active interplay between mathematics and the financial markets.

EXHIBIT 1
Notation—Market and Model Parameters and Normal Distribution

S	stock price or stock price process
r	risk free interest rate
q	dividend yield (continuously paid)
σ	volatility of one stock, or volatility matrix of several stocks
ς	correlation in the two-asset market model
t	date of evaluation ("today")
T	date of maturity
$\tau = T - t$	time to maturity of an option
x	stock price at time t
$f(\cdot)$	payoff function
$v(x, t, \dots)$	value of an option
k	strike of an option
B	barrier of an option
v_x	partial derivation of v with respect to x (and analogous)

$$n(x) \triangleq \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \tag{95}$$

$$\mathcal{N}(x) \triangleq \int_{-\infty}^{x} n(t)\, dt \tag{96}$$

$$n_2(x, y; \varsigma) \triangleq \frac{1}{2\pi\sqrt{1-\varsigma^2}} \exp\left(-\frac{x^2 - 2\varsigma xy + y^2}{2(1-\varsigma^2)}\right) \tag{97}$$

$$\mathcal{N}_2(x, y; \varsigma) \triangleq \int_{-\infty}^{x}\int_{-\infty}^{y} n_2(u, v; \varsigma)\, du\, dv \tag{98}$$

EXHIBIT 2
The Greeks

Delta	Δ	v_x	
Gamma	Γ	v_{xx}	
Theta	Θ	v_t	
Rho	ρ	v_r	
Rhoq	ρ_q	v_q	
Vega	Φ	v_σ	
Kappa	κ	v_ϱ	correlation sensitivity (two-stock model)
Leverage/Gearing	Λ	$\frac{x}{v}v_x$	
Vomma / Volga	Φ'	$v_{\sigma\sigma}$	
Speed		v_{xxx}	
Charm		v_{xt}	
Color		v_{xxt}	
Vanna		$v_{x\sigma}$	
Forward Delta	Δ^F	v_F	
Driftless Delta	Δ^{dl}	$\Delta e^{q\tau}$	
Dual Theta	DualΘ	v_T	
Strike Delta	Δ^k	v_k	
Strike Gamma	Γ^k	v_{kk}	
Barrier Delta	Δ^B	v_B	
Barrier Gamma	Γ^B	v_{BB}	

EXHIBIT 3
Relations of Greeks for European Claims in One-Dimensional Black-Scholes Model

$$0 = \tau\Theta + r\rho + q\rho_q + \frac{1}{2}\sigma\Phi \quad \text{scale invariance of time} \tag{99}$$

$$v = x\Delta + k\Delta^k \quad \text{price homogeneity and strikes} \tag{100}$$

$$x^2\Gamma = k^2\Gamma^k \quad \text{price homogeneity and strikes} \tag{101}$$

$$x\Delta = -B\Delta^B \quad \text{price homogeneity and barriers} \tag{102}$$

$$x^2\Gamma + x\Delta = B^2\Gamma^B + B\Delta^B \quad \text{price homogeneity and barriers} \tag{103}$$

$$\rho = -\tau(v - x\Delta) \quad \text{delta-rho relationship} \tag{104}$$

$$\rho + \rho_q = -\tau v \quad \text{rates symmetry} \tag{105}$$

$$rv = \Theta + (r - q)x\Delta + \frac{1}{2}\sigma^2x^2\Gamma \quad \text{Black-Scholes PDE} \tag{106}$$

$$qv = \Theta + (q - r)k\Delta^k + \frac{1}{2}\sigma^2k^2\Gamma^k \quad \text{dual Black-Scholes (strike)} \tag{107}$$

$$rv = \Theta + (q - r + \sigma^2)B\Delta^B + \frac{1}{2}\sigma^2B^2\Gamma^B \quad \text{dual Black-Scholes (barrier)} \tag{108}$$

$$\rho_q = -\tau x\Delta \quad \text{delta-rho relationship} \tag{109}$$

$$\rho = -\tau k\Delta^k \quad \text{combination of (109) and (100)} \tag{110}$$

$$\Phi = \sigma\tau x^2\Gamma \quad \text{gamma-vega relationship} \tag{111}$$

EXHIBIT 4
Overview of All Relations Among Greeks

equation	v	Δ	Δ^k	Δ^B	ρ	ρ_q	Γ	Γ^k	Γ^B	Φ	Θ
		Greeks $\in G_1$					*Greeks* $\in G_2$				
(99)					X	X				X	X
(100)	O	O	O								
(101)							O	O			
(102)		O		O							
(103)		X		X			X		X		
(104)	O	O			O						
(105)	O				O	O					
(106)	X	X					X				X
(107)	X		X					X			X
(108)	X			X					X		X
(110)			O		O						
(111)							O			O	
(109)		O				O					

X or O denote that the Greek appears in the relation. The relations marked X show that there is a relation between Greeks of G_1 and G_2. The relations marked O show that this relation concerns only the Greeks of one set. See Theorem 7 for the significance of the sets G_1 and G_2.

EXHIBIT 5
Relations of Greeks for European Claims in Two-Dimensional Black-Scholes Model

$$0 = \rho_{q_1} + S_1(0)\tau\Delta_1, \tag{112}$$
$$0 = \rho_{q_2} + S_2(0)\tau\Delta_2, \tag{113}$$
$$0 = q_1\rho_{q_1} + q_2\rho_{q_2} + \frac{1}{2}\sigma_1\Phi_1 + \frac{1}{2}\sigma_2\Phi_2 + r\rho - \tau\Theta, \tag{114}$$
$$\begin{aligned} 0 = {} & \Theta - rv - (r - q_1)S_1(0)\Delta_1 + (r - q_2)S_2(0)\Delta_2 \\ & + \frac{1}{2}\sigma_1^2 S_1(0)^2\Gamma_{11} + \varsigma\sigma_1\sigma_2 S_1(0)S_2(0)\Gamma_{12} + \frac{1}{2}\sigma_2^2 S_2(0)^2\Gamma_{22}, \end{aligned} \tag{115}$$
$$\kappa = \sigma_1\sigma_2\tau S_1(0)S_2(0)\Gamma_{12}, \tag{116}$$
$$0 = \varsigma\kappa - \sigma_1\Phi_1 + \sigma_1^2\tau S_1(0)^2\Gamma_{11}, \tag{117}$$
$$0 = \varsigma\kappa - \sigma_2\Phi_2 + \sigma_2^2\tau S_2(0)^2\Gamma_{22}, \tag{118}$$
$$0 = \sigma_1\Phi_1 - \sigma_2\Phi_2 - \sigma_1^2\tau S_1(0)^2\Gamma_{11} + \sigma_2^2\tau S_2(0)^2\Gamma_{22}, \tag{119}$$
$$\rho = -\tau\left(v - S_1(0)\Delta_1 - S_2(0)\Delta_2\right), \tag{120}$$
$$0 = \tau v - \rho_{q_1} + \rho_{q_2} + \rho. \tag{121}$$

- (112) and (113). These relations are a justification for the rough way to deal with dividends. One subtracts the dividends from the actual spot price and prices the option with this price and without dividends. This relation is not effected by the two-dimensionality of the problem.
- (114). This is the two-dimensional version of the general invariance under time scaling.
- (115). This is the Black-Scholes differential equation. This relation must hold, because we concentrated on European claims. It turns out, that the dynamic of an option price is described by the market model and that the price of the option is defined as a boundary problem.
- (116). This is the cross-gamma-correlation-risk relationship; it is remarkable, that this relationship has such a simple structure.
- (117) and (118). These are the gamma-vega relationships. Notice that one can determine κ only by knowledge of some derivatives with respect to parameters which concern only one stock. Of course, there is no difference between the first and the second stock. These relations are valid in the one-dimensional case with $\kappa = 0$.
- (119) follows from (116).
- (120). This is the delta-rho relationship. The interest rate risk is well known to be the negative product of duration and the amount of money invested. The term in the parentheses is exactly the amount of money one would have to invest in the cash bond in order to delta-hedge the option.
- (121). This relation is the two-dimensional rates symmetry, an extension of equation (105). It follows from (120), (112) and (113).

ENDNOTE

*Computing coefficients in such a way leads to correct results as can be proved: see http://www.wias-berlin.de/preprints/584 for a preprint of this article with an extended appendix.

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