A Numerical Approach to American Currency Option Valuation

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This research introduces an approach to valuing American options on a foreign currency using a short rate model. We discretize the diffusion equation using an implicit Euler method in time and a radial basis function approximation in space. The methodology is general enough to accommodate any Heath-Jarrow-Morton-consistent term structure.

The results are compared to a theoretical solution for European options and to the solution of a finite-element scheme for American options. The numerical results indicate that the approach generates quite accurate approximations of the true option values.

Amin and Jarrow [1991] introduce a general framework for valuing an option on a foreign currency with stochastic interest rates. Their model allows domestic and foreign term structures of interest rates to follow stochastic processes of the Heath, Jarrow, and Morton (HJM) [1992] class. Accordingly, it can price many derivatives whose values are sensitive to the term structure of interest rates such as currency warrants and cross-rate swaps. Amin and Jarrow also obtain closed-form solutions for European options when the volatility functions governing the term structure are deterministic. For American options, Amin and Bodurtha [1995] consider a discrete-time implementation of the Amin and Jarrow framework, using a multinomial version of the lattice technique of Cox, Ross, and Rubinstein [1979].

The Amin and Jarrow model is a multi-factor model in which there are at least three different sources of uncertainty, those associated with the domestic interest rate, the foreign interest rate, and currency exchange rate. The model's multidimensionality makes its numerical implementation computationally burdensome. In the multinomial lattice approach to the three-factor model adopted in Amin and Bodurtha [1995], for example, the number of states (nodes in the multinomial tree) increases at the rate of 8^n for path-dependent models and at the rate of $(n + 1)^3$ for path-independent models as the time step n increases.

We introduce a numerical approach to valuing American options on a foreign currency consistent with the Amin and Jarrow [1991] formulation. Our numerical approach is relatively simple, but it is more accurate with fewer than 1,000 nodes as compared to approximately 1 million nodes in the Amin and Bodurtha [1995] model. We derive the partial differential equation (PDE) associated with the Amin and Jarrow model of currency options using short rate processes, and implement a numerical approximation to the option value using a radial basis function (RBF) methodology.

Similar to finite-difference methods, the RBF technique approximates the true option value by discretizing the diffusion equation or PDE, but it employs radial functions as the basis of the approximation. It requires only that the form of the radial approximating function be selected and a mesh generated, independent of the dimensionality or geometry of the problem.[1] The option value is then approximated by a linear combination of the basis functions with time-varying coefficients.

As the basis function depends solely on the mesh, and only the coefficients are allowed to be time-varying, discretization in time and space can be applied separately. The numerical implementation of RBF is therefore much more simple than traditional finite-element and finite-difference formulations for multidimensional cases.[2] Furthermore, because the RBF function is differentiable with respect to state variables in space, the RBF technique has the great advantage of directly obtaining the option greeks such as option deltas and option rhos from the numerical results.

Our approach uses a short rate model of the term structure of interest rates as opposed to the HJM forward rate processes in the Amin and Jarrow currency option model. Short rate models require an appropriate parameterization to match the observed current term structure. Given a specified HJM forward rate process, there is a corresponding short rate process consistent with the forward process (see Baxter and Rennie [1996]). Therefore, matching the current term structure itself is not a limitation of short rate models.

In order to facilitate comparison with Amin and Jarrow [1991] and with the numerical work of Amin and Bordutha [1995], we use the Vasicek [1977] class of interest rate volatility structures. That is, forward rate volatilities are exponentially declining in their maturities, so the volatility of the spot rate is a constant, independent of its level. Our approach is general enough to accommodate any short rate model, such as Vasicek [1977], Cox, Ingersoll, and Ross [1985], and Black and Karasinski [1991].

One difficulty associated with PDE-based numerical approaches is specifying appropriate boundary conditions, particularly for multidimensional applications. For example, in the case of a one-factor model, boundary conditions are imposed at the two end-points of a state variable at given mesh points. (See, for example, Schwartz [1977].)[3]

For multifactor models, the boundary conditions may be imposed on any combination of mesh points where at least one state variable lies at an end-point. The development of boundary conditions thus represents a significant difficulty for field equation methods in general.

Marcozzi, Choi, and Chen [2001] examine, for one-dimensional cases, the effect of boundary conditions on the performance of both the finite-difference and RBF methods. They find that the RBF, without imposition of the artificial boundary conditions, performs at least as well as with boundary conditions. Indeed, our numerical results indicate that a highly accurate spatial approximation to the solution is achieved without imposing boundary conditions.

I. AMIN AND JARROW CURRENCY OPTION MODEL

The key contribution of Amin and Jarrow [1991] is formulation of the "global" economy through the stochastic representation of the entire term structure of both domestic and foreign interest rates by using the forward rate processes of HJM. In their model of a global economy, the foreign exchange rate follows geometric Brownian motion, and the domestic and foreign term structures evolve as HJM:[4]

$$dZ_t = \eta^Z(t)Z_t dt + \sum_{i=1}^{3} \phi_i^z Z_t d\tilde{W}_t^i \tag{1-A}$$

$$df^d(t,T) = \eta^d(t,T)dt + \sum_{i=1}^{3} \phi_i^d(t,T)d\tilde{W}_t^i \tag{1-B}$$

$$df^f(t,T) = \eta^f(t,T)dt + \sum_{i=1}^{3} \phi_i^f(t,T)d\tilde{W}_t^i \tag{1-C}$$

where Z_t is the exchange rate in terms of domestic currency per unit of foreign currency at time t; $f^d(t, T)$ and $f^f(t, T)$ are the time t instantaneous forward rates of domestic and foreign borrowing rates at date T, respectively; and $d\tilde{W}^i_t$ are independent Brownian motions for $i = 1, 2, 3$ under a probability measure $\tilde{Q}$. The terms η^i are the drifts of the process of each state variable for $i = \{Z, d, f\}$. The parameters ϕ_i^j, $i = 1, 2, 3$, and $j = z, f, d$ represent the volatilities of the respective state variables such that ϕ_i^d and ϕ_i^f may depend on time and the level of each state variable. By definition, the instantaneous short rate processes for domestic and foreign economies are $r_t^i = f^i(t, t)$ for $i = d, f$.

Dropping the functional dependencies for simplicity, the variance-covariance matrix of $d\log(Z_t)$, $df^d(t, T)$, and $df^f(t, T)$ at time t is given by:

$$\begin{bmatrix} \sigma_{zz} & \sigma_{zd} & \sigma_{zf} \\ \sigma_{dz} & \sigma_{dd} & \sigma_{df} \\ \sigma_{fz} & \sigma_{fd} & \sigma_{ff} \end{bmatrix} =$$

$$\begin{bmatrix} \phi_1^z & \phi_2^z & \phi_3^z \\ \phi_1^d & \phi_2^d & \phi_3^d \\ \phi_1^f & \phi_2^f & \phi_3^f \end{bmatrix} \begin{bmatrix} \phi_1^z & \phi_2^z & \phi_3^z \\ \phi_1^d & \phi_2^d & \phi_3^d \\ \phi_1^f & \phi_2^f & \phi_3^f \end{bmatrix}' =$$

$$\begin{bmatrix} \sum_i \phi_i^{z^2} & \sum_i \phi_i^z \phi_i^d & \sum_i \phi_i^z \phi_i^f \\ \sum_i \phi_i^d \phi_i^z & \sum_i \phi_i^{d^2} & \sum_i \phi_i^d \phi_i^f \\ \sum_i \phi_i^f \phi_i^z & \sum_i \phi_i^f \phi_i^d & \sum_i \phi_i^{f^2} \end{bmatrix} \tag{2}$$

Given the three state variables in (1) and the assumption that a domestic investor holds foreign currency only in the form of foreign bonds or units of the foreign money market account, Amin and Jarrow choose three tradable securities. These are dollar bonds $P^d(t, T)$ that pay one domestic dollar at time T, the dollar worth of foreign bonds $Z_t P^f(t, T)$ that pay one unit of foreign currency at time T, and the dollar worth of a foreign cash account $Z_t B^f(t)$ that has a value of one unit of foreign currency at time 0, where:

$$P^i(t, T) = \exp\left(-\int_t^T f^i(t, s)ds\right), \quad i = d, f$$

and

$$B^f(t) = \exp\left(\int_0^t r_s^f ds\right)$$

Given that this economy is complete and arbitrage-free, Amin and Jarrow show that there is a unique risk-neutral probability measure, Q, so that the value of each of these three securities relative to the dollar cash account, $B^d(t) = \int_0^t r_s^d ds$, is Q-martingale. In this case, the fourth asset is the dollar cash account, used as a numeraire. Accordingly, in this economy, there are a total of four tradable securities with three state variables: Z_t, $f^d(t, T)$, and $f^f(t, T)$.

Under the measure Q, the instantaneous expected rate of return in domestic dollar terms on every traded security equals the domestic instantaneous spot interest rate. This implies that under Q, there exists a Q-Brownian motion $W' = (W_t^{'1}, W_t^{'2}, W_t^{'3})$ so that the drift terms in system (1) must satisfy a system of risk-neutral processes:

$$dZ_t = \left(r_t^d - r_t^f\right) Z_t dt + \sum_{i=1}^{3} \phi_i^z Z_t dW_i' \tag{2-A}$$

$$df^d(t, T) = \left(\sum_{i=1}^{3} \left[\phi_i^d(t, T) \int_t^T \phi_i^d(t, u) du \right] \right) dt + \sum_{i=1}^{3} \phi_i^d(t, T) dW_i' \tag{2-B}$$

$$df^f(t, T) = \left(\sum_{i=1}^{3} \left[\phi_i^f(t, T) \int_t^T \phi_i^f(t, u) du - \phi_i^z(t) \right] \right) dt + \sum_{i=1}^{3} \phi_i^f(t, T) dW_i' \tag{2-C}$$

where the terms in brackets are the drift terms corresponding to $\eta^z(t)$, $\eta^d(t, T)$, and $\eta^f(t, T)$ in (1).

For any additional securities that depend on the same uncertainties associated with the three state variables, the real-world probability measure, $\widetilde{Q}$, is irrelevant for their valuation. As the dollar value of a foreign bond relative to the dollar cash account, $P^f(t, T) Z_t / B_t^d$, is a Q-martingale, the value of an option, such as a European put option on the spot exchange rate with a strike price of K, $V(Z_t, T, K)$, at time t is given by:

$$V(Z_t, T, K) = E_Q\left(B^d(t) \frac{(K - P^f(T, T) Z_T)^+}{B^d(T)} \mid \Im_t \right)$$
$$= E_Q\left(\exp\left(-\int_t^T r_s^d ds\right) (K - Z_T)^+ \mid \Im_t \right) \tag{3}$$

where $\Im_t$ denotes the augmented filtration with respect to the Brownian motion, dW_t', under Q; $P^f(T, T) = 1$ by definition; and $(K - Z_T)^+ = \max(K - Z_T, 0)$. Similarly, the value of an American put option, $v(Z_T, T, K)$, at time t is given by:

$$v(Z_T, T, K) = \sup_{t^* \in [t, T]} E_Q\left(B_t B_{t^*}^{-1} (K - Z_{t^*})^+ \mid \Im_t \right)$$
$$= \sup_{t^* \in [t, T]} E_Q\left(\exp\left(-\int_t^{t^*} r_s^d ds\right) (K - Z_{t^*})^+ \mid \Im_t \right) \tag{4}$$

The analytical solution for the European put option is given by

$$V(Z_t,\ T,\ K) = KP^d(t,\ T)\Phi(-h+\varsigma) - P^f(t,\ T)Z_t\Phi(-h) \qquad (5)$$

where

$$h = \frac{\ln\left(\dfrac{P^f(t,\ T)Z_t}{KP^d(t,\ T)}\right) + \dfrac{1}{2}\varsigma^2}{\varsigma}$$

and

$$\varsigma^2 = \mathrm{var}\left[\sum_{i=1}^{3}\int_t^T \left(a_i^f(s,\ T) + \phi_i^z(s) - a_i^d(s,\ T)\right)dW_s^i\right] = \sum_{i=1}^{3}\int_t^T \left(a_i^f(s,\ T) + \phi_i^z(s) - a_i^d(s,\ T)\right)^2 ds$$

such that $a_i^j(t,\ T) = -\int_t^T \phi_i^j(s,\ T)ds$, for $i = 1, 2, 3$, and $j = d, f$. The term ζ can be computed if the volatility terms ϕ_i^j are deterministic.

II. PARTIAL DIFFERENTIAL EQUATION

We derive the partial differential equation associated with the option values as described, using the short rate induced by the forward rate processes in (2). To this end, it is always necessary to specify the volatility structures of the processes. For illustration, we choose the Vasicek [1977] specification that may be suitable for practical applications.

The Vasicek specification of the volatility structure takes the form:

$$\phi_i^j(t,T) = \phi_i^j e^{-\beta^j(T-t)} \text{ for } i = 1,\ 2,\ 3 \text{ and } j = d,\ f$$

where β^j is a non-negative constant. Under this specification, the variance-covariance of $d\log(Z_t)$, $df^d(t,\ T)$, and $df^f(t,\ T)$ at time t is:

$$\sigma_{kj} = \left[\sum_{i=1}^{3}\left(\phi_i^k\phi_i^j\right)\right]e^{-(\beta^k+\beta^j)(T-t)} \text{ for } k,\ j = z,\ d,\ f \qquad (6)$$

where $\beta^z = 0$.[5]

When $\beta^d = \beta^f = 0$, Equation (6) reduces to the volatility structure in the Ho and Lee [1986] term structure model, where the volatilities of all forward rates are equal and constant. Given the forward rate process of (2-B) under Q, Baxter and Rennie [1996, p. 153] derive the corresponding short rate process as follows:

$$dr_t^d = (\theta_t^d - \beta^d r_t^d)dt + \sum_{i=1}^{3}\phi_i^d dW_t^i \qquad (7)$$

with the initial domestic forward rate curve:

$$f^d(0,\ T) = r_0^d e^{-\beta^d T} + \int_0^T \theta_s^d e^{-\beta^d(T-s)}ds - \frac{\sigma_{dd}}{2(\beta^d)^2}(1-e^{-\beta^d T})^2 \qquad (7')$$

Note that θ_t^d is calculated at time 0 from the initial term structure.[6]

Similarly, the foreign short rate process under measure Q also follows:

$$dr_t^f = (\theta_t^f - \sigma_{fz} - \beta^f r_t^f)dt + \sum_{i=1}^{3}\phi_i^f dW_t^i \qquad (8)$$

with the initial foreign forward rate curve given by

$$f^f(0,T) = r_0^f e^{-\beta^f T} + \int_0^T(\theta_s^f - \sigma_{fz})e^{-\beta^f(T-s)}ds - \frac{\sigma_{dd}}{2(\beta^f)^2}(1-e^{-\beta^f T})^2 \qquad (8')$$

Combining (2-A), (7), and (8) in a compact matrix form, we obtain the short rate version of the Amin and Jarrow model of interest rate and foreign currency dynamics under measure Q:

$$\begin{bmatrix} dz_t \\ dr_t^d \\ dr_t^f \end{bmatrix} = \begin{bmatrix} r_t^d - r_t^f - \frac{1}{2}\sigma_{zz} \\ \theta_t^d - \beta^d r_t^d \\ \theta_t^f - \beta^f r_t^f - \sigma_{fz} \end{bmatrix} dt + \begin{bmatrix} \phi_1^z & \phi_2^z & \phi_3^z \\ \phi_1^d & \phi_2^d & \phi_3^d \\ \phi_1^f & \phi_2^f & \phi_3^f \end{bmatrix}\begin{bmatrix} dW_t^1 \\ dW_t^2 \\ dW_t^3 \end{bmatrix} \qquad (9)$$

where $z_t = \log(Z_t)$. The stochastic process of z_t is obtained by applying Ito's lemma, given the process of Z_t in (2-A).

This arbitrage-free economy is characterized by three state variables, z_t, r_t^d and r_t^f with four tradable securities: the domestic cash account, the foreign cash account, the domestic bond, and the foreign bond. In a complete and arbitrage-free market, the stochastic processes of any additional securities that depend on some or all of the three state variables may be obtained by Ito's lemma. The instantaneous expected rates of return on the securities must be the instantaneous risk-free rate under measure Q.

Consider the European put option value, V_t, in (3). By Ito's lemma, V_t follows the diffusion processes:

$$dV_t = \mu_t V_t dt + \sum_{i=1}^{3} \left(\frac{\partial V_t}{\partial z_t} \phi_i^z + \frac{\partial V_t}{\partial r_t^d} \phi_i^d + \frac{\partial V_t}{\partial r_t^f} \phi_i^f \right) dW_t^i \tag{10}$$

where

$$\begin{aligned} \mu_t V_t = {} & \frac{\partial V_t}{\partial t} + \frac{1}{2}\sigma_{zz} \frac{\partial^2 V_t}{\partial z^2} + \frac{1}{2}\sigma_{dd} \frac{\partial^2 V_t}{\partial r^{d^2}} + \\ & \frac{1}{2}\sigma_{ff} \frac{\partial^2 V_t}{\partial r^{f^2}} + \sigma_{dz} \frac{\partial^2 V_t}{\partial r^d \partial z} + \sigma_{fz} \frac{\partial^2 V_t}{\partial r^f \partial z} + \\ & \sigma_{df} \frac{\partial^2 V_t}{\partial r^d \partial r^f} + (r_t^d - r_t^f - \frac{1}{2}\sigma_{zz}) \frac{\partial V_t}{\partial z} + \\ & (\theta_t^d - \beta^d r_t^d) \frac{\partial V_t}{\partial r^d} + (\theta_t^f - \beta^f r_t^f - \sigma_{fz}) \frac{\partial V_t}{\partial r^f} \end{aligned} \tag{11}$$

Under the measure Q, the drift term must be $r_t V_t$ since the instantaneous rate of return on V_t must be the instantaneous risk-free rate of return. Equating the two terms yields the partial differential equation:

$$\mu_t V_t - r_t^d V_t = 0 \tag{12}$$

where $\mu_t V_t$ is given by (11). Given the terminal condition, $V_T = (K - e^{z_T})^+$, the European put option value at time 0 may be obtained by numerical methods.

Similarly, the American put option value, v_t, may be represented as a solution of the variational inequality:

$$\min\left\{ \mu_t v_t - r_t^d v_t,\ v_t - (K - e^{z_t})^+ \right\} = 0 \tag{13}$$

subject to the terminal condition, where $\mu_t v_t$ is expressed as in (11) with V_t replaced by v_t. Note that (12) and (13) have the time-varying (deterministic) coefficients, θ_t^d and θ_t^f.

III. RADIAL BASIS FUNCTION APPROXIMATION

We can apply radial basis functions to obtain a numerical approximation to the solution of the PDE (12) and the variational inequality (13). Suppose we choose N_1, N_2, and N_3 as the number of mesh points for a sufficiently wide interval for the exchange rate, domestic short rate, and foreign short rate, respectively. Let $Z = [z_1, z_2, ..., z_{N_1}]$, $R^d = [r_1^d, r_2^d, ..., r_{N_2}^d]$, and $R^f = [r_1^f, r_2^f, ..., r_{N_3}^f]$ represent the vectors of the resulting mesh points. Therefore there are $N = N_1 \times N_2 \times N_3$ points, $x \in Z \times R^d \times R^f$, in the three-dimensional space. Let $\mathbf{X}$ be the $N \times 3$ matrix such that $\mathbf{X} = (x_1, x_2, ..., x_N)'$. For a given mesh, $\mathbf{X}$, suppose the unknown European option value, $V(x, t)$, at x and time t can be approximated by the RBF, φ:

$$V(x,t) \approx \sum_{j=1}^{N} \alpha_j(t) \varphi(\| x - x_j \|) \tag{14}$$

where α_j are unknown coefficients depending on time, t, and $\varphi(\|x - x_j\|)$ is an RBF such that

$$\varphi(\| x - x_j \|) = \sqrt{(z - z_j)^2 + (r^d - r_j^d)^2 + (r^f - r_j^f)^2 + c^2}$$

where c is a constant known as the shape parameter.

This RBF is known as Hardy's multiquadratic (MQ). With a uniform mesh with spacing $h = |x_j - x_{j-1}|$, we choose $c = 4h$ as suggested by Hardy [1971], although c may be optimized for particular problems.

Collocating at the same N points, x_i, and substituting (14) into (12), we obtain a system of N linear equations, where the partial derivatives in (9) are, for $i = 1, 2, ..., N$:

$$\frac{\partial V(x_i,t)}{\partial t} = \sum_{j=1}^{N} \frac{d\alpha_j(t)}{dt} \varphi(\| x_i - x_j \|)$$

$$\frac{\partial V(x_i,t)}{\partial z_i} = \sum_{j=1}^{N} \alpha_j(t) \frac{\partial \varphi(\| x_i - x_j \|)}{\partial z_i}$$

$$= \sum_{j=1}^{N} \alpha_j(t) \frac{(z_i - z_j)}{\varphi(\| x_i - x_j \|)}$$

$$\frac{\partial^2 V(x_i,t)}{\partial z_i^2} = \sum_{j=1}^{N} \alpha_j(t) \frac{\partial^2 \varphi(\| x_i - x_j \|)}{\partial z_i^2}$$

$$= \sum_{j=1}^{N} \alpha_j(t) \left[\frac{1}{\varphi(\| x_i - x_j \|)} - \frac{(z_i - z_j)^2}{\left[\varphi(\| x_i - x_j \|)\right]^3} \right]$$

and

$$\frac{\partial^2 V(x_i,t)}{\partial z_i \partial r_i^d} = \sum_{j=1}^{N} \alpha_j(t) \frac{\partial^2 \varphi(\| x_i - x_j \|)}{\partial z_i \partial r_i^d}$$

$$= \sum_{j=1}^{N} \alpha_j(t) \left[- \frac{(z_i - z_j)(r_i^d - r_j^d)}{\left[\varphi(\| x_i - x_j \|)\right]^3} \right]$$

Other partial derivatives, $\partial V(x_i, t)/\partial r_i^d$, $\partial V(x_i, t)/\partial r_i^f$, $\partial^2 V(x_i, t)/\partial r_i^{d2}$, $\partial^2 V(x_i, t)/\partial r_i^{f2}$, $\partial^2 V(x_i, t)/\partial r_i^d \partial r_i^f$, and $\partial^2 V(x_i, t)/\partial z_i \partial r_i^f$, are obtained in analogous fashion. It should be noted that the RBF does not depend on time, and the time derivative of V is given in terms of the time derivatives of $\alpha_j(t)$. Furthermore, the partial derivatives of any order with respect to x_i can be computed from the RBF.

The resulting system of N linear equations can be written succinctly in a matrix form. Denote $\mathbf{A}^d$ and $\mathbf{A}^f$ to be $N \times N$ diagonal matrices whose diagonal elements are the second and third column of the matrix $\mathbf{X}$, which are vectors of domestic interest rates and foreign short rates, respectively. Let Φ be the $N \times N$ matrix such that the (i, j)-th element is $\varphi(|x_i - x_j|)$, and let $\alpha(t)$ be the N-dimensional column vector whose j-th element is $\alpha_j(t)$. Then the N-dimensional column vector of option values, $\mathbf{V}_t$, at time t, evaluated at all $x_i \in \mathbf{X}$ is:

$$\mathbf{V}_t = \Phi\alpha(t) \tag{15}$$

Dropping the time notation, we can write Equation (12) as

$$\Phi\dot{\alpha} = \mathbf{B}\alpha \tag{16}$$

such that

$$\begin{aligned} \mathrm{B} \equiv\ & \mathbf{A}^d\mathbf{\Phi} - \frac{1}{2}\sigma_{zz}\mathbf{\Phi}_{zz} - \frac{1}{2}\sigma_{dd}\sigma_{\mathbf{dd}} - \frac{1}{2}\sigma_{ff}\mathbf{\Phi}_{\mathrm{ff}} - \\ & \sigma_{dz}\mathbf{\Phi}_{\mathrm{dz}} - \sigma_{fz}\mathbf{\Phi}_{\mathrm{fz}} - \sigma_{df}\mathbf{\Phi}_{\mathrm{df}} - \\ & (\mathbf{A}^d - \mathbf{A}^f - \frac{1}{2}\sigma_{zz}\mathbf{I})\mathbf{\Phi}_z - \\ & (\theta^d\mathbf{I} - \beta^d\mathbf{A}^d)\mathbf{\Phi}_{\mathrm{d}} - [(\theta^f - \sigma_{fz})\mathbf{I} - \beta^f\mathbf{A}^f]\mathbf{\Phi}_{\mathrm{f}} \end{aligned}$$

where the dot denotes the time derivatives, and $\mathbf{\Phi}_{\mathbf{i}}$ and $\mathbf{\Phi}_{\mathbf{ij}}$ are the $N \times N$ matrix of $\partial\varphi/\partial i$ and $\partial^2\varphi/\partial i \partial j$, respectively, for $i, j \in \{z, r^d, r^f\}$. Given fixed collocation point x_j, (16) constitutes a system of first-order homogeneous ordinary differential equations in $\alpha(t)$. In particular, the matrix Φ is symmetric-positive definite, hence, invertible. The matrix $\mathbf{B}$ is a known constant.[7]

Given the terminal condition at time T, $V(z_T, T) = (K - e^{Z_T})^+$, $\alpha(T)$ is uniquely determined by (16). With an appropriate time discretization, the system (16) may be recursively solved using various backward time integration methods. We use the implicit backward time integration scheme such that, for $m = 1, 2, \ldots, M$:

$$\frac{d\alpha_j(t)}{dt} \approx \frac{\alpha_j^{(m)} - \alpha_j^{(m-1)}}{\Delta\tau} \tag{17}$$

where $\Delta\tau > 0$, $\alpha_j^{(m)} \equiv \alpha_j(T - m\Delta\tau)$, and $M\Delta\tau \equiv T$. Substituting (17) into (16), and rearranging the terms, we obtain

$$\mathbf{V}^{(\mathbf{m})} = \Phi(\Phi - \Delta\tau\mathbf{B})^{-1}\mathbf{V}^{(\mathbf{m}-1)} \tag{18}$$

Thus, the European put solution, $\mathbf{V}^{(M)} = \Phi\alpha^{(M)}$, may be obtained by successive iteration of (18), given the terminal condition of $\mathbf{V}^{(0)} = V(z_T, T) = (K - e^{Z_T})^+$. This iteration procedure requires inverting the matrix, $(\Phi - \Delta\tau\mathbf{B})$, once. We use Gaussian elimination in our computation to invert the matrix.

EXHIBIT 1

RBF Approximation Solutions Under Constant Volatilities of Interest Rates

Option Type	Number of Time Steps	Option Prices Number of Mesh Points 13	25	31	51	Exact Value
European	360	0.0998	0.1114	0.1138	0.1143	0.1144
		(16.34)	(75.62)	(127.44)	(432.88)	
American	360	0.1022	0.1143	0.1166	0.1171	0.1171
		(16.46)	(75.77)	(127.56)	(433.61)	
European	36	0.0993	0.1110	0.1134	0.1138	0.1144
		(16.00)	(70.76)	(117.03)	(401.92)	
American	36	0.1017	0.1137	0.1160	0.1165	0.1171
		(16.02)	(70.87)	(117.15)	(402.90)	

Put option prices are at $Z = 1$. Parameter values used are: $K = 1$, $T = 1$. $\theta_t^d = \theta_t^f = 0.005$, $(\sigma_z, \sigma_d, \sigma_f) = (0.303, 0.0505, 0.0505)$, and $\rho_{zd} = \rho_{zf} = \rho_{fd} = 0.2059$. Number of mesh points refers to the number of exchange rate mesh points. Numbers in parentheses represent CPU time in seconds of calculation. Numerical implementation is by MATLAB on a 600 MHz Pentium III.

For American options, the approach needs to be slightly modified due to the presence of constraints. As an American option can be exercised any time before expiration, there may be optimal exercise boundaries (exchange rates in our context)—but the boundaries are normally unknown. In a lattice approach, the option value is obtained by calculating the value of the option when exercised and the value when not exercised, and using the greater of the two. With a finite number of time steps, the resulting option value is actually the value of a Bermudan option that allows a finite number of exercise opportunities.

Although the value approaches the true American option value as the time step size shrinks, the value of a Bermudan option is theoretically less than the value of an American option. In an implicit time discretization using a finite-difference or finite-element approach, a more accurate American option value may be obtained, for example, by using a technique of projected successive over-relaxation (see Wilmot, Howison, and Dewynne [1995]) or Lemke's algorithm. For simplicity, we use the Bermudan approximation technique.

The updating procedure for American option valuation is as follows. Starting with the terminal value, update $\mathbf{V}^{(m)}$ using (18). For $i = 1, 2, ..., N$, update the i-th element $\mathbf{V}^{(m)}(i) = \max\{K - e^{Z}, \mathbf{V}(m)(i)\}$. Repeat this procedure until $\mathbf{V}^{(M)}$ is obtained. A particularly interesting feature of this approach is that we do not impose any boundary conditions other than terminal conditions.

IV. NUMERICAL RESULTS

To examine the accuracy of our approach, we first assume that domestic and foreign interest rates have constant volatility so that $\beta^d = \beta^f = 0$ in (9). The current domestic and foreign term structures of interest rates are assumed to be consistent with the constant values for θ_t^d and θ_t^f in (9).

Given the fixed computational domain for the log exchange rate, z = [–2.25 and 2.25], we experiment with different numbers of uniform mesh points, N_1, for the exchange rates and with different time step sizes. The five uniform mesh points, N_1, for domestic and foreign short rates (N_2 and N_3) are chosen so that the middle point for each short rate is 0.05 and its mesh size is the same as that of the log exchange rates. Thus, the computational domains for the short rates vary depending on the mesh size chosen for the log exchange rates.

We follow this specific procedure in order to maintain a uniform mesh size in any direction. This procedure for choosing the mesh points could lead to inclusion of negative interest rates in the domain. Although the economic sense of evaluating the equation at negative interest rates is unclear, Equation (12) and variational inequality (13) hold mathematically.

Exhibit 1 reports the values of at-the-money put options for the strike price, $K = 1$ and the expiration, $T = 1$. We use the parameter values: $\theta^d = \theta^f = 0.005$, $(\sigma_z, \sigma_d, \sigma_f) = (0.303, 0.0505, 0.0505)$, and $\rho_{zd} = \rho_{zf} = \rho_{fd} = 0.2059$. The exact value for the European case is obtained

Exhibit 2

RBF Approximation Solutions Under Exponentially Declining Interest Rates—T = 1

Option Moneyness	Exchange Rate	Foreign Short Rate	Option Prices: RBF Approximation Value European	RBF Approximation Value American	Exact Value European	Exact Value American
IN	0.8		0.1832	0.2102	0.1832	0.2089
IN	0.9		0.1075	0.1219	0.1079	0.1220
AT	1.0	0.04	0.0539	0.0592	0.0545	0.0596
OUT	1.1		0.0231	0.0246	0.0236	0.0251
OUT	1.2		0.0086	0.0089	0.0089	0.0093
IN	0.8		0.1896	0.2132	0.1895	0.2122
IN	0.9		0.1132	0.1261	0.1135	0.1260
AT	1.0	0.05	0.0579	0.0628	0.0585	0.0632
OUT	1.1		0.0254	0.0268	0.0260	0.0273
OUT	1.2		0.0097	0.0100	0.0100	0.0104
IN	0.8		0.1960	0.2165	0.1959	0.2159
IN	0.9		0.1192	0.1305	0.1193	0.1302
AT	1.0	0.06	0.0624	0.0665	0.0627	0.0669
OUT	1.1		0.0281	0.0291	0.0284	0.0297
OUT	1.2		0.0111	0.0112	0.0113	0.0117

Parameter values used are: $K = 1$, $T = 1$, $\theta_i^d = \theta_i^f = 0.065$, $(\sigma_z, \sigma_d, \sigma_f) = (0.15, 0.08, 0.08)$, and $\rho_{zd} = -0.01$, $\rho_{zf} = 0.06$, $\rho_{fd} = 0.08$. The computational domain for the log exchange rate, $z = [-1, 1]$, and the number of time steps is $M = 360$. Number of mesh points N_1 for the exchange rates, and N_2 and N_3 for the domestic and foreign interest rates are 31, 5, and 5, respectively. Exact European option value is obtained from Equation (5), and exact American option values are obtained with $N_1 = 51$, $N_2 = N_3 = 9$. CPU times identical to those for corresponding number of mesh points in Exhibit 1.

by Equation (5) of Amin and Jarrow [1991], and the exact value for the American case is from Choi and Marcozzi [2000], who use the finite-element scheme.

The number of mesh points given in Exhibit 1 is the number of mesh points, N_1, for exchange rates. Thus, when $N_1 = 13$, the total number of mesh points becomes $13 \times 5 \times 5 = 325$. As the number of mesh points increases, the RBF solutions approach the true values. The approximation errors become less than 1% with a total number of mesh points of $31 \times 5 \times 5 = 775$.

By way of comparison, note that Amin and Bodurtha [1995] obtain a similar accuracy by using approximately 1 million nodes in their multinomial lattice approach. The approximation errors for fewer time steps appear to be relatively small. Increasing the number of time steps from 36 to 360 improves the accuracy by 0.35% for both European and American options.

The numbers in parentheses in Exhibit 1 represent the CPU time in seconds of the associated calculation. We perform the numerical implementation using MATLAB on a 600 MHz Pentium III.

Our second experiment is based on the Vasicek specification of volatility structure. We choose $K = 1$, $\theta^d = \theta^f = 0.065$, $(\sigma_z, \sigma_d, \sigma_f) = (0.15, 0.08, 0.08)$, $\rho_{zd} = -0.01$, $\rho_{zf} = 0.06$, and $\rho_{fd} = 0.08$. Once again, the current domestic and foreign term structures of interest rates are assumed to be consistent with the constant values of θ^d and θ^f in (9).

Exhibit 2 reports the results obtained with $N_1 = 31$ when $T = 1$ so that the total number of mesh points is $31 \times 5 \times 5 = 775$. The computational domain for the log exchange rates is $z = [-1, 1]$, and the number of time steps is $M = 360$. All option values used in Exhibit 2 are based on the domestic short rate, $r^d = 0.05$. The RBF approximations to the put option values at $(Z, r^d, r^f) = (1, 0.05, 0.05)$ are obtained directly from the iterative procedures since that is one of the N mesh points chosen for the experiment.

All option values other than those mesh points are obtained by interpolating through the RBF function as follows. Given the option values $\mathbf{V}^{(M)}$ at all chosen mesh points, x_j, $\alpha^{(M)}$ is calculated by $\alpha^{(M)} = \Phi^{-1}\mathbf{V}^{(M)}$. Using this

Exhibit 3

RBF Approximation Solutions Under Exponentially Declining Interest Rates —T = 5

Option Moneyness	Exchange Rate	Foreign Short Rate	Option Prices: RBF Approximation Value, European	RBF Approximation Value, American	Exact Value, European	Exact Value, American
IN	0.8		0.1466	0.2652	0.1475	0.2649
IN	0.9		0.1260	0.2023	0.1268	0.1985
AT	1.0	0.04	0.1088	0.1545	0.1092	0.1512
OUT	1.1		0.0947	0.1229	0.0941	0.1191
OUT	1.2		0.0830	0.1025	0.0814	0.0965
IN	0.8		0.1536	0.2721	0.1546	0.2722
IN	0.9		0.1326	0.2107	0.1336	0.2073
AT	1.0	0.05	0.1151	0.1635	0.1156	0.1595
OUT	1.1		0.1006	0.1312	0.1002	0.1267
OUT	1.2		0.0886	0.1097	0.0870	0.1035
IN	0.8		0.1606	0.2792	0.1618	0.2797
IN	0.9		0.1394	0.2193	0.1406	0.2161
AT	1.0	0.06	0.1216	0.1726	0.1223	0.1679
OUT	1.1		0.1068	0.1397	0.1065	0.1345
OUT	1.2		0.0944	0.1172	0.0929	0.1107

Computational domain for the log exchange rate, $z = [-3.5, 3.5]$, and the number of time steps is $M = 360 \times 5$. CPU times for RBF approximation values in this table are 132.25 seconds for the European and 132.97 seconds for the American options.

$\alpha^{(M)}$, the option value at a point, say, $x = (Z, r^d, r^f) = (0.8, 0.05, 0.04)$, is obtained by calculating a vector of $\{\varphi(|x - x_j|)\}$ and post-multiplying it by $\alpha^{(M)}$. The exact American option values are the values obtained by choosing $N_1 = 51$, $N_2 = N_3 = 9$. The approximation error is under 1% relative to the exact solutions, except for deep out-of-the-money options.

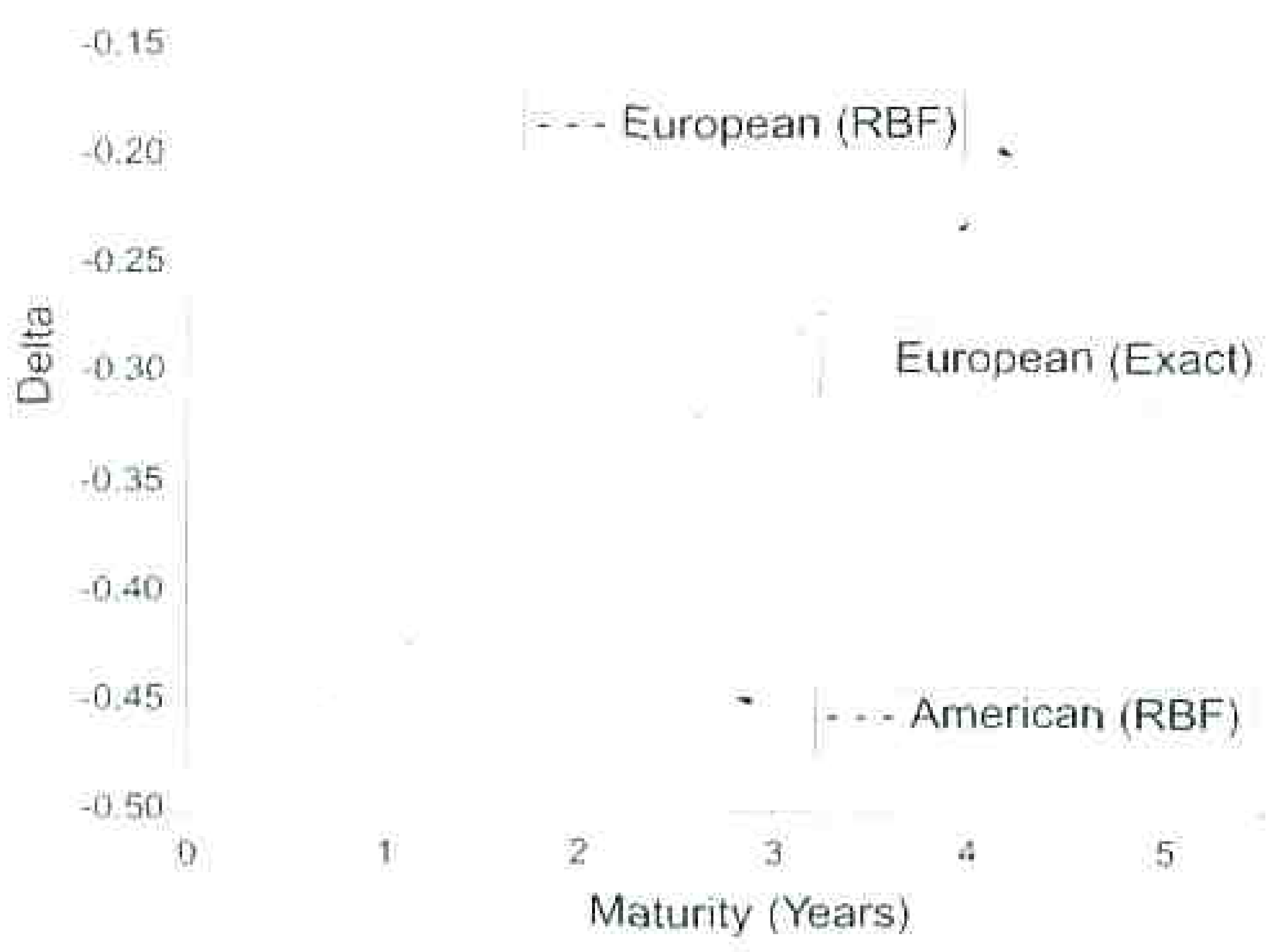
Exhibit 3 reports the results for a longer-term option with $T = 5$. All other parameter values are the same as in Exhibit 2 except the computation domain, where $z = [-3.5, 3.5]$. As the option life is longer, we choose a larger computational domain. Again, the approximation error is under 1% relative to the exact solutions, except for deep out-of-the-money options.

Because the RBF function, MQ, is infinitely differentiable with respect to state variables, we can easily calculate the option deltas (dV/dz) and rhos (dV/dr^d or dV/dr^f). For example, the delta for an American put option is obtained by taking the derivative of Φ with respect to point Z and multiplying by $\alpha^{(M)}$.

Exhibits 4 and 5 show the option deltas and rhos evaluated at at-the-money exchange rates for different

Exhibit 4

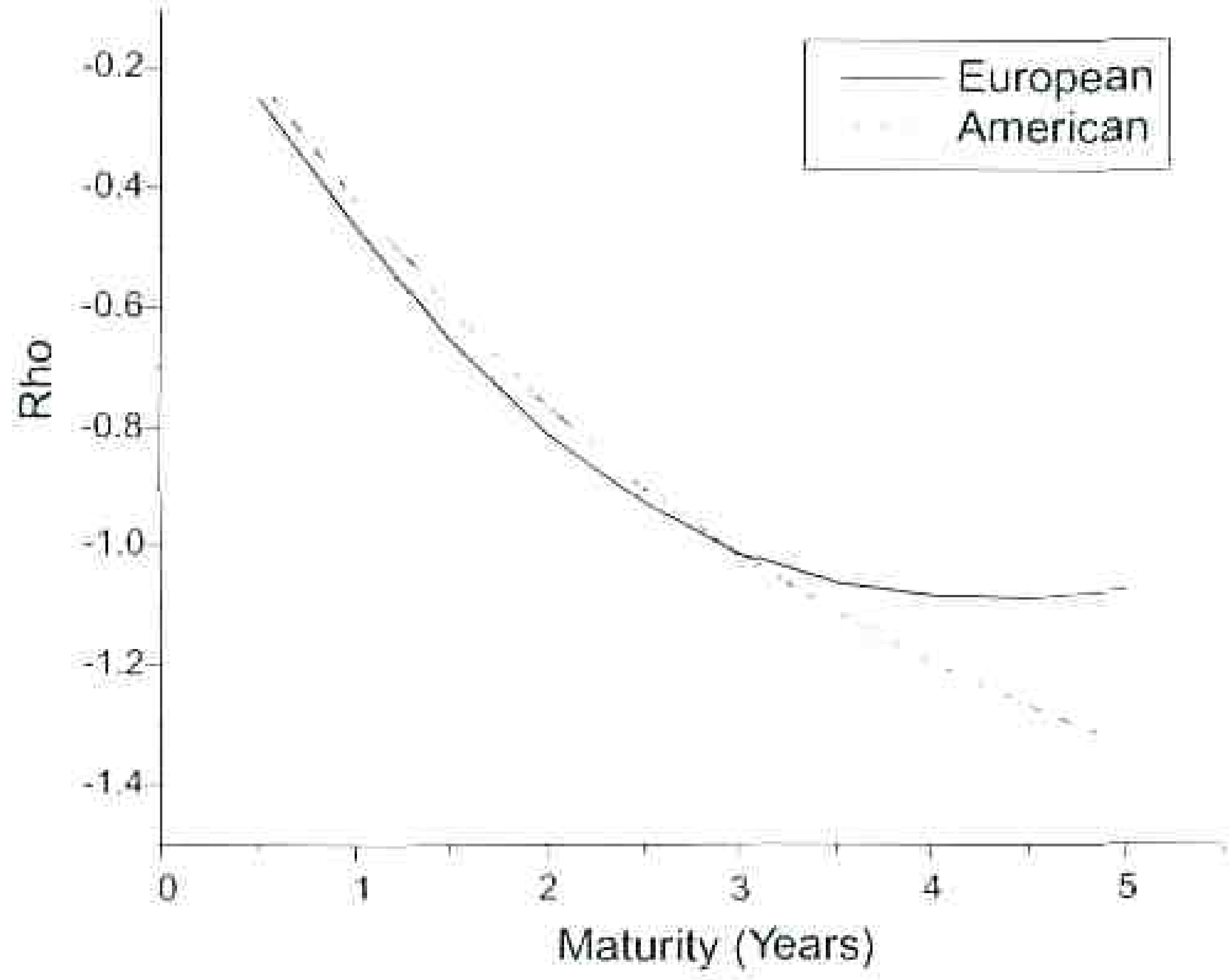
Option Deltas



Option delta values are evaluated at-the-money and with $r^d = r^f = 0.05$. The parameter values used are: $K = 1$, $\theta_l^d = \theta_l^f = 0.065$, $(\sigma_z, \sigma_d, \sigma_f) = (0.15, 0.08, 0.08)$, and $\rho_{zd} = -0.01$, $\rho_{zf} = 0.06$, $\rho_{fd} = 0.08$. $Z = [-3.5, 3.5]$. $M = 360 \times 5$, and $N_1 = 31$, $N_2 = N_3 = 5$.

Exhibit 5

Option Rhos



Option rho values are evaluated at-the-money and with $r^d = r^f = 0.05$. The parameter values used are: $K = 1$, $\theta_i^d = \theta_i^f = 0.065$, $(\sigma_z, \sigma_d, \sigma_f) = (0.15, 0.08, 0.08)$, and $\rho_{zd} = -0.01$, $\rho_{zf} = 0.06$, $\rho_{df} = 0.08$. $Z = [-3.5, 3.5]$, $M = 360 \times 5$, and $N_f = 31$, $N_z = N_t = 5$.

expirations using the same parameters as in Exhibit 3. In the European case, the exact solution for delta is obtained by taking the derivative of (5) with respect to Z.

As shown in Exhibit 4, the option deltas obtained from the RBF approximations are very close to the true option deltas. Note that the difference in delta between the European and the American options is greater for a longer-term option. In fact, the American option delta is twice the size of the European for terms of three years or longer. Accordingly, approximating the American option delta value by the European formula can result in a very different hedging position from the position actually desired.

Finally, we experiment with time-varying the drift parameters θ_i^d and θ_i^f in the short rate processes of (7) and (8), so that they exactly match the current domestic and foreign term structure. In this case, a difficulty arises as the matrix **B** also becomes time-varying. In practice, however, because we observe discrete term structures, we obtain a finite number of θ_i^d and θ_i^f at a finite number of time points. It may be convenient to assume that θ_i^d and θ_i^f are constant between the two time points.

In the numerical experiment, given $T = 1$, we assume that the true forward rate curve is linear, and that we observe five forward rates at time $t \in J_1 = \{t_1, t_2, ..., t_5\} = \{0.1, 0.3, 0.5, 0.7, 0.9\}$. We then find the corresponding θ_i^d and θ_i^f at those time points by (7) and (8).

Let $\theta^d = \{\theta_1^d, \theta_2^d, ..., \theta_5^d\}$ and $\theta^f = \{\theta_1^f, \theta_2^f, ..., \theta_5^f\}$ be the observed parameter values. Assume that θ_i is constant over the interval $[t_i - 0.1, t_i + 0.1)$.[8] In the recursive backward updating procedure at time step, $t = t_i - 0.1$ for all i, and **B** in (17) is updated with new θ_i^d and θ_i^f. This **B** is then used to update option values by (18).

We compare these results with those obtained when **B** is updated at every time step with new θ_i^d and θ_i^f obtained from linear forward rate curves that are assumed to be the true underlying structures. The difference between the two results is so small that we do not report them (under 0.5% for both the European and American cases).

V. CONCLUSION

We have derived a partial differential equation associated with the Amin and Jarrow [1991] currency option model based on the short rate processes. We then implement a numerical approximation technique to obtain the option values. We use a method of implicit time differencing based on a new class of approximating functions, radial basis functions, specifically Hardy's MQ.

Our numerical results indicate that good accuracy of solution is achieved with fewer than 1,000 nodes and imposing no boundary conditions except the terminal values. Their inherent ease of implementation and their global span make RBF techniques particularly suitable for option valuation. The advantage is especially pronounced for multidimensional problems.

ENDNOTES

[1]We use the multiquadratic (MQ) developed by Hardy [1971] to approximate two-dimensional geographic surfaces. Kansa [1990a, 1990b] applies this approach to solve partial differential equations of elliptic, parabolic, and hyperbolic types.

[2]The suitability of RBF techniques for option valuation is explored in Marcozzi, Choi, and Chen [2001] in a one-dimensional setting.

[3]As used here, the term, "boundary conditions" does not include the values of derivatives at expiration.

[4]Amin and Jarrow [1991] present their model with four sources of uncertainty. Without loss of generality, we can use three independent sources of uncertainties.

[5]The volatility structure of (2) is a multifactor version of the Vasicek specification. While volatility declines exponentially with maturity, the correlation between the domestic and foreign forward rates is the same for all maturities. See Ritchken

and Sankarasubramanian [1995] for the properties of the Vasicek specification.

[6]Vasicek [1977] presents the short rate model with constant θ. Hull and White [1990] extend the Vasicek model by allowing θ to be time-varying to match the current term structure.

[7]The matrix **B** becomes time-varying if the terms, $\theta_t^{\prime}$ and $\theta_t^{\prime}$, are time-varying.

[8]The lattice approach implicitly assumes that $f(0, T)$ is a step function in T over time step sizes. Likewise, θ_t may be assumed to be constant over some time interval, but constant θ_t does not imply constant $f(0, T)$ except in the Ho-Lee volatility structure.

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