How Well Can Options Complete Markets?

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Investors with lognormal beliefs and constant relative risk aversion achieve a specific wealth profile when markets are complete. Since the state space is infinite, complete markets can be achieved only through continuous trading strategies or an infinite number of options. Both are approximations of actual markets.

This article looks at how well a finite number of options can complete markets using a least squares approximation of the optimal complete market wealth profile. For all sensible measures of closeness of the finite option/incomplete market to the complete market case, we find that it takes relatively few options to make the difference between incomplete and complete markets negligible.

The celebrated option pricing models of Black and Scholes [1973] and Merton [1973] are based on the replication of option payoffs using dynamic strategies involving the underlying asset and a risk-free bond. When asset prices follow geometric Brownian motion, continuous trading strategies complete markets. That is, for any contingent claim, there is a continuous self-financing trading strategy with a terminal payoff equal to that claim. The key insight is that derivative payoffs are mapped to self-financing trading strategies and, by no arbitrage, the current derivative price must be equal to the value of the replicating stock-bond portfolio.

In a sense, Ross [1976] turns the problem around and shows how options can complete markets. Any Arrow-Debreu state-contingent payoff can be generated by options written on a portfolio of securities. This is static in that there is no need to adjust the option portfolio. Combining these two important results, any payoff can be generated by either a dynamic trading strategy or a portfolio of options.

This has a natural relevance to portfolio choice under utility maximization. If markets are complete, either by continuous trading or by existence of a sufficient number of options, an investor to maximize utility of terminal wealth can choose an optimal terminal wealth profile (as a function of the stock price) subject to a budget constraint. The budget constraint can be described as a stochastic differential equation that a self-financing strategy must satisfy; this is the standard dynamic programming problem as in Merton [1973].

Alternatively, the budget constraint can be expressed in reference to state prices that are usually described by martingale methods. In this martingale approach, developed in Cox and Huang [1989], the cost of a wealth profile is the discounted risk-neutral expectation; one can then derive the continuous trading strategy that generates this optimal wealth profile. Or, one can describe the complete market state prices in terms of option prices, as in Breeden and Litzenberger [1978], and solve for an optimal *static* portfolio of options, as in Carr and Madan [2001].

Hence, with complete markets the investor can achieve an optimal wealth profile with either a continuous trading strategy in the stock and bond or a static portfolio of options (which could be infinite if the state space is infinite).

This equivalence between static positions in options and dynamic trading strategies is accepted in the portfolio insurance literature, as in Leland [1980] and Brennan and Solanki [1981], who examine conditions under which investors would desire convex optimal wealth profiles. Convex wealth profiles are interpreted as portfolio insurance in that the payoff function would be created by long positions in the underlying asset and put options or, by put-call parity, long positions in call options and bonds. This is equivalent to a dynamic replication strategy in the underlying asset and the risk-free bond; hence, portfolio insurance is the trend-following strategy of selling/buying the asset as the price falls/rises. Again, markets are assumed complete in that any wealth profile can be attained.

Given that continuous time and a continuous menu of options are both abstractions, how well options can approximate optimal wealth profiles is an interesting question both theoretically and empirically. By definition, with incomplete markets, an arbitrary function may no longer be attainable. Instead of looking at derivatives pricing (and optimal hedging) under incomplete markets, I will take derivatives prices as given.[1]

Two studies are very similar in spirit to this one. Benninga and Blume [1985] examine conditions under which an optimal wealth profile has a minimum floor (i.e., portfolio insurance). They show that, under dynamically complete markets, the degree of relative risk aversion would have to be time-varying and approach infinity. They also look at incomplete markets and examine a static portfolio consisting of a stock, a bond, and a single put. They go on to show that for a certain set of parameters, unless the investor is prohibited from investing in the risk-free asset, there would be no portfolio insurance (the investor would like to *short* the put option).

More recent work by Ahn et al. [1999] looks at option portfolios and optimal put strike prices in a Black-Scholes setting where "optimal" is with respect to a value at risk management metric. I also look at static option positions but include more than one option. As in Benninga and Blume [1985], I am more concerned with relating this problem to the investor's utility under incomplete versus complete market models.

Specifically, I will ask how well a wealth profile achieved by a static portfolio of a *finite* number of options approximates the optimal, complete markets, wealth profile. Of course, the metric one uses for approximation is crucial in interpreting the results. For all sensible metrics, options do a remarkable job of completing markets.[2]

I. COMPLETE MARKETS

A partial equilibrium approach looks at an investor who wants to maximize the expected utility of terminal wealth (i.e., there is no intermediate consumption). Mathematically, the investor will solve:

$$\max \mathbb{E}\left(u\left(w\right)\right) \tag{1}$$

subject to w, a random variable at time T, being in a given choice set. The economic environment consists of a single risky asset, a risk-free bond, and, for a static decision problem, standard options based on the risky asset.

We make several assumptions about the investor's beliefs and preferences. The investor has constant relative risk aversion $\gamma > 0$, i.e., $u(w) = w^{1-\gamma}/(1-\gamma)$, and believes the single risky asset's price $\equiv S_t$ follows geometric Brownian motion:

$$dS_t = \mu S dt + \sigma S dW_t \tag{2}$$

In addition, the investor can borrow and lend at a constant continuous interest rate r.

Note that Equation (2) will be relevant for continuous trading strategies, but we also use it for the static option portfolio strategy in the assumption that option prices satisfy Black-Scholes (i.e., all options have an implied volatility equal to the investor's volatility belief σ). With continuous (dynamic) trading strategies, the Black-Scholes assumption is necessary; otherwise, the investor would have unbounded expected utility by taking arbitrage trading strategies. With static option portfolios, the Black-Scholes assumption is not necessary.

Finally, using our assumption of constant relative risk aversion and the property that option prices are homogeneous of degree one with respect to the stock price and strike price, we can without loss of generality normalize initial wealth and the initial stock price to 1 (i.e., $w_0 = S_0 = 1$).

We derive the optimal wealth profile for the case of complete markets, and review the two strategies (dynamic and static) investors can use in order to achieve the optimal wealth profile.

The Complete Market Optimal Wealth Profile

Assuming no arbitrage opportunities and complete markets, there is a unique risk-neutral martingale measure, $\mathbb{Q}$, such that a random contingent claim at time T, call it $f(S_T)$, has a value of $\mathbb{E}_{\mathbb{Q}}[f(S_T)/B_T]$ where B_T is the future value of \$1 invested in a locally risk-free bank account. Under our assumptions that the risk-free rate is constant and we are in a Black-Scholes world, $B_T = e^{rT}$, and the martingale measure is found by using a lognormal density:

$$\ln(S_T) \sim \mathcal{N}\left(\left(r - \sigma^2/2\right) T, \sigma^2 T\right) \tag{3}$$

We can transform the utility maximization problem into

$$\max_{w} \mathbb{E}\left(\frac{w(S_T)^{1-\gamma}}{1-\gamma}\right) \tag{4}$$

$$\text{subject to } \mathbb{E}_{\mathbb{Q}}\left(w(S_T)\right) = e^{rT} \tag{5}$$

The solution to this problem is the optimal wealth profile:[3]

$$w^*(S_T) = e^{kT} S_T^{\lambda} \tag{6}$$

where $\lambda \equiv (\mu - r)/(\gamma\sigma^2)$, and $k \equiv (1-\lambda)(r + \sigma^2\lambda/2)$. We see that the wealth profile is convex if and only if $\lambda > 1$ or, equivalently, $(\mu - r)/\sigma^2 > \gamma$. It can be shown that the expected value of optimal wealth is

$$\mathbb{E}(w^*) = \exp\left(\left(r + \lambda(\mu - r)\right)T\right) \tag{7}$$

and the expected utility is:[4]

$$\mathbb{E}(u(w^*)) = \frac{\exp(\varphi T)}{1-\gamma} \tag{8}$$

where

$$\varphi \equiv \frac{(1-\gamma)\lambda(\mu - r)}{2} + r(1-\gamma) \tag{9}$$

Dynamic and Static Replication

The investor has two alternatives in order to achieve this optimal complete market wealth profile. We first review the continuous trading strategy that generates the optimal wealth profile. Merton [1971] uses dynamic programming techniques, and Cox and Huang [1989] use the martingale approach.

Given the ability to trade the stock continuously, which follows the diffusion Equation (2), we can replicate the optimal wealth profile as we would a European contingent claim. To generate the optimal wealth profile, $w^*(S_T) = e^{kT}S_T^{\lambda}$, we can use the dynamic hedging approach, which is to hold $\partial w^*(S, t)/\partial S$ shares of stock (the "delta" of w^*) where

$$w^*(S, t) \equiv e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left(w^*(S_T)\,|\,\mathcal{F}_t\right) \tag{10}$$

This produces a dynamic trading strategy of holding a constant proportion, λ, of wealth in the risky asset. It implies that the proportion of wealth invested in the risk- free asset will be greater/less than 100% if the wealth profile is convex/concave. Hence, convex wealth profiles are generated by highly leveraged dynamic portfolios.

Instead of assuming a continuous trading strategy, we could assume the investor can invest in an uncountable menu of options with strike prices $K \in (0, \infty)$. As in Breeden and Litzenberger [1978], we can establish a correspondence between the state price density, i.e., the probability density $q(S)$ of the martingale measure $\mathbb{Q}$, and the option prices of a continuum of strike prices:

$$e^{-rT}q(K) = \begin{cases} \frac{\partial^2 P_0(K)}{K^2} & \text{for } K \le S_0 \\ \frac{\partial^2 C_0(K)}{K^2} & \text{for } K \ge S_0 \end{cases} \tag{11}$$

Under Black-Scholes, $q(S)$ is the lognormal density; i.e., *under $\mathbb{Q}$, q is the density function for Equation (3).*

Carr and Madan [2001] establish the important result that the investor can achieve any well-behaved wealth profile with a static portfolio of the underlying asset, a risk-free bond, and a continuum of out-of-the-money options. Particularly, they show $f(S_T)$ can be achieved with $f'(1)$ shares of stock; $[f(1) - f'(1)]$ risk-free unit discount bonds; and $f''(K)dK$ out-of-the-money options. Note that the original stock price is normalized to one, so $K < 1$

implies put options and $K > 1$ implies call options. By put-call parity, out-of-the-money options can be used without loss of generality.

Under our Black-Scholes assumption, we can see that if $\lambda > 1$, our optimal wealth profile will be convex, and our investor achieves the optimal wealth profile by purchasing all options; this is the standard portfolio insurance result as in Brennan and Solanki [1981] and Leland [1980]. Alternatively, if $\lambda < 1$, the investor will be writing all options. When $\lambda = 1$, the optimal dynamic strategy is to invest 100% of wealth in the risky asset; therefore there will be no dynamic adjustment and no use for options.

It is interesting to note that the number of options of strike price K that generates the wealth profile will be

$$\frac{d^2 w^*(K)}{dK^2} = e^{kT} \lambda (\lambda - 1) K^{\lambda - 2} \tag{12}$$

Hence, unless $\lambda = 1$, total option demand (where we integrate over the strike prices) will be unbounded, and, unless $\lambda = 2$, the option demand will be unbounded for K either close to zero ($\lambda < 2$) or for K very large (when $\lambda > 2$).

II. INCOMPLETE MARKETS AND THE LEAST SQUARES APPROACH

The complete market environment, i.e., continuous trading or an infinite number of securities, is a convenient theoretical abstraction. We want to consider the economic consequences of relaxing one of these assumptions, that of the infinite number of securities (in this case, options).

We assume the investor at time $t = 0$ can choose n call options with strike prices $\{K_i\}_{i=1}^n$ where $K_{i+1} > K_i$. It is convenient to denote the risky asset as the first call option with a strike price $K_1 = 0$, so that there are actually $n - 1$ standard call options. As before, let w^* be the optimal complete markets wealth profile.

To arrive at an analytical solution, we state the problem as a choice of a (finite) portfolio $\{\theta_i\}_{i=1}^n$, i.e., $\theta \in \mathbb{R}^n$, where θ_i is the proportion of wealth invested in risky security i, which is the best least squares estimate of the optimal wealth function. In general, convex (concave) wealth profiles will be approximated by long (short) positions in the call options since the change in the slope (for an option portfolio) is proportional to θ_i at K_i.

Specifically, let $R_f \equiv e^{rT}$, and let R_i be the simple gross return of the risky security i. Our incomplete market wealth profile is denoted as:

$$\begin{aligned} \hat{w} &\equiv R_f + \sum_{i=1}^{n} \theta_i (R_i - R_f) \\ &= R_f + \theta' (R - 1_n R_f) \end{aligned} \tag{13}$$

where 1_n is an n vector of ones. The least squares problem is:

$$\min_{\theta} \mathbb{E}\left([w^* - \hat{w}]^2 \right) \tag{14}$$

In order to preclude bankruptcy we will add the zero borrowing constraint, $\theta 1_n \leq 1$. This condition is necessary but not sufficient, for we could have the wealth profile decreasing over other regions. For the majority of parameter specifications, this single constraint will be sufficient.

Let Σ denote the comoment matrix of excess returns; specifically, Σ has elements $\Sigma_{ij} = \mathbb{E}[(R_i - R_f)(R_j - R_f)]$. Also let y be the comoment vector of excess returns of assets with excess return of optimal wealth, i.e., $y_i = \mathbb{E}[(R_i - R_f)(w^* - R_f)]$

Given this specification, the problem amounts to a simple quadratic programming problem with a Lagrangian:

$$-\theta' y + \frac{1}{2}\theta' \Sigma \theta + \nu (\theta \cdot 1_n - 1) \tag{15}$$

which yields a solution:

$$\nu^* = \max\left(\frac{1_n' \Sigma^{-1} y - 1}{1_n' \Sigma^{-1} 1_n}, 0 \right) \tag{16}$$

$$\theta^* = \Sigma^{-1} (y - \nu^* 1_n) \tag{17}$$

The only thing that remains is to determine the expressions for Σ and y. The terms are

$$\Sigma_{ij} = \frac{\mathbb{E}(X_i X_j)}{p_i p_j} - \left(\frac{\mathbb{E}(X_i)}{p_i} + \frac{\mathbb{E}(X_j)}{p_j} \right) R_f + R_f^2 \tag{18}$$

and

$$y_i = \frac{\mathbb{E}(w^* X_i)}{p_i} - \left(\frac{\mathbb{E}(X_i)}{p_i} + \mathbb{E}(w^*) \right) R_f + R_f^2 \tag{19}$$

where X_i is the i-th option's payoff, $\max(S - K_i, 0)$, and p_i is its price. The expressions for Σ_{ij} and y_i show that the least squares problem amounts to finding the means of payoffs and the optimal wealth function as well as the comoments between them. These can be determined by a useful property of lognormally distributed random variables: If Z is normally distributed with mean μ and variance σ^2, then

$$\mathbb{E}\left(e^{nZ}; Z \geq M\right) = \exp\left(n\mu + n^2\sigma^2/2\right) N(d) \tag{20}$$

where $d = (\mu + n\sigma^2 - M)/\sigma$.

Since S_T is lognormally distributed, it is now possible to compute terms of Σ and y since they include terms of the form $\mathbb{E}[S_T^\eta; S_T > K_i] = \mathbb{E}[e^{\eta y}; e^{y} > K_i]$ where y is normally distributed with mean $(\mu - \sigma^2/2)T$ and variance $\sigma^2 T$.

III. EXAMPLES

Before considering the different numerical measures of how well options can approximate complete markets, we use a few examples to illustrate how well the finite option portfolio can fit the complete market's optimal wealth profile. The parameter values are: μ = 15%, r = 5%, σ = 25%, and T = 1. The focus is on varying the degree of risk aversion, γ. Besides being reasonable parameters, the results will generally hold for most changes although as T becomes longer the results do weaken. Since we are looking at static strategies, a one-year time horizon may be unrealistically long.

When λ = 1, the wealth profile is linear, and there is no use for options as the investor will always invest 100% of wealth in the risky asset. With our parameters, λ = 1 when γ = 8/5. Hence the wealth profile will be convex/concave if the degree of relative risk aversion, γ, is lower/higher than 1.60. To see how an increasing number of options can converge to a non-linear optimal wealth profile, consider the sequence of strike price meshes: κ_1 = [0], κ_2 = [0, 1.0], κ_3 = [0, 0.5, 1.0, 1.5], and κ_4 = [0, 0.5, 0.75, 1.0, 1.5, 2.0]. Given the menu of strike prices κ_i, let w_i be the least squares wealth profile estimate, and let θ be the portfolio vector (with i elements).

To see how the least squares approach works with a convex profile, assume γ = 4/5. In this case λ = 2.

Exhibit 1 shows the five wealth profiles as well as the investor's lognormal density. We see general improvement in the approximations. Exhibit 2 describes the convex portfolios from each of the least squares scenarios. The investor is 100% invested in the risky securities and is generally long call options. For θ^4, the holdings of the deep out-of-the-money option (i.e., K = 2), 0.1% of total wealth, seem quite small. Since its price is so low, though, it happens to be the largest in terms of number of units held. The deep out-of-the-money calls are a cheap way to achieve convexity.

Returning to Exhibit 1, we see that just five options enable the investor to match (almost perfectly) the complete market wealth profile from the approximate interval [0.5, 2.5]. This region has a likelihood of 99.86%. In

EXHIBIT 1
Convex Wealth Profiles: γ = 0.8

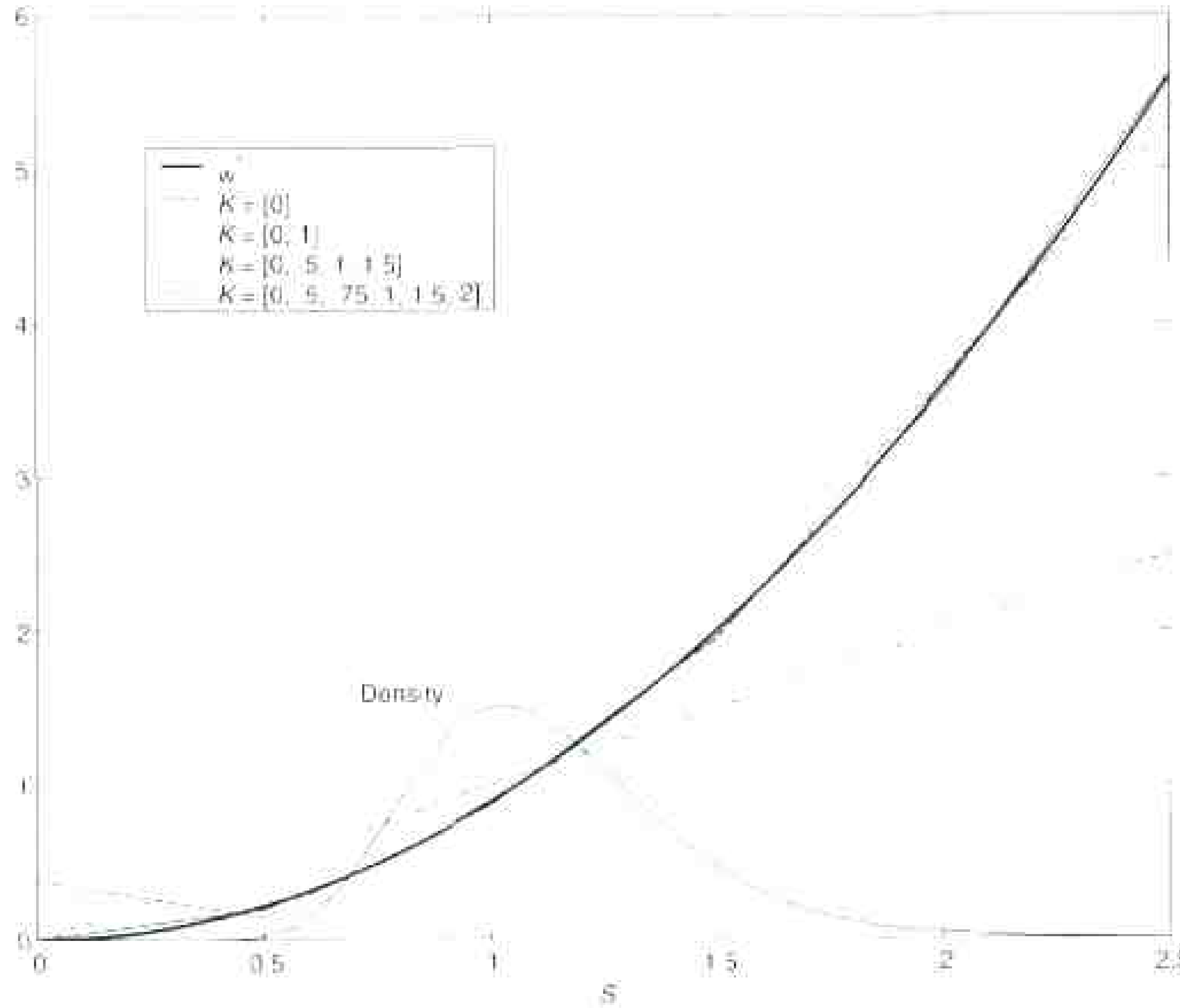


EXHIBIT 2
Four Least Squares Portfolios for Convex and Concave Wealth Profile Examples

	Convex Profile: $\gamma = 0.8$				Concave Profile: $\gamma = 3.2$			
$K =$	θ^1	θ^2	θ^3	θ^4	θ^1	θ^2	θ^3	θ^4
0	1.0000	0.7720	−0.4562	0.4038	0.4706	0.5987	1.0984	0.8619
0.50			0.9907	0.4186			−0.2745	−0.1170
0.75				0.0762				−0.0225
1.00		0.2280	0.0908	0.0912		−0.0200	−0.0134	−0.0118
1.50			0.0121	0.0096			−0.0009	−0.0008
2.00				0.0006				−0.00003

general, the least squares profile will yield higher-than-optimal wealth when S_T is low and lower-than-optimal wealth when S_T is high. As will be shown below, this amounts to lower-than-optimal skewness and kurtosis in portfolio returns (but of neglible economic significance).

To consider how well the portfolio does with concave wealth profiles, consider the case when $\gamma = 3.2$ (i.e., $\lambda = 0.50$). Using the same sequence of strike price meshes, we arrive at the wealth profiles shown in Exhibit 3. For the concave investor, options can do a remarkable job of matching complete markets. The wealth profile of the five-option menu (the dashed line) is virtually indistinguishable in an approximate region $S_T \in [0.25, 2.9]$ an interval with a probability of 99.995%.

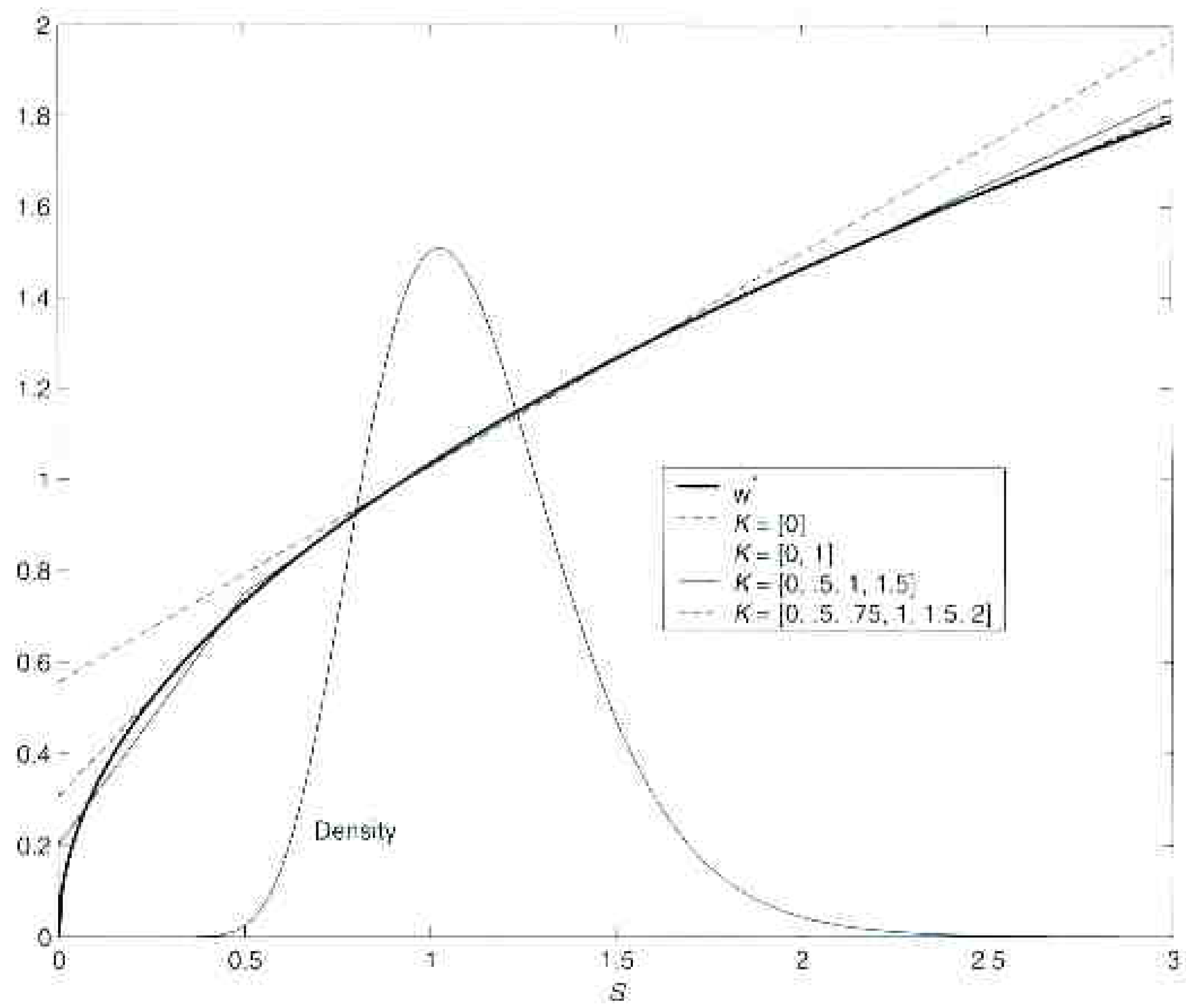
These portfolios are also shown in Exhibit 2. Note that the least squares portfolios all agree with the complete markets scenario in that concave investors will generally write covered calls.

We can see from Exhibit 3 that the least squares portfolios tend to pay off more than the optimal wealth profile at the tails. This means the least squares option portfolio usually leads to higher skewness and kurtosis than the complete markets case. Once again, the economic significance of this is negligible.

IV. MOMENT RESULTS

To get a more precise and constructive measure for how well options can approximate complete market

EXHIBIT 3
Concave Wealth Profiles: $\gamma = 3.2$



wealth profiles, we can analyze the standardized central moments of the least squares wealth profiles, $\mu_i \equiv \mathbb{E}(\hat{w}_i)$, $\sigma_i \equiv \sqrt{\mathbb{E}(\hat{w}_i - \mu_i)^2}$, $SKEW_i \equiv \mathbb{E}(\hat{w}_i - \mu_i)^3/\sigma_i^3$, and $KURT_i \equiv \mathbb{E}(\hat{w}_i - \mu_i)^4/\sigma_i^4$, and compare them to the corresponding moments of w^*: μ^*, σ^*, $SKEW^*$, and $KURT^*$. All four moments of w^* quickly fall as γ is increased (i.e., when we go from convex to concave wealth profiles).

EXHIBIT 4
Mean Ratio

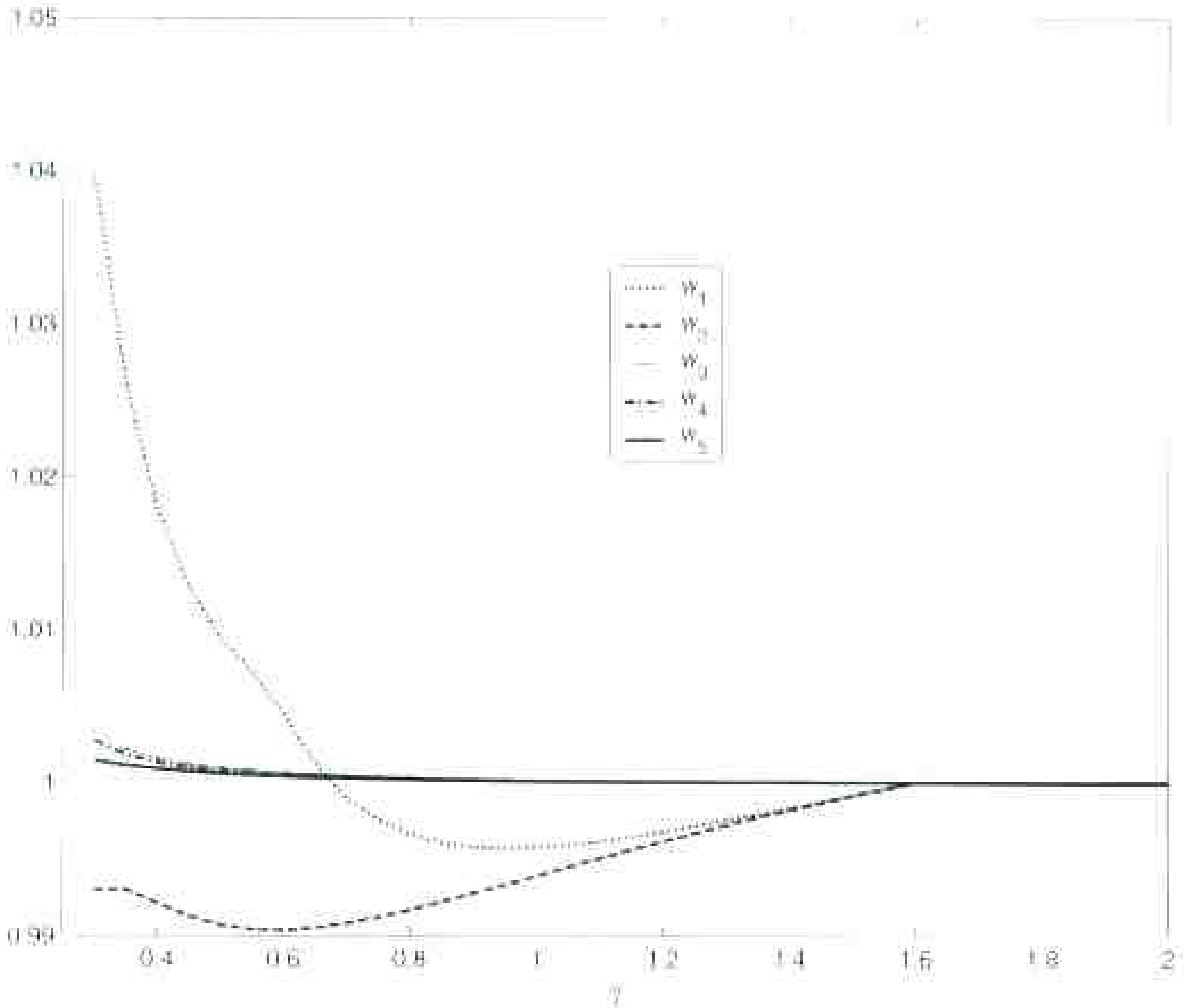
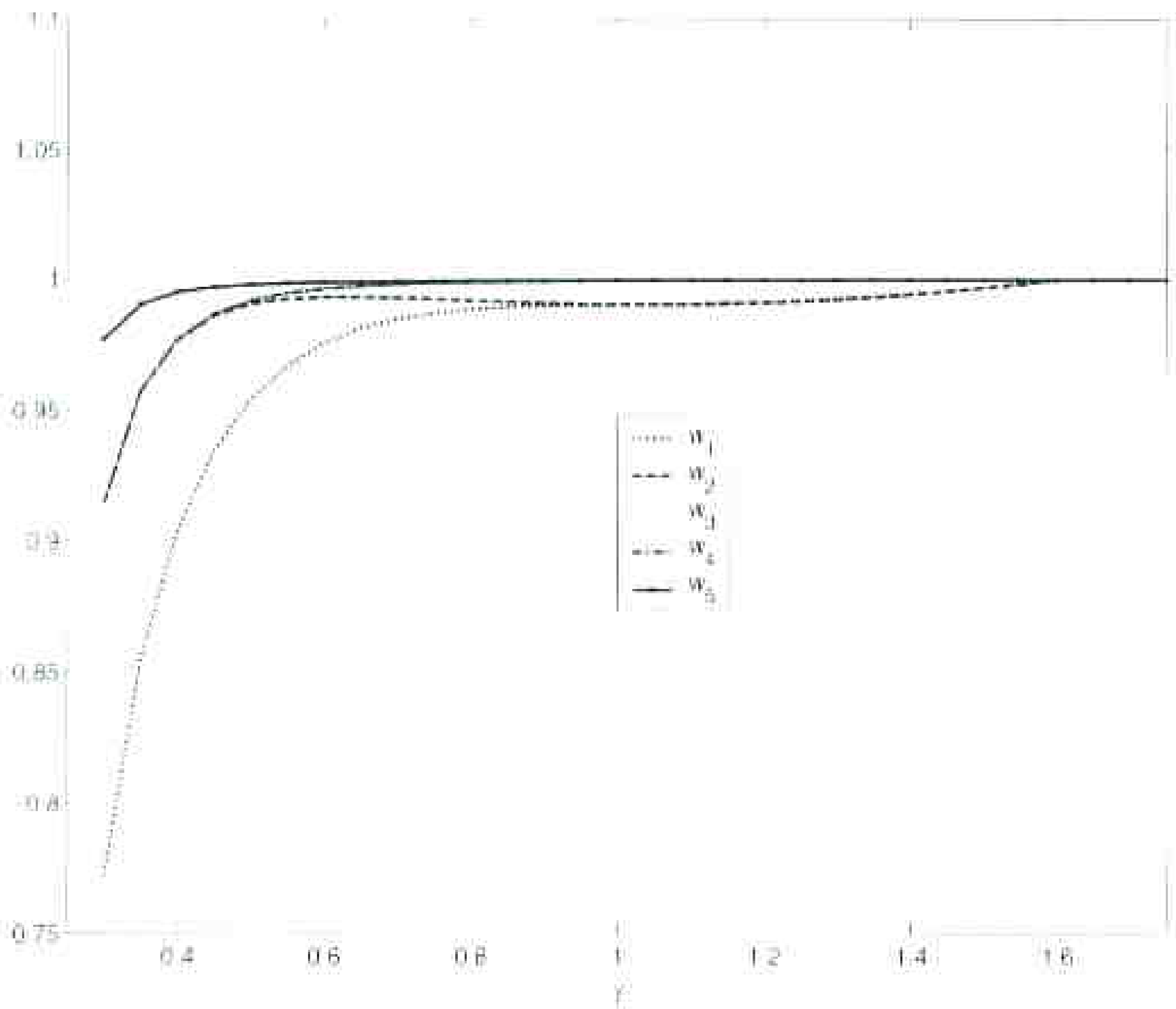


EXHIBIT 5
Volatility Ratio



The moments of the least squares option portfolios can be computed using Equation (20) as the portfolio moments are weighted averages of return comoments. For example:

$$SKEW_i = \sum_k \sum_j \sum_i \theta_i \theta_j \theta_k \times$$
$$\mathbb{E}\left[(R_i - \mathbb{E}(R_i))(R_j - \mathbb{E}(R_j)) \times (R_k - \mathbb{E}(R_k))\right] \quad (21)$$

To see how the least squares options portfolios are able to match these moments, we look at the ratios such as μ_i/μ^*. We show that portfolios with one or two options will closely match the first two moments; the higher two moments require a few more options.

We consider an increasing sequence of strike meshes, starting with [0, 1.0, 1.5], and then add strike prices of 0.5, 0.75, 2.0, and 1.25. This leads to a total of five increasing meshes. Using each mesh, we look at the least squares option portfolio for a given level of risk aversion and generate the moment ratios (e.g., $SKEW_i/SKEW^*$).

Exhibits 4 through 7 show the moment ratios as a function of risk aversion. First note that, with the addition of each strike price, the four moment errors diminish for all degrees of risk aversion. The sole exception is an increase in the first moment error for convex wealth profiles when the first (at-the-money) option is added (see Exhibit 4). We also find, in Exhibits 4 and 5 (μ_i/μ^* and σ_i/σ^*), that for concave wealth profiles, i.e., $\gamma > 1.6$, the least squares option portfolio matches the mean and standard deviation of the w^* for *all* the option menus.

In addition, for relatively few options (four to six), the mean can be matched closely (less than 0.5% error) for even highly convex wealth profiles. With six options, the standard deviation can also be matched for the convex wealth profiles, but when γ is close to 0.4 ($\lambda = 4$), the standard deviation error rises to over 2.5%. Remembering that λ represents the optimal proportion of wealth invested in the risky asset using a continuous time trading strategy, $\lambda = 4$ is a rather extreme example.

In Exhibits 6 and 7, we find that the least squares portfolios generate excess skewness and kurtosis for concave wealth profiles but lower skewness and kurtosis than the complete market's wealth profile. We also see that, with regard to matching skewness and kurtosis, the six-option menu, κ_5, is far superior to the other strike menus. It yields minimal excess skewness for concave wealth profiles and matches skewness remarkably well for the plausible restriction $\gamma > 1/2$ (i.e., $\lambda < 3.2$). Kurtosis is a bit more of a problem with an approximately 20% error for the highly convex wealth profile when $\gamma = 1/2$.

EXHIBIT 6
Skewness Ratio

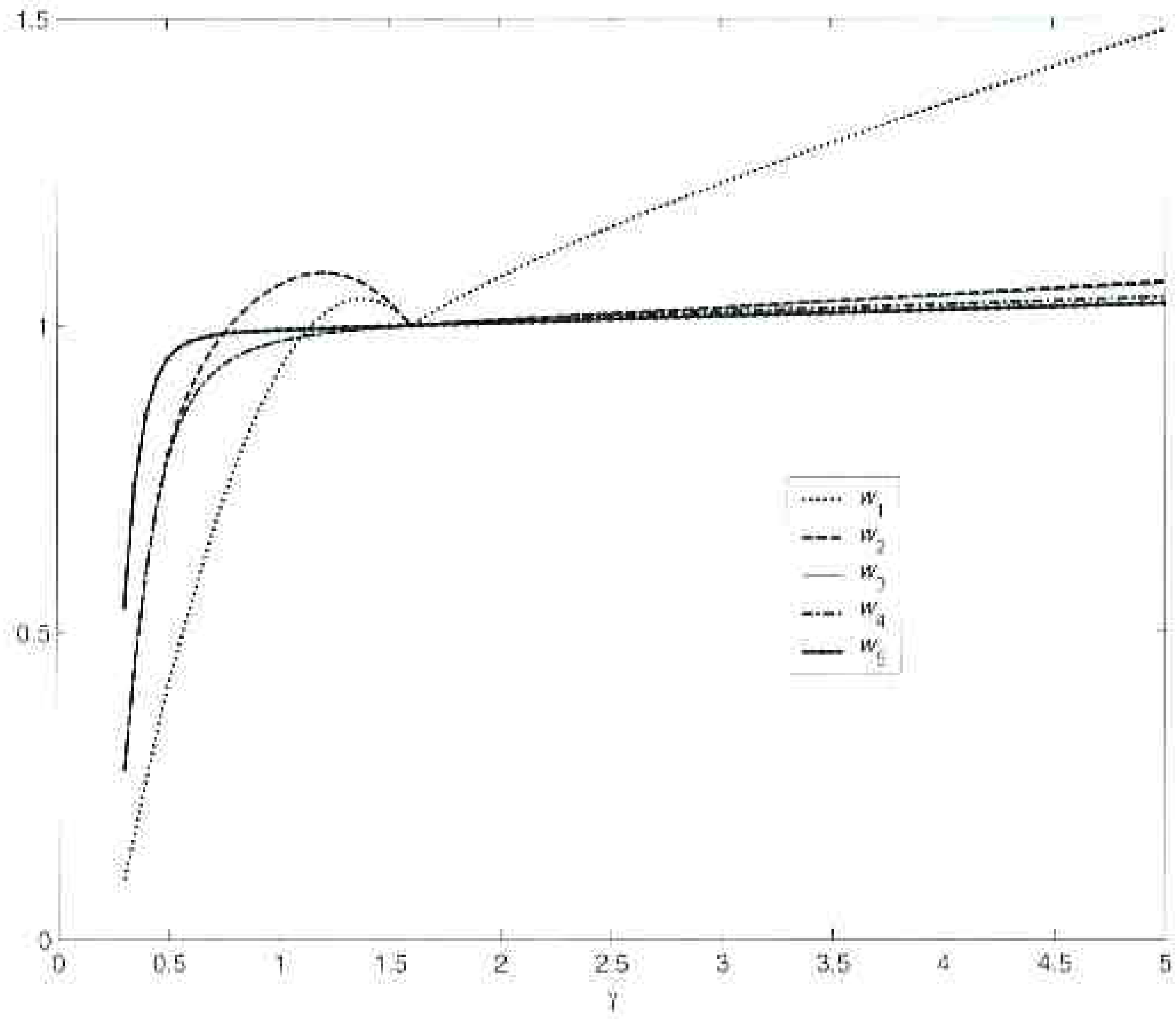
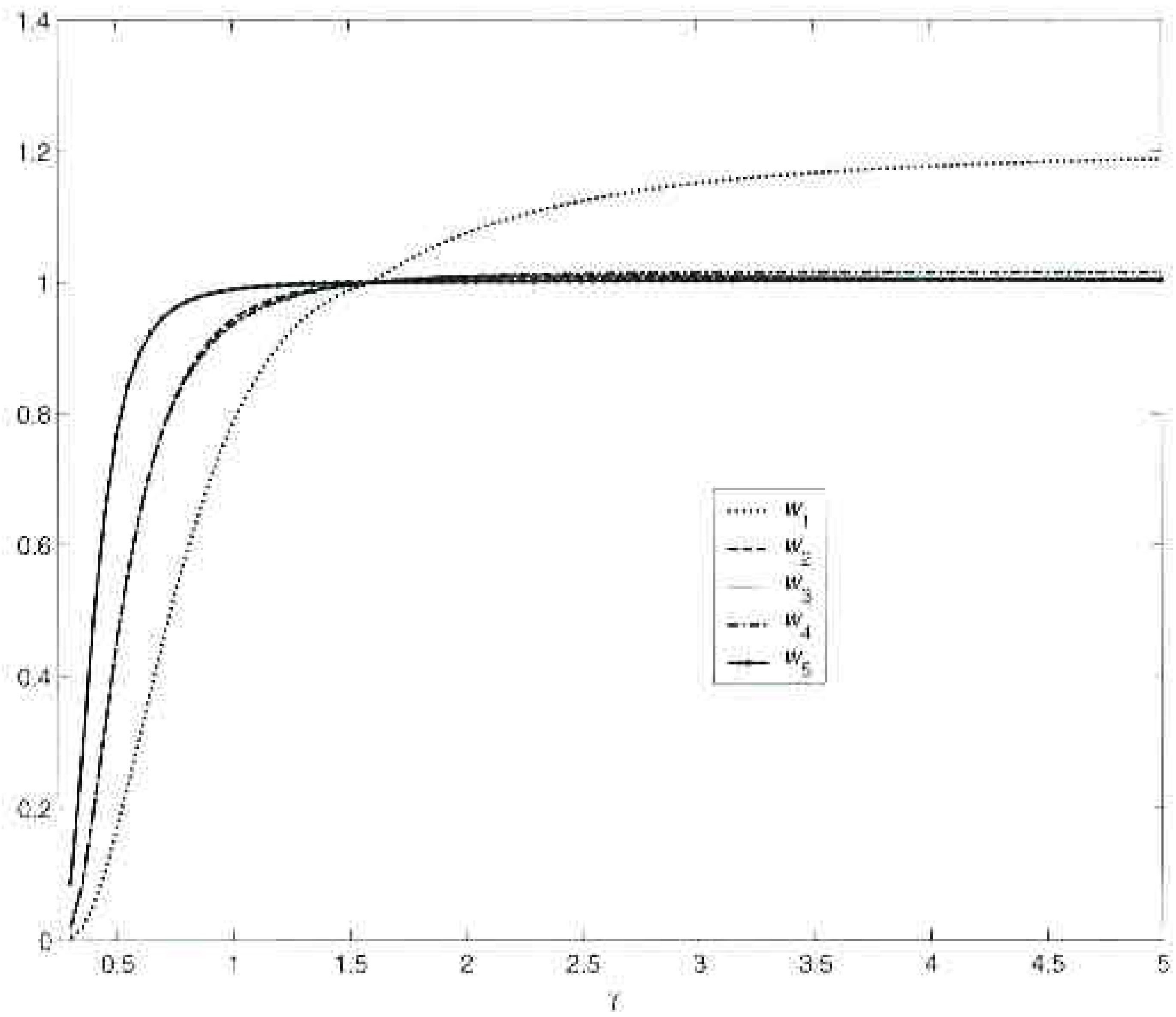


EXHIBIT 7
Kurtosis Ratio



V. ECONOMIC SIGNIFICANCE

We have shown that adding relatively few options can give the investor "nearly" complete markets in the sense that the difference between the incomplete market least squares wealth profile and the complete market wealth profile is negligible on a set of measures very close to 1.0. In addition, the least squares method has some nice statistical properties; that is, for a wide range of risk aversion, the first four standardized central moments converge with relatively few options. While one could find a portfolio using four options to exactly match the four moments of w^*, this is a much more complicated procedure, and it ignores all fifth or higher moments.

In fact, for highly convex wealth profiles (e.g., when $\gamma = 0.5$), where some of the errors are large, the four moments are inadequate to measure expected utility.[5] This is actually promising, in that when the least squares profiles do a poor job of matching the fourth and lower moments, this may be when the higher-order moments matter the most.

Since there is no closed-form solution to the least squares utilities, we must use Monte Carlo techniques to estimate the actual expected utility of the least squares option portfolio, $\mathbb{E}[u(\hat{w})]$. Once we have this estimate, we can ask if it is economically significant.

In order to make this precise, we estimate how much the investor would be willing to pay for complete markets. Define $V(\varepsilon) = J(1 - \varepsilon)$, where J is the indirect utility function under complete market's; i.e., $J(w_0)$ is the complete market's expected utility with initial wealth w_0. Hence, $V(\varepsilon)$ is the expected utility under complete markets when ε is deducted from current wealth.

One can show that for $\gamma \neq 1$:

$$V(\varepsilon) = \exp(\varphi T) \frac{(1-\varepsilon)^{1-\gamma}}{1-\gamma} \tag{22}$$

where

$$\varphi = (1-\gamma)(r + \lambda(\mu - r)/2) \tag{23}$$

We solve for the ε so that the complete markets utility is equal to the expected utility using the least squares portfolio. That is:

$$V(\varepsilon) = \mathbb{E}(u(\hat{w})) \tag{24}$$

The interpretation is that the investor would be willing to pay up to ε for the complete market's wealth profile. For $\gamma \neq 1$, Equation (24) yields the solution

$$\varepsilon = 1 - \exp\left(\frac{rT - \ln\left(\mathbb{E}\left(u\left(w\right)\right)\left(1-\gamma\right)\right)}{\gamma - 1}\right) \tag{25}$$

For the case of logarithmic utility ($\gamma = 1$), the solution is

$$\varepsilon = 1 - \exp\left(\mathbb{E}\left(u\left(w\right)\right) + r\left(\frac{\mu}{\sigma^2} - 1\right)T - \frac{\mu^2 + r^2}{2\sigma^2}T\right) \tag{26}$$

Using simulations, we can get a 95% confidence interval for $\mathbb{E}[u(w)]$ and then translate that to a 95% confidence interval for ε. Specifically, running N simulations, there will be N values for S_T and hence N values for w. This will translate to N values for $u(w)$ which yields a 95% confidence interval for $\mathbb{E}[u(w)]$ defined by

$$\left[\bar{u} - 1.96\frac{s}{\sqrt{N}}, \bar{u} + 1.96\frac{s}{\sqrt{N}}\right] \equiv [u_l, u_h] \tag{27}$$

where $\bar{u}$ and s are the sample mean and standard deviation, respectively, of the N values of $u(w)$. We can then use Equations (25) and (26) to estimate the true value of ε. We will define $\bar{\varepsilon}$ and ε_h as the values of Equations (25) and (26) when $\bar{u}$ and u_l are used. Using an increasing sequence of meshes, $\kappa_1 = [0]$, $\kappa_2 = \kappa_1 \cup [1.0]$, $\kappa_3 = \kappa_2 \cup [1.5]$, $\kappa_4 = \kappa_3 \cup [0.75]$, $\kappa_5 = \kappa_4 \cup [0.5]$, $\kappa_6 = \kappa_5 \cup [2.0]$, and $\kappa_7 = \kappa_6 \cup [1.25]$, we arrive at Exhibit 8 ($\bar{\varepsilon}$ and ε_h for selected degrees of risk aversion).

Even for a conservative estimate of the willingness-to-pay parameter, ε_h, the additional benefit of completing markets is negligible for concave wealth profiles (less than 0.1% of total wealth). For all but highly convex wealth profiles, our estimate of willingness to pay $\bar{\varepsilon}$ is essentially zero when we have six options (i.e., κ_7). Looking at the change in $\bar{\varepsilon}$ or ε_h when we add a single option, we find that for all wealth profiles a single at-the-money option is very effective at reducing the willingness to pay.

As an example, when $\gamma = 0.8$ ($\lambda = 2$), a single at-the-money option reduces the willingness to pay (using ε_h) from 2.56% to 0.44% of total wealth. Even for the highly convex case ($\gamma = 0.4$, i.e., $\lambda = 4$), we are able to lower ε_h dramatically. Introducing six options reduces the costs of incomplete markets from over 10% of wealth to just 0.4% of wealth. This implies that an investor with \$10,000 would be willing to pay no more than \$40 to achieve complete markets (remember that ε_h is the *conservative* upper bound estimate of the willingness to pay).

Exhibit 8

Mean and Upper Bound on Willingness-to-Pay Parameter with Increasing Sequence of Option Menus

Mean Estimate: $\bar{\varepsilon}$ in %							
γ	κ_1	κ_2	κ_3	κ_4	κ_5	κ_6	κ_7
.4	10.559	8.003	0.656	0.176	0.190	0.131	0.008
.6	5.064	0.574	0.443	0.054	0.032	0.021	0.001
.8	2.75	0.266	0.278	0.026	0.009	0.005	0.000
1	1.127	0.133	0.149	0.012	0.003	0.001	0.000
2	0.011	0.001	0.001	0.000	0.000	0.000	0.000
3	0.030	0.002	0.002	0.000	0.000	0.000	0.000
5	0.034	0.003	0.002	0.000	0.000	0.000	0.000
Upper bound Estimate: ε_h in %							
.4	10.639	8.443	1.045	0.572	0.584	0.523	0.402
.6	5.149	0.826	0.678	0.297	0.275	0.263	0.244
.8	2.562	0.443	0.448	0.205	0.188	0.184	0.180
1	1.215	0.271	0.285	0.155	0.146	0.144	0.143
2	0.081	0.073	0.073	0.072	0.072	0.072	0.072
3	0.075	0.051	0.050	0.049	0.049	0.049	0.049
5	0.060	0.032	0.031	0.030	0.029	0.030	0.029

In addition, these highly convex wealth profiles represent unrealistically leveraged investors in a continuous trading environment. For example, if $\lambda = 4$, an investor with \$10,000 would initially be borrowing \$30,000 and investing \$40,000 in the risky asset.

We should stress that our least squares portfolio, which we are using for analytical convenience, is suboptimal (i.e., we are not directly optimizing expected utility). Given that we have found little economic significance for a suboptimal portfolio and are using upper bounds on the willingness-to-pay estimate, these results can be improved.

These results will hold for most reasonable parameter specifications. The only caveat is that the results will be weaker, the longer we make the investment horizon, and stronger, the shorter we make the horizon. We have chosen an investment horizon, $T = 1$ year, that is sufficiently long.

VI. CONCLUSION

There are a few important implications and applications of this analysis. There are also some generalizations that would be of further interest. One practical implication is that there is a straightforward mapping from investor beliefs and risk preferences to near-optimal option portfolios. Instead of frequent adjustments of stock and bond positions, an investor can use a *static* portfolio of options that closely matches the properties of the desired wealth profile. Rational no-bankruptcy continuous strategies that generate highly convex wealth profiles are easily generated with options, while the discrete-time leveraged portfolios have a chance of bankruptcy.

A related issue concerns the main assumption that the Black-Scholes model holds. There is nothing in the least squares analysis that relies on the Black-Scholes model. We could apply a different price vector for our options, derive an optimal complete market wealth profile, and then implement the least squares profile.

For example, assume an investor believes the volatility of the risky asset is 25% but there are options with implied volatilities of 30%. Using a dynamic optimizing trading strategy approach, the investor could implement (on as great a scale as possible) an arbitrage strategy of writing options, purchasing the stock, and borrowing money.

There are two reasons this is implausible. The first is that continuous trading strategies are impossible in the presence of transaction costs and in imperfectly liquid markets. This is the standard problem of hedging options with incomplete markets. What is perhaps a more important reason is that the investor doesn't know the volatility is 25%; rather, 25% is a proxy for some prior distribution the investor has.

In this example, the least squares approach would not lead to a nonsensical optimal strategy, but it will rather put relatively more weight on the "mispriced options." This approach is much simpler than trying to solve a complicated (albeit interesting) Bayesian decision problem. Determining the historical success of these least squares portfolios is an interesting empirical question. One could examine whether static option portfolios based on volatility smiles are more successful than alternative dynamic trading strategies.

There is a more abstract issue concerning the determinants of trading volume. With the introduction of option markets (as well as futures markets), the results here would imply a drastic *reduction* in trading volume. If we take away the speculative nature of markets, relaxing the assumptions would not come close to explaining trading volume. For example, introducing possibly path-dependent intermediate consumption would complicate the problem greatly, but would likely lead to the same result.

We could, as in the static hedging of exotic derivatives, find a static position of options with different maturities (see, for example, Derman, Ergener, and Kani [1994] and Carr, Ellis, and Gupta [1998]). The result would be the same; "liquidity" traders would never have to adjust their portfolio. This suggests we should interpret noise traders as irrational traders as opposed to liquidity traders. As shown in Milgrom and Stokey [1982] and Tirole [1982], rational expectations with common priors will imply no speculation-based trade.

If options can closely approximate complete markets, as my results suggest, casual observation of the huge amount of trading volume rejects common priors and rational expectations.

ENDNOTES

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[1]For a summary of derivatives pricing in incomplete markets, see Chapter 4 of Musiela and Rutkowski [1997].

[2]Haugh and Lo [2001] have independently produced similar results.

[3]Derivations of these standard results are available from the author.

[4]For log utility, i.e., $\gamma = 1$, expected utility is equal to $(r - \lambda(\mu - r) - \lambda^2\sigma^2/2)T$.

[5]For example, using w^*, if $\gamma = 0.5$, a fourth-order Taylor series around the first moment approximates expected utility as 0.5986 when the actual expected utility is 2.221.

REFERENCES

Ahn, Dong-Hyun, Jacob Boudoukh, Matthew Richardson, and Robert Whitelaw. "Optimal Risk Management Using Options." *Journal of Finance*, 54, 1 (1999), pp. 359-375.

Benninga, Simon, and Marshall Blume. "On the Optimality of Portfolio Insurance" *Journal of Finance*, 40, 5 (1985), pp. 1341-1352.

Black, Fischer, and Myron Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, 81, 3 (1973), pp. 637-54.

Breeden, Douglas, and Robert Litzenberger. "Prices of State-contingent Claims Implicit in Option Prices." *Journal of Business*, 51, 4 (1978), pp. 621-651.

Brennan, Michael J., and R. Solanki. "Optimal Portfolio Insurance." *Journal of Financial and Quantitative Analysis*, 16, 3 (1981), pp. 279-300.

Carr, Peter, Katrina Ellis, and Vishal Gupta. "Static Hedging of Exotic Options." *Journal of Finance*, 53, 3 (1998), pp. 1165-1190.

Carr, Peter, and Dilip Madan. "Optimal Positioning in Derivative Securities." *Quantitative Finance*, 1, 1 (2001), pp. 19-37.

Cox, James, and Chi Fu Huang. "Optimum Consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process." *Journal of Economic Theory*, 49 (1989), pp. 33-83.

Derman, Emanuel, Deniz Ergener, and Iraj Kani. "Forever Hedged." *Risk*, 7 (1994), pp. 139-145.

Haugh, Martin B., and Andrew W. Lo. "Asset Allocation and Derivatives." *Quantitative Finance*, 1, 1 (2001), pp. 45-72.

Leland, Hayne E. "Who Should Buy Portfolio Insurance?" *Journal of Finance*, 35, 2 (1980), pp. 581-594.

Merton, Robert. "Optimum Consumption and Portfolio Rules in a Continuous Time Model." *Journal of Economic Theory*, 3 (1971), pp. 373-413.

——. "Theory of Rational Option Pricing." *Bell Journal of Economics*, 4, 1 (1973), pp. 141-183.

Milgrom, P., and N. Stokey. "Information, Trade and Common Knowledge." *Journal of Economic Theory*, 26, 1 (1982), pp. 17-27.

Musiela, Marek, and Marek Rutkowski. *Martingale Methods in Financial Modeling*. Berlin and Heidelberg: Springer-Verlag, 1997.

Ross, Stephen. "Options and Efficiency." *Quarterly Journal of Economics*, 90, 1 (1976), pp. 75-89.

Tirole, J. "On the Possibility of Speculation under Rational Expectations." *Econometrica*, 50, 5 (1982), pp. 1163-1181.