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A Half Century of Returns on Levered and Unlevered Portfolios of Stocks, Bonds, and Bills, with and without Small Stocks*

This paper applies multiperiod portfolio theory to the construction and rebalancing of portfolios composed of U.S. stocks, corporate bonds, government bonds, and a risk-free asset, with small stocks included as a separate investment vehicle. Probability assessments are based on the past, joint empirical distribution. Our principal findings are (1) small stocks, while totally ignored at times, entered even the most risk-averse portfolios most of the time and (2) small stocks, when chosen, tended to replace common stocks, except in the 1970s and early 1980s, when they were primarily held in lieu of the risk-free asset.

I. Introduction

In two earlier papers (Grauer and Hakansson 1982; Grauer and Hakansson 1984), we applied the multiperiod portfolio theory of Mossin (1968), Hakansson (1971, 1974), Leland (1972), Ross (1974), and Huberman and Ross (1983) to the construction and rebalancing of portfolios composed of U.S. stocks, corporate bonds, government bonds, and a risk-free asset. Borrowing was ruled out in the first article, while margin purchases were permitted in the second. The probability distributions used were naively estimated from past realized returns in Ibbotson

* Presented at the Australian Graduate School of Management and at Macquarie University, both in Sydney, Southern Methodist University, Duke University, the University of Southern California, the London Graduate School of Business, and the annual meeting of the Western Financo Association in Scottsdale, Arizona. The authors thank the participants of these seminars, especially Michael Brennan and Ehud Ronn, for helpful comments. Financial support from the Social Sciences and Humanities Research Council of Canada and the Financial Research Foundation of Canada is gratefully acknowledged. The authors also wish to thank Changwoo Lee for research assistance and are greatly indebted to Frederick Shen for computational assistance.

(*Journal of Business*, 1986, vol. 59, no. 2, pt. 1)

0021-9398/86/5902-0005$01.50

and Sinquefield (1982) and in their Center for Research in Security Prices (CRSP) 1926–83 data base, and both annual and quarterly holding periods were employed from the mid-1930s forward. The results of both papers revealed that the gains from active diversification among the major asset categories were substantial, especially for the highly risk averse strategies. In addition, we found evidence of substantial use of, and gains from, margin purchases for the more risk-tolerant strategies from the mid-1930s to the mid-1960s.

In the present paper, small stocks are included as a separate investment vehicle. Thus the opportunity set is expanded to five categories: common stocks, government bonds, corporate bonds, small stocks, and a risk-free asset. The resulting sequential portfolio problem is solved both with and without leverage. When used, all borrowing is assumed to bear interest at the call money rate +1% and to be limited in size by applicable initial margin requirements. Our principal findings are (1) small stocks, while totally ignored at times, entered even the most risk-averse portfolios most of the time, (2) small stocks, when chosen, tended to replace common stocks, except in the 1970s and early 1980s, when they were primarily held in lieu of the risk-free asset, and (3) small stocks had a salutary effect on the geometric means of realized returns, especially from the mid-1930s to the early 1950s in the presence of margin purchases. In the 1934–47 subperiod with leverage, for example, small stocks raised the geometric mean return for the power .5 strategy from 9.09% to 25.95% while reducing variability.

II. Theory

Despite explosive development over three decades and extensive application to the construction of equity portfolios, modern portfolio theory has found relatively little use in the larger portfolio context, namely, the choice of the proportions to be held in the major categories of common stocks and in different types of bonds, money market instruments, real estate, and foreign securities. There are several reasons for this. First, extant portfolio theory, being principally based on the mean-variance model, is fundamentally single-period in nature, whereas the larger problem accents the multiperiod, sequential nature of investment decisions. On top of this, since the universe of interest extends well beyond common stocks, extant betas are too narrowly defined to be useful, and the appropriate betas are not easily estimated because of data problems concerning the market weights of bonds, for example. At the other extreme, continuous-time portfolio theory is somewhat intractable in a world of nontrivial transaction costs. Finally, extant models rely heavily on narrow classes of theoretical (stationary) return distributions, with limited ability to capture the richness of joint, real-world stochastic processes.

There is, however, a middle category of investment models, usually

classified under the heading of discrete-time multiperiod portfolio theory, which has been largely ignored in portfolio selection applications. This is so despite the fact that these models have a strong foundation in theory and lend themselves naturally to the problem of rebalancing portfolios over many periods (200 quarters in the present study). An additional virtue of these models is that they can handle return distributions of every conceivable shape and are unfazed by such things as nonstationary returns or distributions that are just plain unknown beyond the period just ahead.

To review, consider the simplest reinvestment problem, in which the market is perfect and returns are independent over time but otherwise arbitrary and not necessarily stationary. The investor has a preference function U_0 (with $U_0' > 0$, $U_0'' < 0$) defined on wealth w_0 at some terminal point (time 0). Let w_n denote the investor's wealth with n periods to go, r_{in} the return on asset i in period n, z_{in} the amount invested in asset i in period n (with $i = 1$ being the safe asset), and $U_n(w_n)$ the relevant (unknown) utility of wealth with n periods to go. At the end of period n (time $n - 1$), the investor's wealth is

$$w_{n-1}(z_n) = \sum_{i=2}^{M} (r_{in} - r_{1n})z_{in} + w_n(1 + r_{1n}),$$

where $z = (z_{2n}, \ldots, z_{Mn})$ and M is the number of securities.

Consider the portfolio problem with one period to go. The investor, with w_1 to invest, must now solve

$$\max_{z_1|w_1} E[U_0(w_0(z_1))] \equiv U_1(w_1).$$

Clearly, $U_1(w_1)$ represents the highest attainable expected utility level from capital level w_1 at time 1 and thus the "derived" utility of w_1. Employing the induced utility function $U_1(w_1)$, the portfolio problem with two periods to go becomes

$$U_2(w_2) \equiv \max_{z_2|w_2} E[U_1(w_1(z_2))].$$

Thus with n periods to go we obtain (the recursive equation)

$$U_n(w_n) = \max_{z_n|w_n} E[U_{n-1}(w_{n-1}(z_n))], \; n = 1, 2, \ldots$$

Examining the above system, it is evident that the induced utility of current wealth, $U_n(w_n)$, generally depends on "everything," namely, the terminal utility function U_0, the joint distribution functions of future returns, and future interest rates. There is, however, a special case in which $U_n(w_n)$ depends only on U_0: this occurs (Mossin 1968) if and only if $U_0(w_0)$ is isoelastic; that is, if and only if

$$U_0(w_0) = \frac{1}{\gamma} w^{\gamma}, \quad \gamma < 1.$$

(Note that for $\gamma = 0$, $U_0[w_0] = \ln w_0$.) The variable $U_n(w_n)$ is now a positive linear transformation of $U_0(w_n)$; that is, we can write

$$U_n(w_n) = \frac{1}{\gamma} w_n^{\gamma}. \tag{1}$$

For these preferences, the optimal investment policy $z^*_{n_\gamma}$ is proportional to wealth; that is,

$$z^*_{in_\gamma}(w_n) = x^*_{in_\gamma} w_n, \quad \text{all } i, \tag{2}$$

where the $x^*_{in_\gamma}$ are constants. It is also completely myopic since it only depends on U_0 and the current period's return structure and not on returns beyond the current period. Both of these properties hold only for family (1), which is also the only class of preferences exhibiting constant relative risk aversion.[1] Finally, (2) also implies that the utility of wealth relatives, $V_n(1 + r_n)$, is of the same form only for this family; that is,

$$U_n(w_n) = \frac{1}{\gamma} w_n^{\gamma} <=> V_n(1 + r_n) = \frac{1}{\gamma}(1 + r_n)^{\gamma}.$$

While the above properties are interesting, they are clearly rather special. However, the isoelastic family's influence extends far beyond its numbers. As shown by Leland (1972), Hakansson (1974), Ross (1974), and Huberman and Ross (1983), there is a very broad class of terminal utility functions $U_0(w_0)$ for which the induced utility functions U_n converge to an isoelastic function, that is, for which

$$U_n(w_n) \to \frac{1}{\gamma} w_n^{\gamma} \quad \text{for some } \gamma < 1. \tag{3}$$

Hakansson (1974) has also shown that (3) is usually accompanied by convergence in policy; that is,

$$z^*_n \to x^*_{n_\gamma} w_n.$$

Thus the objectives given by (1) are quite robust and encompass a broad variety of different goal formulations for investors with intermediate- to long-term investment horizons.[2] In particular, class (1) spans a continuum of risk attitudes all the way from risk neutrality ($\gamma = 1$) to infinite risk aversion ($\gamma = -\infty$).[3]

Having selected our model, we turn next to what we need to operate it. The major input to the model is an estimate of next period's joint return distribution for the various asset categories.[4] As in our previous

1. This measure is defined as $-wU''_n(w)/U'_n(w)$ and equals $1 - \gamma$ for the class (1).
2. The simple reinvestment formulation does ignore consumption of the course.
3. A plot of the functions $(1/\gamma)(1 + r)^{\gamma}$ for several values of γ was given in Grauer and Hakansson (1982, p. 42).
4. For a comprehensive overview of the issues and problems associated with the estimation of return distributions, see Bawa, Brown, and Klein (1979).

studies, we based this estimate on the so-called simple probability assessment approach. In this approach, the realized returns of the most recent n periods are recorded; each of the n joint realizations is then assumed to have probability $1/n$ of occurring in the coming period. Thus estimates are obtained on a moving basis and used in raw form without adjustment of any kind. Since the whole joint distribution is specified and used, there is no information loss; all moments—marginal and conditional, for example—and every last shred of correlation are implicitly taken into account. It may be noted that the empirical distribution of the past n periods is optimal if the investor has no information about the form and parameters of the true distribution but believes that this distribution went into effect n periods ago.[5] Consequently, we have, as a starting point, resisted all temptations to parameterize the input distributions.

III. Calculations

The model used can be summarized as follows. At the beginning of each period t, the investor chooses a portfolio, x_t, on the basis of some member, γ, of the family of utility functions for returns r given by

$$V(1 + r) = \frac{1}{\gamma}(1 + r)^{\gamma}. \tag{4}$$

This is equivalent to solving the following problem in each period t:

$$\begin{aligned} &\max_{x_t} E\left[\frac{1}{\gamma}(1 + \sum_i x_{it}r_{it} + x_{Lt}r_{Lt} + x_{Bt}r_{Bt}^{d})^{\gamma}\right] \\ &= \max_{x_t} \sum_s \pi_{ts} \frac{1}{\gamma}(1 + \sum_i i_t r_{its} + x_{Lt}r_{Lt} + x_{Bt}r_{Bt}^{d})^{\gamma} \end{aligned} \tag{5a}$$

subject to

$$x_{it} \geqslant 0, \quad x_{Lt} \geqslant 0, \quad x_{Bt} \leqslant 0, \qquad \text{all } i, t, \tag{6}$$

$$\sum_i x_{it} + x_{Lt} + x_{Bt} = 1, \quad \text{all } t, \tag{7}$$

$$\sum_i m_{it}x_{it} \leqslant 1, \quad \text{all } t, \tag{8}$$

$$\text{pr}(1 + \sum_i x_i r_{it} + x_{Lt}r_{Lt} + x_{Bt}r_{Bt}^{d} \geqslant 0) = 1, \tag{9}$$

where

$\gamma \leqslant 1$ = a parameter that remains fixed over time;
$x_t \equiv (x_{1t}, \ldots, x_{nt}, x_{Lt}, x_{Bt})$;
x_{it} = the amount invested in risky asset category i in period t as a fraction of own capital;

5. See Bawa et al. 1979, p. 100.

x_{Lt} = the amount invested in the risk-free asset in period t as a fraction of own capital;
x_{Bt} = the amount borrowed in period t as a fraction of own capital;
r_{it} = the anticipated return on asset category i in period t;
r_{Lt} = the return on the risk-free asset of period t;
r^d_{Bt} = the borrowing rate at the time of the decision at the beginning of period t;
m_{it} = the initial margin requirement for asset category i in period t expressed as a fraction; and
π_{ts} = the probability of event s in period t, in which case the random return r_{it} will assume the value r_{its}.

As noted, when γ equals zero, (4) reduces to the logarithmic utility function:

$$V = \ln(1 + r).$$

The equivalent of (5a) is then

$$\max_{x_t} E[\ln(1 + \sum_i x_{it}r_{it} + x_{Lt}r_{Lt} + x_{Bt}r^d_{Bt})]. \tag{5b}$$

Constraint (6) rules out short positions, and (7) is the budget constraint. Constraint (8) serves to limit borrowing (when desired) to the maximum permissible under the margin requirements that apply to the various asset categories. Finally, since

$$\sum_i x_{it}r_{it} + x_{Lt}r_{Lt} + x_{Bt}r^d_{Bt}$$

is the (ex ante) return distribution for portfolio x_t, the expressions in parentheses in (5a) and (5b) represent the (ex ante) wealth relative in period t.

On the basis of the probability estimation method described earlier, the (sequential) solution of the portfolio problem may be pictured as follows. Suppose quarterly revision is used. Then, at the beginning of quarter t, the portfolio problem (5a) or (5b) for that quarter uses the following inputs—the (observable) risk-free return for quarter t; the (observable) call money rate +1% at the beginning of quarter t; and the (observable) realized returns for common stocks, government bonds, corporate bonds, and small stocks for the previous n quarters. Each joint realization in quarters $t - n$ through $t - 1$ is given probability $1/n$ of occurring in quarter t.

With these inputs in place, the portfolio weights for the various asset categories and the proportion of assets borrowed are calculated by solving system (5a)–(9) (or [5b]–[9]) via nonlinear programming methods.[6] At the end of quarter t, the realized returns on stocks, gov-

6. The nonlinear programming algorithm employed is described in Best (1975).

ernment bonds, corporate bonds, and small stocks are observed, along with the realized borrowing rate r'_{Bt} (which may differ from the decision borrowing rate r^d_{Bt}).[7] Then, using the weights selected at the beginning of the quarter, the realized return on the portfolio chosen for quarter t is recorded. The cycle is then repeated in all subsequent quarters.[8]

All reported returns are gross of transaction costs and taxes and assume that the investor in question had no influence on prices. There are several reasons for this approach. First, since this is one of the first studies in this area, we wish to keep the complications to a minimum. Second, the Ibbotson and Sinquefield series used as inputs and for comparisons also exclude transaction costs (for reinvestment of interest and dividends) and taxes. Third, many investors are tax-exempt, and various techniques are available for keeping transaction costs low. Finally, since the proper treatment of these items is nontrivial, they are better left to a later study.

IV. Data

The data used to estimate the probabilities of next period's returns and to calculate each period's realized returns on risky assets are the total monthly and annual returns on common stocks, long-term government bonds, long-term corporate bonds, and small stocks for 1926–83 in the Ibbotson and Sinquefield CRSP data file; both dividends and capital appreciation are taken into account. The risk-free asset used for quarterly revision was assumed to be 90-day U.S. Treasury bills maturing at the end of the quarter; we used the *Survey of Current Business* and the *Wall Street Journal* as sources. In the annual portfolio revision case, the risk-free return was obtained from the yield, as of the beginning of the year, on that U.S. government obligation (note or bond) that matured on the date closest to the end of the year in question; we obtained the 1926–76 data privately from Roger Ibbotson and the remainder from the *Wall Street Journal*.

Margin requirements for stocks were obtained from the *Federal Reserve Bulletin*. Initial margins were set at 10% for government bonds and at 35% for corporate bonds. These levels are on the conservative side and are designed to compensate for the absence of maintenance requirements.[9]

7. The realized borrowing rate r'_{Bt} was calculated as a monthly average.

8. Note that, if $n = 32$ under quarterly revision, then the first quarter for which a portfolio can be selected is the first quarter of 1934 since the period 1926–33 is required to develop the estimated return distributions used for that quarter's portfolio choice.

9. There was no practical way to take maintenance margins into account in our programs. In any case, it is evident from the results that they would come into play only for the more risk tolerant strategies, and even for them only occasionally, and that the net effect would be relatively neutral.

TABLE 1 Composition of Fixed Weight Portfolios

	Proportion in:				
	Stocks	Government Bonds	Corporate Bonds	Small Stocks	Risk-free Asset
Portfolio A	...	.40	.40	...	.20
Portfolio B	.20	.30	.30	...	.20
Portfolio C	.45	.20	.25	...	.10
Portfolio D	.55	.15	.20	...	.10
Portfolio E	.65	.10	.15	...	.10
Portfolio F	.75	...	.15	...	.10
Portfolio G	.90	.05	.05	...	...
Portfolio H	.70	.05	.05	.20	...
Portfolio I	.50	.05	.05	.40	...
Portfolio J	.30	.05	.05	.60	...

As noted, the borrowing rate was assumed to be the call money rate +1%; for decision purposes (but not for rate of return calculations), the applicable beginning of period rate, r^d_{Bt}, was viewed as persisting throughout the period and thus as risk free.[10] For 1934–76, the call money rates were obtained from the *Survey of Current Business;* for later periods, the *Wall Street Journal* was the source.

V. Results

Because of space limitations, only a portion of the results can be reported here. However, tables 2–6 below and figures 1 and 2 provide a fairly representative sample of our findings.

For comparison, we have calculated and included the returns for 10 fixed-weight portfolios. The compositions of these fixed-weight portfolios are shown in table 1.

Portfolio Returns: The No-Leverage Case

Table 2 compares the geometric means of the annual returns, along with the standard deviations of $\ln(1 + r_t)$, with and without small stocks for the subperiods 1936–47, 1948–65, and 1966–83 and for the full 1936–83 period, under quarterly portfolio revision (with a 40-quarter estimating period) in the absence of leverage. Note that the realized returns without small stocks differed substantially between the three subperiods. For example, the highest geometric mean was 7.75% for 1936–47, 15.69% for 1948–65, 7.38% for 1966–83, and 10.11% for

10. It may be noted that the minimum differential between the borrowing and the lending rate was 1.4% (1935), while the maximum was 5.0% (1981). The differential was 2% or less through 1952 and again in 1964–65. For most of the 1950s, 1960s, and 1970s, and again in 1983, the differential was between 2% and 3%; the exceptions were 1974 and 1980, when it was 4.5%, and, as already mentioned, 1981.

the whole 1936–83 period. In contrast, the realized returns were much more uniform with small stocks included: the highest geometric means were then 15.79% for 1936–47, 15.27% for 1948–65, 14.97% for 1966–83, and 14.54% for the full period.

For the 1936–47 subperiod, table 2 shows that the presence of the small stock category increased both the standard deviation and the geometric mean for each of the 16 strategies. These increases are relatively small for powers −75 through −10 but quite substantial for the others, especially powers −1 and up. The situation for 1966–83 is similar but less dramatic. In the 1948–65 subperiod, on the other hand, the presence of the small stock category again led to uniformly higher standard deviations but also to uniformly lower geometric means. The differences are quite small, however. Standard deviations also increased across the board for the full period, as did the geometric means.

Portfolio Returns: The Leverage Case

Table 3 shows the geometric means of the annual returns, and the standard deviations of $\ln(1 + r_t)$, with and without small stocks for the leverage case. This table is based on quarterly portfolio revision and a 32-quarter estimating period. Again it is evident that the presence of small stocks tended to increase both the geometric means and the standard deviations of returns. This was uniformly so in the 1966–83 subperiod as well as for the full 1934–83 period. This pattern was broken in only two cases: in the 1934–47 subperiod, when powers 0, .25, and .50 achieved lower standard deviations in the presence of the small stock category, and in the 1948–65 subperiod, when powers −3 through 1 attained higher geometric means in the absence of small stocks.

Table 3 also shows that small stocks had a rather uneven effect across the three subperiods. In the 1934–47 subperiod, they provided an enormous lift to most geometric means with very little effect on standard deviations. On the other hand, their effect in the 1948–65 subperiod was minimal. The effect in the last subperiod was fairly modest, while the effect of small stocks on returns over the full 50 years was rather substantial.

Portfolio Compositions

An examination of the strategies used reveals some interesting patterns. In the no-borrowing case, small stocks were ignored by all the active strategies, including the risk-neutral one, from the third quarter of 1955 through the first quarter of 1962, in the third quarter of 1963, and again in the first quarter of 1979. The rest of the time, however, small stocks tended to be an important investment outlet, mostly at the expense of common stocks. Even the ultraconservative power −75

TABLE 2 **Comparison of Geometric Means and Standard Deviations of Annual Returns with and without Small Stocks, 1936–83: No-Leverage Case (Quarterly Portfolio Revision, 40-Quarter Estimating Period)**

A. 1936–47

	Without Small Stocks		With Small Stocks	
Portfolio	Geometric Mean	Standard Deviation*	Geometric Mean	Standard Deviation*
Common stocks	6.71	22.28	6.71	22.28
Government bonds	3.46	3.62	3.46	3.62
Corporate bonds	3.25	2.26	3.25	2.26
Small stocks	...	...	14.65	41.68
Risk free	.24	.18	.24	.18
Power 1	4.78	23.74	14.65	41.68
Power .75	5.19	20.98	12.64	41.72
Power .50	6.95	18.21	14.29	35.77
Power .25	7.75	14.92	15.79	30.19
Power 0	7.72	12.92	15.48	26.11
Power −1	6.18	8.31	9.88	14.29
Power −2	5.27	6.04	7.67	9.87
Power −3	4.79	4.96	6.54	7.71
Power −5	4.31	3.96	5.43	5.63
Power −7	4.03	3.39	4.94	4.66
Power −10	3.73	2.99	4.48	3.81
Power −15	3.53	2.68	4.03	3.24
Power −20	3.11	2.49	3.50	2.92
Power −30	2.52	2.13	2.78	2.51
Power −50	2.09	1.94	2.28	2.27
Power −75	1.74	1.65	2.07	2.14
Portfolio A	2.74	2.19	2.74	2.19
Portfolio B	3.78	5.45	3.78	5.45
Portfolio C	5.13	10.58	5.13	10.58
Portfolio D	5.47	12.63	5.47	12.63
Portfolio E	5.75	14.71	5.75	14.71
Portfolio F	5.97	16.76	5.97	16.76
Portfolio G	6.59	20.13	6.59	20.13
Portfolio H	...	...	8.75	23.70
Portfolio I	...	...	10.68	27.45
Portfolio J	...	...	12.36	31.32

TABLE 2 (*Continued*)

B. 1948–65

Portfolio	Without Small Stocks		With Small Stocks	
	Geometric Mean	Standard Deviation*	Geometric Mean	Standard Deviation*
Common stocks	15.67	14.54	15.67	14.54
Government bonds	1.94	4.96	1.94	4.96
Corporate bonds	2.52	4.11	2.52	4.11
Small stocks	...	...	15.61	19.36
Risk free	2.29	.94	2.29	.94
Power 1	15.67	14.54	14.03	17.12
Power .75	15.67	14.54	14.64	17.14
Power .50	15.67	14.54	14.70	17.16
Power .25	15.67	14.54	14.83	17.04
Power 0	15.67	14.54	14.90	16.98
Power −1	15.69	14.54	15.27	16.21
Power −2	15.52	14.48	15.25	15.54
Power −3	15.18	14.42	14.93	14.72
Power −5	13.65	14.45	13.24	14.60
Power −7	11.85	13.58	11.59	13.79
Power −10	9.48	10.91	9.35	11.09
Power −15	7.27	7.60	7.15	7.81
Power −20	6.12	5.68	6.06	5.86
Power −30	4.98	3.73	4.95	3.87
Power −50	3.92	2.19	3.91	2.31
Power −75	3.39	1.46	3.38	1.56
Portfolio A	2.26	3.56	2.26	3.56
Portfolio B	5.00	3.02	5.00	3.02
Portfolio C	8.41	5.99	8.41	5.99
Portfolio D	9.75	7.46	9.75	7.46
Portfolio E	11.09	9.01	11.09	9.01
Portfolio F	12.44	10.63	12.44	10.63
Portfolio G	14.37	12.90	14.37	12.90
Portfolio H	...	...	14.44	13.55
Portfolio I	...	...	14.47	14.41
Portfolio J	...	...	14.46	15.46

TABLE 2 (*Continued*)

C. 1966–83

Portfolio	Without Small Stocks		With Small Stocks	
	Geometric Mean	Standard Deviation*	Geometric Mean	Standard Deviation*
Common stocks	7.62	17.03	7.62	17.03
Government bonds	4.13	9.91	4.13	9.91
Corporate bonds	4.90	10.82	4.90	10.82
Small stocks	...	...	16.11	27.78
Risk free	7.37	2.86	7.37	2.86
Power 1	7.38	12.03	14.97	27.29
Power .75	6.65	11.33	13.06	25.72
Power .50	6.40	11.09	12.85	24.50
Power .25	6.34	10.85	11.86	24.30
Power 0	6.35	10.39	11.48	23.98
Power −1	6.84	9.07	10.24	22.37
Power −2	7.08	7.79	10.01	19.65
Power −3	7.18	6.22	9.89	15.98
Power −5	7.28	4.58	9.23	10.88
Power −7	7.31	3.86	8.84	8.35
Power −10	7.33	3.37	8.49	6.33
Power −15	7.35	3.05	8.17	4.74
Power −20	7.36	2.94	7.99	4.00
Power −30	7.36	2.86	7.80	3.36
Power −50	7.37	2.84	7.63	2.99
Power −75	7.37	2.83	7.55	2.88
Portfolio A	5.20	8.27	5.20	8.27
Portfolio B	6.03	7.96	6.03	7.96
Portfolio C	6.61	10.08	6.61	10.08
Portfolio D	6.92	10.99	6.92	10.99
Portfolio E	7.20	12.06	7.20	12.06
Portfolio F	7.48	13.38	7.48	13.38
Portfolio G	7.44	15.62	7.44	15.62
Portfolio H	...	...	9.41	16.86
Portfolio I	...	...	11.25	18.73
Portfolio J	...	...	12.95	21.06

TABLE 2 (*Continued*)

D. 1936–83

Portfolio	Without Small Stocks		With Small Stocks	
	Geometric Mean	Standard Deviation*	Geometric Mean	Standard Deviation*
Common stocks	10.34	17.64	10.34	17.64
Government bonds	3.14	6.96	3.14	6.96
Corporate bonds	3.59	7.12	3.59	7.12
Small stocks	...	...	15.56	28.66
Risk free	3.64	3.40	3.64	3.40
Power 1	9.74	16.69	14.54	27.97
Power .75	9.57	15.63	13.54	27.45
Power .50	9.93	14.65	13.90	24.97
Power .25	10.11	13.65	13.94	23.11
Power 0	10.11	13.03	13.75	21.78
Power −1	9.91	11.77	12.01	18.14
Power −2	9.70	11.10	11.35	16.06
Power −3	9.50	10.56	10.89	13.93
Power −5	8.86	9.97	9.74	11.63
Power −7	8.15	9.12	8.87	10.24
Power −10	7.21	7.33	7.79	8.11
Power −15	6.35	5.33	6.74	5.92
Power −20	5.82	4.33	6.13	4.79
Power −30	5.24	3.52	5.46	3.82
Power −50	4.73	3.13	4.88	3.30
Power −75	4.44	3.06	4.59	3.16
Portfolio A	3.48	5.67	3.48	5.67
Portfolio B	5.08	5.82	5.08	5.82
Portfolio C	6.91	8.80	6.91	8.80
Portfolio D	7.60	10.19	7.60	10.19
Portfolio E	8.27	11.70	8.27	11.70
Portfolio F	8.93	13.34	8.93	13.34
Portfolio G	9.77	15.93	9.77	15.93
Portfolio H	...	...	11.10	17.50
Portfolio I	...	...	12.30	19.51
Portfolio J	...	...	13.37	21.84

* Standard deviation is for the variables $\ln(1 + r_t)$.

TABLE 3 **Comparison of Geometric Means and Standard Deviations of Annual Returns with and without Small Stocks, 1934–83: Leverage Case (Quarterly Portfolio Revision, 32-Quarter Estimating Period)**

A. 1934–47

	Without Small Stocks		With Small Stocks	
Portfolio	Geometric Mean	Standard Deviation*	Geometric Mean	Standard Deviation*
Common stocks	8.59	22.42	8.59	22.42
Government bonds	4.03	3.72	4.03	3.72
Corporate bonds	4.42	3.61	4.42	3.61
Small stocks	...	...	16.98	38.76
Risk free	.24	.17	.24	.17
Power 1	−5.86	56.91	11.57	88.59
Power .75	5.30	40.76	19.95	48.99
Power .50	9.09	38.63	25.95	35.10
Power .25	9.19	34.75	23.43	31.01
Power 0	10.67	31.30	22.58	28.25
Power −1	10.19	22.95	17.47	24.83
Power −2	9.32	17.86	13.94	20.89
Power −3	8.83	14.91	12.33	17.41
Power −5	7.17	10.36	9.44	11.87
Power −7	6.24	7.67	7.85	8.67
Power −10	5.53	5.59	6.58	6.36
Power −15	4.85	4.43	5.69	5.06
Power −20	3.92	3.60	4.55	4.18
Power −30	2.99	2.64	3.39	3.02
Power −50	2.42	2.09	2.63	2.31
Power −75	1.86	1.64	2.29	1.98
Portfolio A	3.44	2.72	3.44	2.72
Portfolio B	4.68	5.60	4.68	5.60
Portfolio C	6.40	10.74	6.40	10.74
Portfolio D	6.83	12.80	6.83	12.80
Portfolio E	7.22	14.88	7.22	14.88
Portfolio F	7.57	16.96	7.57	16.96
Portfolio G	8.38	20.28	8.38	20.28
Portfolio H	...	...	10.63	23.01
Portfolio I	...	...	12.64	26.09
Portfolio J	...	...	14.40	29.40

TABLE 3 (*Continued*)

B. 1948–65

Portfolio	Without Small Stocks		With Small Stocks	
	Geometric Mean	Standard Deviation*	Geometric Mean	Standard Deviation*
Common stocks	15.67	14.54	15.67	14.54
Government bonds	1.94	4.96	1.94	4.96
Corporate bonds	2.52	4.11	2.52	4.11
Small stocks	...	...	15.61	19.36
Risk free	2.29	.94	2.29	.94
Power 1	24.70	24.79	23.40	28.61
Power .75	24.70	24.79	22.92	28.01
Power .50	24.70	24.79	23.31	27.60
Power .25	24.70	24.79	23.54	27.49
Power 0	24.70	24.79	23.75	27.41
Power −1	22.79	25.18	21.93	27.25
Power −2	20.17	23.67	19.49	24.30
Power −3	17.97	22.31	17.79	22.82
Power −5	15.04	19.35	15.08	19.73
Power −7	12.38	16.07	12.49	16.28
Power −10	9.96	12.82	10.09	13.00
Power −15	8.00	9.45	8.17	9.62
Power −20	6.73	7.16	6.81	7.37
Power −30	5.36	4.71	5.39	4.88
Power −50	4.17	2.73	4.19	2.87
Power −75	3.56	1.77	3.57	1.90
Portfolio A	2.26	3.56	2.26	3.56
Portfolio B	5.00	3.02	5.00	3.02
Portfolio C	8.41	5.99	8.41	5.99
Portfolio D	9.75	7.46	9.75	7.46
Portfolio E	11.09	9.01	11.09	9.01
Portfolio F	12.44	10.63	12.44	10.63
Portfolio G	14.37	12.90	14.37	12.90
Portfolio H	...	...	14.44	13.55
Portfolio I	...	...	14.47	14.41
Portfolio J	...	...	14.46	15.46

TABLE 3 (*Continued*)

C. 1966–83

Portfolio	Without Small Stocks		With Small Stocks	
	Geometric Mean	Standard Deviation*	Geometric Mean	Standard Deviation*
Common stocks	7.62	17.03	7.62	17.03
Government bonds	4.13	9.91	4.13	9.91
Corporate bonds	4.90	10.82	4.90	10.82
Small stocks	...	...	16.11	27.78
Risk free	7.37	2.86	7.37	2.86
Power 1	9.22	15.82	13.26	33.56
Power .75	10.16	14.98	11.80	34.55
Power .50	9.91	13.58	10.11	32.42
Power .25	9.74	11.93	11.54	31.12
Power 0	9.25	11.44	12.17	30.16
Power −1	9.32	10.42	10.87	25.07
Power −2	8.96	8.68	10.51	19.69
Power −3	8.41	7.24	10.33	16.11
Power −5	8.05	5.46	9.54	11.13
Power −7	7.89	4.58	9.08	8.61
Power −10	7.75	3.92	8.66	6.58
Power −15	7.64	3.45	8.29	4.97
Power −20	7.58	3.25	8.08	4.21
Power −30	7.51	3.07	7.86	3.53
Power −50	7.46	2.96	7.67	3.11
Power −75	7.43	2.92	7.58	2.97
Portfolio A	5.20	8.27	5.20	8.27
Portfolio B	6.03	7.96	6.03	7.96
Portfolio C	6.61	10.08	6.61	10.08
Portfolio D	6.92	10.99	6.92	10.99
Portfolio E	7.20	12.06	7.20	12.06
Portfolio F	7.48	13.38	7.48	13.38
Portfolio G	7.44	15.62	7.44	15.62
Portfolio H	...	...	9.41	16.86
Portfolio I	...	...	11.25	18.73
Portfolio J	...	...	12.95	21.06

TABLE 3 (*Continued*)

D. 1934–83

Portfolio	Without Small Stocks		With Small Stocks	
	Geometric Mean	Standard Deviation*	Geometric Mean	Standard Deviation*
Common stocks	10.73	17.84	10.73	17.84
Government bonds	3.31	6.88	3.31	6.88
Corporate bonds	3.90	7.14	3.90	7.14
Small stocks	...	...	16.17	28.22
Risk free	3.51	3.40	3.51	3.40
Power 1	9.89	35.87	16.32	52.70
Power .75	13.74	28.00	17.98	36.62
Power .50	14.78	26.70	19.09	31.49
Power .25	14.75	24.96	19.05	29.63
Power 0	14.99	23.59	19.13	28.44
Power −1	14.25	20.67	16.61	25.62
Power −2	12.97	18.09	14.64	21.60
Power −3	11.88	16.31	13.53	18.95
Power −5	10.26	13.39	11.47	14.88
Power −7	9.01	10.87	9.95	11.87
Power −10	7.91	8.57	8.58	9.28
Power −15	6.98	6.48	7.51	6.97
Power −20	6.24	5.19	6.63	5.61
Power −30	5.45	3.97	5.71	4.23
Power −50	4.84	3.27	4.99	3.41
Power −75	4.45	3.12	4.63	3.16
Portfolio A	3.64	5.62	3.64	5.62
Portfolio B	5.28	5.81	5.28	5.81
Portfolio C	7.19	8.89	7.19	8.89
Portfolio D	7.91	10.31	7.91	10.31
Portfolio E	8.59	11.85	8.59	11.85
Portfolio F	9.27	13.50	9.27	13.50
Portfolio G	10.16	16.12	10.16	16.12
Portfolio H	...	...	11.54	17.51
Portfolio I	...	...	12.79	19.39
Portfolio J	...	...	13.90	21.60

* Standard deviation is for the variables $\ln(1 + r_t)$.

strategy, for example, had an uninterrupted presence in small stocks from the beginning of 1942 through the first quarter of 1952, from the fourth quarter of 1963 through the second quarter of 1974, and again from the third quarter of 1980 on.

Table 4 gives a comparison of the quarter-by-quarter portfolio compositions and returns for powers 0 and −15 with and without small stocks in the leverage case. For each power, the first five columns show the proportions invested in common stocks, government bonds, corporate bonds, small stocks, and the risk-free asset. The sixth column reports the fraction in borrowing and the last column the portfolio's return for the quarter. This is then repeated with small stocks excluded.

Turning first to the logarithmic investor, we note that he stayed away from small stocks in the first quarter of 1934, from the beginning of 1953 through mid-1960, during the second quarter of 1961, and from the beginning of 1974 through mid-1977 with the exception of three quarters. During these periods, portfolio holdings and returns were thus unaffected by the opportunity to invest in small stocks. In the 1930s, small stocks tended to replace holdings in governments and, to a lesser extent, common stocks. In the 1940s and early 1950s, and again in the 1960s and early 1970s, it was principally common stocks that were being crowded out by small stocks for the logarithmic investor. In the late 1970s through the third quarter of 1982, it was primarily the risk-free asset that gave way for small stocks, followed by common stocks one more time.

The logarithmic investor's use of leverage was remarkably similar with and without the small stock category. Its presence did not always increase leverage; while they markedly did so in 1982 and 1983, small stocks sharply reduced the use of leverage in 1946 and 1947.

Turning now to the rather conservative power −15 strategy, we first observe that this investor was a heavy and repeated user of the risk-free asset beginning with the third quarter of 1962. Prior to the second quarter of 1951, however, the risk-free asset was practically ignored. In fact, from the beginning of 1940 through the third quarter of 1946, the power −15 strategy was a consistent user of leverage, with holdings of government bonds on margin.

Small stocks first entered the power −15 portfolio in the last quarter of 1939, growing gradually in importance to a 19% allocation in the first quarter of 1950, then declining and disappearing as of mid-1951, 47 quarters later. As table 4 shows, these holdings were primarily at the expense of common stocks. After a nearly total hiatus during the rest of the 1950s, small stocks reappeared for 55 consecutive quarters beginning with the second quarter of 1960, reaching a maximum allocation of 30% in the second quarter of 1962, in a streak that lasted through 1973. These positions were also essentially at the expense of common

stocks, except in the second half of 1973, when they replaced the risk-free asset. When the power −15 investor returned to small stocks at the beginning of 1978, they first replaced principally investments in the risk-free asset and, beginning in the last quarter of 1982, holdings in common stocks.

Many investigators have noted that small stocks have, over long periods, produced what is usually referred to as excess risk-adjusted returns (e.g., Banz 1981; and Reinganum 1981). This suggests that the empirical (past) distribution used in our model should, at a minimum, have found small stocks a consistently attractive outlet. While this was obviously the case most of the time, small stocks were, as noted, totally ignored in the middle and late 1950s and for intervals in the middle 1970s even by the more risk-tolerant strategies.

Other Results

We also examined the case in which portfolios were revised only once a year rather than quarterly, with 8- and 10-year estimating periods. As in the previous studies, the differences in returns and portfolio compositions were fairly small. The use of 24-quarter and 40-quarter estimating periods also led to only minor differences in the results.

VI. Tests

The returns under quarterly reinvestment with leverage in the presence of small stocks over the 1934–47 subperiod are shown as round dots in figure 1, while the returns over the full 1934–83 period are similarly depicted in figure 2. The same figures also plot the returns for the higher powers in the absence of small stocks (see diamonds) as well as for the fixed-weight portfolios in table 1 (see triangles). The geometric means of the annual returns are measured on the vertical axis and the standard deviations of $\ln(1 + r_t)$ on the horizontal.

The attained returns reflect clearly the benefits from diversification among the five (or four) asset categories (represented in the figures by the square points RL [risk-free asset], GB [government bonds], CB [corporate bonds], CS [common stocks], and SS [small stocks]). In other words, the portfolios selected by the model have enabled investors to travel in a distinctly "northwesterly" direction, confirming the results of the previous studies.

Recall that terminal wealth w_0 in terms of beginning wealth w_n is given by

$$\begin{aligned} w_0 &= w_n(1 + r_n)(1 + r_{n-1}) \ldots (1 + r_1) \\ &= w_n \exp\left[\sum_{t=1}^{n} \ln(1 + r_t)\right]. \end{aligned}$$

TABLE 4 Portfolio Composition and Realized Returns for Powers 0 and −15 with and without Small Stocks, 1934–83 (Quarterly Revision, with Leverage, 32-Quarter Estimating Period)

	Power 0												
	With Small Stocks							Without Small Stocks					
	Inv. Fractions							Inv. Fractions					
Period	CS	GB	CB	SS	RL	B	r_t	CS	GB	CB	RL	B	r_t
1934:1	.29	4.74	1.34	...	...	−5.37	33.21	.29	4.74	1.34	...	−5.37	33.21
1934:2	.32	5.23	1.16	.04	...	−5.75	15.46	.38	5.29	1.13	...	−5.80	15.52
1934:3	.04	5.78	1.10	.14	...	−6.07	−17.50	.25	5.98	1.01	...	−6.23	−17.65
1934:4	...	4.39	1.53	.13	...	−5.05	18.89	.14	4.61	1.46	...	−5.21	18.56
1935:1	...	4.26	1.50	.11	...	−4.87	14.94	.03	4.48	1.54	...	−5.05	17.56
1935:2	...	4.82	1.42	.05	...	−5.29	7.95	...	4.92	1.45	...	−5.37	7.62
1935:3	...	6.28	1.03	.03	...	−6.34	−5.75	...	6.34	1.05	...	−6.39	−6.27
1935:4	...	5.06	1.31	.08	...	−5.44	10.30	...	5.23	1.36	...	−5.59	7.91
1936:1	...	2.73	1.85	.18	...	−3.76	15.90	.10	3.02	1.87	...	−3.98	10.90
1936:2	...	3.34	1.59	.24	...	−4.18	−.71	.11	3.79	1.64	...	−4.53	4.03
1936:3	...	3.09	1.75	.17	...	−4.02	7.84	.08	3.42	1.77	...	−4.28	5.98
1936:4	...	4.90	1.25	.16	...	−5.31	15.04	.04	5.29	1.30	...	−5.62	12.61
1937:1	...	5.27	1.09	.17	...	−5.52	−18.69	...	5.69	1.23	...	−5.92	−24.08
1937:2	...	5.84	.80	.25	...	−5.88	−4.75	...	6.32	1.05	...	−6.37	3.19
1937:3	...	4.98	1.15	.18	...	−5.31	−2.24	...	5.33	1.33	...	−5.66	1.95
1937:4	...	5.60	1.03	.14	...	−5.77	6.40	...	5.88	1.18	...	−6.06	12.40
1938:1	...	4.03	1.50	.18	...	−4.71	−5.63	...	4.54	1.56	...	−5.10	.11
1938:2	...	4.47	1.53	.04	...	−5.05	15.33	...	4.59	1.54	...	−5.14	13.11
1938:3	...	4.35	1.35	.24	...	−4.93	2.90	...	5.07	1.41	...	−5.48	2.75
1938:4	...	4.13	1.35	.28	...	−4.77	12.11	...	4.98	1.43	...	−5.42	7.97
1939:1	...	4.16	1.22	.39	...	−4.77	−1.89	.15	5.15	1.22	...	−5.52	9.92
1939:2	...	6.23	.86	.19	...	−6.28	14.67	...	6.80	.92	...	−6.71	15.88
1939:3	...	6.32	.78	.24	...	−6.34	−30.65	.04	7.02	.80	...	−6.86	−48.79
1939:4	...	5.59	.47	.69	...	−5.75	32.70	.79	5.67	.33	...	−5.80	37.90
1940:1	...	7.28	...	.68	...	−6.96	20.44	.67	7.32	...	...	−6.99	9.62
1940:2	...	7.04	...	.74	...	−6.78	−28.20	.75	7.01	...	...	−6.76	−21.87
1940:3	...	6.84	...	.79	...	−6.63	15.40	1.03	5.89	...	...	−5.92	16.87
1940:4	...	7.21	.03	.67	...	−6.91	20.70	.75	6.25	.21	...	−6.22	16.92
1941:1	...	6.84	...	.79	...	−6.63	−8.89	.88	6.48	...	...	−6.36	−12.80
1941:2	...	6.74	...	.82	...	−6.55	12.38	1.00	6.00	...	...	−6.00	11.50
1941:3	...	7.88	...	.53	...	−7.41	6.58	.48	6.71	.39	...	−6.58	1.65
1941:4	...	7.45	...	.64	...	−7.09	−22.65	.75	7.01	...	...	−6.76	−17.81
1942:1	...	7.98	...	.50	...	−7.49	14.96	.45	8.21	...	...	−7.66	7.46
1942:2	...	8.11	...	.47	...	−7.58	−.10	.41	8.13	.06	...	−7.61	2.65
1942:3	...	7.86	...	.54	...	−7.39	12.18	.60	7.60	...	...	−7.20	5.74
1942:4	...	7.56	...	.61	...	−7.17	5.12	.61	7.56	...	...	−7.17	6.86
1943:1	...	7.43	...	.64	...	−7.07	41.32	.75	6.99	...	...	−6.75	14.04
1943:2	...	6.28	...	.93	...	−6.21	23.69	1.12	5.53	...	...	−5.65	12.80
1943:3	...	6.16	...	.96	...	−6.12	−7.91	1.00	5.98	...	...	−5.99	−2.25
1943:4	...	6.61	...	.85	...	−6.46	−.82	.81	6.77	...	...	−6.58	−3.29
1944:1	...	6.93	...	.77	...	−6.69	15.42	.69	7.25	...	...	−6.93	4.74
1944:2	...	7.04	...	.74	...	−6.78	11.72	.68	7.27	...	...	−6.95	6.68
1944:3	...	6.45	...	.89	...	−6.34	1.72	.78	6.88	...	...	−6.66	1.60
1944:4	...	6.66	...	.84	...	−6.49	12.71	.72	7.12	...	...	−6.84	5.99
1945:1	...	6.68	...	.83	...	−6.51	16.37	.67	6.39	.27	...	−6.32	14.14
1945:2	...	7.08	...	.58	...	−6.66	40.15	.39	8.07	...	...	−7.45	32.01
1945:3	...	5.79	...	.56	...	−5.36	.54	.27	7.99	...	...	−7.26	−1.76
1945:4	...	4.76	...	.70	...	−4.46	33.27	.38	7.16	...	...	−6.54	30.68
1946:1	...	2.83	...	.96	...	−2.79	11.22	.47	6.44	...	...	−5.91	3.77
1946:2	...	1.31	...	.87	...	−1.18	5.38	.37	6.26	...	...	−5.64	−6.59
1946:3	...	1.49	...	.85	...	−1.34	−26.30	.39	6.14	...	...	−5.53	−19.88
1946:4	...	.75	...	.93	...	−.67	1.81	.50	5.00	...	...	−4.50	7.38

TABLE 4 (*Continued*)

Power −15

With Small Stocks							Without Small Stocks					
Inv. Fractions							Inv. Fractions					
CS	GB	CB	SS	RL	B	r_t	CS	GB	CB	RL	B	r_t
...	.15	.83	...	.02	...	5.83	...	.15	.83	.02	...	5.83
...	.15	.85	...	...	...	3.52	...	.15	.85	...	...	3.52
...	.14	.86	...	...	...	−.04	...	.14	.86	...	...	−.04
...	.08	.92	...	...	...	3.36	...	.08	.92	...	...	3.36
...	.05	.95	...	...	...	3.95	...	.05	.95	...	...	3.95
...	...	1.00	...	...	...	2.68	...	...	1.00	...	...	2.68
...	...	1.00	...	...	...	.69	...	...	1.00	...	...	.69
...	...	1.00	...	...	...	1.95	...	...	1.00	...	...	1.95
...	...	1.00	...	...	...	2.20	...	...	1.00	...	...	2.20
...	...	1.00	...	...	...	1.49	...	...	1.00	...	...	1.49
...	...	1.00	...	...	...	1.46	...	...	1.00	...	...	1.46
...	...	1.00	...	...	...	1.44	...	...	1.00	...	...	1.44
...	...	1.00	...	...	...	−1.36	...	...	1.00	...	...	−1.36
...	.05	.95	...	...	...	1.58	...	.05	.95	...	...	1.58
...	...	1.00	...	...	...	.47	...	...	1.00	...	...	.47
...	...	1.00	...	...	...	2.02	...	...	1.00	...	...	2.02
...	...	1.00	...	...	...	−.39	...	...	1.00	...	...	−.39
...	...	1.00	...	...	...	2.45	...	...	1.00	...	...	2.45
...	.01	.99	...	...	...	1.55	...	.01	.99	...	...	1.55
...	.03	.97	...	...	...	2.38	...	.03	.97	...	...	2.38
...	...	1.00	...	...	...	1.08	...	...	1.00	...	...	1.08
...	.11	.89	...	...	...	1.61	...	.11	.89	...	...	1.61
...	.15	.85	...	...	...	−3.11	...	.15	.85	...	...	−3.11
...	.14	.81	.02	.03	...	4.07	...	.03	.95	.02	...	3.98
...	...	2.64	.04	...	−1.68	2.92	...	...	2.42	...	−1.42	2.17
...	...	2.70	.05	...	−1.75	−1.84	...	...	2.46	...	−1.46	−.60
...	...	2.79	.06	...	−1.85	2.83	.06	...	2.69	...	−1.74	2.83
...	...	2.79	.06	...	−1.85	1.74	.06	...	2.72	...	−1.78	1.59
...	...	2.79	.06	...	−1.85	−1.17	.07	...	2.67	...	−1.74	−1.44
...	...	2.78	.06	...	−1.85	4.44	.08	...	2.63	...	−1.71	4.26
...	...	2.79	.06	...	−1.85	4.10	.08	...	2.68	...	−1.76	3.44
...	...	2.78	.07	...	−1.85	−3.92	.09	...	2.67	...	−1.75	−3.46
...	...	2.72	.06	...	−1.77	1.31	.05	...	2.43	...	−1.47	.45
...	...	2.69	.06	...	−1.74	.71	.03	...	2.35	...	−1.39	.92
...	...	2.58	.06	...	−1.64	2.28	.04	...	2.27	...	−1.31	1.39
...	...	2.70	.07	...	−1.77	1.40	.06	...	2.34	...	−1.40	1.49
...	...	2.61	.07	...	−1.68	5.53	.07	...	2.25	...	−1.33	2.43
...	...	2.54	.08	...	−1.62	4.52	.09	...	2.16	...	−1.25	3.28
...	...	2.51	.08	...	−1.58	−.27	.09	...	2.13	...	−1.22	.21
...	...	2.43	.07	...	−1.50	−.26	.07	...	2.09	...	−1.16	−.36
...	...	2.26	.07	...	−1.32	2.80	.05	...	1.92	...	−.98	1.69
...	...	2.21	.06	...	−1.27	1.67	.05	...	1.85	...	−.90	1.11
...	...	2.13	.07	...	−1.21	1.25	.06	...	1.78	...	−.83	1.10
...	...	2.07	.07	...	−1.14	4.80	.05	...	1.71	...	−.76	3.59
...	...	2.11	.07	...	−1.17	2.75	.04	...	1.75	...	−.80	2.24
...	...	2.67	.06	...	−1.73	1.88	.05	...	2.29	...	−1.33	.73
...	.21	2.24	.08	...	−1.53	.31	.07	...	2.20	...	−1.27	.56
...	.24	2.22	.10	...	−1.56	6.73	.10	...	2.20	...	−1.30	4.56
...	.75	1.25	.13	...	−1.13	3.82	.14	.26	1.68	...	−1.09	3.69
...	.33	2.31	.16	...	−1.80	...	.21	.05	2.25	...	−1.51	−.29
...	.21	2.35	.16	...	−1.71	−8.51	.20	...	2.28	...	−1.48	−7.35
...	.59	.29	.12	...	...	1.41	.11	.20	.78	...	−.09	1.53

TABLE 4 *(Continued)*

	Power 0												
	With Small Stocks							Without Small Stocks					
	Inv. Fractions							Inv. Fractions					
Period	CS	GB	CB	SS	RL	RB	RP	CS	GB	CB	RL	RB	RP
1947:1	...	1.47	...	.85	...	−1.32	−.03	.47	5.30	...	...	−4.77	−.89
1947:2	...	...	...	1.33	...	−.33	−14.08	.96	2.78	...	...	−2.74	.30
1947:3	...	...	...	1.33	...	−.33	11.44	1.17	1.19	...	...	−1.37	1.04
1947:4	...	.97	...	1.20	...	−1.18	−.60	...	10.00	...	...	−9.00	−45.20
1948:1	...	...	...	1.33	...	−.33	−0.61	1.33	...	...	...	−.33	−.46
1948:2	...	...	...	1.33	...	−.33	20.06	1.33	...	...	...	−.33	16.55
1948:3	...	...	...	1.33	...	−.33	−14.46	1.33	...	...	...	−.33	−8.54
1948:4	...	...	...	1.33	...	−.33	−6.34	1.33	...	...	...	−.33	−.01
1949:1	...	...	...	1.33	...	−.33	3.81	1.33	...	...	...	−.33	.59
1949:2	...	...	...	2.00	...	−1.00	−20.04	2.00	...	...	...	−1.00	−9.04
1949:3	...	...	...	2.00	...	−1.00	28.94	2.00	...	...	...	−1.00	22.72
1949:4	...	...	...	2.00	...	−1.00	23.58	2.00	...	...	...	−1.00	19.98
1950:1	...	...	...	2.00	...	−1.00	13.00	2.00	...	...	...	−1.00	8.80
1950:2	...	...	...	2.00	...	−1.00	−3.72	2.00	...	...	...	−1.00	7.66
1950:3	...	...	...	2.00	...	−1.00	34.02	2.00	...	...	...	−1.00	23.20
1950:4	...	...	...	2.00	...	−1.00	24.14	2.00	...	...	...	−1.00	15.14
1951:1	...	...	...	2.00	...	−1.00	6.77	2.00	...	...	...	−1.00	11.95
1951:2	...	...	...	1.33	...	−.33	−7.01	1.33	...	...	...	−.33	−.77
1951:3	...	...	...	1.33	...	−.33	16.22	1.33	...	...	...	−.33	16.24
1951:4	...	...	...	1.33	...	−.33	−3.75	1.33	...	...	...	−.33	5.27
1952:1	...	...	...	1.33	...	−.33	.50	1.33	...	...	...	−.33	4.94
1952:2	.19	...	...	1.14	...	−.33	−2.14	1.33	...	...	...	−.33	5.24
1952:3	1.25	...	...	.09	...	−.33	−1.03	1.33	...	...	...	−.33	−1.03
1952:4	1.28	...	...	.05	...	−.33	12.76	1.33	...	...	...	−.33	12.99
1953:1	1.33	...	...	...	...	−.33	−5.14	1.33	...	...	...	−.33	−5.14
1953:2	2.00	...	...	...	...	−1.00	−6.91	2.00	...	...	...	−1.00	−6.91
1953:3	2.00	...	...	...	...	−1.00	−5.24	2.00	...	...	...	−1.00	−5.24
1953:4	2.00	...	...	...	...	−1.00	15.18	2.00	...	...	...	−1.00	15.18
1954:1	2.00	...	...	...	...	−1.00	18.93	2.00	...	...	...	−1.00	18.93
1954:2	2.00	...	...	...	...	−1.00	18.80	2.00	...	...	...	−1.00	18.80
1954:3	2.00	...	...	...	...	−1.00	22.48	2.00	...	...	...	−1.00	22.48
1954:4	2.00	...	...	...	...	−1.00	25.00	2.00	...	...	...	−1.00	25.00
1955:1	1.67	...	...	...	...	−.67	3.77	1.67	...	...	...	−.67	3.77
1955:2	1.67	...	...	...	...	−.67	21.53	1.67	...	...	...	−.67	21.53
1955:3	1.43	...	...	...	...	−.43	10.00	1.43	...	...	...	−.43	10.00
1955:4	1.43	...	...	...	...	−.43	7.15	1.43	...	...	...	−.43	7.15
1956:1	1.43	...	...	...	...	−.43	10.43	1.43	...	...	...	−.43	10.43
1956:2	1.43	...	...	...	...	−.43	−3.56	1.43	...	...	...	−.43	−3.56
1956:3	1.43	...	...	...	...	−.43	−4.32	1.43	...	...	...	−.43	−4.32
1956:4	1.43	...	...	...	...	−.43	4.94	1.43	...	...	...	−.43	4.94
1957:1	1.43	...	...	...	...	−.43	−7.06	1.43	...	...	...	−.43	−7.06
1957:2	1.43	...	...	...	...	−.43	11.49	1.43	...	...	...	−.43	11.49
1957:3	1.43	...	...	...	...	−.43	−14.31	1.43	...	...	...	−.43	−14.31
1957:4	1.43	...	...	...	...	−.43	−7.31	1.43	...	...	...	−.43	−7.31
1958:1	1.43	...	...	...	...	−.43	8.54	1.43	...	...	...	−.43	8.54
1958:2	2.00	...	...	...	...	−1.00	15.87	2.00	...	...	...	−1.00	15.87
1958:3	2.00	...	...	...	...	−1.00	22.18	2.00	...	...	...	−1.00	22.18
1958:4	1.43	...	...	...	...	−.43	15.59	1.43	...	...	...	−.43	15.59
1959:1	1.11	...	...	...	...	−.11	1.22	1.11	...	...	...	−.11	1.22
1959:2	1.11	...	...	...	...	−.11	6.84	1.11	...	...	...	−.11	6.84
1959:3	1.11	...	...	...	...	−.11	−2.34	1.11	...	...	...	−.11	−2.34
1959:4	1.11	...	...	...	...	−.11	6.71	1.11	...	...	...	−.11	6.71
1960:1	1.11	...	...	...	...	−.11	−7.72	1.11	...	...	...	−.11	−7.72
1960:2	1.11	...	...	...	...	−.11	3.98	1.11	...	...	...	−.11	3.98

TABLE 4 (*Continued*)

Power −15

With Small Stocks							Without Small Stocks					
Inv. Fractions							Inv. Fractions					
CS	GB	CB	SS	RL	B	r_t	CS	GB	CB	RL	B	r_t
...	.74	.15	.12	...	...	.40	.12	.38	.51	...	...	.55
...	.34	.52	.14	...	...	−1.24	.15	.15	.70	...	...	.60
...	.08	.78	.13	...	...	−.17	.16	...	.84	...	...	−1.44
...	.89	...	.11	...	...	−3.21	.07	1.08	...	...	−.15	−4.16
...	...	.87	.13	...	...	1.51	.13	.30	.58	...	...	1.30
...	...	.87	.13	...	...	1.58	.13	.11	.76	...	...	1.48
...	.12	.69	.18	...	...	−1.79	.25	.40	.35	...	...	−1.49
...	...	.83	.17	...	...	1.25	.21	.29	.50	...	...	1.66
...	...	.84	.16	...	...	1.18	.21	...	.79	...	...	.78
...	...	.84	.16	...	...	−.31	.22	.19	.59	...	...	.30
...	...	.85	.15	...	...	3.53	.20	.24	.55	...	...	3.59
...	.08	.78	.15	...	...	1.39	.21	.37	.42	...	...	2.29
.03	.17	.61	.19	...	...	1.78	.31	.52	.17	...	...	1.43
.20	...	.70	.11	...	...	.70	.37	.04	.59	...	...	1.58
.19	...	.70	.11	...	...	4.61	.36	.10	.53	...	...	4.70
.25	...	.68	.07	...	...	3.36	.37	.05	.58	...	...	3.31
.18	...	.71	.11	...	...	−.27	.37	...	.63	...	...	.67
.38	.01	.03	.01	.57	...	−.05	.40	.03	.02	.56	...	−.01
.38	...	...	...	.62	...	4.93	.38	...	...	.62	...	4.93
.40	...	...	...	.60	...	1.91	.40	...	...	.60	...	1.91
.42	...	...	...	.58	...	1.88	.42	...	...	.58	...	1.88
.42	...	...	...	.58	...	1.97	.42	...	...	.58	...	1.97
.41	...	...	...	.59	...	.05	.41	...	...	.59	...	.05
.41	...	...	...	.59	...	4.33	.41	...	...	.59	...	4.33
.41	...	...	...	.59	...	−1.18	.41	...	...	.59	...	−1.18
.38	...	...	...	.62	...	−.78	.38	...	...	.62	...	−.78
.35	...	...	...	.65	...	−.38	.35	...	...	.65	...	−.38
.34	...	...	...	.66	...	2.96	.34	...	...	.66	...	2.96
.34	...	...	...	.66	...	3.61	.34	...	...	.66	...	3.61
.35	...	.09	...	.57	...	3.57	.35	...	.09	.57	...	3.57
.35	...	.25	...	.39	...	4.48	.35	...	.25	.39	...	4.48
.92	...	.08	...	...	...	11.98	.92	...	.08	...	...	11.98
.92	...	.08	...	...	...	2.40	.92	...	.08	...	...	2.40
.93	...	...	...	.07	...	12.35	.93	...	...	.07	...	12.35
.94	...	...	...	.06	...	6.87	.94	...	...	.06	...	6.87
.92	...	...	...	.08	...	4.99	.92	...	...	.08	...	4.99
.92	...	...	...	.08	...	7.09	.92	...	...	.08	...	7.09
.94	...	...	...	.06	...	−1.97	.94	...	...	.06	...	−1.97
.90	...	...	...	.10	...	−2.31	.90	...	...	.10	...	−2.31
1.00	...	...	...	...	...	3.86	1.00	...	...	...	...	3.86
1.00	...	...	...	...	...	−4.53	1.00	...	...	...	...	−4.53
.90	...	...	...	.10	...	7.72	.90	...	...	.10	...	7.72
.98	...	...	.02	...	...	−9.58	1.00	...	...	...	...	−9.60
.65	...	...	...	.35	...	−2.75	.65	...	...	.35	...	−2.75
.61	...	...	...	.39	...	4.13	.61	...	...	.39	...	4.13
.69	...	.26	...	.05	...	6.29	.69	...	.26	.05	...	6.29
.70	...	.30	...	...	...	6.47	.70	...	.30	...	...	6.47
.61	...	...	...	.39	...	7.18	.61	...	...	.39	...	7.18
.61	...	...	...	.39	...	1.02	.61	...	...	.39	...	1.02
.59	...	...	...	.41	...	4.02	.59	...	...	.41	...	4.02
.61	...	...	...	.39	...	−.88	.61	...	...	.39	...	−.88
.53	...	...	...	.47	...	3.74	.53	...	...	.47	...	3.74
.52	...	...	...	.48	...	−2.99	.52	...	...	.48	...	−2.99
.43	...	...	.04	.52	...	2.20	.47	...	...	.53	...	2.20

TABLE 4 (*Continued*)

	Power 0												
	With Small Stocks							Without Small Stocks					
	Inv. Fractions							Inv. Fractions					
Period	CS	GB	CB	SS	RL	B	r_f	CS	GB	CB	RL	B	r_f
1960:3	.79	...	...	.32	...	−.11	−5.66	1.11	...	...	...	−.11	−5.93
1960:4	.90	...	...	.52	...	−.43	9.92	1.43	...	...	...	−.43	13.11
1961:1	1.13	...	...	.30	...	−.43	20.71	1.43	...	...	...	−.43	17.71
1961:2	1.43	...	...	...	...	−.43	−.48	1.43	...	...	...	−.43	−.48
1961:3	.88	...	...	.55	...	−.43	1.92	1.43	...	...	...	−.43	5.09
1961:4	.89	...	...	.54	...	−.43	11.86	1.43	...	...	...	−.43	10.95
1962:1	...	...	...	1.43	...	−.43	4.91	1.43	...	...	...	−.43	−3.59
1962:2	...	...	...	1.43	...	−.43	−34.28	1.43	...	...	...	−.43	−30.05
1962:3	...	...	...	1.43	...	−.43	4.34	1.43	...	...	...	−.43	4.67
1962:4	...	...	...	2.00	...	−1.00	13.26	2.00	...	...	...	−1.00	25.18
1963:1	1.64	...	...	.36	...	−1.00	12.98	2.00	...	...	...	−1.00	11.30
1963:2	1.76	...	...	.24	...	−1.00	8.96	2.00	...	...	...	−1.00	8.64
1963:3	.34	...	...	1.66	...	−1.00	6.32	2.00	...	...	...	−1.00	6.82
1963:4	...	...	...	2.00	...	−1.00	.20	2.00	...	...	...	−1.00	9.84
1964:1	...	...	...	1.43	...	−.43	12.01	1.43	...	...	...	−.43	8.07
1964:2	...	...	...	1.43	...	−.43	5.38	1.43	...	...	...	−.43	5.41
1964:3	...	...	...	1.43	...	−.43	10.62	1.43	...	...	...	−.43	4.81
1964:4	...	...	...	1.43	...	−.43	.87	1.43	...	...	...	−.43	1.67
1965:1	...	...	...	1.43	...	−.43	16.55	1.43	...	...	...	−.43	2.75
1965:2	...	...	...	1.43	...	−.43	−7.92	1.43	...	...	...	−.43	−2.94
1965:3	...	...	...	1.43	...	−.43	20.17	1.43	...	...	...	−.43	10.40
1965:4	...	...	...	1.43	...	−.43	22.89	1.43	...	...	...	−.43	4.60
1966:1	...	...	...	1.43	...	−.43	11.86	1.43	...	...	...	−.43	−4.57
1966:2	...	...	...	1.43	...	−.43	−10.16	1.43	...	...	...	−.43	−6.77
1966:3	...	...	...	1.43	...	−.43	−18.42	1.43	...	...	...	−.43	−13.41
1966:4	...	...	...	1.43	...	−.43	5.61	1.17	...	...	...	−.17	6.68
1967:1	...	...	...	1.43	...	−.43	43.99	1.00	...	...	...	...	13.21
1967:2	...	...	...	1.43	...	−.43	16.72	1.43	...	...	...	−.43	1.13
1967:3	...	...	...	1.43	...	−.43	22.06	1.43	...	...	...	−.43	10.02
1967:4	...	...	...	1.43	...	−.43	9.98	1.43	...	...	...	−.43	.12
1968:1	...	...	...	1.43	...	−.43	−10.31	1.43	...	...	...	−.43	−8.92
1968:2	...	...	...	1.43	...	−.43	36.96	1.43	...	...	...	−.43	15.26
1968:3	...	...	...	1.25	...	−.25	7.14	1.25	...	...	...	−.25	4.39
1968:4	...	...	...	1.25	...	−.25	10.34	1.25	...	...	...	−.25	1.99
1969:1	...	...	...	1.25	...	−.25	−10.37	1.25	...	...	...	−.25	−2.38
1969:2	...	...	...	1.25	...	−.25	−8.76	1.00	...	...	...	...	−3.00
1969:3	...	...	...	1.25	...	−.25	−8.92	1.00	...	...	...	...	−3.92
1969:4	...	...	...	1.25	...	−.25	−8.97	.72	...	...	.28	...	.27
1970:1	...	...	...	1.25	...	−.25	−7.12	...	...	...	1.00	...	1.98
1970:2	...	...	...	1.25	...	−.25	−41.44	.50	...	...	.50	...	−8.20
1970:3	...	...	...	1.05	...	−.05	29.38	.75	...	...	.25	...	13.01
1970:4	...	...	...	1.37	...	−.37	.66	1.00	...	...	...	...	10.43
1971:1	...	...	...	1.45	...	−.45	37.53	1.00	...	...	...	...	9.69
1971:2	...	...	...	1.54	...	−.54	−11.34	1.54	...	...	...	−.54	−.61
1971:3	...	...	...	1.44	...	−.44	−4.21	1.01	...	...	...	−.01	−.61
1971:4	...	...	...	1.35	...	−.35	1.27	1.00	...	...	...	...	4.66
1972:1	...	...	...	1.47	...	−.47	18.42	1.19	...	...	...	−.19	6.54
1972:2	...	...	...	1.51	...	−.51	−6.34	1.17	...	...	...	−.17	.53
1972:3	...	...	...	1.37	...	−.37	−8.46	1.00	...	...	...	...	3.91
1972:4	...	...	...	1.12	...	−.12	1.87	1.00	...	...	...	...	7.56
1973:1	...	...	...	1.09	...	−.09	−15.27	1.00	...	...	...	...	−4.89
1973:2	...	...	...	.75	.25	...	−11.86	.62	...	...	.38	...	−2.98
1973:3	...	...	...	.32	.68	...	7.21	...	...	...	1.00	...	2.00
1973:4	...	...	...	.48	.52	...	−8.20	...	...	...	1.00	...	1.79

TABLE 4 (*Continued*)

Power −15

With Small Stocks							Without Small Stocks					
Inv. Fractions							Inv. Fractions					
CS	GB	CB	SS	RL	B	r_t	CS	GB	CB	RL	B	r_t
.40	...	.21	.15	.25	...	−1.90	.52	...	.15	.33	...	−2.03
.36	.39	...	.19	.06	...	4.86	.51	.30	...	.19	...	5.58
.37	.43	...	.19	.01	...	9.29	.52	.34	...	.14	...	6.95
.53	.42	...	.06	...	...	...	.58	.42	...	...	...	.02
.47	.40	...	.13	...	...	2.14	.60	.40	...	...	...	2.88
.47	.37	...	.16	...	...	5.05	.62	.38	...	...	...	4.75
.41	.37	...	.23	...	...	1.27	.63	.36	...	.01	...	−.07
.31	.39	...	.30	...	...	−13.30	.60	.39	...	.01	...	−12.18
.10	...	.07	.13	.70	...	1.49	.24	...	.07	.70	...	1.54
.10	...	.13	.13	.64	...	2.94	.24	...	.13	.64	...	3.76
.20	...	.11	.03	.66	...	2.21	.23	...	.11	.66	...	2.06
.22	...	.16	.02	.60	...	1.74	.24	...	.16	.60	...	1.71
.12	...	.09	.10	.69	...	1.43	.22	...	.10	.69	...	1.48
.03	...	.04	.17	.76	...	.96	.20	...	.04	.76	...	1.80
.08	...	...	.11	.80	...	2.19	.20	...	...	.80	...	1.90
.04	...	.03	.15	.78	...	1.52	.20	...	.03	.78	...	1.54
.05	...	.11	.16	.68	...	2.18	.22	...	.11	.67	...	1.54
.06	...	.23	.18	.53	...	1.06	.25	...	.25	.50	...	1.18
.03	...	.30	.20	.46	...	3.24	.25	...	.32	.44	...	1.32
.03	...	.23	.21	.53	...	−.57	.26	...	.22	.52	...	.13
...	...	.42	.22	.36	...	3.58	.24	...	.39	.37	...	2.20
.03	...	.46	.25	.26	...	3.73	.31	...	.44	.25	...	.67
...	...	...	.29	.71	...	3.35	.29	...	...	.71	...	.02
...	...	...	.28	.72	...	−1.06	.27	...	...	.73	...	−.29
...	...	...	.25	.75	...	−2.15	.22	...	...	.78	...	−1.02
...	...	...	.18	.82	...	1.90	.13	...	...	.87	...	1.96
...	...	...	.17	.83	...	6.43	.14	...	...	.86	...	2.84
...	.01	...	.20	.79	...	3.15	.19	...	...	.81	...	1.02
...	...	...	.21	.79	...	4.19	.17	...	...	.83	...	2.15
...	...	...	.23	.77	...	2.60	.19	...	...	.81	...	1.04
...	...	...	.22	.78	...	−.50	.16	...	...	.84	...	.16
...	...	...	.22	.78	...	6.74	.16	...	...	.84	...	2.89
...	...	...	.23	.77	...	2.44	.17	...	...	.83	...	1.79
...	...	...	.25	.75	...	3.15	.21	...	...	.79	...	1.46
...	...	...	.24	.76	...	−.77	.17	...	...	.83	...	1.03
...	...	...	.21	.79	...	−.13	.12	...	...	.88	...	.98
...	...	...	.18	.82	...	.23	.08	...	...	.92	...	1.28
...	...	...	.17	.83	...	.33	.05	...	...	.95	...	1.66
...	...	...	.13	.87	...	1.05	...	...	...	1.00	...	1.98
...	...	...	.13	.87	...	−2.87	.03	...	...	.97	...	1.00
...	...	...	.10	.90	...	4.15	.05	...	...	.95	...	2.35
...	...	...	.12	.88	...	1.43	.11	...	...	.89	...	2.50
...	...	...	.13	.87	...	4.28	.15	...	...	.85	...	2.44
...	...	...	.14	.86	...	−.13	.18	...	...	.82	...	.81
...	...	...	.11	.89	...	.94	.12	...	...	.88	...	1.13
...	...	...	.11	.89	...	1.13	.13	...	...	.87	...	1.56
...	...	...	.12	.88	...	2.31	.16	...	...	.84	...	1.62
...	...	...	.12	.88	...	.37	.15	...	...	.85	...	.89
...	...	...	.11	.89	...	.27	.12	...	...	.88	...	1.36
...	...	...	.09	.91	...	1.24	.10	...	...	.90	...	1.82
...	...	...	.08	.92	...	.05	.10	...	...	.90	...	.69
...	...	...	.05	.95	...	.72	.04	...	...	.96	...	1.28
...	...	...	.02	.98	...	2.33	...	...	...	1.00	...	2.00
...	...	...	.03	.97	...	1.16	...	...	...	1.00	...	1.79

TABLE 4 *(Continued)*

	Power 0												
	With Small Stocks							Without Small Stocks					
	Inv. Fractions							Inv. Fractions					
Period	CS	GB	CB	SS	RL	RB	RP	CS	GB	CB	RL	RB	RP
1974:1	...	...	...	...	1.00	...	1.94	...	...	...	1.00	...	1.94
1974:2	...	...	...	...	1.00	...	2.06	...	...	...	1.00	...	2.06
1974:3	...	...	...	...	1.00	...	1.94	...	...	...	1.00	...	1.94
1974:4	...	...	...	...	1.00	...	1.81	...	...	...	1.00	...	1.81
1975:1	...	...	...	...	1.00	...	1.62	...	...	...	1.00	...	1.62
1975:2	...	...	...	.02	.98	...	1.72	...	...	...	1.00	...	1.42
1975:3	...	...	...	.08	.92	...	.68	.03	...	...	.97	...	1.17
1975:4	...	...	...	...	1.00	...	1.52	...	...	...	1.00	...	1.52
1976:1	...	...	.62	...	.38	...	3.07	...	...	.62	.38	...	3.07
1976:2	.23	...	.35	.23	.19	...	.27	.49	...	.35	.16	...	1.52
1976:3	.27	...	...	...	.73	...	1.48	.27	...	...	.73	...	1.48
1976:4	.14	...	.64	...	.22	...	5.55	.14	...	.64	.22	...	5.55
1977:1	...	...	1.00	...	...	...	−2.31	...	...	1.00	...	...	−2.31
1977:2	...	...	1.01	...	...	−.01	2.94	...	...	1.01	...	−.01	2.94
1977:3	...	...	.90	.10	...	...	.99	...	...	1.00	...	...	1.09
1977:4	...	...	.85	.15	...	...	.51	...	...	1.00	...	...	−.82
1978:1	...	...	.69	.31	...	...	3.75	...	...	1.00	...	...	.03
1978:2	...	...	.36	.64	...	...	8.90	...	...	1.00	...	...	−1.08
1978:3	...	...	...	1.28	...	−.28	20.47	.36	...	.64	...	...	5.10
1978:4	...	...	...	1.00	...	...	−17.37	.14	...	...	.86	...	1.05
1979:1	...	...	...	.73	.27	...	16.92	...	...	...	1.00	...	2.43
1979:2	...	...	...	.71	.29	...	7.19	...	...	...	1.00	...	2.37
1979:3	...	...	...	1.00	...	...	5.64	...	...	...	1.00	...	2.15
1979:4	...	...	...	1.00	...	...	1.70	...	...	...	1.00	...	2.52
1980:1	...	...	...	.81	.19	...	−10.24	...	...	...	1.00	...	2.96
1980:2	...	...	...	.05	.95	...	4.58	...	...	...	1.00	...	3.68
1980:3	...	...	...	1.00	...	...	25.10	...	...	...	1.00	...	1.97
1980:4	...	...	...	1.00	...	...	7.49	...	...	...	1.00	...	2.85
1981:1	...	...	...	.85	.15	...	13.13	...	...	...	1.00	...	3.73
1981:2	...	...	...	1.08	...	−.08	8.72	...	...	...	1.00	...	3.24
1981:3	...	...	...	1.06	...	−.06	−18.98	...	...	...	1.00	...	3.73
1981:4	...	...	...	1.00	...	...	11.29	...	...	...	1.00	...	3.77
1982:1	...	...	...	1.74	...	−.74	−12.76	...	...	...	1.00	...	2.86
1982:2	...	...	...	1.11	...	−.11	−.86	...	...	...	1.00	...	3.46
1982:3	...	...	...	1.51	...	−.51	13.78	...	...	...	1.00	...	3.33
1982:4	...	...	...	2.00	...	−1.00	44.04	1.00	...	....	...	...	18.14
1983:1	...	...	...	2.00	...	−1.00	36.76	1.00	...	...	...	...	10.05
1983:2	...	...	...	2.00	...	−1.00	39.57	1.25	...	...	...	−.25	13.25
1983:3	...	...	...	2.00	...	−1.00	−5.90	1.00	...	...	...	...	−.14
1983:4	...	...	...	2.00	...	−1.00	−7.32	1.78	...	...	...	−.78	−1.59

NOTE.—CS = common stocks, GB = government bonds, CB = corporate bonds, SS = small stocks, RL = risk-free asset, B = the fraction in borrowing, and r_t = return for the quarter.

TABLE 4 (*Continued*)

Power −15

With Small Stocks							Without Small Stocks					
Inv. Fractions							Inv. Fractions					
CS	GB	CB	SS	RL	RB	RP	CS	GB	CB	RL	RB	RP
...	...	...	...	1.00	...	1.94	...	...	...	1.00	...	1.94
...	...	...	...	1.00	...	2.06	...	...	...	1.00	...	2.06
...	...	...	...	1.00	...	1.94	...	...	...	1.00	...	1.94
...	...	...	...	1.00	...	1.81	...	...	...	1.00	...	1.81
...	...	...	...	1.00	...	1.62	...	...	...	1.00	...	1.62
...	...	...	...	1.00	...	1.44	...	...	...	1.00	...	1.42
...	...	...	...	1.00	...	1.49	...	...	...	1.00	...	1.52
...	...	...	...	1.00	...	1.52	...	...	...	1.00	...	1.52
...	...	.04	...	.96	...	1.35	...	...	.04	.96	...	1.35
.02	...	.02	.01	.95	...	1.17	.03	...	.02	.95	...	1.24
.02	...	...	...	.98	...	1.33	.02	...	...	.98	...	1.33
.01	...	.04	...	.95	...	1.50	.01	...	.04	.95	...	1.50
...	.07	.20	...	.73	...	−.05	...	.07	.20	.73	...	−.05
...	...	.24	...	.76	...	1.56	...	...	.24	.76	...	1.56
...	...	.22	...	.78	...	1.21	...	...	.22	.78	...	1.21
...	...	.18	...	.82	...	1.05	...	...	.18	.82	...	1.05
...	...	.21	.01	.78	...	1.29	...	...	.23	.77	...	1.18
...	...	.06	.04	.91	...	1.93	...	...	.13	.87	...	1.26
...	...	...	.09	.91	...	3.15	.02	...	.09	.89	...	1.99
...	...	...	.08	.92	...	.44	.01	...	...	.99	...	1.93
...	...	...	.04	.96	...	3.31	...	...	...	1.00	...	2.43
...	...	...	.04	.96	...	2.66	...	...	...	1.00	...	2.37
...	...	...	.07	.93	...	2.38	...	...	...	1.00	...	2.15
...	...	...	.06	.94	...	2.47	...	...	...	1.00	...	2.52
...	...	...	.05	.95	...	2.15	...	...	...	1.00	...	2.96
...	...	...	...	1.00	...	3.74	...	...	...	1.00	...	3.68
...	...	...	.07	.93	...	3.64	...	...	...	1.00	...	1.97
...	...	...	.07	.93	...	3.18	...	...	...	1.00	...	2.85
...	...	...	.05	.95	...	4.31	...	...	...	1.00	...	3.73
...	...	...	.09	.91	...	3.73	...	...	...	1.00	...	3.24
...	...	...	.11	.89	...	1.36	...	...	...	1.00	...	3.73
...	...	...	.07	.93	...	4.30	...	...	...	1.00	...	3.77
...	...	...	.14	.86	...	1.70	...	...	...	1.00	...	2.86
...	...	...	.10	.90	...	3.08	...	...	...	1.00	...	3.46
...	...	...	.12	.88	...	4.19	...	...	...	1.00	...	3.33
...	...	...	.21	.79	...	6.56	.17	...	...	.83	...	4.70
...	...	...	.23	.77	...	6.16	.18	...	...	.82	...	3.44
...	...	...	.23	.77	...	6.45	.14	...	...	.86	...	3.44
...	...	...	.22	.78	...	1.42	.12	...	...	.88	...	1.98
...	...	...	.24	.76	...	1.16	.18	...	...	.82	...	1.90

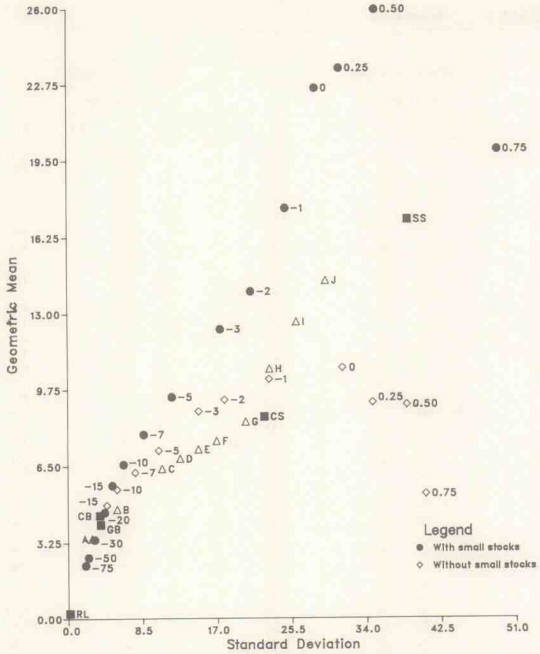

FIG. 1.—Geometric means and standard deviations of annual returns with and without small stocks, 1934–47 (quarterly revision, with leverage, 32-quarter estimating period).

Since the returns themselves are not additive but compound multiplicatively, we employed the paired *t*-test for dependent observations to the quarterly (and additive) variables $\ln(1 + r_t)$ to test whether the presence of small stocks improved returns significantly.[11] Thus to compare the return series $r_1^1, \ldots, r_n^1$ with the return series $r_1^2, \ldots, r_n^2$ for two different strategies, we calculate the statistic

$$t = \frac{\bar{d}}{\sigma(d)/\sqrt{n}},$$

11. This test was also employed by Fama and MacBeth (1974).

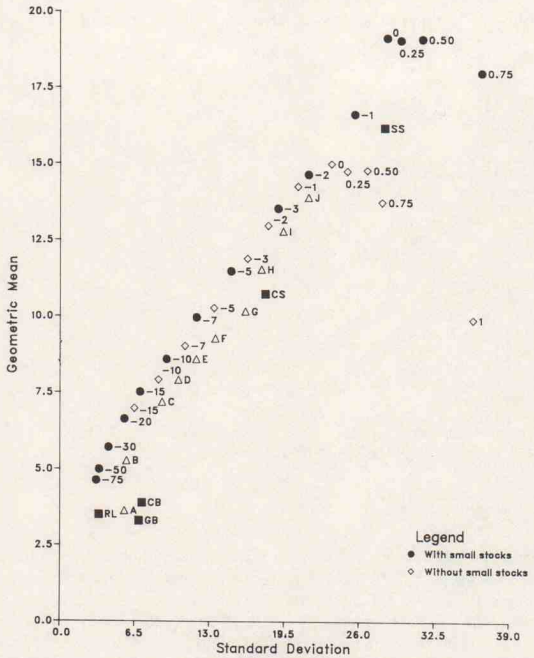

FIG. 2.—Geometric means and standard deviations of annual returns with and without small stocks, 1934–83 (quarterly revision, with leverage, 32-quarter estimating period).

where

$$\bar{d} = \sum_{t=1}^{n} \frac{\ln(1 + r_t^1) - \ln(1 + r_t^2)}{n}$$

and $\sigma(d)$ is the standard deviation of the differences between $\ln(1 + r_t^1)$ and $\ln(1 + r_t^2)$. In each case, the null hypothesis is that

$$E[\ln(1 + r_t^1)] = E[\ln(1 + r_t^2)]$$

and the alternative hypothesis is that

$$E[\ln(1 + r_t^1)] > E[\ln(1 + r_t^2)].$$

TABLE 5 Paired *t*-Tests, 1934–47 Subperiod, with Leverage

Comparison	$\bar{d}$	$\sigma(d)$	t
Power −1 vs. common stocks	.0196	.1267	1.16
Power 0 vs. portfolio I	.0378	.1438	1.97*
Power .5 with small stocks vs. power .5 without small stocks	.0360	.1813	1.48
Power .25 with small stocks vs. power .25 without small stocks	.0306	.1333	1.72*
Power .25 with small stocks vs. power 0 without small stocks	.0273	.1157	1.76*
Power 0 with small stocks vs. power 0 without small stocks	.0256	.1068	1.79*
Power −1 with small stocks vs. power −1 without small stocks	.0160	.0591	2.03**
Power −2 with small stocks vs. power −2 without small stocks	.0103	.0389	1.99**
Power −3 with small stocks vs. power −3 without small stocks	.0079	.0311	1.90*
Power −5 with small stocks vs. power −5 without small stocks	.0052	.0211	1.85*
Power −7 with small stocks vs. power −7 without small stocks	.0038	.0156	1.81*
Power −10 with small stocks vs. power −10 without small stocks	.0025	.0104	1.79*
Power −15 with small stocks vs. power −15 without small stocks	.0020	.0073	2.06**

* Significant at the 10% level.
** Significant at the 5% level.

The results for selected pairs with comparable standard deviations in the 1934–47 period (56 quarters) are given in table 5, while table 6 gives the results for the full 200-quarter 1934–83 period.

Table 5 suggests that the presence of small stocks was especially significant in improving returns for the "middle" strategies (powers −15–.25) from the early 1930s to the late 1940s. Statistically, the improvements were less impressive for the full period (fig. 2 and table 6). The active strategies performed extremely well when compared to bonds and bills, attaining highly significant improvements, through diversification, in the geometric means without increasing variability. Figures 1 and 2 and tables 5 and 6 also reveal that the active strategies performed surprisingly well when compared to the 10 fixed-weight portfolio strategies.

VII. Concluding Remarks

By way of summary, several conclusions emerge. First, the present study confirms that the simple probability assessment approach, which uses only the past to (naively) forecast the future, is apparently not without merit when combined with multiperiod investment theory. Not

TABLE 6 Paired *t*-Tests, 1934–83 Period, with Leverage

Comparison	$\bar{d}$	$\sigma(d)$	t
Power −3 vs. common stocks	.0062	.0758	1.16
Power −15 vs. corporate bonds	.0085	.0452	2.67***
Power −15 vs. government bonds	.0100	.0458	3.08***
Power −50 vs. risk-free asset	.0036	.0102	4.94***
Power −10 vs. portfolio C	.0032	.0419	1.09
Power −20 vs. portfolio B	.0032	.0311	1.45
Power −20 vs. portfolio A	.0071	.0356	2.82***
Power 0 with small stocks vs. power .75 without small stocks	.0116	.1073	1.52

*** Significant at the 1% level.

only did it produce respectable results, but the logarithmic policy also always came out at or near the top in the geometric mean dimension, suggesting an absence of clear biases in the estimating method; the exception occurs in the last subperiod, when the approach appears to have been somewhat too conservative.[12] Second, small stocks entered even the most risk-averse portfolios most of the time. On the other hand, they were ignored by all the strategies some of the time (especially in the mid- and late 1950s and mid-1970s). Third, small stocks, when chosen, tended to replace common stocks, except in the 1970s and early 1980s, when they were primarily held in lieu of the risk-free asset. This was true even when (additional) margin purchases would have been possible. Consequently, the presence of small stocks had relatively little effect on the use of leverage; in fact, the effect was not always in the positive direction. Thus the use of, and gains from, margin purchases occurred, as in our previous paper (Grauer and Hakansson 1982), primarily for the more risk-tolerant strategies from the mid-1930s to the mid-1960s.

Fourth, small stocks had a notable positive effect on returns, not only for the risk-tolerant strategies but also for conservative investors. This effect was especially significant from the mid-1930s to the early 1950s in the presence of margin purchases. Fifth, the strong gains from diversification among the major asset categories reported in the previous papers are, if anything, strengthened when small stocks are included.

Sixth, the performances of the active strategies, when measured against those of the fixed-weight strategies, were surprisingly strong. Could it be that the naive, simple-minded empirical distribution contains the kind of information money managers usually pay good money for?

12. Under perfectly valid distributions, power $\gamma = 1/T$, where T is the number of periods, has the highest asymptotic probability of attaining the largest geometric mean as T increases.

Finally, one cannot escape the conclusion that U.S. financial markets offered a generous environment during the half-century we studied. As a point of reference, consider the following. The maximum employee-portion of the social security contribution for an individual in 1984 was $2,532.60. The purchasing power of this amount (in January 1, 1984, dollars) at the beginning of 1934 was $328.78. Under a naive logarithmic policy in an environment with access to small stocks and leverage, this initial contribution alone would have grown to $2,078,925 as of the end of 1983.

Of course, the reader should also be reminded of the limitations of the study. The model used is focused on sequential reinvestments only, without concern for intermediate consumption; even though its birth occurred in the mid-1970s, it was applied as far back as 1934. The latter statement also applies at least partially to the data base used. The joint probability estimates were based on periods ranging from the most recent 5–10 years only. All investors were assumed to be strict price takers. Transactions costs and taxes were ignored (as in the underlying returns series). Finally, maintenance margins were ignored whenever leverage was used.

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