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Bias in Small Sample Tests of Stock Price Rationality*

This paper demonstrates a severe small sample bias in the use of sample variances to test stock price variance bounds, under the maintained assumption that prices and dividends follow stationary and ergodic stochastic processes. This bias is due to the smoothness of the "perfect foresight" price, which is used to construct the variance bound. However, the small sample bias demonstrated here seems insufficient to explain the empirical apparent gross violations of the stock price variance inequality, in contrast to the result for variance bounds tested with bonds.

I. Introduction

While Keynes (1936, chap. 12) regarded stock prices as the outcome of "the mass psychology of a large number of ignorant individuals," the empirical success of modern financial economics, especially concerning "efficient market" models (see, e.g., Fama 1970), has dominated the literature in the last two decades. However, recent research on "variance bounds" by Grossman and Shiller (1981), LeRoy and Porter (1981), and Shiller (1981*a*, 1981*b*, 1981*c*), among others, questions whether the large variation over time in stock prices can be explained in the context of rational valuation models. The model most often used in this literature is the dividend valuation model,

$$p_t = \sum_{\tau=1}^{\infty} \frac{E(d_{t+\tau}|\Phi_t)}{(1 + r)^{\tau}}, \tag{1}$$

where r is an assumed constant discount rate and $(X|\Phi)$ denotes the conditional distribution of the

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random variable X given the information Φ. A new variable p_t^*[1] is defined as

$$p_t^* \equiv \sum_{\tau=1}^{\infty} \frac{d_{t+\tau}}{(1+r)^\tau}. \tag{2}$$

A comparison of (1) and (2) shows

$$p_t = E(p_t^* | \Phi_t), \tag{3}$$

which implies the variance bound

$$\mathrm{var}(p_t) \leq \mathrm{var}(p_t^*). \tag{4}$$

The basis for this inequality is that, since the price rationally forecasts p_t^* by (3), the variance of the forecast (the price) should be less than that of the variable being forecast. However, when the inequality is tested using sample variances of price and p_t^* under the assumption of stationary and ergodic processes for prices and dividends, it appears to be "grossly violated." For example, Shiller (1981*b*) reports that the ratio of the sample standard deviation of price to the sample standard deviation of p_t^* for Standard and Poor's data is 5.59. Subsequent research that has relaxed the assumption of constant discount rates in (1) has not explained these results (see, e.g., Shiller 1981*a*), although the violations are very sensitive to the assumption of stationary prices and dividends (see Kleidon in press; and Marsh and Merton 1984).[2]

Similar results are found from variance bounds tests based on interest rates. Shiller (1979) and Singleton (1980) report apparent excess volatility of long-term interest rates relative to short-term rates when the rates are modeled as stationary processes. These tests also appear sensitive to the assumption of stationarity, and Shiller (1979, p. 1214) reports that the relevant variance bounds are not violated if the rates are treated as nonstationary. Since empirically it is not possible to rule out stationarity of interest rates, the question arises as to whether problems other than nonstationarity can account for the violation of the bounds. For interest rates, the answer is yes. Flavin (1982, 1983) assumes stationarity and demonstrates a small sample bias in the rele-

1. This variable is called the "perfect foresight" price in much of the variance bound literature (e.g., see Shiller 1981*c*, p. 202), although it is not necessarily the price that would prevail under perfect foresight since the discount rate r is not necessarily appropriate under these conditions. For more discussion on this point, see Kleidon (in press).

2. Other issues include the interpretation of time series plots of prices and p_t^* (Kleidon in press) and the effects of dividend smoothing (shown by Lintner [1956] and Fama and Babiak [1968] and discussed in Kleidon [1983, in press] and Marsh and Merton [1984]). The current paper addresses small sample bias, which may also occur because of the "peso problem," as in Krasker (1980) and Hart and Kreps (1984), who argue that a low probability event may rationally affect expectations yet not appear in a small sample. See LeRoy (1984) for a summary of criticisms of variance bounds tests.

vant tests. She concludes (1983, p. 931): "Much of the evidence of excess volatility in the bond market disappears when the tests are corrected for small sample bias."

This paper assumes stationarity in prices and dividends and demonstrates a small sample bias in stock price variance bounds tests similar to that shown in Flavin (1982, 1983) for bonds.[3] It is shown here that, even if prices are rationally set by (1) and follows a stationary first-order autoregressive (AR[1]) process (as assumed in papers reporting gross violation of the variance inequality), a small sample bias results in incorrect apparent rejection of the inequality (4) in a large proportion of tests based on sample variances, with the severity of the bias increasing with the AR(1) parameter. For large parameter values, the point estimates frequently appear to reject the bound even in samples as large as 3,000.

However, when one examines the amount by which stock prices appear to violate the relevant bound, as measured by the ratio of the sample standard deviation of price to the sample standard deviation of p_t^*, the gross violations found in variance bounds tests seem too large to be explained by the small sample bias alone. The tests of the inequality (4) based on stock prices must therefore be distinguished from the variance bounds tests on bonds since the latter can apparently be explained by the small sample bias if interest rates are stationary.

The paper is organized as follows. Section II specifies the assumed stationary and ergodic dividend process, derives the implied price and "perfect foresight" price p_t^* given by (1) and (2), and discusses the properties of these processes relevant for testing the inequality (4). Section III relates these results to the estimation of population variances and demonstrates a severe small sample bias in stock price variance bounds tests. The bias can be traced to different time series properties of the price and p_t^* series, which particularly affect the use of sample means as estimators of population means when calculating sample variances. The results of Monte Carlo simulations of the stationary series for different sample sizes and parameter values show that, although the small sample bias is severe, it does not seem sufficient to account for the reported gross violations of the inequality (4). Section IV concludes with a discussion of the implications of these results.

3. I became aware of Marjorie Flavin's earlier similar conclusions for interest rates after realizing the small sample bias in the stock price case. As noted below, some of my procedures were changed to conform more closely to her's. See Kleidon (1982) for alternate procedures with similar results. Mankiw, Romer, and Shapiro (1985) also discuss small sample bias in variance bounds tests, although their results give point estimates but no indication of sampling variation (see table 2, p. 686), which will be substantial given the high autocorrelation in the relevant time series, especially of p_t^*. See, esp., Merton 1985, pp. 29–38.

II. Dividend, Price, and Perfect Foresight Price Processes

To investigate the small sample bias in variance bounds tests of stock price variability under the maintained hypothesis that dividends follow a stationary stochastic process, it is assumed (consistent with current variance bounds literature) that the dividend process is first-order autoregressive (AR[1]); that is,

$$d_t = \mu_d + \rho d_{t-1} + \epsilon_t, \tag{5}$$

where $\mu_d \equiv$ a constant, $|\rho| < 1.0$, and ϵ_t is independent and identically distributed (i.i.d.) $(0, \sigma_\epsilon^2)$. Given this assumption about the process generating dividends, this section shows the stochastic processes governing the behavior of actual prices, p_t, under the rational valuation model (1), and the stochastic process generating "perfect foresight" prices, p_t^*, whose sample variance is compared with that of p_t in tests of the variance bound (4). Some key results are then given concerning the conditional and unconditional variances of p_t and p_t^* and the estimation of population means of p_t and p_t^*, which are important in understanding the small sample bias in stock price variance bounds tests.

A. *Price and* P_t^* *Processes*

Proposition 1 first derives the price process implied by the dividend process (5) and the valuation model (1), assuming that the information Φ_t in (1) comprises current and past dividends. Proposition 2 then derives the p_t^* process if dividends are given by (5).

PROPOSITION 1. Assume the valuation model (1), the dividend process (5), and that the information Φ_t comprises current and past dividends. Then prices follow a stationary AR(1) process with the same autoregressive parameter as dividends,

$$p_t = \mu_p + \rho p_{t-1} + \delta_t, \tag{6}$$

where $\mu_p \equiv \mu_d/r$ and $\delta_t \equiv \epsilon_t\rho/(1 + r - \rho)$.

Proof. Given the assumed information, we have:

$$\begin{aligned} E(d_{t+\tau}|\Phi_t) &= \mu_d(1 + \rho + \ldots + \rho^{\tau-1}) + \rho^\tau d_t \\ &= \mu_d\left(\frac{1-\rho^\tau}{1-\rho}\right) + \rho^\tau d_t. \end{aligned} \tag{7}$$

Substitution from (7) into (1) and simplification shows that the implied rational price process is

$$p_t = \frac{\mu_d(1 + r)}{r(1 + r - \rho)} + \frac{d_t\rho}{(1 + r - \rho)}, \tag{8}$$

which can be rewritten to give (6).

PROPOSITION 2. Given the dividend process (5) and assuming $|1/(1 + r)| < 1.0$, p_t^* follows the stationary AR(2) process:

$$(1 - \phi_1 B - \phi_2 B^2)\, p_t^* = \mu_{p^*} + \psi_t, \tag{9}$$

where

$$\phi_1 \equiv \rho + \left(\frac{1}{1+r}\right),$$

$$\phi_2 \equiv -\rho\left(\frac{1}{1+r}\right),$$

$$\mu_{p^*} \equiv \frac{\mu_d}{(1+r)},$$

$$\psi_t \equiv \frac{\eta_{t+1}}{(1+r)},$$

and where $E(\eta_t) = E(\epsilon_t)$, $\text{var}(\eta_t) = \text{var}(\epsilon_t) \equiv \sigma_\epsilon^2$, and $B \equiv$ the backshift operator.

Proof. From the definition (2), we have

$$p_{t+1}^* = p_t^*(1 + r) - d_{t+1} \tag{10}$$

or, using the forward shift operator F,

$$\left[1 - \frac{F}{(1+r)}\right] p_t^* = \frac{d_{t+1}}{(1+r)}. \tag{11}$$

Substitution in (11) from the assumed dividend process (5) gives

$$\left[1 - \frac{F}{(1+r)}\right] p_t^* = \frac{\mu_d}{(1-\rho B)(1+r)} + \frac{\epsilon_{t+1}}{(1-\rho B)(1+r)},$$

that is,

$$(1 - \rho B)\left[1 - \frac{F}{(1+r)}\right] p_t^* = \frac{\mu_d}{(1+r)} + \frac{\epsilon_{t+1}}{(1+r)}. \tag{12}$$

Equation (12) can be rewritten in the more usual form by exploiting the multiplicity of autoregressive moving average (ARMA) models:[4]

$$(1 - \rho B)\left[1 - \frac{B}{(1+r)}\right] p_t^* = \frac{\mu_d}{(1+r)} + \frac{\eta_{t+1}}{(1+r)},$$

from which (9) follows by definition.

Note that, for p_t^* to be stationary, we need $|\rho| < 1.0$, which is true by assumption from the dividend process (5), and that $|1/(1 + r)| < 1.0$. The second condition holds for $r > 0$, although, strictly, the prohibited values of r are $-2 \le r \le 0$.

4. That is, we can regard the (stationary) series in terms of past observations, with an implied set of innovations η_t, or equally well in terms of future observations and a (different) set of innovations ϵ_t. The stochastic properties of the stationary process and the innovations are the same in both cases. (See Box and Jenkins 1976, pp. 195 ff.)

B. *Population Unconditional Moments of Price and* p_t^*

This section gives the unconditional means and variances of the price and p_t^* processes in (6) and (9), respectively, and shows that the variance inequality (4) holds in terms of the population unconditional variances of price and p_t^*.

The price process (6) has a well-defined unconditional mean,

$$E(p_t) = \frac{\mu_d}{r(1 - \rho)},$$

and unconditional variance,

$$\mathrm{var}(p_t) = \left(\frac{\rho}{1 + r - \rho}\right)^2 \frac{\sigma_\epsilon^2}{(1 - \rho^2)}. \tag{13}$$

The unconditional mean of p_t^* from the AR(2) process (9) is

$$E(p_t^*) = \frac{\mu_{p^*}}{(1 - \phi_1 - \phi_2)}$$

$$= \frac{\mu_d}{r(1 - \rho)},$$

which equals the unconditional mean for price. The unconditional variance of p_t^* in (9) is (Box and Jenkins 1976, eq. 3.3.28, p. 62)

$$\mathrm{var}(p_t^*) = \left(\frac{1 - \phi_2}{1 + \phi_2}\right) \frac{\sigma_\psi^2}{[(1 - \phi_2)^2 - \phi_1^2]}$$

$$= \frac{\sigma_\epsilon^2(1 + r + \rho)}{(1 + r - \rho)(2r + r^2)(1 - \rho^2)}. \tag{14}$$

The inequality (4) is readily verified for the population unconditional variances given by (13) and (14). We have

$$\Delta \equiv \mathrm{var}(p_t^*) - \mathrm{var}(p_t)$$

$$= \frac{\sigma_\epsilon^2(1 + r)^2}{(1 + r - \rho)^2(2r + r^2)}.$$

Clearly, $\Delta > 0$ for $r > 0$, $\sigma_\epsilon^2 > 0$, which confirms (4).

C. *Conditional Variances of Price and* p_t^*

Although the variance inequality found in the literature is almost always given in terms of unconditional variances as in (4), Kleidon (in press) shows that similar inequalities hold for conditional variances; that is,

$$\mathrm{var}(p_t^*|\Phi_{t-k}) \geq \mathrm{var}(p_t|\Phi_{t-k}), \quad k = 0, \ldots, \infty, \tag{15}$$

where Φ_{t-k} is information at $t - k$ and is included in information at t (traders do not forget) and the limit as $k \to \infty$ of the conditional variances in (15) gives the unconditional variances in (4).[5] Of more immediate consequence is the additional result that the "smoothness" of time series plots of price and p_t^* is also determined by conditional variances, although they differ from those in (15) and do not satisfy an inequality such as (15). Instead, the conditional variance that determines the smoothness of the p_t^* series implies less short-term variation in p_t^* than that implied for the price series by the corresponding conditional variance of p_t. The resulting greater smoothness of the p_t^* series is the basic result underlying the small sample bias in stock price variance bounds tests.

The smoothness or amount of short-term variation in price and p_t^* is taken here to be determined by the variance conditional on past values of the series, that is, $\text{var}(p_t|p_{t-k})$ and $\text{var}(p_t^*|p_{t-k}^*)$.[6] Note that $\text{var}(p_t|p_{t-k})$ is equivalent to the conditional variance of price in (15) if information comprises current and past dividends and prices are given by the process (6). However, the conditional variance $\text{var}(p_t^*|p_{t-k}^*)$, which determines the smoothness of the time series of p_t^*, is *not* equivalent to $\text{var}(p_t^*|\Phi_{t-k})$ since, by the definition of p_t^* in (2), past values of p_t^* depend on *future* outcomes of dividends, which are not known at or prior to time t. Consequently, there is no necessity for the conditional variances $\text{var}(p_t|p_{t-k})$ and $\text{var}(p_t^*|p_{t-k}^*)$ to satisfy an inequality such as (15), and indeed they do not. Kleidon (in press, propositions 2, 5) shows for the price process (6):

$$\text{var}(p_t|p_{t-k}) = \frac{\sigma_\epsilon^2\rho^2(1 - \rho^{2k})}{(1 + r - \rho)^2(1 - \rho^2)}, \tag{16}$$

while for the p_t^* process (9) (assuming ϵ_t in the dividend process [5] is normally distributed) we have

$$\text{var}(p_t^*|p_{t-k}^*) = \text{var}(p_t^*)(1 - \rho_k^2), \tag{17}$$

where

$$\rho_k \equiv \frac{\text{cov}(p_t^*, p_{t-k}^*)}{\text{var}(p_t^*)} = \frac{\rho^{k+1}(2r + r^2) - (1 - \rho^2)(1 + r)^{1-k}}{(1 + r + \rho)(\rho + r\rho - 1)}$$

and where $\text{var}(p_t^*)$ is the unconditional variance given by (14).

Examination of (16) and (17) show that the limit as $k \to \infty$ of these

5. These conditional variances are exploited in Kleidon (in press) to give valid tests even if dividends and prices follow nonstationary random walks since in that case the population conditional variances are well defined even though the population unconditional variances are not.

6. An early version of Kleidon (in press) drew the distinction between the unconditional variances in (4) and conditional variances similar to these, and this distinction was adopted in LeRoy (1984) in terms of these conditional variances.

conditional variances gives the unconditional variances in (13) and (14), respectively. Hence for k large enough, $\text{var}(p_t^*|p_{t-k}^*)$ must exceed $\text{var}(p_t|p_{t-k})$ since the bound (4) holds for unconditional variances. However, the key insight is that short-term variances show the opposite result. For k small,

$$\text{var}(p_t^*|p_{t-k}^*) < \text{var}(p_t|p_{t-k}),$$

and this can hold for quite large k, depending on the parameter ρ in the dividend process.

This result and its implications for the interpretation of time series plots of price and p_t^* are discussed in detail in Kleidon (in press), but the fundamental intuition for the greater smoothness of p_t^* relative to prices is that the former series gives the present value of the ex post dividend outcomes by the definition (2) while actual prices give the present value of *expected* dividends. A change in current dividends from past levels rationally implies a change in all future expected dividends if dividends are given by the AR(1) process (5). Consequently the price should change by the present value of those changes in expected future dividends, which will typically be much larger than the current dividend change itself. However, since p_t^* by definition already uses full knowledge of the ex post dividend stream, there are no revisions in expectations as a result of changes in current dividends. In fact, (2) implies that consecutive changes in p_t^* are limited to the capital gain that, together with current dividends, will give exactly the return r, and this change in p_t^* will typically be smaller than the change in prices.

The value of this result for the current paper is that the time series of p_t^* will be smoother than that of prices by construction, which has important implications for the estimation of the population unconditional variances in the inequality (4). Results are now given concerning the estimation of population means of price and p_t^*, which of course are used in the corresponding unconditional variances, while Section III directly examines the problem of estimating the unconditional variances with small samples.

D. *Estimation of Population Means of Price and* p_t^*

Although the population unconditional means of price and p_t^* are equal, the two series have quite different time series properties, with p_t^* showing less short-term variation than prices. The result is that, although the large sample means of p_t and p_t^* are unbiased estimators of the population means, the variance of the sample mean of p_t^* ($\text{var}[\bar{p}^*]$) exceeds that of p_t ($\text{var}[\bar{p}]$) even in large samples. Propositions 3 and 4 establish $\text{var}(\bar{p})$ and $\text{var}(\bar{p}^*)$, respectively, while proposition 5 shows that $\text{var}(\bar{p}^*) > \text{var}(\bar{p})$.

PROPOSITION 3. Given the price process (6), the variance of the sample mean of price in large samples, $\mathrm{var}(\bar{p})$, is given by

$$\mathrm{var}(\bar{p}) \approx \frac{\rho^2 \sigma_\epsilon^2}{T(1 + r - \rho)^2 (1 - \rho)^2}, \tag{18}$$

where T is the sample size.

Proof. See the Appendix.

PROPOSITION 4. Given the p_t^* process (9), the variance of the sample mean of p_t^* in large samples, $\mathrm{var}(\bar{p}^*)$, is given by

$$\mathrm{var}(\bar{p}^*) \approx \frac{\sigma_\epsilon^2}{T(1 - \rho)^2 r^2}. \tag{19}$$

Proof. See the Appendix.

PROPOSITION 5. For all $|\rho| < 1$ and $r > 0$,

$$\mathrm{var}(\bar{p}^*) < \mathrm{var}(\bar{p}).$$

Proof. Equations (18) and (19) imply:

$$\mathrm{var}(\bar{p}^*) = \frac{\mathrm{var}(\bar{p})}{a^2 r^2},$$

where $a \equiv \rho/(1 + r - \rho)$. If $|\rho| > 1$ and $r > 0$, then $r\rho < 1 + r - \rho$ or $ar < 1$, and hence for all relevant stationary processes $\mathrm{var}(\bar{p}^*) > \mathrm{var}(\bar{p})$.

III. Variance Bounds Tests in Small Samples

This section links the time series properties of the price and p_t^* series discussed in Section II to the problem of the estimation of population unconditional variances used in the inequality (4) when only a finite number of time series observations are available. First, Section III*A* gives time series plots of price and p_t^* for simulated data that are constructed from the valuation model (1) and the dividend process (5), and these plots illustrate the relevant characteristics discussed in Section II and the consequent small sample bias when sample variances are used to test the inequality (4). In particular, the bias is largely attributable to problems with estimation of the population mean in small samples.

This is not the only problem, however. Section III*B* relates the result $\mathrm{var}(p_t^*|p_{t-1}^*) < \mathrm{var}(p_t|p_{t-1})$ to the problem of constructing a time series of p_t^* using only a finite stream of dividends rather than the infinite stream used in the definition of p_t^* given by (2). The solution typically used in current work (and followed in the simulated plots of Sec. III*A*) is to set an arbitrary terminal value for p_T^*, frequently setting $p_T^* \equiv p_T$. This procedure guarantees that small sample variances are closely linked to the conditional variances $\mathrm{var}(p_t|p_{t-1})$ and $\mathrm{var}(p_t^*|p_{t-1}^*)$, and, consequently, it is not surprising if the sample variance of p_t exceeds that of p_t^*. This terminal condition problem and the difficulties in estimation of population means become less severe as the sample size increases, and Section III*C* examines the properties of sample variances of p_t^* and p_t for various parameter values and sample sizes via simulation.

A. *Simulated Plots of Time Series of Price and* p_t^*

The time series properties of p_t and p_t^* are illustrated with simulated data, which show how the smoothness of p_t^* relative to p_t results in a small sample bias in tests of the inequality (4) using sample variances of price and p_t^*. In all the simulations, a series of independent and identically normally distributed (n.i.d.) disturbances ϵ_t is generated by the International Mathematical and Statistical Library, Incorporated (IMSL), subroutine GGNPM,[7] and the dividend and corresponding rational price series are generated by (5) and (6). The p_t^* series is generated by the recursion (10), with the terminal condition $p_T^* \equiv p_T$ as in Grossman and Shiller (1981). The initial dividend is determined by a draw from the unconditional distribution d_t.[8]

The parameter values are chosen to give sample values for samples of size 100 corresponding to Standard and Poor's statistics reported in Shiller (1981*b*, p. 431), namely, $E(p) = 145.5$, $E(d) = 6.989$, and $\sigma(p) = 50.12$.[9] Shiller estimates r by

$$r = \frac{E(p_t)}{E(d_t)},$$

and this procedure is also adopted in the simulations. The implied innovation variance σ_ϵ^2 is calculated by inverting (13), which gives the variance of the price process (6) in terms of ρ, r, and σ_ϵ^2.[10]

7. For details of this subroutine, see Kleidon (1983, app. C). The results were checked against those from the IMSL subroutine GGNML, which produced similar results. Further, diagnostic checks show that Standard and Poor's price data are consistent with an assumption of normality of the innovations ϵ_t in (6), although such an assumption does not appear crucial for the conclusions reached here. The tests were repeated using an alternate Student *t*-distribution with kurtosis of 9 (i.e., extremely fat tails), and although there is a slight increase in the ratios of sample standard deviation of price to that of p_t^* relative to those produced under the normality assumption, the conclusions are unchanged.

8. An earlier version of this paper reports the effect of the distance from the population mean of this initial draw (see Kleidon 1982). The procedure here is similar to that in Flavin (1983).

9. This assumes that the unconditional population variance of price is constant over the relevant interval. Although there is some evidence that conditional variances are nonconstant for Standard and Poor's series, this is not sufficient to rule out constant unconditional variances (see, e.g., Blattberg and Gonedes 1974). Further, the simulations are conservative since they ignore the apparent smoothing in Standard and Poor's dividend series, which shows much lower innovation variance since 1950 than either prices or earnings.

10. Note that this procedure results in inferred innovation variances σ_ϵ^2 that decrease in ρ since they are generated so that the resulting price variance in a sample of 100

However, the small sample bias that this paper addresses creates a problem when processes are simulated to produce the sample values found in Standard and Poor's data. If one were to simulate a series with population values equal to those given above, the resulting *sample* values for the simulated samples of 100 observations may be different from the population values. In particular, setting the population innovation variance such that the population price standard deviation is 50.12 results in samples of size 100 with *lower* sample standard deviation. The population innovation variance was increased in these simulations so that the resulting mean sample variance for samples of 100 matched that for the reported Standard and Poor's sample.[11]

Since the relative smoothness of the series and the degree of the subsequent small sample bias is an increasing function of the AR(1) parameter ρ in the price process (6), this section gives plots of the price and p_t^* series for $\rho = 0.80$ and $\rho = 0.99$, and the simulations in Section III*C* give results for $\rho = 0.80$, 0.90, 0.95, and 0.99. To illustrate the effect of ρ on the relative smoothness of price and p_t^*, assume $r = .065$, and note that, for $\rho = 0.80$, the population values from (16) and (17) for conditional variances indicating the smoothness of price and p_t^* are $\text{var}(p_t|p_{t-1}) = 9.11\,\sigma_\epsilon^2$ and $\text{var}(p_t^*|p_{t-1}^*) = 2.02\,\sigma_\epsilon^2$, while, for $\rho = 0.99$, these values are $174.24\,\sigma_\epsilon^2$ and $6.49\,\sigma_\epsilon^2$, respectively. Clearly, there is greater relative smoothness of p_t^* compared with p_t for $\rho = 0.99$ than for $\rho = 0.80$. Similar differences are apparent in the small sample biases reported in Section III*C*, where the greater relative smoothness of p_t^* for large ρ results in a larger bias.

It is natural to ask what the empirical value of ρ is for the data used to test the bound (4). The cited tradition in financial economics, dating from at least the results of Working (1934; cited in Roberts 1959, p. 2) is that prices follow a nonstationary random walk; that is, $\rho = 1.0$.[12] Kleidon (in press, sec. 3.3) tests this hypothesis directly and cannot reject the null hypothesis $\rho = 1.0$ for Standard and Poor's annual (deflated) price series from 1926–79 for this series augmented with Cowles Commission data from 1871–1925 (giving a total series from 1871–1979) or for Standard and Poor's quarterly series from 1947:1 to

corresponds to the observed Standard and Poor's price sample variance, and equation (13) shows that the innovation variance must decrease in ρ to imply the same fixed price variance.

11. The procedure followed was initially to set the population price variance as 2,512.01 and then to calculate the average sample price variance for samples of size 100 across 1,000 replications, for each value of ρ examined. Since the small sample bias is a function of ρ, the resulting mean sample variances, although all less than 2,512.01, were not uniformly downward biased. The population price variance was then increased by the ratio of the desired sample variance (2,512.01) to the initial mean sample variances. In the resulting simulations, the average sample price variances for all values of ρ appear sufficiently close to the desired value of 2,512.01 to justify the simple adjustment used.

12. Often a geometric random walk is assumed, i.e., a random walk in logs, which has the desirable property that the expected capital gain rate is not inversely related to price level as in the arithmetic random walk.

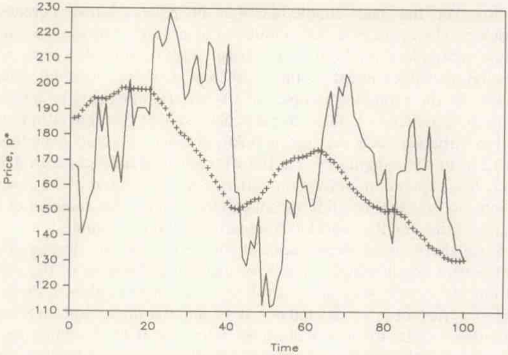

FIG. 1.—Simulated series of rational, stationary AR(1) prices (solid line) perfect foresight prices, $\rho = 0.99$. Note that the relevant processes are

$$d_t = \mu_d + \rho d_{t-1} + \epsilon_t,$$
$$p_t = \mu_p + \rho p_{t-1} + a\epsilon_t,$$
$$p_t^* \equiv (p_{t+1}^* + d_{t+1})/(1 + r),$$
$$p_T^* \equiv p_T,$$

where ϵ_t i.i.d. $N(0, \sigma^2)$, $a \equiv \rho/(1 + r - \rho)$, and r is a constant discount rate. The parameter values are chosen to give sample values for samples of size 100 corresponding to Standard and Poor's statistics in Shiller (1981*b*), namely, $E(p) = 145.5$, $E(d) = 6.989$, $\sigma(p) = 50.12$, and $r \equiv E(d)/E(p)$.

1978:4. Consistent with the well-known small sample problem in the estimation of ρ, which causes sample estimates $\hat{\rho}$ to be centered at values less than 1.0 when the population $\rho = 1.0$, the point estimates $\hat{\rho}$ for the three samples examined in Kleidon (in press) are 0.90, 0.91, and 0.97, respectively. Consequently, the results based on $\rho = 0.99$ are regarded as empirically more relevant than those for $\rho = 0.80$, which are given for comparison.[13]

Figures 1 and 2 give plots of price and p_t^* for $\rho = 0.99$ and $\rho = 0.80$, respectively. The most obvious feature of figure 1 is that the series p_t^* is much smoother than the price series (the solid line), which is to be expected, given the previous analysis of the conditional variances $\text{var}(p_t|p_{t-k})$ and $\text{var}(p_t^*|p_{t-k}^*)$ when prices are constructed from the valuation model (1) with constant discount rates. Although figure 2 also

13. Other empirical difficulties with the specification $\rho = 0.80$ are also evident. For example, the high innovation variance σ_ϵ^2 for the $\rho = 0.80$ case necessary to reconcile the process (6) with the observed data (see nn. 10 and 11 above) sometimes results in $p_t^* < 0$.

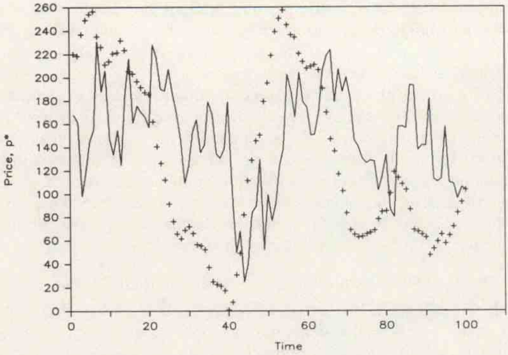

FIG. 2.—Simulated series of rational, stationary AR(1) prices (solid line) and perfect foresight prices, $\rho = 0.80$. Note that the relevant processes are

$$d_t = \mu_d + \rho d_{t-1} + \epsilon_t$$
$$p_t = \mu_p + \rho p_{t-1} + a\epsilon_t$$
$$p_t^* \equiv (p_{t+1}^* + d_{t+1})/(1 + r)$$
$$p_T^* \equiv p_T,$$

where ϵ_t i.i.d. $N(0, \sigma^2)$, $a \equiv \rho/(1 + r - \rho)$, and r is a constant discount rate. The parameter values are chosen to give sample values for sample size 100 corresponding to Standard and Poor's statistics in Shiller (1981*b*), namely, $E(p) = 145.5$, $E(d) = 6.989$, $\sigma(p) = 50.12$, and $r \equiv E(d)/E(p)$.

shows less short-term variation in p_t^* than in price, the phenomenon is much less pronounced than in figure 1.

Section II*D* shows that it is more difficult to estimate the population mean for p_t^* than for p_t even in large samples because of the time series properties of the two series. The problem is exacerbated in small samples and strongly affects attempts to estimate unconditional variances. The issues are conveniently illustrated with reference to figure 1. Suppose that only the first 15 observations from figure 1 were used to construct sample variances for both p_t (the solid line) and p_t^*. Clearly, a sample variance calculated as squared deviations from the sample mean would be considerably lower for this sample of p_t^* than for the corresponding sample of prices p_t. In contrast, assume deviations are taken around the population mean rather than the sample mean. Since all the first 15 observations of p_t^* in figure 1 are greater than 185, while the population mean is 145, each observation contributes substantially to the true unconditional variance—something that would be badly misestimated using just the sample mean. On the other hand, although the price series for the first 15 observations has larger deviations around its sample mean than does p_t^*, the sum of squared deviations from the population mean of 145 is less for these price series observations than for the corresponding p_t^*.

Clearly tests of the bound (4) based on sample variances can be very misleading. For this sample, exactly the opposite conclusions would be reached using estimators of the unconditional variances based on squared deviations from the sample versus the true (but unknown) population mean. The former estimates would suggest that the bound (4) is violated in the sample, while the latter would not. The use of sample variances to test (4) is further complicated by the arbitrary terminal condition $p_T^* \equiv p_T$, which is now considered.

B. *The Terminal Condition* $\mathrm{p}_T^* \equiv \mathrm{p}_T$ *and Conditional Variances*

As shown in Section II*C* above, $\mathrm{var}(p_t^*|p_{t-k}^*)$ is less than $\mathrm{var}(p_t|p_{t-k})$ for k small.[14] This section examines the implications of the consequent smoothness of p_t^*, and the terminal condition $p_T^* \equiv p_T$, for tests of the inequality (4) based on sample variances. At one extreme, if the sample size is very large, the relative smoothness of the p_t^* series and the particular value of p_T^* have negligible effects on the use of sample variances as estimators of population unconditional variances for the stationary price and p_t^* processes examined here. At the other extreme, for a sample of two observations,[15] this section shows analytically that the terminal condition combined with the greater smoothness of p_t^* results in a very severe small sample bias in tests of (4) with sample variances. An understanding of the two-observation case makes the issues (if not the exact analytic results) clear as the sample size grows, and Section III*C* completes the analysis with simulation results for various sample sizes and parameter values.

Assume observations of dividends and price at $t = 0$ and $t = 1$, with the terminal condition $p_1^* \equiv p_1$. The sample variance of p_t will exceed that of p_t^* if $|p_0 - p_1| > |p_0^* - p_1^*|$. Clearly, the probability that sample variances will show this apparent violation of (4) is directly linked to the conditional distributions of p_0 and p_0^*, given $p_1 = p_1^*$. We have the following result for reasonable values of r and ρ:

PROPOSITION 6. Assume two observations of dividends d_0 and d_1 given by the AR(1) dividend process (5), prices p_0 and p_1 given by (6), and p_t^* given by (9) subject to the terminal condition $p_1^* \equiv p_1$; that $r > 0$ and $1 > \rho > 0.5$; and that, in (5), $\epsilon_t > 0$ and $\epsilon_t < 0$ each occur with probability .5. Then the sample variance of price will exceed that of p_t^* with probability greater than .5 and increasing in ρ.

14. "Small" is relative since Kleidon (in press, fig. 6) shows that, if $\rho = 0.99$ in the dividend process (5), $\mathrm{var}(p_t|p_{t-k})$ exceeds $\mathrm{var}(p_t^*|p_{t-k}^*)$ for k as large as 60.

15. I am grateful to Fischer Black for suggesting an examination of samples of this size.

Proof. See the Appendix.

This result is driven by the greater smoothness of p_t^*, which implies that the (absolute) difference between p_0 and p_1 tends to exceed that for p_t^*. Further, the small sample bias increases in ρ since the relative smoothness of p_t^* increases in ρ. The effect of proposition 6 is strengthened by the simulation results in III*C* below, which show that for relevant parameter values the probability of apparent rejection of (4) using sample variances approaches 1.0 for the two-observation case.

The analysis gets more complicated as the sample size increases,[16] but the issues are clear. Observations p_{T-k} and p_{T-k}^* for $k > 1$ are drawn from the corresponding conditional distributions, given $p_T^* \equiv p_T$, but the sample variances are now functions of outcomes from several conditional distributions, which also result in different sample means of p_t and p_t^*. As the sample size increases, the arbitrary terminal condition becomes less important, the sample means become better estimators of the population means, and, consequently, the sample variances become better estimators of the unconditional variances used in the bound (4). What is not yet clear, but is essential for an interpretation of the empirical results reported in the literature, is the behavior of sample variances for sample sizes used in the current tests. The next section shows via simulation the extent of the problem in samples of different sizes for various parameter values.

C. Simulation Results

Section II derives a consistent set of stationary dividend, price, and p_t^* processes:

$$d_t = \mu_d + \rho d_{t-1} + \epsilon_t, \tag{5}$$

$$p_t = \mu_p + \rho p_{t-1} + \delta_t, \tag{6}$$

$$p_{t+1}^* = p_t^*(1 + r) - d_{t+1}, \tag{10}$$

where ϵ_t is assumed i.i.d. $N(0, \sigma_\epsilon^2)$, $\mu_p \equiv \mu_d/r$, $\delta_t \equiv \epsilon_t[\rho/(1 + r - \rho)]$.

This section reports the results of Monte Carlo simulations of these processes. First, the small sample bias derived above for a sample of two observations persists in larger samples and for relevant parameter values is severe. Second, the problem is related to difficulty in estimating the population (unconditional) mean, which is greater for p_t^* than for p_t. Third, although the problem is severe, it does not appear to explain the gross violations reported in the literature.

The details of the simulation procedures are given in Section III*A* above. To investigate the importance of the estimation of the population mean in the estimation of population variances by sample vari-

16. Kleidon (1982) shows the analysis for a sample of three observations.

ances, we calculate "true mean" variances defined as mean squared deviations about the population mean rather than the sample mean. For both usual sample variances and mean squared deviations about the population mean, the variance bound

$$\mathrm{var}(p_t) \leq \mathrm{var}(p_t^*) \tag{4}$$

is deemed violated if the sample point estimates do not satisfy this (weak) inequality. Table 1 reports the frequency of violations (across 100 replications) for both the usual sample variances and "true mean" variances about the population mean, for various sample sizes (from 2 to 3,000) and various values of ρ (from 0.80 to 0.99).

Table 1 shows three main results. First, for all values of ρ, the use of sample variances results in frequent rejection of the bound (4) in small samples, even when the price series is constructed to satisfy the valuation model (1). Second, the bias is clearly a function of ρ. For large ρ, the bias persists in even very large samples and is severe for a sample size of 100, while for $\rho = 0.80$ the problem appears negligible for samples of 100. Third, the problem of estimation of the mean is a major contributor to the bias. When population means are used in the "true mean" variances, the degree of incorrect rejection of the bound (4) is reduced substantially.[17]

The bias is investigated in more detail in table 2, which reports the mean and standard deviation across 100 replications of the sample means and sample variances of p_t and p_t^*, for $\rho = 0.99$. The higher sampling variance of the mean of p_t^* relative to that of p_t, discussed in Section II*D* above, is apparent in table 2. Further, the sample variance is a downward-biased estimator of the population variance in small samples of both p_t and p_t^*, but the problem is more severe for p_t^* with a consequent bias in the test (4). For example, for samples of 100, the mean sample variances are 2,385.70 and 1,614.17 for p_t and p_t^*, respectively, whereas the population variances are 9,135.32 and 11,214.57. A similar result holds for other values of ρ, but again the bias is an increasing function of ρ.

Although tables 1 and 2 establish the existence of the bias, they do not address the issue of the magnitude by which the inequality is typically violated. Given the gross violations reported in the literature—Shiller (1981*b*) reports that the ratio of the sample standard deviation of price to the sample standard deviation of p_t^* for Standard and Poor's data is 5.59—table 3 presents summary statistics for this ratio across 1,000 replications for various values of ρ. It is evident from table 3 that the small sample bias alone does not seem to explain the apparent gross

17. For the two-observation case, analysis similar to proposition 6 shows that the probability of violation of (4) based on "true mean" variances is less than .5 and increasing in ρ.

TABLE 1 **Percentage* of Violations of Sample and True Mean† Variance Bounds for Simulated Rational AR(1) Price Series‡**

	$\rho = .80$		$\rho = .90$		$\rho = .95$		$\rho = .99$	
Sample Size (T)	Sample Variance	True Mean Variance	Sample Variance	True Mean Variance	Sample Variance	True Mean Variance	Sample Variance	True Mean Variance
2	84	40	92	47	94	47	99	51
5	81	34	95	43	99	45	100	49
10	71	28	91	34	99	41	100	42
50	11	1	40	14	77	37	99	50
100	1	0	11	3	51	12	82	35
200	0	0	2	2	15	6	78	34
1,000	0	0	0	0	0	0	13	5
2,000	0	0	0	0	0	0	6	2
3,000	0	0	0	0	0	0	2	2

* Based on 100 replications.
† "True mean variances" are calculated as mean squared deviations from the population mean, not the sample mean as in "sample variances."
‡ The relevant processes are

$$d_t = \mu_d + \rho d_{t-1} + \epsilon_t,$$

ϵ_t i.i.d. $N(0, \sigma^2)$, and

$$p_t = \mu_p + \rho p_{t-1} + a\epsilon_t,$$

where $a \equiv \rho/(1 + r - \rho)$ and r = a constant discount rate. The parameter values are chosen to give sample values for samples of size 100 corresponding to Standard and Poor's statistics in Shiller (1981*b*), namely, $E(p) = 145.5$, $E(d) = 6.989$, $\sigma(p) = 50.12$, and $r \equiv E(d)/E(p)$. The "perfect foresight" series is defined as

$$p_t^* = (p_{t+1}^* + d_{t+1})/(1 + r),$$
$$p_T^* \equiv p_T.$$

TABLE 2 Percentage* of Violations of Variance Bounds and Summary Statistics of Distribution of Sample Mean and Sample Variance of Simulated Rational Price Series with First Order Autocorrelation (ρ) .99 and Constructed "Perfect Foresight" Series†

		Rational Price p				Perfect Foresight p^*				Ratio of Sample S.D. (p) to Sample S.D. (p^*)	
		Sample Mean ($\bar{p}$)		Sample Variance (var[p])		Sample Mean ($\bar{p}^*$)		Sample Variance (var[p^*])			
Sample Size (T) ($\rho = 0.99$)	Violations (%)	Average‡	S.D.‡	Average	S.D.	Average	S.D.	Average	S.D.	Average	S.D.
5	100	145.55	54.68	198.38	181.99	145.87	60.31	1.62	2.50	16.60	10.36
10	100	143.75	57.99	378.09	312.86	139.89	70.66	12.19	15.88	8.31	4.88
50	99	137.47	67.13	1,289.60	1,042.81	134.94	83.03	412.61	593.78	2.29	.90
100	82	132.90	69.91	2,385.70	1,782.87	126.98	90.41	1,614.17	1,722.06	1.47	.50
200	78	134.14	70.62	3,597.69	1,891.22	132.18	87.04	3,242.19	2,457.88	1.19	.30
1,000	13	142.76	38.26	7,773.82	3,733.63	142.21	46.28	9,200.47	5,342.19	.94	.06
2,000	6	143.55	27.78	8,274.78	2,626.09	143.24	33.77	9,953.71	3,737.17	.92	.04
3,000	2	143.65	22.38	8,590.04	2,272.17	143.37	27.19	10,388.41	3,235.92	.92	.03
Population parameter		145.5		9,135.32		145.5		11,214.57			
Large sample value		145.5	$1{,}348/\sqrt{T}$			145.5	$1{,}647/\sqrt{T}$				

* Based on 100 replications.

† The relevant processes are

$$d_t = \mu_d + \rho d_{t-1} + \epsilon_t,$$

ϵ_t i.i.d. $N(0, \sigma^2)$, and

$$p_t = \mu_p + \rho p_{t-1} + a\epsilon_t,$$

where $a \equiv \rho/(1 + r - \rho)$ and r = a constant discount rate. The parameter values are chosen to give sample values for samples of size 100 corresponding to Standard and Poor's statistics in Shiller (1981*b*), namely, $E(p) = 145.5$, $E(d) = 6.989$, $\sigma(p) = 50.12$, and $r \equiv E(d)/E(p)$. The "perfect foresight" series is defined as

$$p_t^* = (p_{t+1}^* + d_{t+1})/(1 + r),$$
$$p_T^* \equiv p_T.$$

‡ All summary statistics are for the distribution across 100 replications.

TABLE 3 **Summary Statistics of the Distribution of the Ratio of Sample Standard Deviation of Price to Sample Standard Deviation of "Perfect Foresight" Price, across 1,000 Replications for Sample Size 100, for Simulated Rational AR(1) Price Process***

Assumed ρ	Violations (%): Ratio > 1.0	Ratio Mean†	Ratio Standard Deviation	Minimum Ratio	Maximum Ratio	Percentile 50th (Median)	Percentile 90th	Percentile 95th
.80	.4	.53	.14	.26	1.12	.51	.72	.79
.90	14.0	.77	.21	.38	1.98	.73	1.05	1.18
.95	44.2	1.02	.29	.48	2.86	.96	1.38	1.57
.99	88.0	1.50	.49	.76	3.86	1.42	2.15	2.50

* The relevant processes are

$$d_t = \mu_d + \rho d_{t-1} + \epsilon_t,$$

ϵ_t i.i.d. $N(0, \sigma^2)$, and

$$p_t = \mu_p + \rho p_{t-1} + a\epsilon_t,$$

where $a \equiv \rho/(1 + r - \rho)$ and r = a constant discount rate. The parameter values are chosen to give sample values for samples of size 100 corresponding to Standard and Poor's statistics in Shiller (1981*b*), namely, $E(p) = 145.5$, $E(d) = 6.989$, $\sigma(p) = 50.12$, and $r \equiv E(d)/E(p)$. The "perfect foresight" series is defined as

$$p_t^* = (p_{t+1}^* + d_{t+1})/(1 + r),$$
$$p_T^* \equiv p_T.$$

† All summary statistics are for the distribution across 1,000 replications.

violations of (4). Even for $\rho = 0.99$, the maximum in 1,000 simulations is only 3.86.

As a further check on the probability of observing a ratio of 5.59 in samples of 100 observations if the underlying processes were those simulated here, the experiment was replicated 100,000 times. In all cases the ratio was less than 5.59, and the highest ratios for the four values of ρ examined here were 1.84, 2.33, 3.35, and 5.42. The observed ratio of 5.59 is indeed extremely unlikely, *given* the maintained hypothesis of this paper that dividends and prices follow the stationary and ergodic processes (5) and (6).

IV. Conclusions

This paper derives the rational price and "perfect foresight" p_t^* processes, given a stationary AR(1) dividend process (and assuming that information comprises current and past dividends). Although the variance bound (4) holds in terms of the unconditional population variances of price and p_t^*, the time series properties of the two series are sufficiently different that inferences from time series observations about the validity of the inequality (4) can be very misleading. This paper focuses on a small sample bias in tests of (4) based on sample variances of price and p_t^*, even if by construction dividends follow a stationary and ergodic process and prices are given by the valuation model (1). This in turn can be traced to the lower short-term variation that is to be expected in time series observations of p_t^* relative to p_t if prices are set by (1).

However, the results of table 3 show that the small sample bias, at least by itself, does not seem sufficient to account for the reported gross violations of the bound. One alternative possible explanation concerns the assumption of a constant discount rate r in (1), as discussed, for example, by LeRoy and LaCivita (1981) and Michener (1982), although this does not seem to explain the violations (see Shiller 1981*a*). Other potential problems (cited in n. 2 above) include dividend smoothing and the "peso problem," both of which imply that the sample of observed dividends may not allow adequate estimation of the variances in the inequality (4). But perhaps the most important issue is that the assumption in this paper of a stationary price process is not the only one consistent with the data. As shown in Kleidon (in press), one cannot reject the null hypothesis that prices follow a nonstationary random walk. In that case, the use of sample variances to estimate population unconditional variances is invalid, and Kleidon (in press) demonstrates that the apparent gross violations of the inequality (4) are to be expected if procedures that assume stationarity are inadvertently applied to nonstationary series.

Consequently, the stock price variance bounds tests provide distinct results from those based on bonds, which apparently may be caused by either nonstationarity of interest rates or small sample bias. At least for stock prices, the explanation for the variance bound violations does not seem to lie in small sample bias alone.

Appendix

This appendix provides proofs of propositions 3, 4, and 6.

PROPOSITION 3. Given the price process (6), the variance of the sample mean of price in large samples, $\mathrm{var}(\bar{p})$, is given by

$$\mathrm{var}(\bar{p}) \approx \frac{\rho^2\sigma_\epsilon^2}{T(1 + r - \rho)^2(1 - \rho)^2}, \tag{18}$$

where T is the sample size.

Proof. By definition,

$$\begin{aligned}\mathrm{var}(\bar{p}) &= \frac{1}{T^2}\,\mathrm{var}\sum_{t=1}^{T} p_t \\ &= \frac{1}{T^2}\,\boldsymbol{\iota}'\boldsymbol{\Omega}\boldsymbol{\iota},\end{aligned} \tag{A1}$$

where $\boldsymbol{\iota}' \equiv (1\ 1 \ldots 1)$ (a $1 \times T$ vector) and $\boldsymbol{\Omega} \equiv$ the $T \times T$ covariance matrix of prices. The term $\boldsymbol{\iota}'\boldsymbol{\Omega}\boldsymbol{\iota}$ can be expanded as

$$\boldsymbol{\iota}'\boldsymbol{\Omega}\boldsymbol{\iota} = [T\gamma_0 + 2(T - 1)\gamma_1 + 2(T - 2)\gamma_2 + \ldots + 2\gamma_{T-1}], \tag{A2}$$

where $\gamma_k \equiv$ covariance (p_t, p_{t-k}). Since prices follow the stationary AR(1) process (6), we have

$$\begin{aligned}\gamma_k &= \rho^k \frac{\sigma_\delta^2}{(1 - \rho^2)} \\ &\approx 0 \text{ for } k \text{ large},\end{aligned}$$

where $\sigma_\delta^2 \equiv$ the variance of δ_t in (6). Hence in large samples we can approximate (A2)[18] by

$$\begin{aligned}\boldsymbol{\iota}'\boldsymbol{\Omega}\boldsymbol{\iota} &\approx T(\gamma_0 + 2\gamma_1 + 2\gamma_2 + \ldots) \\ &= T\sum_{-\infty}^{\infty} \gamma_k.\end{aligned} \tag{A3}$$

It is convenient to evaluate (A3) by use of the autocovariance generating function (Box and Jenkins 1976, eq. 3.1.11, p. 49):

$$\gamma(B) = \sum_{-\infty}^{\infty} \gamma_k B^k. \tag{A4}$$

18. The precision of the approximation is a function of ρ as well as the sample size, and although the large sample variance of the sample mean derived here is verified for all values of ρ examined in the simulations reported in Section III*C*, the required sample size increases in ρ.

If we write a general stationary and invertible ARMA process (ignoring irrelevant means) as

$$\phi(B)p_t = \theta(B)a_t, \tag{A5}$$

where

$$\phi(B) \equiv (1 - \phi_1 B - \phi_2 B^2 - \ldots)$$

and

$$\theta(B) \equiv (1 - \theta_1 B - \theta_2 B^2 - \ldots),$$

then we can equivalently give the infinite moving average (MA) representation of (A5) as

$$\begin{aligned} p_t &= \phi^{-1}(B)\theta(B)a_t \\ &= \pi(B)a_t \\ &= \sum_{k=0}^{\infty} \pi_k a_{t-k}, \end{aligned} \tag{A6}$$

where

$$\begin{aligned} \pi(B) &\equiv \frac{\theta(B)}{\phi(B)} \\ &\equiv (1 + \pi_1 B + \pi_2 B^2 + \ldots). \end{aligned}$$

Equation (A6) implies the following covariances:

$$\gamma_k = \sigma_a^2 \sum_{j=0}^{\infty} \pi_j \pi_{k+j}, \tag{A7}$$

where $\sigma_a^2 \equiv$ the variance of a_t in (A5). Substitution from (A7) into (A4) gives (Box and Jenkins 1976, app. A3.1, p. 81):

$$\gamma(B) = \sigma_a^2 \pi(B)\pi(B^{-1}). \tag{A8}$$

Equation (A8) allows easy evaluation of the approximation to $\boldsymbol{\iota}'\boldsymbol{\Omega}\boldsymbol{\iota}$ given by (A3). For the AR(1) process (6), $\pi(B) = (1 - \phi_1 B)^{-1}$. Hence (A4) and (A8) give

$$\begin{aligned} \sum_{-\infty}^{\infty} \gamma_k &= \gamma(1) \\ &= \sigma_a^2 \pi(1)\pi(1) \\ &= \frac{\sigma_a^2}{(1 - \phi_1)^2}. \end{aligned}$$

For the AR(1) price process given by (6), $\phi_1 = \rho$ and $\sigma_a^2 = \rho^2\sigma_\epsilon^2/(1 + r - \rho)^2$. Hence (A3) gives

$$\boldsymbol{\iota}'\boldsymbol{\Omega}\boldsymbol{\iota} \approx \frac{T\,\rho^2\sigma_\epsilon^2}{(1 + r - \rho)^2(1 - \rho)^2}, \tag{A9}$$

and substitution from (A9) for $\boldsymbol{\iota}'\boldsymbol{\Omega}\boldsymbol{\iota}$ in (A1) gives (18).

PROPOSITION 4. Given the p_t^* process (9), the variance of the sample mean of p_t^* in large samples, $\text{var}(\bar{p}^*)$, is given by

$$\text{var}(\bar{p}^*) \approx \frac{\sigma_\epsilon^2}{T(1 - \rho)^2 r^2}. \tag{19}$$

Proof. The proof is similar to that for proposition 3. We must evaluate the covariance matrix of p_t^* corresponding to $\mathbf{\Omega}$ in (A1) for prices p_t. Since p_t^* follows the AR(2) process in (9), we have:

$$\pi(B) = (1 - \phi_1 B - \phi_2 B^2)^{-1},$$

and using (A4) and (A8) as before gives

$$\text{var}(\bar{p}^*) \approx \frac{\sigma_a^2}{T(1 - \phi_1 - \phi_2)^2},$$

which for the parameters in (9) gives (19).

PROPOSITION 6. Assume two observations of dividends d_0 and d_1 given by the AR(1) dividend process (5), prices p_0 and p_1 given by (6), and p_t^* given by (9) subject to the terminal condition $p_1^* \equiv p_1$; that $r > 0$ and $1 > \rho > 0.5$; and that, in (5), $\epsilon_t > 0$ and $\epsilon_t < 0$ each occur with probability .5. Then the sample variance of price will exceed that of p_t^* with probability greater than .5 and increasing in ρ.

Proof. Define $d_0 - E(d_t) \equiv \beta_0$, where $E(d_t)$ is the population (unconditional) mean. Given the assumed conditions, it can be verified that $a \equiv \rho/(1 + r - \rho) > 1$ and that

$$p_0 = E(p_t) + a\beta_0, \tag{A10}$$

$$d_1 = E(d_t) + \rho\beta_0 + \epsilon_1, \tag{A11}$$

$$p_1 = E(p_t) + a(\rho\beta_0 + \epsilon_1), \tag{A12}$$

$$p_0^* = E(p_t) + a\left(\beta_0 + \frac{\epsilon_1}{\rho}\right). \tag{A13}$$

The sample variance of p_t exceeds that of p_t^* if $|p_1 - p_0| > |p_1 - p_0^*|$, given $p_1 \equiv p_1^*$. From (A10) to (A13) it follows that

$$|p_1 - p_0| = |a[\beta_0(1 - \rho) - \epsilon_1]|, \tag{A14}$$

$$|p_1 - p_0^*| = \left|a(1 - \rho)\left(\beta_0 + \frac{\epsilon_1}{\rho}\right)\right|. \tag{A15}$$

We proceed for the case $\beta_0 > 0$, and note that the analysis is symmetric for $\beta_0 < 0$.

Since (A14) and (A15) involve ϵ_1, the first period innovation, we examine separately the cases $\epsilon_1 > 0$ and $\epsilon_1 < 0$.

CASE A: $\epsilon_1 > 0$.

We have

$$|p_1 - p_0| = \begin{cases} a[\beta_0(1 - \rho) - \epsilon_1] & \text{if } \beta_0(1 - \rho) \geq \epsilon_1, \\ a[\epsilon_1 - \beta_0(1 - \rho)] & \text{otherwise;} \end{cases}$$

$$|p_1 - p_0^*| = a(1 - \rho)\left(\beta_0 + \frac{\epsilon_1}{\rho}\right).$$

Hence the bound (4) is violated if either

$$a[\beta_0(1 - \rho) - \epsilon_1] > a(1 - \rho)\left(\beta_0 + \frac{\epsilon_1}{\rho}\right),$$

which is impossible for $a > 0$, given $\beta_0 > 0$ and $\epsilon_1 > 0$, or

$$a[\epsilon_1 - \beta_0(1 - \rho)] > a(1 - \rho)\left(\beta_0 + \frac{\epsilon_1}{\rho}\right),$$

that is,

$$\epsilon_1(2\rho - 1) > 2\beta_0\rho(1 - \rho),$$

which is possible (but not necessary). Since $\beta_0(1 - \rho) < \epsilon_1$ to reach this inequality, violations increase in ρ.

CASE B: $\epsilon_1 < 0$.

We have

$$|p_1 - p_0| = a[\beta_0(1 - \rho) - \epsilon_1],$$

$$|p_1 - p_0^*| = \begin{cases} a(1 - \rho)\left(\beta_0 + \dfrac{\epsilon_1}{\rho}\right) & \text{if } \beta_0\rho \geq -\epsilon_1, \\ a(1 - \rho)\left(-\beta_0 - \dfrac{\epsilon_1}{\rho}\right) & \text{otherwise.} \end{cases}$$

Hence (4) appears violated if

$$a[\beta_0(1 - \rho) - \epsilon_1] > a(1 - \rho)\left(\beta_0 + \frac{\epsilon_1}{\rho}\right),$$

which is always true for $a > 0$, given $\beta_0 > 0$ and $\epsilon_1 < 0$, or if

$$a[\beta_0(1 - \rho) - \epsilon_1] > a(1 - \rho)\left(-\beta_0 - \frac{\epsilon_1}{\rho}\right),$$

that is,

$$2\beta_0\rho(1 - \rho) > \epsilon_1(2\rho - 1),$$

which is always true for $1.0 > \rho > 0.5$, given $\beta_0 > 0$ and $\epsilon_1 < 0$. Hence for $\beta_0 > 0$ (equivalently, $p_0 > E[p_t]$), we have

(*a*) $\epsilon_1 > 0$: violation of bound if $\epsilon_1(2\rho - 1) > 2\beta_0\rho(1 - \rho)$ and increasing in ρ; and

(*b*) $\epsilon_1 < 0$: always violates if $1 > \rho > 0.50$.

The analysis is symmetric for $\beta_0 < 0$.

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