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Option Bounds in Discrete Time: Extensions and the Pricing of the American Put*

This paper generalizes and tightens the discrete-time bounds for European options and develops similar bounds for American puts. The stock return has a distribution with a finite range and is subject to the restriction that the stock be "nonnegative beta." The bounds are derived from arbitrage portfolios using the stock, the option, and the riskless asset. They converge to the previous ones when the range coincides with the nonnegative real line, and they both become equal to the binomial option model under a two-state stock return. Upper and lower bounds for the American put are derived recursively by the same method, with arbitrage portfolios involving the stock, the riskless asset, and American and European puts. An example is provided with a trinomial return distribution.

I. Introduction

In a recent paper (Perrakis and Ryan 1984), upper and lower bounds on European option prices were derived in discrete time. These bounds were based on an extension of the discrete-time derivation of Black-Scholes pricing, first presented in the work of Black and Scholes (1973). The bounds are functions of the stock and exercise prices, the riskless rate of interest, the time to maturity, and the entire distribution of stock returns. In other words, they need more information than Black-Scholes for their computation, and the derived results are weaker.[1] On the other hand, they are significantly more general in their assumptions, insofar as they do not rely on the continuous-time riskless hedge of the original Black-Scholes derivations or on the assumption of a binomial distribution of stock returns used in Cox, Ross, and Rubinstein (1979) and Rendleman and Bartter (1979).

For this reason they are likely to be particularly useful in valuing options on assets, for which the Black-Scholes methodology is clearly

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1. Note, however, that this is also true of other Black-Scholes generalizations, such as Merton (1976).

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improper. Such assets are, for instance, thinly traded stocks whose one-period returns have, at best, a trinomial distribution, up, down, and stay the same. Risky ventures are another example; in them there is a positive probability of an absorbing state (bankruptcy) after one period. In both cases exact pricing is infeasible in discrete time, as already noted in Cox et al. (1979) for the trinomial example.

In this paper the method and results of Perrakis and Ryan (1984) are extended along two different dimensions. First of all, the bounds presented in Perrakis and Ryan are tightened up significantly under the same assumptions and analysis, provided that the range of the distribution of the one-period returns per dollar invested in the optioned stock is finite and has a strictly positive lower limit. These restrictions are eminently reasonable for many (and perhaps most) stock returns over "short" time periods, such as the time to expiration of most traded options. Under these restrictions new, tighter bounds hold for European option prices under the same assumptions as in Perrakis and Ryan, bounds that coincide with the earlier results when the lower and upper limits of the possible stock returns become zero and infinity, respectively. Finally, these new, improved bounds converge to a single value when the one-period stock return distribution assumes exactly two possible values, as in Cox et al. (1979) and Rendleman and Bartter (1979). Thus it is established that the results of Perrakis and Ryan, as extended here, are a valid generalization of the two-state option models of Cox et al. and Rendleman and Bartter.

Second, the methodology and European option bounds are used to develop similar bounds on American put options. Ever since Merton's (1973) paper, it has been known that American puts, unlike American calls, cannot be priced like their European counterparts, even if the stock pays no dividends until expiration of the option.[2] Even under this latter condition the American put has a positive probability of exercise before expiring. Hence, its value will be higher than that of the corresponding European option. For these reasons applications of the Black-Scholes methodology to American put options have not yielded, until now, analytical formulas, even though approximate solutions to the problem of their valuation were provided in the work of Brennan and Schwartz (1977) and Parkinson (1977) by numerical methods. In this paper, bounds for American put options are developed within the European option model of Perrakis and Ryan, as extended in this paper.

The organization of this paper is as follows. Section II summarizes

2. Under the assumption of no dividends till expiration (or of payout protected stocks) it was shown in Merton (1973) that it would never pay to exercise an American call before expiration, which means that its value would then be equal to that of the European call.

the assumptions, methodology, and basic results of Perrakis and Ryan (1984) and develops the improved European option bounds. Section III derives the bounds for American puts, while Section IV presents a numerical example with a trinomial distribution.

II. The Improved Bounds

The following notation, broadly similar to that in Perrakis and Ryan (1984), is used:

S = the initial price of the underlying asset;
X = the exercise price of the option;
i = the riskless rate of interest per period (assumed constant without loss of generality);
V = the random stock return per dollar invested per period, with range $[\underline{v}, \bar{v}]$, where $1 + i > \underline{v} \geq 0$ and $1 + i < \bar{v} \leq \infty$;
$G(v)$ = the distribution of V,[3] assumed independent of S;[4]
$\left.\begin{matrix} C_n(S, X) \\ P_n(S, X) \end{matrix}\right\}$ = the European call and put prices with n periods to expiration;
$P_n^a(S, X)$ = the American put option price with n periods to expiration.

The basic assumptions of our model, as well as of the model of Perrakis and Ryan, are an extension of those of Rubinstein (1976, pp. 408–11):

1. the single-price law of markets;
2. nonsatiation;
3. perfect, competitive, and Pareto-efficient financial markets;
4. rational, time-additive tastes;
5. weak aggregation or the existence of an average investor;[5]
6. no taxes, dividends, or transactions costs.

Assumption 6 implies that the stock price one period later is equal to SV. Assumptions 1–5 were shown by Rubinstein to imply the existence of a positive random variable, Z, equal to the marginal utility for consumption of the average investor normalized by its expectation so that $E(Z) = 1$ and such that the current value, $V_o(\cdot)$, of any random cash

3. Hereafter V denotes the random variable and v its realized value.
4. This is a standard assumption in most models, such as those of Black and Scholes (1973) and Cox et al. (1979). It is not necessary for all results since the bounds presented in Perrakis and Ryan (1984) did not use it, but it does simplify the presentation significantly and produces tighter bounds.
5. The same results can also be derived by replacing assumptions 3, 4, and 5 by the assumption that no arbitrage is possible in equilibrium, as was also done in Rubinstein (1976).

flows received at the end of one period is equal to the discounted expectation of the product of these cash flows times Z. In particular, for the stock price, S, we have

$$S = V_o(SV) = \frac{S}{1 + i}[\hat{v} + \text{cov}(V, Z)] = \frac{S}{1 + i}E(VZ), \quad (1)$$

where $\hat{v} \equiv E(V)$.

The next assumption, 7, is particular to the model of Perrakis and Ryan and will be used in this paper as well. Let $F(v, z)$ denote the joint distribution of V and Z, and $f(v, z) \equiv \partial^2 F/\partial v \partial z$ the corresponding density, assumed to exist without loss of generality. The following restriction is assumed on F:

7. The conditional mean, $\bar{Z}(v) \equiv E(Z|V = v) = \int_o^\infty[zf(z, v)dz]/\int_o^\infty[f(t, v)dt]$, is either monotone nonincreasing or monotone nondecreasing for all values of $v \in (\underline{v}, \bar{v})$.[6]

As discussed in Perrakis and Ryan (1984), the above assumption includes as special cases the assumptions of bivariate lognormality and bivariate normality of the discrete-time contingent claims models of Brennan (1979) and Rubinstein (1976). Assumption 7 is sufficiently general to accommodate all possible forms of $G(v)$, the marginal distribution of stock returns. It implies that, if $\bar{Z}(v)$ is monotone nonincreasing (nondecreasing), then for any nondecreasing contingent claim, $h(SV)$, we have from (1) that

$$V_o(h) \underset{(\geq)}{\leq} \frac{E(h)}{1 + i}$$

or that the consumption betas of the asset and its contingent claims are nonnegative (nonpositive).[7] Hereafter it will be assumed that $\bar{Z}(v)$ is nonincreasing, given that it was shown in Blume (1971) that the overwhelming majority of stocks have positive betas.

Assumptions 1–7 were used in the work of Perrakis and Ryan (1984) to derive the bounds for European calls. The methodology involved special applications of the following general result that will also be used extensively in this paper.

THEOREM. Let $h_1(V)$ and $h_2(V)$ denote two contingent claims, functions of the one-period stock return, V, with $V_o(h_1) = V_o(h_2)$. If the graphs of these two functions intersect only once at a value v_1 of V,

6. In Perrakis and Ryan (1984) it was assumed that $\bar{Z}(v)$ was monotone nonincreasing. However, similar results may be derived if $\bar{Z}(v)$ is monotone nondecreasing, as discussed below.

7. The covariance between two positive functions of a single random variable is nonnegative if both functions are nondecreasing or nonincreasing and nonpositive if one is nonincreasing and the other nondecreasing.

such that $v_1 \in (\underline{v}, \bar{v})$ and $0 < G(v_1) < 1$, and if $h_1(v) \geq h_2(v)$ for all $v \in (\underline{v}, v_1)$, the inequality being strict for at least one value of v of nonzero measure, then, under assumptions 1–7, $E[h_2(V)] \geq E[h_1(V)]$.

Proof. From (1) we have $V_o[h_i(V)] = 1/(1 + i)\, E[h_i(V)Z]$, $i = 1, 2$, from which we get $\int_{\underline{v}}^{v_1}[h_1(v) - h_2(v)]\bar{Z}(v)dG(v) = \int_{v_1}^{\bar{v}}[h_2(v) - h_1(v)]Z(v)dG(v)$ by the equality of the current values. In the relation above, the left-hand side (LHS) decreases and the right-hand side (RHS) increases if $\bar{Z}(v_1)$ replaces $\bar{Z}(v)$ under each integral, given assumption 7 and the fact that the differences of the $h(v)$'s are nonnegative in their respective intervals. The inequality of the expectations follows directly. Q.E.D.

The pairs of functions A–B in figures 1 and 3 below are examples of contingent claim functions satisfying the theorem. On the basis of this, and starting from a given initial wealth, arbitrage portfolios were constructed by Perrakis and Ryan (1984) involving the stock, the option, and the riskless asset, whose terminal wealth conditions satisfied the theorem above. The inequalities on the expected wealths were eventually converted into upper and lower bounds on call option prices, as shown below in our notation:

$$C_n(S, X) \leq S - \frac{X}{\hat{v}^n} + \frac{S}{\hat{v}^n}\int_{\underline{v}^n}^{L} G_n(v)dv \equiv \bar{H}_n(S, X), \qquad (2)$$

$$C_1(S, X) \geq \max\left[0,\ S - \frac{X}{1 + i} + \frac{S}{1 + i}\int_{\underline{v}}^{L} G(v)dv\right] \equiv \underline{H}_1(S, X), \quad (3)$$

$$C_n(S, X) \geq \max\left\{0,\ S\left(1 - \frac{\hat{v}}{1 + i}\right) + \frac{1}{1 + i}E[\underline{H}_{n-1}(SV, X)]\right\}, \qquad (4)$$

where $L \equiv X/S$, and $G_1(v) \equiv G(v)$, $G_n(v) = \int_{\underline{v}}^{\bar{v}}G_{n-1}(v/t)dG(t)$, $n > 1$. In other words, G_n, the n-fold multiplicative convolution of G with itself, is the distribution of the stock return over n periods.

The bounds in (2)–(4), however, are not the best ones that can be derived from assumptions 1–7. It is shown below that, whenever $\underline{v} > 0$ or $\bar{v} < \infty$ or both, there exist optimal arbitrage portfolios involving the stock, the option, and the riskless asset that yield tighter bounds than those of (2)–(4), given the same assumptions and applying the theorem above.[8] These tighter bounds, denoted by $\bar{C}_n(S, X)$ and $\underline{C}_n(S, X)$ for the upper and lower bounds, respectively, collapse to the expressions in (2)–(4) if both $\underline{v} = 0$ and $\bar{v} = \infty$.

If we have a single period until the expiration of the option, then the theorem can be applied by forming the following two portfolios, starting from \$1.00 initial wealth.

8. The bounds presented in Perrakis and Ryan (1984) are still applicable, however, when $\underline{v} > 0$ and/or $\bar{v} < \infty$.

Portfolio A. The entire amount invested in the stock.

Portfolio B. A proportion $\alpha \in (0, 1)$ invested in one-period call options, with the remaining $1 - \alpha$ invested in the riskless asset.

The terminal wealth of A is V, while that of B is $(1 - \alpha)(1 + i) + [\alpha S/C(S, X)]M(V)$, where $M(V) \equiv \max(V - L, 0)$, and the time subscript was suppressed in the function C for notational convenience. These two portfolios are generalized versions of the ones used in Perrakis and Ryan (1984): for $\alpha = 1$ we get the entire amount invested in the call option, as in the derivation of the upper bound; for $\alpha = C/S$ we get the portfolio used in the derivation of the lower bound.

Figure 1 shows the terminal wealths of A and B as functions of the stock return V. It is clear that these terminal wealths have two intersection points, 1 and 2 in the diagram. At least one of these points should be within the interval $(\underline{v}, \bar{v})$; otherwise it will be impossible for B and A to have the same current value of \$1.00.

If both intersection points are within $(\underline{v}, \bar{v})$, then the theorem cannot be applied, and no option bounds can be derived from assumptions 1–7. If, however, only one intersection point lies in the interior of $(\underline{v}, \bar{v})$, then the theorem can be applied, and the expected terminal wealth of one of the two portfolios, A and B, exceeds that of the other. This gives rise to an upper or lower bound on the option price, depending on whether A or B is highest in figure 1 in a nonzero measure neighborhood to the right of $\underline{v}$.

As sketched in figure 1, the terminal wealth of A lies initially above that of B. As long, therefore, as the intersection point 1 lies on or to the left of $\underline{v}$, the expected value of B will exceed that of A, according to the theorem. Hence, we have

$$\hat{v} \leq (1 - \alpha)(1 + i) + \frac{\alpha S E[M(V)]}{C(S, X)}, \tag{5}$$

from which

$$C(S, X) \leq \frac{\alpha S E[M(V)]}{\hat{v} - (1 - \alpha)(1 + i)}. \tag{6}$$

Differentiating the RHS of (6) with respect to α, we find that the derivative is nonnegative, given that $\hat{v} \geq 1 + i$ for $\bar{Z}(V)$ nonincreasing. Hence, the lowest upper bound for the call option price is found by choosing an α as low as possible, consistent with intersection point 1 being at or to the left of $\underline{v}$. This is obviously the value of α, at which 1 coincides with $\underline{v}$, as in figure 1. In other words, $\alpha = (1 + i - \underline{v})/(1 + i)$. Substituting in (6), we get

$$\begin{aligned} C(S, X) &\leq \frac{(1 + i - \underline{v}) S E[M(V)]}{(1 + i)(\hat{v} - \underline{v})} \\ &= \frac{(1 + i - \underline{v}) S E[\max(V - L, 0)]}{(1 + i)(\hat{v} - \underline{v})}. \end{aligned} \tag{7}$$

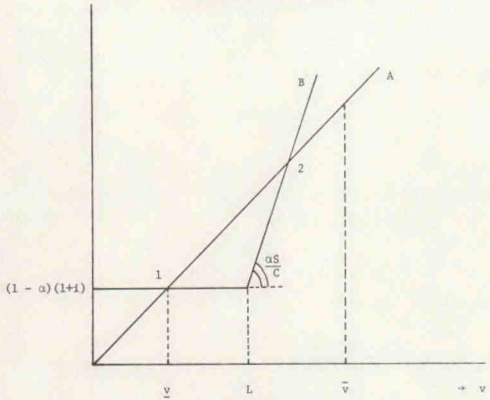

FIG. 1

Evaluating the expectation at the RHS of (7), we get

$$C(S, X) \leq S\,\frac{1 + i - \underline{v}}{1 + i} - \frac{S(1 + i - \underline{v})}{(1 + i)(\hat{v} - \underline{v})} \int_{\underline{v}}^{L} [1 - G(v)]dv. \tag{8}$$

It is easy to see that (8) coincides with the previous upper bound (2) for $n = 1$ if and only if either $\underline{v} = 0$ or $\hat{v} = 1 + i$. Otherwise (8) yields a lower upper bound than (2).

The bound at the RHS of (8) is valid as long as the situation depicted in figure 1 prevails. In other words, S and X must be such that $L \in (\underline{v}, \bar{v})$; otherwise either the call will be worthless (if $L \geq \bar{v}$), or its terminal wealth will always be $SV - X$ if $L \leq \underline{v}$. It is easy to see that the RHS of (8) becomes zero for $L = \bar{v}$ and $S - X/(1 + i)$ for $L = \underline{v}$. Hence, if we denote by $\bar{C}_1(S, X)$ the new single-period upper bound for the call option price, we have

$$C(S, X) \leq \left(\begin{array}{l} 0,\ S \leq \dfrac{X}{\bar{v}} \\[2ex] \text{RHS of (8) for } \dfrac{X}{\bar{v}} < S \leq \dfrac{X}{\underline{v}} \\[2ex] S - \dfrac{X}{1 + i},\ S > \dfrac{X}{\underline{v}} \end{array} \right) \equiv \bar{C}_1(S, X). \tag{9}$$

The derivation of the lower bound proceeds along similar lines. In figure 1 the parameter α is now chosen such that the intersection point

2 is at or to the right of $\bar{v}$, which implies automatically that point 1 is within $(\underline{v}, \bar{v})$. Then, inequality (5) is reversed since portfolio B's terminal wealth is initially above that of A's. Hence, inequality (6) is also reversed, and we must have

$$C(S, X) \geq \max\left(\frac{\alpha S E[M(V)]}{\hat{v} - (1 - \alpha)(1 + i)} \mid \text{point 2 at or to the right of } \bar{v}\right). \tag{10}$$

Since the maximand in (10) is increasing in α, the largest lower bound occurs when intersection point 2 is at $\bar{v}$. This yields a value of

$$\alpha = \frac{C(S, X)[\bar{v} - (1 + i)]}{S\bar{v} - X - (1 + i)C(S, X)}. \tag{11}$$

Substituting in (10) and rearranging, we get

$$\begin{aligned} C(S, X) \geq & \frac{S[\bar{v} - (1 + i)]}{\bar{v} - \hat{v}} \frac{E[M(V)]}{1 + i} \\ & - \left(\frac{\hat{v} - (1 + i)}{\bar{v} - \hat{v}}\right)\left(\frac{S\bar{v} - X}{1 + i}\right), \end{aligned} \tag{12}$$

which becomes, after incorporating the limited liability condition and evaluating $E[M(V)]$,

$$C(S, X) \geq \max\left[0, S - \frac{X}{1 + i} + \frac{\bar{v} - (1 + i)}{\bar{v} - \hat{v}} \frac{S}{1 + i} \int_{\underline{v}}^{L} G(v) dv\right]. \tag{13}$$

It is clear that this new lower bound exceeds the one found in the work of Perrakis and Ryan (1984) and given in (3) by the factor $[\bar{v} - (1 + i)]/(\bar{v} - \hat{v})$ times the discounted integral within the braces in the RHS of (13). This term is always greater than 1, and it becomes equal to 1 when either $\bar{v} \to \infty$ or $\hat{v} = 1 + i$.[9]

The bound in (13) becomes equal to $S - X/(1 + i)$ when $L \leq \underline{v}$ or when $S \geq X/\underline{v}$. For "low" values of S the limited liability condition in (13) becomes dominant, even though the value of S, at which the second term within braces in (13) becomes zero, is lower than the corresponding value in (3). This value, S_1 of S, is given by the solution of the equation in L

$$1 + i - \underline{v} = \int_{\underline{v}}^{L}\left[1 - \frac{\bar{v} - (1 + i)}{\bar{v} - \hat{v}} G(v)\right] dv. \tag{14}$$

Similarly, for $L = \bar{v}$ or $S = X/\bar{v}$, the bound becomes zero, but its derivative in S is initially negative, implying that $S_1 > X/\bar{v}$. Summariz-

9. For $\hat{v} = 1 + i$ all bounds coincide.

ing all these properties, we get the following expression for the single-period call option lower bound, denoted by $\underline{C}_1(S, X)$:

$$C_1(S, X) \geq \left(\begin{array}{l} 0,\ S \leq S_1,\ S_1 \text{ given by (13)},\ S_1 > \dfrac{X}{\bar{v}} \\[2ex] \text{RHS of (13)},\ S_1 < S \leq \dfrac{X}{\underline{v}} \\[2ex] S - \dfrac{X}{1 + i},\ S \geq \dfrac{X}{\underline{v}} \end{array} \right) \equiv \underline{C}_1(S, X). \tag{15}$$

It is clear that the properties of convexity and linear homogeneity of the bounds derived in Perrakis and Ryan are also preserved in those given by (9) and (15). Figure 2 shows the two bounds and the interval within which $C_1(S, X)$ falls.

Suppose now that the random variable, V, can take only two possible values, namely, $\underline{v}$ or $\bar{v}$, with probabilities p and $1 - p$, respectively. Then, for $L \in (\underline{v}, \bar{v})E[M(V)] = p(\bar{v} - L)$, $\hat{v} - \underline{v} = p(\bar{v} - \underline{v})$, and $\bar{v} - \hat{v} = (1 - p)(\bar{v} - \underline{v})$. Substituting in the RHS of (7) and (12), we find that both expressions become equal to $S(1 + i - \underline{v})(\bar{v} - L)/(1 + i)(\bar{v} - \underline{v})$. In other words, both bounds converge under these conditions to the two-state option pricing model of Cox et al. (1979) and Rendleman and Bartter (1979).[10] Hence, the extension of the work of Perrakis and Ryan presented in this paper generalizes the two-state contingent claims pricing approach to any distribution of stock returns.

The derived bounds can be easily extended recursively to any number of periods until expiration of the option. The results here are all derived for $n = 2$, the expressions for higher values of n being similar. As for the single period, we consider again the two investment strategies, A and B, the stock and the option-plus-riskless-asset portfolios. One period later the return per dollar to A is V, while that of B is $(1 - \alpha)(1 + i) + \alpha C_1(SV, X)/C_2(S, X)$. The expectation of one return should exceed that of the other, provided their graphs as functions of v intersect only once within $(\underline{v}, \bar{v})$. As with the single period, it is possible by an appropriate choice of α to get both an upper and a lower bound for $C_2(S, X)$.

As shown in Merton (1973), the call option is increasing and convex in the stock price, with derivative ≤ 1. Hence, if we set $C_1(SV, X) = SC_1(V, L)$, we have A and B as shown in figure 3. To derive the lower bound, the intersection point 2 should be at or to the right of $\bar{v}$, as shown.

10. Note also that the convergence of the two bounds to a single value is maintained for any number of periods until expiration. This can be easily proven recursively, by using (18) and (20) below.

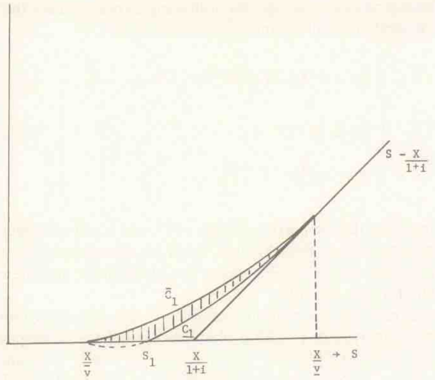

FIG. 2

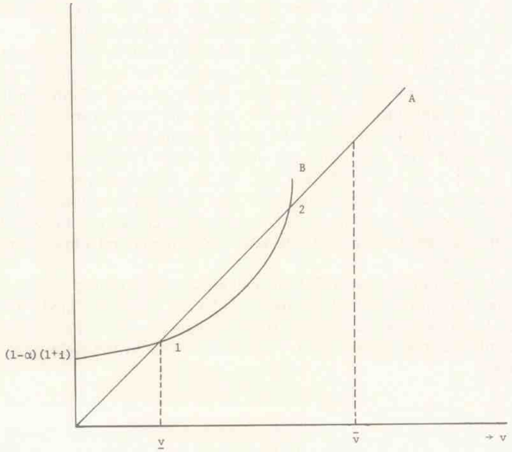

FIG. 3

According to the theorem, the expectation of the terminal wealth of A should exceed that of B. Substituting and rearranging, we get

$$C_2(S, X) \geq \frac{\alpha E[C_1(SV, X)]}{\hat{v} - (1 - \alpha)(1 + i)} \geq \frac{\alpha E[\underline{C}_1(SV, X)]}{\hat{v} - (1 - \alpha)(1 + i)}. \quad (16)$$

Since the RHS of (16) is increasing in α, the largest lower bound for $C_2(S, X)$ is obtained by using the largest value of α consistent with the theorem. This is obviously the case depicted in figure 3, with the terminal wealth of B passing through $\bar{v}$ and with $\underline{C}_1(SV, X)$ replacing $C_1(SV, X)$ in B. The corresponding value of α is easily seen to be equal to

$$\frac{C_2(S, X)[\bar{v} - (1 + i)]}{\underline{C}_1(S\bar{v}, X) - (1 + i)C_2(S, X)},$$

which, when replaced in the RHS of (16) and simplified, yields

$$C_2(S, X) \geq \frac{[\bar{v} - (1 + i)]E[\underline{C}_1(SV, X)] - [\hat{v} - (1 + i)]\underline{C}_1(S\bar{v}, X)}{(1 + i)(\bar{v} - \hat{v})}. \quad (17)$$

Since the procedure is obviously valid for any value of n, the lower bound becomes, after incorporation of the limited liability condition[11]

$$C_n(S, X) \geq$$

$$\max\left\{0, \frac{[\bar{v} - (1 + i)]E[\underline{C}_{n-1}(SV, X)] - [\hat{v} - (1 + i)]\underline{C}_{n-1}(S\bar{v}, X)}{(1 + i)(\bar{v} - \hat{v})}\right\}, \quad (18)$$

which is the counterpart of (4).

For the upper bound one must have the terminal wealth of B intersect the 45 degree line for the first time (point 1 of figure 3) at $\underline{v}$ or to the left of it and for the second time within $(\underline{v}, \bar{v})$.

The value of α yielding the lowest upper bound is when A and B intersect at $\underline{v}$ (as in figure 4) for all possible call option functions. Hence, α is found from the equation $\underline{v} = (1 - \alpha)(1 + i) + \alpha\bar{C}_1(S\underline{v})/C_2(S, X)$. On the other hand, by the theorem, we have $\hat{v} \leq (1 - \alpha)(1 + i) + \alpha E[\bar{C}_1(SV)]/C_2(S, X)$. Substituting for α and rearranging, we get

$$C_2(S, X) \leq \frac{(1 + i - \underline{v})E[\bar{C}_1(SV, X)] + \bar{C}_1(S\underline{v}, X)[\hat{v} - (1 + i)]}{(1 + i)(\hat{v} - \underline{v})}. \quad (19)$$

Summarizing these various cases we get the general upper bound when there are n periods until expiration, in a manner identical to that of $n = 2$.

11. It should be noted that the lower bound, $\underline{C}_n(S, X)$, becomes equal to $S - [X/(1 + i)^n]$ for all $S \geq X/\underline{v}^n$.

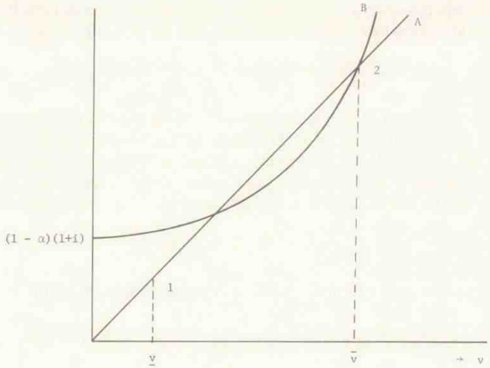

FIG. 4

$$C_n(S, X) \le \bar{C}_n(S, X) =$$

$$\begin{cases} 0, \; S \le \dfrac{X}{\bar{v}^n} \\ \dfrac{(1 + i - \underline{v})E[\bar{C}_{n-1}(SV)] + \bar{C}_{n-1}(S\underline{v})[\hat{v} - (1 + i)]}{(1 + i)(\hat{v} - \underline{v})}, \; \dfrac{X}{\bar{v}^n} < S \le \dfrac{X}{\underline{v}^n} \\ S - \dfrac{X}{(1 + i)^n}, \; S \ge \dfrac{X}{\underline{v}^n} \end{cases} \tag{20}$$

The last part of (20) represents the case in which the stock can never fall below the exercise price of the option until expiration.

In the last section a numerical example will be presented, comparing the bounds presented here with those of Perrakis and Ryan. The improvement turns out to be major for all n.

Given these results it is now possible to derive upper and lower bounds for European put options by means of the put-call parity theorem.

$$P_n(S, X) = C_n(S, X) - S + \frac{X}{(1 + i)^n}. \tag{21}$$

Let $\bar{B}_n(S, X)$ and $\underline{B}_n(S, X)$ denote, respectively, the upper and lower bounds for the n-period European put options. From (18)–(20) these bounds are equal to

$$\bar{B}_n(S, X) = \begin{cases} 0, \; S \geq \dfrac{X}{\underline{v}^n} \\ \dfrac{X}{(1+i)^n} - S + \bar{C}_n(S, X), \; \dfrac{X}{\underline{v}^n} > S \geq \dfrac{X}{\bar{v}^n} \\ \dfrac{X}{(1+i)^n} - S, \; S < \dfrac{X}{\bar{v}^n}, \end{cases} \tag{22}$$

$$\underline{B}_n(S, X) = \begin{cases} 0, \; S \geq \dfrac{X}{\underline{v}^n} \\ \dfrac{X}{(1+i)^n} - S + \underline{C}_n(S, X), \; \dfrac{X}{\underline{v}^n} > S \geq S_n \\ \dfrac{X}{(1+i)^n} - S, \; S < S_n, \; S_n > \dfrac{X}{\bar{v}^n}, \end{cases} \tag{23}$$

where S_n is the largest value of S at which $\underline{C}_n(S, X) = 0$.

From (21) and the properties of call-option pricing functions derived in the work of Merton (1973), it follows that $0 \geq \partial P_n(S, X)/\partial S \geq -1$. These same inequalities are satisfied by the derivatives in S of the bounds $\bar{B}_n(S, X)$ and $\underline{B}_n(S, X)$. For $S \leq X/\bar{v}^n$ both bounds are equal to $X/(1 + i)^n - S$, while their derivatives in S are equal to -1. These properties play a role in the next section, in which the bounds for the American put are derived.

III. The American Put Option

In this section upper and lower bounds $\bar{B}_n^a(S, X)$ and $\underline{B}_n^a(S, X)$ will be derived for the n-period American put. The general methodology used will be to construct pairs of arbitrage portfolios, starting from the same initial wealth and involving the stock, the riskless asset, the American put, and European-type options on the stock. The terminal values of these portfolios will have shapes satisfying the theorem of the previous section, implying an inequality relation on the expected values of the portfolios. This relation will then be converted into a bound for the value of the American put.

Let $\Delta P_n(S, X) \equiv P_n^a(S, X) - P_n(S, X)$ denote the n-period premium that the possibility of early exercise confers on the American put over its European counterpart. From expression (2) of Merton (1973), combined with the put-call parity theorem (21), we have

$$\Delta P_n(S, X) \leq X\left[1 - \frac{1}{(1+i)^n}\right]. \tag{24}$$

In what follows we shall derive a lower bound $\underline{\Delta}_n(S, X)$ on $\Delta P_n(S, X)$.

The lower bound $\underline{B}_n^a(S, X)$ of the American put will then be equal to $\underline{\Delta}_n(S, X) + \underline{B}_n(S, X)$.

The derivation of both bounds will be done recursively, following the approach established by Parkinson (1977). For $n = 1$ the value of the unexercised American put will obviously be equal to that of its European counterpart. Hence, the two values will be different only for $n \geq 2$, and $\Delta P_n(S, X) \geq 0$ for all n. The following auxiliary result will be of use in further derivations.

LEMMA 1. Under assumptions 1–7, $\Delta P_n(S, X)$ is decreasing in S, increasing in X, and linear homogeneous in (S, X). If the American put is prevented from being exercised in period n, then $\Delta P_n(S, X) \leq [X/(1 + i)][1 - 1/(1 + i)^{n-1}]$.

Proof. For $n = 2$ we have, from (1),

$$\Delta P_2(S, X) = \max\left\{X - S - \frac{1}{1 + i}\int_{\underline{v}}^{\bar{v}} P_1(Sv, X)\bar{Z}(v)dG(v),\right.$$
$$\left.\frac{1}{1 + i}\int_{\underline{v}}^{\frac{S_1'}{S}}[X - Sv - P_1(Sv, X)]\bar{Z}(v)dG(v)\right\},$$

where S_1' is defined from the equation $X - S_1' = P_1(S_1', X)$. By direct differentiation it can be shown that $\Delta P_2(S, X)$ possesses the properties of the lemma. The first term within the parentheses corresponds to immediate exercise in period 2, while the second is the bound given that the American put will not be exercised immediately. For $S = X/\bar{v}^2$ this latter term becomes equal to $[X/(1 + i)][1 - 1/(1 + i)]$. Q.E.D. The proof can now proceed recursively.

$$\Delta P_3(S, X) = \max\left[X - S - \frac{1}{1 + i}\int_{\underline{v}}^{\bar{v}} P_2(Sv, X)\bar{Z}(v)dG(v),\right.$$
$$\left.\frac{1}{1 + i}\int_{\underline{v}}^{\bar{v}} \Delta P_2(Sv, X)\bar{Z}(v)dG(v)\right],$$

which also can be shown to satisfy the lemma by direct differentiation. Q.E.D.

The derivation of the lower bound will now be done recursively. The results are expressed by means of the following.

PROPOSITION 1. Under assumptions 1–7 of the previous section, the lower bound $\underline{\Delta}_n(S, X)$ of the n-period premium of the American put over its European counterpart is given by the recursive relations

$$\underline{\Delta}_2(S, X) = \frac{1}{1 + i}\int_{\underline{v}}^{\frac{\hat{S}_1}{S}} [X - Sv - \bar{B}_1(Sv, X)]dG(v) \tag{25}$$

$$\underline{\Delta}_n(S, X) = \frac{1}{1 + i} E\{\max[X - Sv - \bar{B}_{n-1}(Sv, X), \underline{\Delta}_{n-1}(Sv, X)]\}, \tag{26}$$

where the sequence $\overline{B}_n(S, X)$ is given by (22) and $\hat{S}_1$ is the solution of the equation $X - S = \overline{B}_1(S, X)$.

Proof. Suppose that for $n = 2$ we consider the following two investment strategies, given that the American put is not exercised immediately.

Strategy A. A short sale of the two-period European put.
Strategy B. A short sale of the two-period American put.

Then, after one period, the wealth position of A will be

$$P_2(S, X)(1 + i) - P_1(Sv, X),$$

and the wealth position of B will be

$$P_2^a(S, X)(1 + i) - \max[X - Sv, P_1(Sv, X)],$$

since in B the put will be exercised one period later if the value of its European counterpart falls below the exercise value.

The shapes of the two wealth positions are as shown in figure 5.[12] By the theorem, therefore, it follows that the expected wealth of B exceeds that of A. Substituting and rearranging, we get

$$\begin{aligned} P_2^a(S, X) &\geq P_2(S, X) + \frac{1}{1 + i} E\{\max[X - SV - P_1(SV, X), 0]\} \\ &\geq P_2(S, X) + \frac{1}{1 + i} E\{\max[X - SV - \overline{B}_1(SV, X), 0]\}. \end{aligned} \tag{27}$$

Equation (25) now follows directly from (27) and the properties of the upper bound $\overline{B}_1(S, X)$. Q.E.D. By direct differentiation it is easy to see that $\underline{\Delta}_2(S, X)$ is decreasing in S and increasing in X.

The computation can now proceed recursively. If the two investment strategies A and B are undertaken n periods before the expiration of the two put options, then the wealth positions of A and B after one period are, respectively:

$$P_n(S, X)(1 + i) - P_{n-1}(Sv, X)$$

and

$$P_n^a(S, X)(1 + i) - \max[X - Sv, P_{n-1}^a(Sv, X)].$$

It can be easily shown that A and B can cross at most at a single point.[13] Hence, by the theorem, we have

12. By lemma 1, $P_2(S, X)(1 + i) - \{[X/(1 + i)] - Sv\} \geq P_2^a(S, X)(1 + i) - (X - Sv)$. Further, the term $P_1(Sv, X)$ tends to zero for large Sv, so that B eventually exceeds A.

13. For $Sv = X/\bar{v}^{n-1}$ the difference between A and B is equal to $-\Delta P_n(S, X)(1 + i) + X[1 - 1/(1 + i)^{n-1}]$, which is nonnegative by lemma 1. Further, this same difference decreases in Sv, and $-[\partial P_n(S, X)/\partial S] \leq 1$.

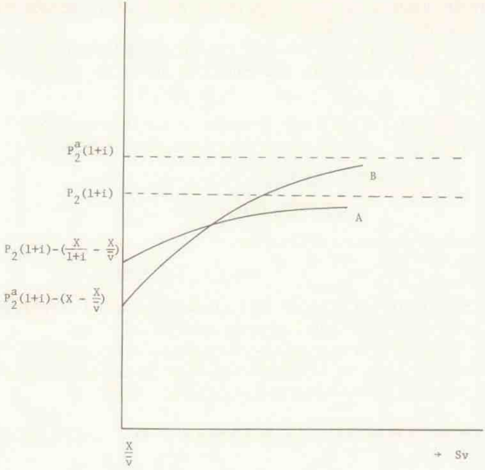

FIG. 5

$$P_n^a(S, X)(1+i) - E\{\max[X - SV, P_{n-1}^a(SV, X)]\} \geq P_n(S, X) - E[P_{n-1}(SV, X), \tag{28}$$

from which

$$\Delta P_n(S, X) \geq \frac{1}{1+i} E\{\max[X - SV - P_{n-1}(SV, X), \Delta P_{n-1}(SV, X)]\} \geq \frac{1}{1+i} E\{\max[X - SV - \overline{B}_{n-1}(SV, X), \underline{\Delta}_{n-1}(SV, X)]\} \equiv \underline{\Delta}_n(S, X). \tag{29}$$

Q.E.D.

Proposition 1 now allows the immediate derivation of the lower bound $\underline{B}_n^a(S, X)$ for the n-period American put.

COROLLARY. The n-period American put is bounded from below by the expression

$$\underline{B}_n^a(S, X) = \max[X - S, \underline{B}_n(S, X) + \underline{\Delta}_n(S, X)], \tag{30}$$

where $\underline{B}_n$ is given by (23) and $\underline{\Delta}_n$ by (26). In the next section this bound will be computed for the case of a trinomial distribution. The upper bound is similarly computed recursively, and its value is expressed by proposition 2.

PROPOSITION 2. Under assumptions 1–7 the upper bound $\overline{B}_n^a(S, X)$ for the n-period American put is given by the following recursive relations.

$$\overline{B}_2^a(S, X) = \begin{cases} \max[X - S, \overline{M}_2(S, X)], \; S < \dfrac{X}{\underline{v}^2} \\ 0, \; S \geq \dfrac{X}{\underline{v}^2}, \end{cases} \tag{31}$$

where $\overline{M}_2(S, X)$ is given by the function in the first part of (32) as long as that function is decreasing in S and kept constant and equal to its minimum value thereafter.

$$\overline{M}_2(S, X) =$$

$$\begin{cases} S\left(\dfrac{\hat{v}}{1 + i} - 1\right) + \dfrac{1}{1 + i} E\{\max[X - SV, \overline{B}_1(SV, X)]\}, \; S \leq \overline{S}_1 \\ \overline{S}_1\left(\dfrac{\hat{v}}{1 + i} - 1\right) + \dfrac{1}{1 + i} E\{\max[X - \overline{S}_1 V, \overline{B}_1(S_1 V, X)]\}, \; S > \overline{S}_1 \\ \dfrac{\partial \overline{M}_2(S, X)}{\partial S}\bigg|_{S = \bar{S}_1} = 0 \end{cases} \tag{32}$$

Similarly, for $n > 2$,

$$\overline{B}_n^a(S, X) = \begin{cases} \max[X - S, \overline{M}_n(S, X)], \; S < \dfrac{X}{\underline{v}^n} \\ 0, \; S \geq \dfrac{X}{\underline{v}^n} \end{cases} \tag{33}$$

$$\overline{M}_n(S, X) = \begin{cases} S\left(\dfrac{\hat{v}}{1 + i} - 1\right) + \dfrac{1}{1 + i} E[\overline{B}_{n-1}^a(SV, X)], \; S \leq \overline{S}_1 \\ \overline{S}_1\left(\dfrac{\hat{v}}{1 + i} - 1\right) + \dfrac{1}{1 + i} E[\overline{B}_{n-1}^a(\overline{S}_1 V, X)], \; S > \overline{S}_1 \\ \dfrac{\partial \overline{M}_n(S, X)}{\partial S}\bigg|_{S = \bar{S}_1} = 0 \end{cases} \tag{34}$$

Proof. I compare the following investment strategies for $n = 2$ and for an initial investment equal to S.

Strategy C. A long position on the stock.
Strategy D. A short sale of the American put.

As shown in figure 6, after one period the respective wealth positions become, for C, Sv, and, for D:[14]

$$[S + P_2^a(S, X)](1 + i) - \max[X - Sv, P_1(Sv, X)].$$

By the theorem of the previous section, the expectation of C must exceed that of D. Substituting and rearranging, we get

$$\begin{aligned} P_2^a(S, X) &\leq S\left(\frac{\hat{v}}{1 + i} - 1\right) + \frac{1}{1 + i}E\{\max[X - SV, P_1(SV, X)]\} \\ &\leq S\left(\frac{\hat{v}}{1 + i} - 1\right) + \frac{1}{1 + i}E\{\max[X - SV, \overline{B}_1(SV, X)]\}, \end{aligned} \tag{35}$$

which forms an upper bound on $P_2^a(S, X)$, given that it is not going to be exercised in period 2.

For $S = 0$ this upper bound is equal to $[X/(1 + i)][1 - 1/(1 + i)]$, which is the bound given in lemma 1. Further, the RHS of (35) is easily seen to be convex in S, while its derivative is equal to -1 for $S = 0$. Since for large S this derivative eventually becomes positive,[15] it follows that the upper bound, $\overline{B}_2^a(S, X)$, derived from (35), is equal to the RHS of (35) as long as this RHS is decreasing and is equal to the minimum value that is reached thereafter, for $S > \overline{S}_1$.[16] Finally, it should be noted that for "low" values of S the put is going to be exercised immediately in period 2, which means that its value can never fall below $X - S$, while for $S \geq X/\underline{v}^2$ the put will become worthless since the stock price cannot exceed the exercise price X.

Figure 7 shows the general shape of the upper bound for the two-period American put. The convexity of $\overline{M}_2(S)$ guarantees its intersection with the line $X - S$ at some value $\overline{S}_2$, so that $\overline{B}_2^a(S, X) = X - S$ for $S \leq \overline{S}_2$, $\overline{B}_2^a(S, X) = \overline{M}_2(S, X)$ for $X/\underline{v}^2 > S > \overline{S}_2$, $\overline{B}_2^a(S, X) = 0$, $S \geq X/\underline{v}^2$. $\overline{S}_2$ may be smaller than $\overline{S}_1$ (as represented in the diagram) or larger than it, depending on the value of the parameters and the shape of $G(v)$. Hence, (31) and (32) hold. Q.E.D.

The computation of the bound for $n > 2$ may now proceed recursively. We form again the portfolios C and D, which have wealth positions one period later of Sv and $[S + P_2^a(S, X)](1 + i) - P_{n-1}^a(Sv, X)$. Applying the theorem, rearranging, and so forth, we get

14. At $\underline{v}$, C is equal to $S\underline{v}$, while D is equal to $(S + P_2^a)(1 + i) - \max[X - S\underline{v}, P_1(S\underline{v}, X)]$. If $X - S\underline{v} \geq P_1(S\underline{v}, X)$, then D exceeds C because $P_2^a(S, X) \geq P_1(S, X) \geq [X/(1 + i)] - S$ for all S. If $P_1(S\underline{v}, X) > X - S\underline{v}$, then D exceeds C because $[S + P_1(S, X)]/(1 + i) > [S\underline{v} + P_1(S\underline{v}, X)]/(1 + i)$, and $P_2^a(S, X) \geq P_1(S, X)/(1 + i)$. Hence D exceeds C always. Q.E.D.

15. The term $\hat{v}/(1 + i) - 1 > 0$ becomes dominant as $Sv \rightarrow \infty$.

16. This is a direct consequence of the fact that $P_2^a(S, X)$ is nonincreasing in S.

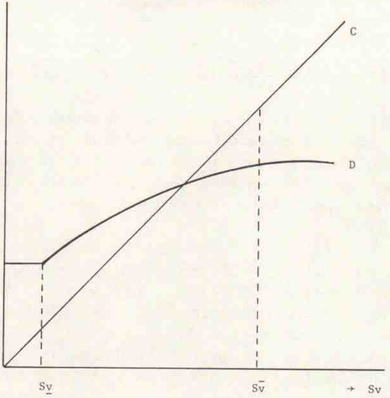
C
D
$S\underline{v}$
$S\bar{v}$
→ Sv

FIG. 6

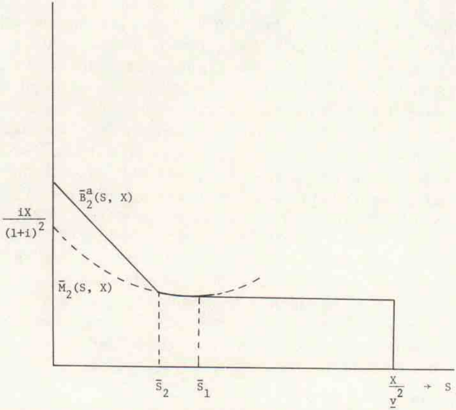
$\frac{iX}{(1+i)^2}$
$\bar{B}_2^a(S, X)$
$\bar{M}_2(S, X)$
$\bar{S}_2$
$\bar{S}_1$
$\frac{X}{\underline{v}^2}$
→ S

FIG. 7

$$
\begin{aligned}
P_n^a(S, X) &\leq S\left(\frac{\hat{v}}{1+i} - 1\right) + \frac{1}{1+i} E[P_{n-1}^a(SV, X)] \\
&\leq S\left(\frac{\hat{v}}{1+i} - 1\right) + \frac{1}{1+i} E[\bar{B}_{n-1}^a(SV, X)],
\end{aligned} \tag{36}
$$

from which (33) and (34) hold for the n-period bound. Q.E.D.

The last general result concerns the probability of exercise of the American put during a given point in time, defined in terms of time periods until expiration. Suppose that we have an unexercised put n periods before expiration. Then the probability of exercise one period later is as follows.

PROPOSITION 3. Let $\bar{S}_2$ and $\bar{S}_3$ denote, respectively, the solutions of the equations in $S : X - S = \bar{M}_{n-1}(S, X)$ and $X - S = \underline{B}_{n-1}(S, X) + \underline{\Delta}_{n-1}(S, X)$. Then the probability that an n-period unexercised American put will be exercised one period from now is at least equal to $G(\bar{S}_2/S)$ and at most equal to $G(\bar{S}_3/S)$.

Proof. One period later the stock price will be equal to Sv. For all $S \leq \bar{S}_2$ it is clear that $\bar{B}_{n-1}(S, X) = \underline{B}_{n-1}(S, X) = P_{n-1}^a(S, X) = X - S$. Hence, the event $(V \leq \bar{S}_2/S)$ corresponds to exercise during period $n - 1$. Similarly, for $Sv > \bar{S}_3$, it is always true that $P_{n-1}^a(X, Sv) > X - Sv$, implying that the put will not be exercised in period $n - 1$. It follows that the probability of exercise during $n - 1$ lies between $G(\bar{S}_2/S)$ and $G(\bar{S}_3/S)$. Q.E.D.

I close this section by examining the derived bounds for the American put in relation to those given by (22)–(23) for its European counterpart. The difference between the two represents the value of the option for early exercise. The comparison between the two put options is done for $n = 2$. Thus the lower bound for the American put is shown by (25) and (26) to exceed that of the European put by the discounted value on early exercise one period later. Similarly, the upper bound in (31)–(32) contains, in addition to the probabilities of early exercise in periods 2 and 1, an amount equal to the discounted risk premium on the stock, $[\hat{v}/(1 + i)] - 1$. We note that the upper bound, given that the put is not going to be exercised in period 2, $\bar{M}_2(S, X)$, is equal to that risk premium plus a term that is equal to the discounted combination of the expectations of the upper bound on the one-period European put and of the exercise value during period one. Both upper bound and exercise value are decreasing functions of the one-period stock return, V. This implies, by assumption 7, that their value one period earlier exceeds their discounted expectation, with the discounted risk premium on the stock yielding an upper limit on this excess.

IV. A Numerical Example

Consider a stock currently trading at \$65. During the next period there is the possibility of a tender offer at \$78, which has a probability assessed at 40%. There is also a 10% chance that the stock will fall to \$52, as well as a 50% chance that its price will not change. Hence, the return V per dollar invested per period has a probability distribution $G(v)$ such that $dG(v) = .10$, $v = .8$; $dG(v) = .50$, $v = 1$; $dG(v) = .40$, $v = 1.2$; $dG(v) = 0$ otherwise, with $\underline{v} = .8$, $\bar{v} = 1.2$, $\hat{v} = 1.06$. Also let $1 + i = 1.05$.

For this particular firm we first compare the new bounds presented in this paper with those of Perrakis and Ryan (1984). Table 1 shows the old and new call option bounds, respectively, $\bar{H}_n(S, X)$, $\underline{H}_n(S, X)$ and $\bar{C}_n(S, X)$, $\underline{C}_n(S, X)$, for at-the-market options ($X = \$65$) and $n = 1, \ldots, 4$.

As shown in the last two columns of table 1, the reduction in the size of the between-bounds interval is quite spectacular. It ranges from about 40% for $n = 1$ to more than 400% for $n = 4$.

Tables 2–5 give bounds for American and European put options for $S = \$65$ and four different exercise prices, \$60, \$65, \$70, and \$75, for up to four periods until expiration.

In these tables the value $\underline{\Delta}_n$ represents a lower bound on the early exercise option of the American put. This option becomes more valuable as the number of periods to expiration increases. As a result, both bounds are nondecreasing in n, even though the corresponding European put bounds decline after some value of n.

The computation of the bounds can be extended recursively to any number of periods until expiration. The main statistical problem in any realistic setting stems from the estimation of the single-period distribution of stock returns, assumed trinomial in this example. In practice, the length of the period is a major consideration in any empirical applications of discrete-time option models, as discussed further in the next section.

V. Summary and Conclusions

This paper presented two important extensions of the discrete-time option pricing bounds put forth in Perrakis and Ryan. First of all, the bounds were tightened significantly whenever the lowest and highest possible single-period stock returns are nonzero and finite, respectively. It was also found that these improved bounds converge to the option models of Cox et al. (1979) and Rendleman and Bartter (1979) when the single-period distribution is binomial. Second, bounds were derived for the American put option by using sequential early exercise possibilities together with the basic arbitrage arguments already used in Perrakis and Ryan. The put bounds are exact to the extent that the stock return distribution used in the computations represents the "true" single-period distribution. Otherwise, the put bounds are approximations to the exact values of the bounds since early exercise is

TABLE 1 Call Bounds for X = \$65

n	$\bar{H}_n(S, X)$	$\underline{H}_n(S, X)$	$\bar{C}_n(S, X)$	$\underline{C}_n(S, X)$	$\bar{H}_n - \underline{H}_n$	$\bar{C}_n - \underline{C}_n$
1	4.9057	4.3333	4.7619	4.4218	.5724	.3401
2	8.7006	7.6703	8.1982	7.8208	1.0303	.3774
3	11.9786	10.4978	11.1018	10.7296	1.4808	.3722
4	14.9644	13.0769	13.7318	13.3645	1.8875	.3673

TABLE 2 Put Bounds for X = \$60

n	$\bar{B}_n$	$\underline{B}_n$	$\underline{\Delta}_n$	$\bar{B}_n^a$	$\underline{B}_n + \underline{\Delta}_n$	$\underline{B}_n^a$
1	1.0256	.8163	...	...	...	...
2	1.2417	1.0246	.1884	1.8694	1.213	1.213
3	1.2989	1.0662	.3541	2.6010	1.4203	1.4203
4	1.2831	1.0427	.5030	3.2680	1.5457	1.5457

TABLE 3 Put Bounds for X = \$65

n	$\bar{B}_n$	$\underline{B}_n$	$\underline{\Delta}_n$	$\bar{B}_n^a$	$\underline{B}_n + \underline{\Delta}_n$	$\underline{B}_n^a$
1	1.6667	1.3265	...	...	...	...
2	2.1551	1.7777	.2948	2.7778	2.0725	2.0725
3	2.2512	1.8790	.5897	3.6367	2.4687	2.4687
4	2.2074	1.8401	.8446	4.3645	2.68472	2.6847

TABLE 4 Put Bounds for X = \$70

n	$\bar{B}_n$	$\underline{B}_n$	$\underline{\Delta}_n$	$\bar{B}_n^a$	$\underline{B}_n + \underline{\Delta}_n$	$\underline{B}_n^a$
1	4.5971	4.3878	...	...	...	...
2	4.3421	3.9990	.5093	5.0855	4.5083	5.00
3	3.9932	3.6109	.9268	5.4905	4.5377	5.00
4	3.6351	3.2365	1.3154	6.0041	4.5519	5.00

TABLE 5 Put Bounds for X = \$75

n	$\bar{B}_n$	$\underline{B}_n$	$\underline{\Delta}_n$	$\bar{B}_n^a$	$\underline{B}_n + \underline{\Delta}_n$	$\underline{B}_n^a$
1	7.5275	7.4490	...	...	...	...
2	6.5292	6.2203	1.5175	10.00	7.7378	10.00
3	5.7413	5.3470	2.4464	10.00	7.7934	10.00
4	5.0741	4.6416	3.1734	10.00	7.8150	10.00

considered less frequently than in reality. This issue has already been discussed in Cox et al. (1979) and Parkinson (1977), and the same remarks apply to this paper.

The bounds derived here are completely general insofar as the stock price distribution is concerned, provided assumptions 1–7 are satisfied. Hence, they may be easily modified to account for fixed dividends or trading commissions. For instance, trading commissions will affect $\underline{B}_n^a(S, X)$ only through the lower bound $\underline{B}_n(S, X)$ of the European put in (30), assuming no difference in commissions between American and European puts. By contrast, the upper bound will be raised by the difference in trading commissions between a long position on the stock and a short position on the put, in addition to the adjustment through the upper bound $\overline{B}_1$ of the European put in the RHS of (35).

Other important extensions of the theory of Perrakis and Ryan that are also applicable to the valuation of American puts and do not necessitate important conceptual modifications are dividend policies that are deterministic functions of the stock return. Suppose, for instance, that the firm has a policy of paying no dividends if the return falls below some minimum level and of paying out all earnings above that level up to a maximum amount if the return is above that minimum. Then bounds for European call or put options that are not payout protected may still be derived by applying the methods of Section II, and the results can then be extended to American puts. More complex payout policies may, however, need a strengthening of assumptions 1–7 and major new theoretical work. The same is true if interest rates or dividend yields are stochastic.

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