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Futures Contracts on Commodities with Multiple Varieties: An Analysis of Premiums and Discounts

In this paper we examine the structure of futures contracts on commodities with heterogeneous varieties. The structure of premiums and discounts allowed on different grades of a commodity is analyzed. We then identify penalty and equivalence systems for quality adjustments. We also explore the implications of the adjustment system for the risk exposure to hedgers using the futures contract. We find that there is no prima facie case for setting contract premiums and discounts equal to actual commercial premiums and discounts.

I. Introduction

In casual conversation, people often speak of a commodity as if it occurs in a single, homogeneous form. A moment's reflection reveals, however, that classifications like "gold" and "wheat" are actually labels for commodity groups that include varieties with different characteristics. For example, gold can differ according to fineness and wheat can differ according to grade. Such quality variations in a commodity rarely present a problem for cash (or spot) market transactions, because buyers and sellers can bargain over price in light of the characteristics of particular lots.[1] The situation is quite different for a futures contract, where price must be negotiated without reference to the specific lot to be delivered. In this case buyers and sellers have an incentive to define carefully those characteristics which distinguish acceptable from unacceptable grades.[2]

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1. See Cox and Wright (1979) for an example of cash market trading (in feeder cattle) on the basis of oral and written descriptions rather than visual inspections.

2. The fact that most futures contracts are liquidated by

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This paper examines futures contracts for commodities with multiple varieties and focuses on the use of such contracts for hedging (or risk transfer). In Section II we discuss some general principles of contract design. We point out that there is a trade-off between enhancing the certainty of what will be delivered on a futures contract and reducing the likelihood of squeezes. Most futures contracts traded in the United States avoid the price pressures associated with inadequate supplies of specific varieties (a squeeze) by allowing sellers to deliver any of several commodity grades. More particularly, contracts permit delivery of high-quality varieties at a premium to the contract price and/or low-quality grades at a discount from the contract price. Specification of the premiums and discounts is important because they influence the hedging characteristics of futures contracts.

Section III describes how existing futures contracts implement quality adjustments via premiums and discounts. This may be viewed as unbundling the intrinsic characteristics of a commodity in the sense of Lancaster (1966*a*, 1966*b*). The analysis suggests two alternative methods for distinguishing among varieties. In some cases, a commodity is valued only for a single underlying characteristic, such as the fineness of gold or the quantity of soybean oil remaining after refining. If relative prices of different varieties reflect only differences in one characteristic, then futures prices can be adjusted to reflect the amount of the characteristic that is delivered. In other cases, differences among varieties do not depend on any single underlying attribute. Examples include wheat of different grades and growths and slaughter steers of different yield and quality. The relative prices of these different varieties fluctuate through time, and price adjustments are more complicated.

The analyses of Sections IV and V evaluate the distribution of risk among hedgers as a function of contract characteristics. We develop a simple model which shows that if two varieties can be delivered on a contract, a hedged position in at least one must bear some residual, unhedgeable risk. A series of Monte Carlo simulations illustrates that the structure of premiums and discounts on a futures contract with multiple delivery options has important implications for the size and allocation of this residual risk. This examination of the hedging characteristics of futures contracts follows earlier work on the same topic by Working (1954); Powers (1967); Crow, Riley, and Purcell (1972); and Sandor (1973) and is related to the issue of contract proliferation examined by Silber (1981).

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offset rather than physical delivery does not reduce the importance of delivery specifications, because those specifications determine the price to which a contract converges on its final settlement date and hence influence the price of the contract prior to final settlement. For an example, see the derivation of equilibrium futures prices in the Appendix to this paper.

II. General Principles

The value of a commodity in the cash market varies according to variety and location. Futures markets can deal with such diversity in a number of ways, each requiring a somewhat different approach to contract design.

One approach to specifying a futures contract is to define acceptable delivery in terms of a narrow set of characteristics at a single location. This specification gives a buyer certainty over what he receives but opens up the possibility of a multiplicity of narrowly defined contracts. This arrangement has two major disadvantages. First, it fragments trading over a large number of contracts and thereby reduces overall market liquidity (see Garbade and Silber 1979; Silber 1981). In addition, narrow contract specification enhances the likelihood of squeezes, during which the price of a particular variety diverges from its normal commercial value.[3] Hoffman (1932, p. 278) observed that the Chicago Board of Trade abandoned narrowly specified contracts in the early 1880s for this reason.[4]

A second approach to designing a futures contract is to specify acceptable delivery in terms of a broad range of commodity grades and delivery locations without providing for any price adjustments. At first glance, a futures contract permitting delivery of several varieties at a number of locations seems to solve the problem of squeezes while enhancing the liquidity of the contract. However, the price of such a contract is most closely related to the "cheapest" of the deliverable grades. For example, suppose two varieties, labeled A and B, are deliverable on a futures contract and that the cash market price of A is, on average, 10% above the cash market price of B. Because A is normally at a premium, market participants will not expect to deliver it, and the price of the futures contract will depend primarily on the current and expected future price of variety B. The price of A would influence futures trading significantly only if changes in supply and demand eliminated the normal 10% premium. Thus, specifying a broader range of acceptable commodity characteristics may not enhance significantly the liquidity of a contract and might provide relief only to the most severe squeezes.

A third approach to designing a futures contract is to allow delivery at the contract price of a "standard" variety at a specific location but also to provide for delivery of nonstandard grades (at alternative loca-

3. See, e.g., the analysis of Treasury bill prices prior to the June 1979 settlement of the 3-month bill contract on the Chicago Mercantile Exchange in Commodity Futures Trading Commission (1981, pt. 3, pp. 56–61).

4. In some cases the certainty derived from narrow delivery specifications may outweigh the potential for squeezes due to narrow specifications. See the description of trading in hard winter wheat futures in Working (1954) and of trading in Treasury debt futures in Commodity Futures Trading Commission (1981, pt. 3, pp. 55, 98–117).

tions) at premiums to and discounts from the contract price. In the preceding example, such a system of "nonpar" delivery might allow variety B to be delivered at the contract price, while A could be delivered at the contract price plus a premium equal to 8% of the contract price. Such nonpar delivery options have two important implications. First, a squeeze in the standard grade will be relieved before it becomes severe. In the example, a squeeze in variety B need only narrow the normal 10% premium on A to 8% before A becomes the cheapest deliverable grade. Second, for the same reason, the price of a futures contract is more likely to reflect the nonstandard variety more of the time.[5]

To appreciate the consequences of nonpar delivery options for hedging positions in different varieties of a commodity, it is necessary to examine how futures contracts presently distinguish among different grades and how premiums and discounts are established. The next section addresses these issues.

III. The Structure of Premiums and Discounts on Nonpar Delivery

Examination of the terms of existing futures contracts suggests there are two types of nonpar pricing rules: (1) "equivalence systems," which specify premiums and discounts based on the quantity of a particular characteristic in a delivered lot relative to a standard lot; and (2) "penalty systems," which specify premiums and discounts without reference to any single underlying characteristic.

The evidence presented below suggests that an equivalence system is used when an underlying characteristic of a commodity can be isolated through a technological extraction process. This permits buyers and sellers to agree on fixed exchange ratios (and, therefore, price ratios) for ostensibly different varieties. Incorporating these exchange ratios into the pricing rule of a futures contract leads to equivalent valuations on different lots of a commodity. Thus, market participants are indifferent to the lots they deliver or receive.

A penalty system is used when objective exchange ratios between different varieties of a commodity do not exist and when market participants may have preferences for a specific variety. In such cases, a futures contract might provide for delivery of nonstandard lots at a penalty. Buyers would normally expect to receive standard lots, and in abnormal circumstances sellers would be paying a penalty to buyers as compensation for delivering nonstandard grades. We now provide

5. See also Commodity Futures Trading Commission (1981, pt. 3, pp. 98–117) and Paul, Kahl, and Tomek (1981, p. 111) for a discussion of the trade-off between certainty of the varieties actually delivered on a futures contract and the pricing of that contract vs. having a "safety valve" to prevent squeezes.

some empirical content for our conjectures on the nature of both the equivalence and penalty adjustment systems.

Illustrations of Nonpar Pricing Systems

Part A of table 1 shows the major futures contracts which provide for nonpar pricing on the basis of an equivalence system. Observe that, with one exception,[6] the price of each commodity is adjusted as a function of a single, objectively measurable characteristic. In every case the adjustment to the contract price is specified as a percentage of that price (or a cash market price), where the percentage is derived from a technological extraction rule. For example, the per ounce price of gold as quoted on Comex is based on 100% fineness. If a bar with 99.8% fineness is delivered, the seller charges the buyer the contract price times 0.998. Similarly, if the delivered bar has a 99.5% fineness, the contract price is multiplied by 0.995. These adjustments are acceptable to both buyers and sellers because the cost of extracting the gold content of the two bars is the same.

Part B of table 1 shows the major futures contracts which provide for nonpar pricing on the basis of a penalty system. In every case, the discounts and premiums are not defined in terms of any single characteristic. For example, grain grades depend on test weight, quantity of damaged kernels, and quantity of foreign material. Similarly, beef quality grades depend on the age of the animal and the firmness, color, and marbling of the rib-eye muscle.

Penalty Systems versus Equivalence Systems

The use of a penalty system rather than an equivalence system stems from the absence of an extraction technology which can generate objective exchange ratios between different grades of a commodity. For example, No. 1 dark northern spring wheat and No. 2 soft red winter wheat do not contain a unique characteristic which can be extracted to produce equivalent valuations of the two varieties. This is illustrated by the historical variability in the relative prices of the two grades. One possibility is to allow delivery of both varieties, with premiums and discounts reflecting normal commercial differences. But this ignores the fact that a buyer may prefer a specific grade even though he would be compensated "fairly" (according to commercial values) for accept-

6. The Chicago Board of Trade soybean meal contract prices two characteristics: protein content and carbohydrate content. The former is done directly by pricing a delivered lot below the contract price if the protein content is less than the 44% standard. The latter is done indirectly by reducing the price on a delivered lot if the fiber content is in excess of the standard. In addition, because protein content is measured on the basis of a standard 12% moisture content, the price on a delivered lot is also reduced below the contract price if the moisture content is in excess of 12%.

TABLE 1 **Premiums and Discounts Based on Quality Differences**

Commodity (Exchange)	Characteristics	Price Adjustments
	A. Equivalence Systems	
Gold (Comex)	Fineness	% of contract price
Sugar (NYCSC)	Polarization (sweetness)	% of contract price
Soybean oil (CBT)	Refining loss	% of cash market price at time of loading
Soybean meal (CBT)	Protein	Based on cash market price per unit of protein at time of loading
	Moisture	% of cash market price on date of shipment
	Fiber content	% of cash market price on date of shipment
	B. Penalty Systems	
Wheat (CBT)	Growth and USDA grade for test weight, damaged kernels, and foreign matter	Cents per bushel
Corn, oats, soybeans (CBT)	USDA grade for test weight, damaged kernels, and foreign matter	Cents per bushel
Live slaughter cattle (CME)	Estimated hot yield	Cents per pound per % below standard
	Estimated USDA yield grade and quality grade of carcass	Cents per pound
	Weight relative to lot average	Cents per pound
Live hogs (CME)	Estimated USDA yield grade and quality grade of carcass	Cents per pound
	Weight relative to external standard	Cents per pound
Frozen pork bellies (CME)	Weight relative to external standard	Cents per pound
	Imperfections	Cents per pound as a function of number of imperfections
Coffee (NYCSC)	Country of origin	Cents per pound
Cocoa (NYCSC)	Country of origin	Dollars per ton
	Weight relative to external standard	Dollars per ton

NOTE.—CBT, Chicago Board of Trade; CME, Chicago Mercantile Exchange; Comex, New York Commodity Exchange; NYCSC, New York Coffee, Sugar, and Cocoa Exchange.

ing another variety.[7] More particularly, a buyer may be reluctant to use a futures contract on which a seller can deliver any of several grades, because this exposes him to the risk that he will have to bear the transactions costs of selling what is delivered and buying what he needs in the cash market.

To promote certainty in what will be delivered, contract premiums and discounts are established such that the standard variety (e.g., No. 2 soft red winter wheat) will be delivered most of the time. Thus, the premium allowed on the contract for delivering No. 1 dark northern spring wheat is somewhat less than the normal commercial premium. Similarly, the contract discount allowed on No. 3 hard red winter wheat is somewhat greater than the normal commercial discount. This structure penalizes the delivery of nonstandard grades and makes delivery of the standard variety the normal state of affairs.

Purposeful divergence between the discounts and premiums allowed on a futures contract compared with normal commercial price differences is the essence of a penalty system. In particular, the smaller the contract premiums relative to commercial premiums, and the larger the contract discounts relative to commercial discounts, the greater is the incentive to deliver the standard variety.

We will show in Section V that the structure of premiums and discounts on a penalty system contract influences the value of the contract for hedging both standard and nonstandard varieties of the commodity. These issues do not arise in the context of equivalence system contracts, because the relative prices of the varieties are stable at the exchange ratios implied by the extraction technology. As long as a futures contract reflects those exchange ratios, different grades of the commodity will be fully hedgeable with the contract.[8]

Location Differences

Table 2 shows the major futures contracts which provide for nonpar pricing as a function of delivery point. With one exception, location

7. For example, a flour miller may have a specific requirement for No. 2 soft red winter wheat (commonly used in cake and cookie flour) that cannot be satisfied by No. 1 dark northern spring wheat (commonly used in bread flour). See *New York Times*, August 3, 1981, Besant (1977, pp. 107–9), and Bean (1978) for comments on the specificity of requirements for different varieties of wheat.

8. The Treasury bond futures of the Chicago Board of Trade and the New York Futures Exchange have pricing rules that are based on incorrect extraction technologies and that do not leave market participants indifferent about the bonds delivered on those contracts. See Commodity Futures Trading Commission (1981, pt. 3, pp. 110–16) and Kilcollin (1982). The CBT rule assumes that different bonds are equivalent if their cash flows have the same present value at an 8% yield to maturity. The NYFE rule assumes bonds are equivalent if their cash flows have the same present value at the yield on the NYFE's 9% par bond. Both are incorrect rules for "extracting" the value of the cash flow on a bond, because bond values depend on coupon rates, tax treatment, and the entire term structure of interest rates.

TABLE 2 **Premiums and Discounts Based on Location**

Commodity (Exchange)	Price Adjustment
At least one nonpar delivery point:	
Wheat (CBT)	Cents per bushel or cents per pound based on city where warehouse of delivery is located
Corn (CBT)	
Oats (CBT)	
Soybeans (CBT)	
Frozen pork bellies (CME)	
Coffee (NYCSC)	
Live hogs (CME)	Cents per pound based on city where stockyard of delivery is located
Soybean meal (CBT)	Dollars per ton based on territory where shipper is located
Soybean oil (CBT)	Add carload freight rate from Decatur, Ill., to New York, subtract carload freight rate from point of delivery to New York
Multiple delivery points all at par:	
Live slaughter cattle (CME)	. . .
Sugar (NYCSC)	. . .
Cocoa (NYCSC)	. . .
Only one delivery point:	
Gold (Comex)	. . .

NOTE.—See note to table 1 for identification of exchange abbreviations.

premiums and discounts fall within the penalty system structure. This is consistent with our analysis, because there is no technological process which will transform, for example, Chicago wheat into Toledo wheat at a uniform price for all market participants. The only candidate for such a process is freight rates, but they are inappropriate unless all alternative delivery points supply a common consumption point.[9] Only in this very special case will market price differentials reflect differences in costs of transportation for all market participants. Thus, to enhance certainty over where a commodity will be delivered, a penalty system for nonpar delivery is used. For example, the Chicago Board of Trade wheat contract permits delivery in Toledo at the normal commercial discount on wheat in that city relative to Chicago, plus a bit more to encourage Chicago delivery in most cases.[10]

Soybean oil is the single exception to the location rule. In reality, it is the exception that proves the rule, since most soybean oil processed by crushers in the Midwest moves to the New York region for further processing into consumer foods products. The New York price of dif-

9. For specific examples of this, see Paul et al. (1981, p. 109) on Maine potatoes and Crow et al. (1972) on live slaughter steers. See also Baldwin (1934, chap. 3).

10. Crow et al. (1972) note that when Guyman, Okla., was added as a delivery point on the CME live slaughter cattle contract, deliveries at that point were discounted $1.00 per hundred weight relative to deliveries in Omaha at the contract price, even though cash market prices in Guyman then averaged only $0.10 per hundred weight below cash market prices in Omaha.

ferent lots of oil must be the same regardless of where those lots originate. Thus, the price of oil at a crushing plant will be lower the further the plant is from New York. The location adjustment rule for soybean oil futures follows this pattern, and can be viewed as an equivalence system based on a technology (transportation) for extracting soybean oil from different delivery points in the Midwest and moving that oil to a common consumption point (New York).

IV. The Behavior of Cash Market Prices for Different Varieties of the Same Commodity

Having explored the basis for premiums and discounts on futures contracts with nonpar delivery options, we would now like to evaluate the usefulness of those contracts for hedging different varieties of a commodity. This requires an analysis of the behavior of prices in the cash markets. In this section we develop a dynamic model of cash market prices for two varieties of a commodity, and estimate the parameters of that model for wheat, corn, and soybeans. The next section links this cash market model to prices on futures contracts, and analyzes the implications of nonpar delivery for hedging positions in the two grades.

Structure of the Model

The model is designed to capture the behavior of prices on two varieties of a commodity which are close substitutes in the long run, but which may be subject to different short-run changes in supply and/or demand. In particular, we focus on grades which are not equivalent through a technological extraction process. (Commodities which are technologically equivalent do not create a hedging problem, as mentioned above.)

Suppose there exist two varieties of a commodity: a premium grade with price P_A and a standard grade with price P_B. The normal commercial premium on A is denoted Q. We assume that, over the long run, the average value of the price ratio P_A/P_B is $1 + Q$.[11] (We specify $Q > 0$, but nothing in our analysis would change for $Q < 0$.) The varieties can differ because of quality (e.g., hard red winter wheat vs. soft red winter wheat) or because of location (e.g., wheat in Toledo vs. wheat in Chicago).

To model the dynamic interaction between P_A and P_B, we assume that the logarithm of the price of the premium variety (reduced by the normal commercial premium) and the logarithm of the price of the standard grade are composed of a common component, denoted E, and

11. For simplicity, we assume the normal commercial premium is stationary. At the cost of greater expostional complexity, the premium could follow a seasonal or cyclical pattern. A similar, but much more complex, analysis can be undertaken for an additive premium.

an "abnormal" difference component, denoted D, which is shared between the two varieties:

$$\ln[P_A/(1 + Q)] = E + \alpha \cdot D \tag{1a}$$

$$\ln[P_B] = E - \beta \cdot D, \tag{1b}$$

where $\alpha + \beta = 1$. Changes in E through time reflect broad changes in supply and/or demand for the commodity as a whole. Changes in D reflect differential changes in supply and/or demand for the two grades.

The specification of the dynamic behavior of the common and difference components will reflect two ideas. First, in the long run the abnormal difference component (D) is expected to be zero, so that the prices of the varieties are determined by the common component (E) and the normal commercial premium on A. Second, in the short run, the ratio of P_A to P_B can differ from $1 + Q$ as a result of transient deviations of the difference component (D) from zero. Note that $P_A/P_B = (1 + Q) \cdot \exp(D)$. Thus, the two grades are close substitutes in the long run (because the abnormal difference component tends to zero) but not in the short run.

From equation (1) the abnormal difference component is

$$D = \ln[P_A/(1 + Q)] - \ln(P_B). \tag{2}$$

To model this difference component as a transitory phenomenon, we specify that it varies through time as a stable first-order process around zero:

$$dD = -\lambda \cdot D \cdot dt + \omega \cdot dz_D, \tag{3}$$

where dz_D is a Gauss-Weiner process and ω^2 is the variance of the random change in the difference component per unit time. We show below that ω^2 is in a range of $(0.005)^2$ per day to $(0.01)^2$ per day for different varieties of grains.

In equation (3), λ is the rate at which the abnormal difference component is expected to decay through time. To see this, note that equation (3) implies the value of D at time $t_0 + t$ is related to the value of D at time t_0 by the stochastic equation

$$D(t_0 + t) = D(t_0) \cdot \exp(-\lambda \cdot t) + d \tag{4a}$$

$$d \sim N\{0, \tfrac{1}{2}\omega^2[1 - \exp(-2 \cdot \lambda \cdot t)]/\lambda\}. \tag{4b}$$

If $\lambda = 0.01$ per day, then the expected value of $D(t_0 + t)$ given $D(t_0)$ is $0.99 \cdot D(t_0)$ after one day. More particularly, 99% of the abnormal difference at time t_0 is expected to persist to time $t_0 + t$. If the decay rate is higher than 0.01, the persistence of the difference component will be smaller than 99%.[12] We estimate below that λ is in the range of 0.03–0.17 per day for different varieties of grain.

From equation (1) the common price component is a weighted average of the prices of the two varieties:

$$E = \beta \cdot \ln[P_A/(1 + Q)] + \alpha \cdot \ln(P_B). \tag{5}$$

We assume that changes in the common component are permanent and persist indefinitely through time. More specifically, we specify that the common component varies as a random walk with zero drift:

$$dE = \sigma \cdot dz_E \tag{6}$$

where dz_E is a Gauss-Weiner process which is uncorrelated with dz_D and σ^2 is the variance of the change in the common component per unit time.

The model of equation (6) is an often-used model of the variation of prices of storable assets such as common stock and commodities. The only wrinkle introduced in this context is our assumption that only the common element of the two prices follows a random walk, and that the difference component follows a transitory zero-mean process. Other models have not had occasion to examine the relationship between prices of two varieties of the same commodity.

Parameter Estimates

A realistic analysis of the consequences of nonpar delivery options for hedging requires reasonable estimates of λ and ω^2. (The values of the other parameters in eqq. [1]–[4] do not affect the qualitative results.) From equations (2) and (4) we have that the change in the difference of the logarithms of the two prices from time t_0 to time $t_0 + t$ can be written as

$$\ln[P_A(t_0 + t)] - \ln[P_B(t_0 + t)] = \delta + \gamma\{\ln[P_A(t_0)] - \ln[P_B(t_0)]\} + u \tag{7}$$

where

$$\delta = \ln(1 + Q) \cdot [1 - \exp(-\lambda \cdot t)] \tag{8a}$$

$$\gamma = \exp(-\lambda \cdot t) \tag{8b}$$

$$\mathrm{var}(u) = \tfrac{1}{2}\omega^2[1 - \exp(-2 \cdot \lambda \cdot t)]/\lambda. \tag{8c}$$

Table 3 reports estimates of the parameters of equation (7) derived from daily data for five different pairs of varieties of wheat, corn, and

12. In the limit, as λ goes to zero in eq. (3), there will be complete persistence of the difference component. In other words, there will be no tendency for the price ratio P_A/P_B to converge to the "normal" ratio, and the two varieties are more accurately characterized as separate commodities instead of as two varieties of the same commodity.

TABLE 3 Estimates of Equation (7), Daily Observations

Nonstandard Variety/ Standard Variety§	Estimated Parameters†			Identified Parameters‡		
	δ	γ	Var(u)	Q	λ	ω^2
Toledo soft wheat/ Chicago soft wheat	−.000835 (.000508)	.9689** (.0113)	$(.00814)^2$	−.0265	.0316 per day	$(.00827)^2$ per day
Toledo soybeans/ Chicago soybeans	.000476* (.000213)	.8716** (.0227)	$(.00424)^2$	.0037	.1734 per day	$(.00453)^2$ per day
Toledo corn/ Chicago corn	.000155 (.000405)	.9049** (.0210)	$(.00836)^2$	.0016	.0999 per day	$(.00878)^2$ per day
St. Louis corn/ Chicago corn	.000770 (.000523)	.9157** (.0190)	$(.01000)^2$	.0092	.0881 per day	$(.01044)^2$ per day
Chicago hard wheat/ Chicago soft wheat	−.00263* (.00085)	.9324** (.0172)	$(.00986)^2$	−.0382	.0700 per day	$(.01021)^2$ per day

§All Toledo and St. Louis prices are F.O.B. truck. All Chicago prices are F.O.B. boxcars on track in the Chicago switching district. Wheat is No. 2 red winter (hard or soft, as noted). Corn is No. 2 yellow corn. Soybeans are No. 1 yellow soybeans. Data are from 1978, 1979, and 1980 statistical annuals of the Chicago Board of Trade.

†Standard errors of estimate in parentheses.

‡$Q = \exp[\delta/(1-\gamma)] - 1$, $\lambda = -\ln(\gamma)$, $\omega^2 = -2 \cdot \ln(\gamma) \cdot \mathrm{var}(u)/(1-\gamma^2)$.

*Significantly different from zero at 99% confidence level.

**Significantly less than unity at 99% confidence level.

soybeans.[13] The table shows that the estimates of γ are all significantly less than unity, implying that λ is greater than zero. Thus, abnormal price differences among the varieties of wheat, corn, and soybeans do not persist through time.

V. Implications of Nonpar Delivery Options for Hedging

Having established reasonable values for the parameters of our model, we turn now to an evaluation of hedging alternative varieties of a commodity. We use Monte Carlo simulation experiments to illustrate the risk on a long cash–short futures position. Similar results would be obtained for a long futures–short cash position. In either case, as long as a futures contract permits delivery of alternative commodity grades, market participants hedging a position in one (and possibly both) of the grades will bear risk. The importance of contract design for enhancing or impairing the contract's usefulness for hedging will be emphasized.

Contract Specification and Pricing

To analyze the consequences of nonpar delivery for hedging a long cash market position, we must first specify the terms of different futures contracts and show how the settlement prices on those contracts are linked to contemporaneous cash market prices. We evaluate two types of futures contracts: (1) a single-delivery (SD) contract on which only the standard variety (B) is deliverable; and (2) a multiple-delivery (MD) contract on which B is deliverable at the contract price or variety A is deliverable at the contract price times a premium of $1 + Q_c$. The premium (Q_c) allowed on the multiple-delivery contract can vary from zero to the normal commercial premium (Q) on grade A. If $Q_c = Q$, the premium grade is deliverable at the contract price times the normal commercial premium. If $Q_c < Q$, grade A is deliverable at a penalty relative to the normal commercial premium. This penalty is typical of the adjustment systems for quality and location differences described in Section III above.

We now turn to the determination of the settlement price on a futures contract. The settlement price depends upon contract design and the structure of prices in the cash market (given that the seller has the option of delivering any of several varieties). Let R denote the rate of interest per unit time (compounded continuously) on debt of all maturities,[14] and let t^* denote the delivery date of the futures contract.

13. Note that corn in St. Louis and soybeans and corn in Toledo are normally at a premium relative to the same grains in Chicago—i.e., the estimated Q's are greater than zero—and that wheat in Toledo is normally at a discount relative to wheat in Chicago. In addition, hard wheat in Chicago is normally at a discount to soft wheat in Chicago.

14. This implies that in this model, as in Black and Scholes (1973), the term structure of interest rates is flat and stationary. This assumption can be relaxed at the cost of greater expositional complexity. See Merton (1973).

As derived in the Appendix, arbitrage between the cash and futures markets will produce the following equilibrium settlement price at time $t < t^*$ on the single-delivery contract (F_{SD}):

$$F_{SD} = P_B \cdot \exp[R \cdot (t^* - t)]. \tag{9}$$

The equilibrium settlement price on the multiple-delivery contract (F_{MD}) will be

$$F_{MD} = \{P_B \cdot [1 - N(d_1)] + P_A \cdot (1 + Q_c)^{-1} \cdot N(d_2)\} \cdot \exp[R \cdot (t^* - t)] \tag{10}$$

where

$$d_1 = \frac{\ln[P_B \cdot (1 + Q_c)/P_A] + \tfrac{1}{2}\omega^2 \cdot (t^* - t)}{\omega \cdot (t^* - t)^{1/2}} \tag{11a}$$

$$d_2 = d_1 - \omega \cdot (t^* - t)^{1/2} \tag{11b}$$

$$N(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp[-\tfrac{1}{2}u^2] \cdot du. \tag{11c}$$

This expression for F_{MD} reflects the fact that a seller has an option to deliver either grade on his contract. (See App. for more details.)

Of crucial interest from our standpoint is that equation (9) implies that the single-delivery contract is always priced in relation to B (which is the only variety deliverable on that contract). On the other hand, equation (10) shows that the multiple-delivery contract is priced in relation to *both* varieties. The reasonableness of this result can be appreciated by considering two extreme cases. If $P_A/(1 + Q_c)$ is much greater than P_B (so that d_1 and d_2 are near zero in eqq. [11a] and [11b]), then F_{MD} will be comparable to F_{SD}. This corresponds to the case where grade A is at such a large premium in the cash market that sellers are unlikely to deliver it on their multiple-delivery futures contracts, so those contracts are essentially single-delivery contracts on B. On the other hand, if $P_A/(1 + Q_c)$ is much less than P_B, then F_{MD} will approximately equal $\{P_A/(1 + Q_c)\} \cdot \exp[R(t^* - t)]$, that is, the settlement price on a contract which permits delivery of variety A only. This case corresponds to the situation where the price of A is sufficiently low that sellers are unlikely to deliver variety B on their multiple-delivery futures contracts, so those contracts behave as if they were single-delivery contracts on A. In less extreme cases, F_{MD} will vary with both P_A and P_B.

Hedging Simulations

To evaluate the implications of alternative contract designs for hedging, we consider a market participant who has a long position in the commodity (variety A or B) that he wants to hedge for 30 days. One way to hedge is to sell out the position and invest in short-term debt securities. The investor then bears no risk as a result of fluctuations in the price of the commodity. Another way to hedge is to short futures contracts on the commodity, hold the short position for 30 days, and then liquidate the short position.[15] We measure the "residual risk" on the futures hedge strategy as the standard deviation of the ratio of the (generally uncertain) net value of the hedged position at the end of 30 days to the (known) value of the debt securities at the end of 30 days. For example, a residual risk of 5% implies approximately a 66% probability that the net value of the hedged position at the end of 30 days would fall within 5% of the value which the debt securities would have reached.[16]

The experiment just described was undertaken for long cash market positions in varieties A and B. The result, for any single delivery or multiple delivery contract, is a pair of numbers characterizing the residual risks on hedged positions in each of the grades.

Figures 1 and 2 show the estimated residual risks on hedged positions in grades A and B for the various types of futures contracts.[17] Figure 1 shows the results for low volatility of the abnormal difference between the prices of the two varieties, and figure 2 shows the results for high volatility. The vertical axis in each figure measures the residual risk on a hedged position in grade A. The horizontal axis measures the residual risk on a hedged position in variety B.

In both figures, the point describing the residual risks on the single-delivery contract (denoted x) is located on the vertical axis because, by construction, the single-delivery contract provides a perfect hedge for a position in variety B (the only grade deliverable under the contract).

For the multiple-delivery contract, points y and z show the risk combinations for the cases where the contract premium (Q_c) is zero and equal to the commercial premium, respectively. The arc connecting points y and z shows risk exposure for intermediate values of the contract premium ($0 < Q_c < Q$). Both figures illustrate that any multi-

15. In our simulations the futures contracts have 180 days to settlement at the beginning of the 30-day hedging interval. The qualitative results are not affected by alternative futurities of settlement. In all cases the hedge ratio of a futures contract is taken to be $\exp[R \cdot (t^* - t)]$, so that the participant holds $\exp[-R \cdot (t^* - t)]$ contracts short to hedge each unit of the commodity held long.

16. The simulations assume that the participant trades to rebalance his hedge, and meets his mark-to-market margin calls, once an hour. Credits resulting from net cash inflows on mark-to-market settlements earn interest at the rate of R, and debits resulting from net cash outflows are charged interest at the rate of R.

17. All simulations assumed $\sigma^2 = (0.015)^2$ per day, $Q = 0.10$, $R = 0.10$ per year, and $\alpha = \beta = 0.5$. The value of E at the start of the hedging interval was set to ln(100) and the value of D at the start of the hedging interval was chosen randomly from a normal distribution with mean zero and variance $\frac{1}{2}\omega^2/\lambda$. As shown by eq. (4), this is the asymptotic distribution of D. All estimates are based on 100 replications.

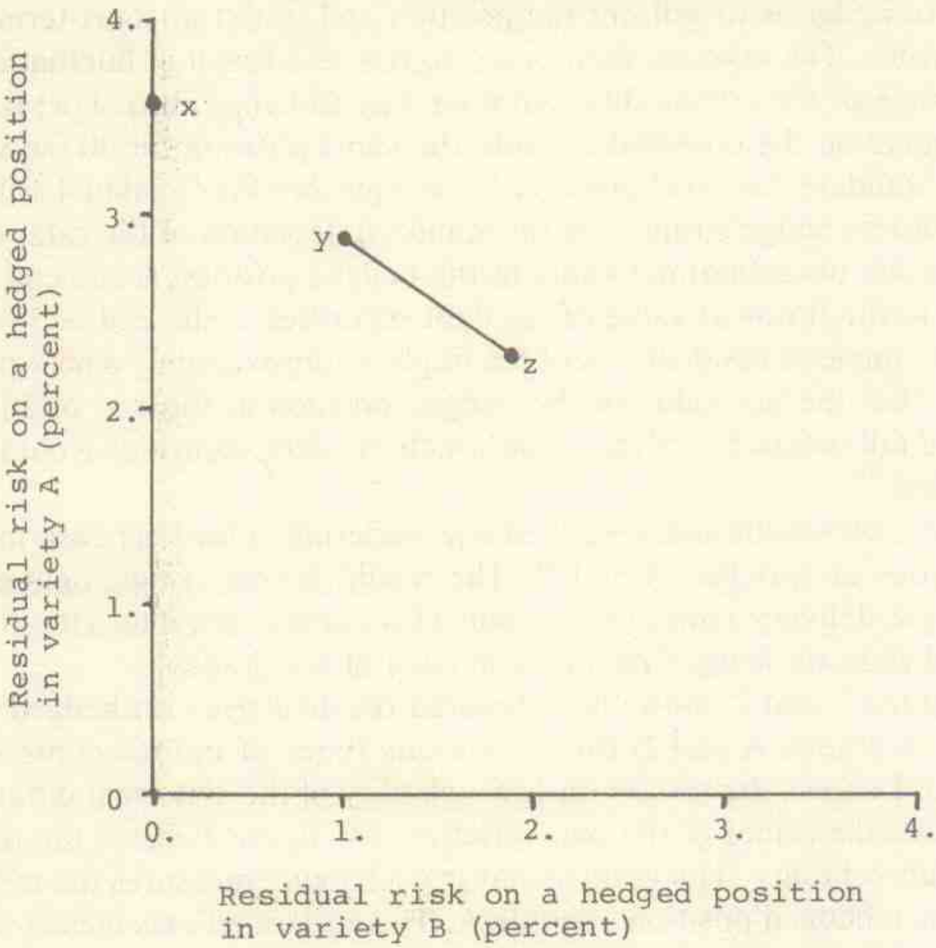

FIG. 1.—Residual risks for low volatility of abnormal price differences, $\omega^2 = (.01)^2$/day, with decay rate of $\lambda = .075$/day. x = single-delivery contract; y = multiple-delivery contract, $Q_c = 0$; z = multiple-delivery contract, $Q_c = Q$.

ple-delivery contract is a poorer hedge than the single-delivery contract for variety B, that is, the line segment y–z is to the right of point x. This result occurs because the multiple-delivery contract allows delivery of type A and hence is influenced by fluctuations in P_A that may not occur in P_B.

Note that the quality of the hedge for variety B becomes worse as the contract premium (Q_c) permitted on A moves toward the commercial premium (Q). This follows from the fact that as Q_c approaches Q the penalty for delivering A becomes smaller and $P_A(P_B)$ will have a larger (smaller) effect on the settlement price more of the time. In this case, of course, the quality of the hedge for variety A improves.

Implications for Contract Design

It is useful to emphasize what might seem to be an intuitively obvious result: permitting delivery of a nonstandard grade (A) on a futures contract imposes risk on those hedging positions in the standard variety (B). This risk exposure increases as the contract premium allowed on A approaches the normal commercial premium. While a contract

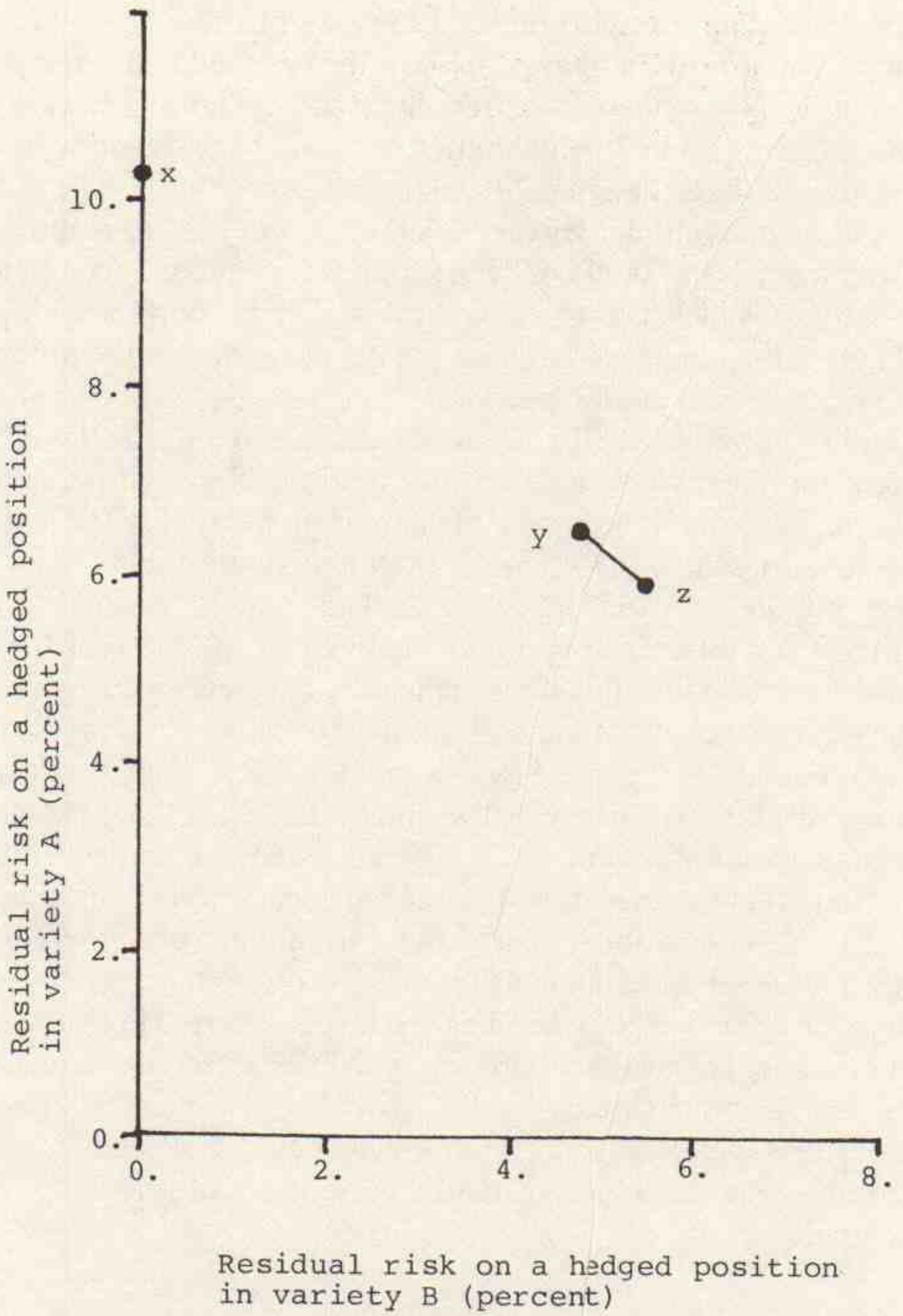

FIG. 2.—Residual risks for high volatility of abnormal price differences, $\omega^2 = (.03)^2$/day, with decay rate $\lambda = .075$/day. x = single-delivery contract; y = multiple-delivery contract, $Q_c = 0$; z = multiple-delivery contract, $Q_c = Q$.

premium equal to the commercial premium reduces the risk exposure of those hedging the nonstandard grade, the aggregate effect may still be a net loss to society if the bulk of the hedging in the commodity is against the standard variety. This implies that casual statements to the effect that contract premiums on nonstandard varieties ought to be "equitable" and equal to normal commercial premiums may not be optimal from a social welfare standpoint.

An important implication of our analysis for contract design is that the decision to allow delivery of the nonstandard variety at *any* premium becomes more important, and the size of the penalty becomes less important, with increases in ω^2, the variance of unexpected changes in the abnormal price difference (D). This follows from a comparison of figures 1 and 2, which differ because the assumed value of ω is three times as large in figure 2 (0.03) as in figure 1 (0.01). The result is that point y (the multiple-delivery contract with $Q_c = 0$) is much further away from x (the single-delivery contract) in figure 2 than in figure 1. Similarly, y is closer to z (the multiple-delivery contract with $Q_c = Q$) in figure 2 than in figure 1. These results mean that as the difference in the prices of two grades becomes more volatile, it becomes more important to consider whether there should be separate single-delivery contracts for the two varieties.[18] In addition, if a multiple-delivery contract is preferred, it becomes less important to identify the standard variety and to select the contract premium or discount (hence the penalty) on the nonstandard variety. This occurs because greater volatility of the difference in prices implies that the price of the nonstandard variety will influence significantly the settlement price of a multiple-delivery contract more of the time for *any* contract premium.

A final simulation exercise suggests that the role of contract design in allocating residual risk becomes less important when there is substantial persistence of abnormal price differences among grades. In terms of equation (4) above, greater persistence is characterized by a smaller value of λ. Ceteris paribus, decreasing λ from 0.075 per day to 0.0075 per day reduces the residual risk exposure on both varieties (for all contracts) to approximately one-half the levels reported in figures 1 and 2.[19] This occurs because, with more stable price differences over time, a hedger can be more certain that the "basis" will remain unchanged.[20] Thus, it is less important to develop special contract designs for different varieties when there is substantial persistence of price differences among grades.

VI. Summary

This paper has examined the structure of futures contracts on commodities with heterogeneous varieties. In general, such contracts allow

18. This was an important factor in the decision of the New York Mercantile Exchange to introduce a new contract on heating oil deliverable at Gulf Coast ports in 1981 rather than to add Gulf Coast deliveries to its existing contract on heating oil deliverable at the Port of New York.

19. In particular, the asymptotic variance of the abnormal difference component, $\frac{1}{2}\omega^2/\lambda$ (see eq. [4]), must be held constant, so that decreasing λ by a factor of 10 (from 0.075 to 0.0075) must be accompanied by a decrease in ω by a factor of $(10)^{1/2} = 3.162$.

20. For example, over a 30-day hedging interval only 10.5% of the initial abnormal difference persists to the end of the interval if $\lambda = 0.075$ per day, $\exp(-0.075 \times 30) = 0.105$, but 79.9% of the difference persists if $\lambda = 0.0075$ per day, $\exp(-0.0075 \times 30) = 0.799$.

delivery of different grades at specified premiums and discounts to reduce the likelihood and severity of squeezes.

The way those premiums and discounts are derived suggests a number of conclusions. If the varieties of a commodity are related through a technological extraction process, buyers and sellers can agree on exchange ratios, and hence relative prices, for different grades. More particularly, broadening delivery options on a contract to reduce the likelihood of squeezes will not create residual, unhedgeable risks.

If the varieties of a commodity are not related through an extraction technology, some of the risk of cash market positions will be unhedgeable with futures contracts that permit nonpar delivery. We demonstrated that the contract premium on a nonstandard variety helps to allocate unhedgeable risks among market participants. We also argued that there is no prima facie case for setting the contract premium equal to the normal commercial premium. Finally, we showed that when the volatility of the abnormal difference in grade prices increases, the contract premium becomes a less powerful tool for allocating residual risks.

The results of this paper suggest that there are important advantages to distinguishing among different varieties of a commodity with an extraction technology rather than with cruder grading systems. The implementation of objective exchange ratios permits broader delivery specifications without increasing the unhedgeable risks of different grades. More particularly, there may be important advantages of unraveling the intrinsic characteristics of commodities in the sense of Lancaster (1966*a*, 1966*b*) and in specifying premiums and discounts on nonstandard varieties in terms of those characteristics.[21]

Appendix

Derivation of Equations (9) and (10)

The settlement prices F_{SD} and F_{MD} specified in equations (9) and (10) for the two types of futures contracts can be derived in a four-step process:

21. In some cases the usefulness of a contract for hedging could be improved by moving toward an equivalence system. Ideally, for example, the price of live slaughter cattle should be based on steer weight with an equivalence adjustment for hot yield and a penalty adjustment for the USDA yield grade and quality grade of the carcass. (Hot yield is the ratio of carcass weight immediately after slaughter to the weight of the live steer.) At the present time, differences in hot yield are adjusted with a penalty system that appears to discourage delivery of high hot yield steers. This has resulted in a reluctance of meat packers to use the live slaughter cattle contract as a long hedge when they have a specific need for high hot yield steers. In other cases, a different grading scheme would more closely approximate the pricing of an extraction technology. See Cox and Wright (1979) for an analysis of grading standards for live feeder steers. It is interesting to note that a proposed technology-based pricing system for cotton was never implemented, because the system omitted some important but qualitative characteristics of cotton. See Hoffman (1932, pp. 299–300).

1. Derive diffusion equations for P_A and P_B from the diffusion equations for E and D.

2. Derive a differential equation for dF_k (k = SD or MD) as a function of dP_A, dP_B, and dt.

3. Derive a partial differential equation for the equilibrium settlement price F_k (k = SD or MD) based on riskless arbitrage with the cash market.

4. Solve that partial differential equation, subject to the respective boundary conditions implied by the two contracts, for the behavior of the equilibrium settlement price.

The methodology used here is similar to that of Black and Scholes (1973) and Black (1976). In particular, we assume a flat and stationary term structure of interest rates, no costs of storage (other than financing costs), no limitation on short sales or margin financing, and an infinitely elastic supply of arbitrage services. (See Garbade and Silber [1983] for an analysis of the behavior of futures prices when arbitrage services are not infinitely elastic.)

Diffusion Equations for P_A *and* P_B

The prices of the two varieties (A and B) in terms of the common and difference components, E and D, respectively, are given by equation (1) in the text:

$$P_A = \exp(E + \alpha \cdot D) \cdot (1 + Q) \quad \text{(A1a)}$$

$$P_B = \exp(E - \beta \cdot D). \quad \text{(A1b)}$$

From Ito's lemma we have

$$\begin{aligned} dP_i = {} & (\partial P_i/\partial E) \cdot dE + (\partial P_i/\partial D) \cdot dD \\ & + ½(\partial^2 P_i/\partial E^2) \cdot (dE)^2 + ½(\partial^2 P_i/\partial D^2) \cdot (dD)^2 \\ & + [\partial^2 P_i/(\partial E \cdot \partial D)] \cdot dE \cdot dD, \qquad i = \text{A or B}. \end{aligned} \quad \text{(A2)}$$

The diffusion processes for the common and difference components are given by equations (3) and (6) in the text:

$$dE = \sigma \cdot dz_E \quad \text{(A3a)}$$

$$dD = -\lambda \cdot D \cdot dt + \omega \cdot dz_D, \quad \text{(A3b)}$$

where dz_E and dz_D are uncorrelated Gauss-Weiner processes.

Substituting (A3) into (A2) for i=A and i=B, and using the multiplication rules of stochastic calculus—$(dt)^2 = 0$, $(dt)(dz_j) = 0$ for $j=E$ or D, $(dz_j)^2 = dt$ for $j=E$ or D, and $(dz_E)(dz_D) = 0$—gives

$$\begin{aligned} dP_A = {} & P_A \cdot \sigma \cdot dz_E + P_A \cdot \alpha \cdot \omega \cdot dz_D \\ & + P_A \cdot (-\alpha \cdot \lambda \cdot D + ½\sigma^2 + ½\alpha^2 \cdot \omega^2) \cdot dt \end{aligned} \quad \text{(A4a)}$$

$$\begin{aligned} dP_B = {} & P_B \cdot \sigma \cdot dz_E - P_B \cdot \beta \cdot \omega \cdot dz_D \\ & + P_B \cdot (\beta \cdot \lambda \cdot D + ½\sigma^2 + ½\beta^2 \cdot \omega^2) \cdot dt. \end{aligned} \quad \text{(A4b)}$$

Differential Equations for dF_k, k = *SD and MD*

In general, the settlement price on a futures contract can be written as a function of time and the contemporaneous prices of the underlying varieties:

$$F_k = F_k(P_A, P_B, t), \qquad k = \text{SD or MD}. \quad \text{(A5)}$$

From Ito's lemma we have

$$\begin{aligned} dF_k = {} & (\partial F_k/\partial P_A) \cdot dP_A + (\partial F_k/\partial P_B) \cdot dP_B + (\partial F_k/\partial t) \cdot dt \\ & + \tfrac{1}{2}(\partial^2 F_k/\partial P_A^2) \cdot (dP_A)^2 + \tfrac{1}{2}(\partial^2 F_k/\partial P_B^2) \cdot (dP_B)^2 \\ & + [\partial^2 F_k/(\partial P_A \cdot \partial P_B)] \cdot dP_A \cdot dP_B, \qquad k = \text{SD or MD}. \end{aligned} \tag{A6}$$

Computing $(dP_A)^2$ and $(dP_B)^2$ from (A4), subject to the multiplication rules noted above, gives

$$\begin{aligned} dF_k = {} & (\partial F_k/\partial P_A) \cdot dP_A + (\partial F_k/\partial P_B) \cdot dP_B + \{(\partial F_k/\partial t \\ & + \tfrac{1}{2}(\partial^2 F_k/\partial P_A^2) \cdot P_A^2 \cdot (\sigma^2 + \alpha^2 \cdot \omega^2) \\ & + \tfrac{1}{2}(\partial^2 F_k/\partial P_B^2) \cdot P_B^2 \cdot (\sigma^2 + \beta^2 \cdot \omega^2) \\ & + [\partial^2 F_k/(\partial P_A \cdot \partial P_B)] \cdot P_A \cdot P_B \cdot (\sigma^2 - \alpha \cdot \beta \cdot \omega^2)\} \cdot dt, \; k = \text{SD or MD}. \end{aligned} \tag{A7}$$

Partial Differential Equation for an Equilibrium Futures Price

Suppose a market participant establishes a long position in $h_A = \partial F_k/\partial P_A$ units of variety A, a long position in $h_B = \partial F_k/\partial P_B$ units of variety B, and a short position in 1 unit of futures contract k, where k = SD or MD. Because the futures contract has no value (see Black [1976]), his net worth is

$$W = h_A \cdot P_A + h_B \cdot P_B. \tag{A8}$$

With continuous mark-to-market settlement of the futures contract (see Black [1976]), the differential of his net worth is

$$dW = h_A \cdot dP_A + h_B \cdot dP_B - dF_k. \tag{A9}$$

In view of equation (A7) and the definitions of h_A and h_B, this becomes

$$\begin{aligned} dW = {} & -(\partial F_k/\partial t) \cdot dt - \tfrac{1}{2}(\partial F_k/\partial P_A^2) \cdot P_A^2 \cdot (\sigma^2 + \alpha^2 \cdot \omega^2) \cdot dt \\ & - \tfrac{1}{2}(\partial F_k/\partial P_B^2) \cdot P_B^2 \cdot (\sigma^2 + \beta^2 \cdot \omega^2) \cdot dt \\ & - [\partial^2 F_k/(\partial P_A \cdot P_B)] \cdot P_A \cdot P_B \cdot (\sigma^2 - \alpha \cdot \beta \cdot \omega^2) \cdot dt. \end{aligned} \tag{A10}$$

Since the differential of net worth in equation (A10) depends only on the passage of time, and not on changes in either cash or futures prices, the position is fully hedged. The rate of change of net worth must, therefore, be equal to the rate of interest on risk-free debt (which we assume is equal to R for debt of all maturities):

$$dW/W = R \cdot dt. \tag{A11}$$

Using equations (A8) and (A10), this implies that

$$\begin{aligned} \partial F_k/\partial t = {} & -R \cdot P_A \cdot (\partial F_k/\partial P_A) - R \cdot P_B \cdot (\partial F_k/\partial P_B) \\ & - \tfrac{1}{2}(\partial^2 F_k/\partial P_A^2) \cdot P_A^2 \cdot (\sigma^2 + \alpha^2 \cdot \omega^2) \\ & - \tfrac{1}{2}(\partial^2 F_k/\partial P_B^2) \cdot P_B^2 \cdot (\sigma^2 + \beta^2 \cdot \omega^2) \\ & - [\partial^2 F_k/(\partial P_A \cdot \partial P_B)] \cdot P_A \cdot P_B \cdot (\sigma^2 - \alpha \cdot \beta \cdot \omega^2), \qquad k = \text{SD or MD}. \end{aligned} \tag{A12}$$

Equation (A12) is the partial differential equation for the equilibrium settlement price which must be satisfied by both of the futures contracts.

Solution to the Partial Differential Equation

The equilibrium prices on the two futures contracts specified in Section V of the text can be found by solving equation (A12) subject to appropriate boundary conditions.

The single-delivery contract allows delivery only of variety B. On the final settlement date t^* we must, therefore, have convergence of F_{SD} and P_B, so that

$$F_{SD}(P_A, P_B, t^*) = P_B. \tag{A13}$$

Equations (A12) and (A13) together imply that, for $t \leq t^*$,

$$F_{SD}(P_A, P_B, t) = \exp[R \cdot (t^* - t)] \cdot P_B. \tag{A14}$$

The multiple-delivery contract allows delivery, at the seller's option, of either variety B at the settlement price or variety A at the settlement price times a contract premium of $1 + Q_c$. On the final settlement date t^* we must have convergence of F_{MD} and the price of the net cheapest deliverable variety, so that

$$F_{MD}(P_A, P_B, t^*) = \min[P_B, P_A \cdot (1 + Q_c)^{-1}]. \tag{A15}$$

Note that if $P_B < P_A \cdot (1 + Q_c)^{-1}$ at t^*, the settlement price will converge to P_B and only variety B will be delivered (at the contract price). If $P_B > P_A \cdot (1 + Q_c)^{-1}$ at t^*, the settlement price will converge to $P_A \cdot (1 + Q_c)^{-1}$ and only variety A will be delivered (at the contract price times the premium $1 + Q_c$, or at P_A). Equations (A12) and (A15) together imply that, for $t \leq t^*$,

$$\begin{aligned} F_{MD}(P_A, P_B, t) = \exp[1R \cdot (t^* - t)]\, \{P_B \cdot [\,- N(d_1)] \\ + P_A \cdot (1 - Q_c)^{-1} \cdot N(d_2)\}, \end{aligned} \tag{A16}$$

where

$$d_1 = \frac{\ln[P_B \cdot (1 + Q_c)/P_A] + \tfrac{1}{2}\omega^2 \cdot (t^* - t)}{\omega \cdot (t^* - t)^{1/2}} \tag{A17a}$$

$$d_2 = d_1 - \omega \cdot (t^* - t)^{1/2} \tag{A17b}$$

$$N[x] = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp[-\tfrac{1}{2}u^2] \cdot du. \tag{A17c}$$

It should be noted that equations (A14) and (A16) can be written in the form

$$F_k = \exp[R \cdot (t^* - t)] \cdot S_k, \qquad k = \text{SD or MD}, \tag{A18}$$

where

$$S_{SD} = P_B \tag{A19a}$$

$$S_{MD} = P_B - C \tag{A19b}$$

$$C = P_B \cdot N(d_1) - P_A \cdot (1 + Q_c)^{-1} \cdot N(d_2). \tag{A19c}$$

Equation (A18) says that the futures price is equal to some spot market price S_k compounded over the time interval remaining to settlement at the interest rate R. In the case of the single-delivery contract, the relevant spot price is simply the price of the deliverable commodity (P_B). For the multiple-delivery contract, however, the spot price is the price of the standard variety (P_B) less the value C of a call option on one unit of that variety where the cost of exercising the option is $P_A \cdot (1 + Q_c)^{-1}$.[22] This shows that the existence of an option to deliver more than one variety of a commodity on a futures contract will generally act to depress the price of the contract below the price of a single-delivery contract. This result is reasonable, because the choice of which variety to deliver rests with the seller. He pays for the right to choose by contracting to sell at a lower price.

It should also be noted that the prices of both the single-delivery and multiple-delivery futures contracts follow the same partial differential equation (A12). What distinguishes the contracts is the boundary conditions of equation (A13) compared to those of equation (A15). These boundary conditions are defined by the delivery specifications of the contracts. This shows, quite directly, that delivery specifications are crucial to the behavior of futures prices even when futures contracts are not liquidated by physical delivery.

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