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Additional Aspects of Rational Insurance Purchasing*

This paper examines the use of the Arrow-Pratt and Ross measures of risk aversion. It is shown that in situations where the individual faces more than one uncertain prospect and can purchase insurance only for a subset of these prospects, then in many cases the Arrow-Pratt and Ross measures of risk aversion are not sufficient to describe the behavior of the individual in purchasing insurance.

I. Introduction

The purchase of insurance provides an individual with the ability to shift risk associated with either timeless or temporal uncertain prospects (see Drèze and Modigliani 1972). Previous insurance decision analysis can be further divided into cases in which the insurance contract is exogenously specified (see Mossin 1968; Smith 1968) and cases in which the optimal form of contract is to be determined (see Raviv 1979; Brennan and Solanki 1981). This note is concerned with extending some of the results for timeless uncertain prospects with the insurance contract exogenously specified.

Many of the properties associated with the purchase of insurance for timeless uncertain prospects are analyzed by Mossin (1968) for different forms of contracts (see also Gould 1969). Utilizing the Arrow-Pratt measure of absolute risk aversion, the effect of wealth upon the optimal amount of insurance is determined for the different forms of contracts. Recent work by Ross (1981) demonstrates that in situations where individuals face more than one (timeless) uncertain prospect and they can purchase insurance only for a subset of these prospects, then a more risk-averse individual in the Arrow-Pratt

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sense will not necessarily purchase more insurance than a less risk-averse individual.

The first objective of this paper is to examine the properties of the optimal amount of insurance an individual purchases when there is more than one element of uncertainty. Two cases are considered. The first case considers the maximum acceptable premium for full coverage, when insurance covers a particular event. For example, an individual purchases insurance against theft. The payment by the insurance company in the event of a theft may be indexed or nonindexed. If it is indexed, the insurance company pays the replacement value of the stolen property, and if nonindexed, it pays a maximum amount. While insurance provides against the contingency of theft, it does not protect the individual against changes in the market value of the work of art. It is shown that the Arrow-Pratt and Ross measures of risk aversion are not sufficient to describe the behavior of individuals purchasing insurance. The second case considers the optimal level of a deductible. It is shown that the Ross measure of risk aversion is sufficient to describe the behavior of individuals choosing the optimal level of deductible.

The second objective of this paper is to examine how information affects an individual's decision to purchase insurance. Information may have value in at least two ways. It may encourage an agent to purchase more insurance or alter the pricing structure of an insurance contract; and it may encourage dissemination of information between different parties, reducing possible informational asymmetries. If information must be purchased, the optimal amount of information for the individual to buy will in general depend on the degree of risk aversion. It is shown that the Arrow-Pratt and Ross measures of risk aversion are not in general sufficient to describe the behavior of individuals purchasing information.

Definitions are given in Section II. Section III examines the properties of the optimal amount of insurance an individual purchases when there is more than one element of uncertainty. The value of information production and dissemination is examined in Section IV. An application is given in Section V.

II. Definitions

Consider two individuals who have von-Neumann Morgenstern utility functions A and B defined on the real line. For simplicity, it will be assumed that all utility functions are strictly monotone, strictly concave, and twice differentiable.

The statement "A is more risk averse than B in the Arrow-Pratt sense" is condensed to $A \supset_{AP} B$ and is defined as follows:

DEFINITION: *$A \supset_{AP} B$ if and only if* $-[A''(x)/A'(x)] \geq -[B''(x)/B'(x)]$ *for all x*. As shown by Pratt (1964), this implies that there exists a function G such that $A = G(B)$, where $G' \geq 0$, $G'' \leq 0$.

The statement "A is more risk averse than B in the Ross sense" is condensed to $A \supset_R B$ and is defined as follows:

DEFINITION: $A \supset_R B$ *if and only if there exists a* $\lambda(>0)$ *such that for all* x_1 *and* x_2, $[A''(x_1)/B''(x_1)] \geq \lambda \geq [A'(x_2)/B'(x_2)]$. As shown by Ross (1981), this implies that there exists a function G such that $A = \lambda B + G, G', G'' \leq 0$.

Consider the family M of cumulative distribution functions $F(x)$ for $x \geq 0$, where x is some random prospect. If $F, G \in M$, the statement "F stochastically dominates G" is written $F \gtrsim G$ and is defined as follows:

DEFINITION: $F \gtrsim G$ *if and only if* $\int_0^y [F(x) - G(x)]dx$ *for all* $y \geq 0$. The above is a definition of second degree stochastic dominance.

III. Maximum Acceptable Premium for Full Coverage

Consider an individual owning a work of art of value V, which is initially assumed to be known with certainty. In addition, the individual owns other assets to a total value of W, which is initially assumed to be known with certainty. It is assumed that there is a probability p that the work of art will be stolen, $1 - p$ being the probability that it will not be stolen.

If the individual takes no insurance, the final wealth will be a stochastic variable, $W + \tilde{h}V$, where $h = 0$ with probability p and $h = 1$ with probability $1 - p$. If the individual purchases insurance, final wealth will be $W + V - \pi$, where π is the insurance premium. The maximum insurance premium the individual would be willing to pay is determined by

$$U(W + V - \pi) = E[U(W + \tilde{h}V)], \tag{1}$$

where U is the utility function of the individual.

Intuitively, if A is more risk averse than B in the Arrow-Pratt sense, then A would be willing to pay more for insurance than B.

LEMMA 3.1 (Pratt 1964): If $A \supset_{AP} B$, then $A \geq B$.

Suppose now that the wealth of the other assets is not known with certainty. The individual can purchase insurance only against the work of art's being stolen. In this case the maximum the individual pays for insurance is given by

$$E[U(\tilde{W} + V - \pi)] = E[U(\tilde{W} + \tilde{h}V)]. \tag{2}$$

In this case, where there is only partial insurance, the Arrow-Pratt measure of risk aversion is not sufficient to describe the behavior of individuals purchasing insurance. If, however, the Ross definition of risk aversion is used, it is possible to obtain an unambiguous result.

LEMMA 3.2 (Ross 1981): If $A \supset_R B$, then $\pi_A \geq \pi_B$.

Now consider a third type of problem. It is again assumed that the value of other assets is known with certainty, so that W is nonstochastic. The value of the work of art is not known with certainty, being described by a random variable $\tilde{V}$. The individual can purchase insurance to protect against the loss if the work of art is stolen, but can not purchase insurance to guard against the consequences of adverse changes in its market value.[1] The final wealth of the individual is $W + \tilde{h}\tilde{V}$. The individual can purchase one of two types of insurance contracts, indexed and nonindexed. An indexed contract is one that pays the full replacement value of the work of art; a nonindexed contract is one that in the event that the work of art is stolen places an upper limit on the amount of repayment. Define $\tilde{V} \equiv L + \tilde{\epsilon}$, where $E(\tilde{\epsilon}) = 0$. The nonindexed contract can be specified as paying L if $\epsilon \geq 0$ and $L + \epsilon$, if $\epsilon < 0$. The maximum amount the individual would pay for the indexed contract is

$$\int_{-L}^{\infty} U(W + L + \epsilon - \pi_I)dF(\epsilon) = pU(W) + (1 - p) \int_{-L}^{\infty} U(W + L + \epsilon)dF(\epsilon) \tag{3}$$

and for the nonindexed contract

$$p\left[U(W + L - \pi_N)\int_0^{\infty} dF(\epsilon) + \int_{-L}^{0} U(W + L + \epsilon - \pi_N)dF(\epsilon)\right] + (1 - p) \int_{-L}^{\infty} U(W + L + \epsilon - \pi_N)dF(\epsilon) = pU(W) + (1 - p) \int_{-L}^{\infty} U(W + L + \epsilon)dF(\epsilon), \tag{4}$$

where $F(\epsilon)$ is the cumulative distribution function for ϵ, and Stieltjes integrals are used. From (3) and (4), it follows that $\pi_N \leq \pi_I$;[2] that is, a risk-averse individual is always willing to pay more for the indexed contract than the nonindexed contract. For the purposes of this study, it suffices to discuss the properties of the index contract only.

The amount the individual is willing to pay for insurance depends upon the risk associated with the value of the work of art.

LEMMA 3.3: If the individual has decreasing risk aversion, then for a mean preserving spread in ϵ, π_I decreases.

Proof: Equation (3) can be written in the form $E[\psi(\tilde{\epsilon})] = pU(W)$, where $\psi(\tilde{\epsilon}) = U(W + L + \tilde{\epsilon} - \pi_I) - (1 - p)\, U(W + L + \epsilon)$; and the right hand side is independent of ϵ. Now $\psi''(\epsilon) = -R(W + L + \epsilon - \pi_I)\, U'(W + L + \epsilon - \pi_I) + (1 - p)\, R(W + L + \epsilon)\, U'(W + L + \epsilon)$, where

1. In a temporal context an individual purchases insurance at $t = 1$ to cover contingencies at $t = 2$. In a world of uncertainty the replacement value of the asset will not in general be known at $t = 1$.

2. The left-hand side of eq. (4) can be written in the form $-p\Delta + \int_{-L}^{\infty} U(W + L + \epsilon - \pi_N)dF(\epsilon)$, where $\Delta = \int_0^{\infty} [U(W + L + \epsilon - \pi_N) - U(W + L - \pi_N)]dF(\epsilon) \geq 0$. Equating the left-hand side of (3) and (4) gives $-p\Delta + E[U(W + L + \tilde{\epsilon} - \pi_N)] = E[U(W + L + \tilde{\epsilon} - \pi_I)]$, implying that $E[U(W + L + \tilde{\epsilon} - \pi_N)] \geq E[U(W + L + \tilde{\epsilon} - \pi_I)]$.

$R = -U''/U'$. If R is a decreasing function, then $\psi''(\epsilon) < R(W + L + \epsilon - \pi_I)[U'(W + L + \epsilon) - U'(W + L + \epsilon - \pi_I) - pU'(W + L + \epsilon)] \leqslant 0$. If ψ is a concave (convex) function in ϵ, then from Rothschild and Stiglitz (1970) a mean preserving spread in ϵ decreases (increases) $E[\psi(\epsilon)]$. As the right hand side is independent of ϵ, this implies π_I decreases. Q.E.D.

As the uncertainty of the value of the work of art increases, individuals with decreasing risk aversion will decrease the amount they spend on insurance. Insurance provides against the contingency of the work of art's being stolen; it does not offer the individual protection against the uncertainty associated with changes in the market value of the work of art.

Given that individuals have only partial insurance, it does not follow, as shown below, that if individual A is more risk averse in the Arrow-Pratt sense than individual B, A will pay more for insurance than B.

Counterexample 1

Suppose $A(W) = -\exp -aW$, $B(W) = -\exp -bW$, where $a = .02$, $b = .001$, implying that $A \supset_{AP} B$. Let $W = 100$, $p = .999$, $1 - p = .001$. The future market value of the work of art is described by the probability distribution: $V = 50$ with a probability ½ or $V = 150$ with a probability ½. As $A \supset_{AP} B$, it might be expected that $\pi_A \geqslant \pi_B$. However, it can be shown that the optimal value $\pi_A < 80$, while for B, $\pi_B \geqslant 95$.

In this example the final wealth of the individual is $W + \tilde{h}\tilde{V}$, where $h = 0$ with probability p if the work of art is stolen and $h = 1$ with probability $1 - p$ if it is not stolen. It is assumed that $\tilde{h}$ and $\tilde{V}$ are independent. Kihlstrom, Romer, and Williams (1981) have shown that for independent additive contingencies, if the utility functions of either A or B exhibit nonincreasing Arrow-Pratt risk aversion, then if $A \supset_{AP} B$, $\pi_A \geqslant \pi_B$. As the negative exponential utility function belongs to the class of nonincreasing Arrow-Pratt risk aversion functions, the above example demonstrates that the Kihlstrom conditions do not hold for multiplicative contingencies.

Given lemma 3.2 it might be expected that an unambiguous result would be obtained with the Ross definition of risk aversion. As will be shown, if $A \supset_R B$, this is not sufficient to imply that $\pi_A \geqslant \pi_B$.

Counterexample 2

Suppose $A(W) = W - \exp -aW$, $B(W) = W - b\exp -aW$, where $a > 0$, and $0 < b < 1$, implying that $A \supset_R B$.[3] Let $W = 100$, $p = .999$, $1 - p = .001$. The future market value of the work of art is described by the

3. For $W > 0$, then $A''/B'' = 1/b > A'/B' = (1 + a\exp -aW)/(1 + ab\exp -aW)$, which verifies that $A \supset_R B$.

probability distribution: $V = 50$ with probability ½ or $V = 150$ with probability ½. As $A \supset_R B$, it might be expected that $\pi_A \geqslant \pi_B$. However, when $b = .21$ and $a = .01$, then the optimal value $\pi_A = 99.85$ while for B, $\pi_B = 99.89$.

The failure to obtain an unambiguous result using the Ross definition stems from the nature of the risk facing the individual and the type of partial insurance contract. In lemma 3.2 the final wealth of the individual is $\tilde{W} + \tilde{h}V$ and the insurance contract provides against the contingency $\tilde{h}$, ignoring the additive contingency $\tilde{W}$. In lemma 3.3 the final wealth of the individual is $W + \tilde{h}\tilde{V}$ and the insurance contract provides against $\tilde{h}$, ignoring the multiplicative contingency $\tilde{V}$.

Optimal Amount of Deductible

Consider an individual who may have to pay a claim, $\tilde{X}$, the amount of the claim not known with certainty. The individual can purchase insurance with a deductible S, so that the individual covers the first S units of any claims while the insurance company pays any excess. The amount paid by the insurance company is

$$Y(X; S) = \begin{cases} \tilde{X} - S & ; \tilde{X} \geqslant S \\ 0 & ; 0 \leqslant \tilde{X} < S. \end{cases}$$

If the initial wealth of the individual is W, the final wealth of the individual if insurance is purchased is

$$W_1 \equiv W - p(S) + \begin{cases} -S & ; \tilde{X} \geqslant S \\ -\tilde{X} & ; 0 \leqslant \tilde{X} < S, \end{cases}$$

where $p(S)$ is the cost of the insurance policy. The situation is analogous to that of the individual who goes short in a stock and purchases a call option to reduce the risk.

It is assumed that the insurance company sets the premium proportional to the expected value of the payoff:

$$p(S) = (1 + \mu) \int_S^\infty (x - S)\, dF(x), \tag{5}$$

where μ is a nonnegative loading factor, and F is the subjective cumulative probability distribution the insurance company ascribes to the random variable $\tilde{X}$. For the individual the expected utility associated with the final wealth is

$$\begin{aligned} E[U(\tilde{W}_1)] = & \int_0^S U[W - x - p(S)]h(x)dx \\ & + \int_S^\infty U[W - S - p(S)]h(x)dx, \end{aligned} \tag{6}$$

where $h(x)$ is the subjective probability density function the individual ascribes to the random variable, X.

Assuming that the individual picks S to maximize the expected utility the first order condition is

$$\frac{d}{dS}E[U(\tilde{W}_1)] = -\frac{dp(S)}{dS}\left\{\int_0^S U'[W - x - p(S)]h(x)dx + U'[W - S - p(S)]\int_S^\infty h(x)dx\right\} - U'[W - S - p(S)]\int_S^\infty h(x)dx, \tag{7}$$

where $dp(S)/dS = -(1 + \mu)\int_S^\infty dF(x)$; and it is assumed that $E[U(\tilde{W}_1)]$ has a maximum at a finite value of S, say S^*.

If individual A is more risk averse in the Arrow-Pratt sense than individual B, then it is expected that A will have a lower deductible than B.

LEMMA 3.5: If $A \supset_{AP} B$ and individuals have the same wealth W and subjective probability density functions, then $S_A^* \leq S_B^*$.

Proof: see Appendix A.

Again, if the wealth of the individual is also stochastic, it is possible to construct a counterexample where, even though individual A is more risk averse in the Arrow-Pratt sense than individual B, A will pick a higher deductible. Kihlstrom et al. (1981) have argued that if $\tilde{W}$ and $\tilde{X}$ are independent and if the utility function of either A or B exhibits nonincreasing Arrow-Pratt risk aersion, then if $A \supset_{AP} B$, A will pick a lower deductible than B. If individual A is more risk averse than individual B in the Ross sense, then without restricting the utility functions of either A or B to exhibit nonincreasing Arrow risk aversion, individual A will pick a lower deductible.

LEMMA 3.6: If $A \supset_R B$ and $\tilde{W}$ and $\tilde{X}$ are statistically independent, then $S_A^* \leq S_B^*$.

Proof: see Appendix B. If $\tilde{W}$ and $\tilde{X}$ are statistically dependent then the result is in general indeterminant.

IV. Information

The extent of an agent's information will be evaluated in terms of stochastic dominance.[4] Assume that a random variable x is described by the cumulative distribution function $\Pi(x)\epsilon M$. An agent is said to have complete or perfect information if the subjective cumulative distribution function $F(x)$, $F(x)\epsilon M$ the agent ascribes to the random prospect is identical to Π. It may be expected however that an agent will not have perfect information and a typical situation will be an ordering $\Pi \gtrsim F$.

Information may have value in at least two ways. First, it may en-

4. The stochastic dominance criterion for defining more information and the characterization of the production of information is similar to that used by Kwon, Fellingham, and Newman (1979).

courage an agent either to purchase more insurance or to alter the pricing structure for policies; and second, it may encourage dissemination of information between the insurer and insuree, reducing possible informational asymmetries.

Consider the case described by equation (3) of an individual's purchasing insurance against theft of a work of art when the market value is not known with certainty. If the individual can gain more information about the market value, this will in general affect the amount the individual is willing to pay for insurance. It is assumed that the production of information can be characterized by a single parameter λ, $0 \leq \lambda \leq 1$, and the cumulative distribution function is revised so that

$$F_\lambda(V) = F(V) + \lambda[\Pi(V) - F(V)], \tag{8}$$

where $\Pi \gtrsim F_\lambda \gtrsim F$. If $\lambda = 1$, the individual is said to have complete information about the distribution function.

Equation (3) can be rewritten in the form

$$\int_0^\infty U(W + V - \pi_I)dF_\lambda(V) = pU(W) + (1 - p)\int_0^\infty U(W + V)dF_\lambda(V), \tag{9}$$

where V is the market value of the art.

LEMMA 4.1: The individual's expected utility increases with an increase in λ, if the individual possesses imperfect information and the utility function is strictly concave.

Proof: The utility function $U^*(V) = pU(W) + (1 - p)\, U(W + V)$ is a nondecreasing concave function of V and $F_\lambda \gtrsim F$, so that the expected utility function $\int_0^\infty U^*(V)\, dF_\lambda(V)$ increases (see Hadar and Russell 1969).[5] Q.E.D.

Lemma 4.1 demonstrates that the individual has an incentive to produce information privately when the dominance criterion holds. This assumes that the information is disseminated so that the market value of the work of art and hence the wealth of the individual increases. The optimal level of information chosen depends on the cost of production, which, for the moment, is ignored.

LEMMA 4.2: An individual with decreasing absolute risk aversion will be willing to pay more for insurance as the individual becomes more informed about the market value V.

Proof: Differentiating (9) with respect to λ and using (8) gives, after simplification,

$$\frac{d\pi_I}{d\lambda}\int_0^\infty U'(W + V - \pi_I)dF_\lambda(V) = \int_0^\infty [U(W + V - \pi_I) - (1 - p)\, U(W + V)]\,[\pi(V) - f(V)]\, dV.$$

5. Hadar and Russell (1969) prove that if $F, G \in M$, and $F \geq G$, then for all nondecreasing concave functions U, $\int_0^\infty U(S)dF(S) \geq \int_0^\infty U(S)dG(S)$.

Now $U(W + V - \pi_I) - (1 - p)\, U(W + V)$ is a nondecreasing concave function,[6] and as $\Pi \gtrsim F$, then the right hand side is positive. Therefore $d\pi_I/d\lambda \geq 0$. Q.E.D.

Lemma 4.2 shows that if an individual possess imperfect information, then as λ increases the individual will be willing to pay more for insurance against theft. This suggests that it might be in the interests of an insurance company to provide the individual with information that increases λ.

From lemma 4.1 the individual has an incentive to produce information privately. If information is costly to produce, the optimal amount of information for the individual to purchase is determined by maximizing

$$H \equiv pU[W - C(\lambda)] + (1 - p) \int_0^\infty u[W + V - C(\lambda)]dF_\lambda(V) \tag{10}$$

with respect to λ, where $C(\lambda)$ is the cost of producing information so that $F_\lambda \gtrsim F$, and it is assumed that $C' > 0$, $C'' > 0$. The first order condition is determined by $dH/d\lambda = 0$, where[7]

$$\begin{aligned}\frac{dH}{d\lambda} &= (1 - p) \int_0^\infty U[W + V - C(\lambda)]\,[\pi(V) - f(V)]\,dV \\ &- C'(\lambda)\left\{pU'[W - C(\lambda)] + (1 - p) \int_0^\infty U'[W + V - C(\lambda)]dF_\lambda(V)\right\}.\end{aligned} \tag{11}$$

Intuitively, it might be expected that the optimal λ depends on the degree of risk aversion, though it is not clear, if an individual A is more risk averse than an individual B in the Arrow-Pratt sense, whether the optimal purchase of information for A is greater than for B ($\lambda_A > \lambda_B$). In fact, it can be shown that if $A \supset_{AP} B$ the result is ambiguous. This result remains unchanged even if $A \supset_R B$, which, given counterexample 2, is perhaps not surprising.[8]

6. A condition for decreasing absolute risk aversion is $U''' \geq 0$. Hence $U''(W + V) \geq U''(W + V - \pi_I)$. As $U'' < 0$ then $(1 - p)U''(W + V) > U''(W + V)$. Therefore, $U(W + V - \pi_I) - (1 - p)U(W + V)$ is a concave function in V.

7. The second order condition is given by

$$\begin{aligned}\frac{d_2H}{d\lambda^2} &= -C''(\lambda)\left\{pU'[W - C(\lambda)] + (1 - p) \int_0^\infty U'[W + V - C(\lambda)]\, f_\lambda(V)\, dV\right\} \\ &\quad -2C'(\lambda)\,(1 - p) \int_0^\infty U'[W + V - C(\lambda)]\,[\pi(V) - f(V)]\,dV \\ &\quad + [C'(\lambda)]^2\left\{pU''[W - C(\lambda)] + (1 - p) \int_0^\infty U''[W + V - C(\lambda)]\, f_\lambda(V)\, dV\right\} \leq 0\end{aligned}$$

given $U'' < 0$, $C' > 0$, and $C'' > 0$.

8. Let λ_A, λ_B denote the optimal amount of information purchased by A and B respectively. As $A \supset_R B$, then $A = \mu B + B$, $\mu > 0$, $G' < 0$, and $G'' < 0$. Let H_A and H_B denote the right-hand side of (10) for individuals A and B, respectively. After rearrangement it

For the case of determining the optimal amount of deductible, the insurance company is assumed to determine the price of the policy for a given level of deductible via equation (5), using the cumulative distribution function $F(x)$, while the individual determines the optimal deductible using the cumulative distribution function $H(x)$ (see eq. [6]). If the insurance company can obtain more information about the random variable X, how does the additional information affect its pricing decision? Similarly, if the individual can obtain more information about X, how does this affect the individual's expected utility and the optimal amount of the deductible? If there are informational asymmetries between the individual and the insurance company, under what conditions will it be in their mutual interests to share information?

The cumulative distribution function for X is denoted by $\Pi(x)$. In the absence of perfect information on the part of the insurance company and the individual, it is assumed that $\Pi \gtrsim F$ and $\Pi \gtrsim H$. The production of information by the insurance company is characterized by λ, where $F_\lambda(x) = F(x) + \lambda[\lambda(x) - F(x)]$ and for the individual by η, where $H_\eta(x) = H(x) + \eta[\Pi(x) - H(x)]$. The optimal level of information depends on the cost of production, which, for the sake of simplicity, is ignored in the present analysis.

LEMMA 4.3: If the insurance company possesses imperfect information, then as λ increases, the effect on the price of its policy is ambiguous.

Proof: Equation (5) can be written in the form

$$p(S^*) = (1 + \mu) \int_0^\infty \psi(x) dF_\lambda(x),$$

where $\psi(x) \equiv \max(0, x - S^*)$. While $\psi(x)$ is a convex function, it is also a nondecreasing function. Therefore, the sign of $\partial p/\partial\lambda\ (S^*)$ is ambiguous. Q.E.D.

This implies that the insurance company may or may not have an incentive to produce information privately when the stochastic dominance criterion holds. Should it be the case that $F_\lambda(x)$ stochastically dominates $F(x)$ in a first degree sense, then $\partial p/\partial\lambda\ (S^*) \geq 0$. In this case

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can be shown that

$$\left.\frac{dH_A}{d\lambda}\right|_{\lambda_B} - \left.\frac{dH_B}{d\lambda}\right|_{\lambda_B} = -(1 - p)\, C'(\lambda_B) \int_0^\infty G'[W + V - C(\lambda_B)]\, \pi(V)\, dV$$
$$+ (1 - p) \int_0^\infty \{G[W + V - C(\lambda_B)] + (1 - \lambda_B)\, C'(\lambda_B)\, G'[W + V - C(\lambda_B)]\}\, [\pi(V) - f(V)]\, dV.$$

The first term on the right-hand side of the equation is positive as $G' < 0$. The second term is negative, given that $\{-G[W + V - C(\lambda_B)] - (1 - \lambda_B)\, C'(\lambda_B)\, G'[W + V - C(\lambda_B)]\}$ is a nondecreasing convex function of V, assuming $G''' < 0$. If $G''' > 0$, the sign of the second term is ambiguous. Hence the sign of the right-hand side is ambiguous.

the insurance company will have an incentive to produce information privately.

LEMMA 4.4: If an individual possesses imperfect information, then $dS^*/d\eta$ is ambiguous.

Proof: Differentiating the optimality condition with respect to μ (setting the right hand side of [7] equal to zero) gives

$$\frac{dS^*}{d\eta}\left\{-\frac{d_2}{dS^2}E[U(\tilde{W}_1)]\bigg|_{S^*}\right\} = (1+\mu)[1-F(S^*)]\int_0^\infty \psi(x)$$

$$[\pi(x)-h(x)]dx - U'(W-S^*-P)\int_{S^*}^\infty[\pi(x)-h(x)]dx,$$

where $\psi(x)$ is a nondecreasing function defined by $\psi(x) = U'(W-x-P)$ for $0 \leqslant x \leqslant S^*$ and $\psi(x) = U'(W-S^*-P)$ for $x \geqslant S^*$. Though $U'(W-S^*-P) > 0$, the sign of $\int_{S^*}^\infty [\pi(x)-h(x)]dx$ is ambiguous. The sign of $\int_0^\infty \psi(x)[\pi(x)-h(x)]dx$ is positive (ambiguous) if $\psi(x)$ is a concave (convex) function which requires $U'''(W-x-P) <(>) 0$. Therefore, $dS^*/d\eta$ is ambiguous in sign. Q.E.D.

If the insurance company is more informed than the individual ($F \gtrsim H$), this lemma implies that it is not necessarily in the interests of the insurance company to disseminate its information to the individual, in the sense that the equilibrium premium paid by the individual may increase or decrease.

V. Application: Portfolio Insurance

Consider an individual with a risky portfolio, $\tilde{S}$, and other assets, $\tilde{W}$, which are also risky. To limit the risk associated with the risky portfolio the individual can acquire insurance by puchasing a put option on the portfolio. The payoff of the put option is $(K-\tilde{S})$ if $0 \leqslant \tilde{S} \leqslant K$ and 0 if $\tilde{S} \geqslant K$, where K is the exercise price of the put option. The price of the put option is denoted by $P(K)$, where $P'(K) > 0$. The final wealth of the individual is $\tilde{W}_1 \equiv \tilde{W} - P(K) + \max(\tilde{S}, K)$, and the individual's expected utility is

$$E[U(\tilde{W}_1)] \equiv E\left\{U[\tilde{W}+K-P(K)]\int_0^k f(S)\,dS\right.$$

$$\left.+\int_k^\infty U[\tilde{W}+S-P(K)]f(S)\,dS\right\},$$

where it is assumed for simplicity that W and S are statistically independent. It can be shown that if $A \supset_R B$, then $K_A^* \geqslant K_B^*$, where K^* denotes the optimal value of K. As the proof is similar to that given in Appendix B, details are omitted. This implies that as $A \supset_R B$, then A will desire to purchase more insurance against adverse changes in the market value of the portfolio than will B. If A wants to purchase a put option with striking price K_A^* (> 0), then in equilibrium there must be another individual in the economy willing to sell (the writer) such an option. Such an individual will be less risk averse, in the Ross sense, than A and/or have different expectations about the portfolio.[9] From lemma 4.4, although an individual may be able to obtain more information about the distribution of $\tilde{S}$, such that $F_\lambda \gtrsim F$, it is not clear whether the individual will increase or decrease the amount of insurance.

Appendix A

The first order condition for B can be written

$$(1 + \mu)[1 - F(S_B^*)]\, E[B'(\tilde{W}_1)|S_B^*] = [1 - H(S_B^*)]\, B'[W - S_B^* - p(S_B^*)], \tag{A1}$$

where H is the cumulative probability distribution the individual ascribes to the random variable X. Let $L = E[A(\tilde{W}_1)]$; it will be shown that $dL/dS|_{S=S_B^*} < 0$, which implies that $S_A^* \leqslant S_B^*$, assuming L is concave over the required range. Using (A1) to eliminate $(1 + \mu)$, as $A = G(B)$, gives

$$\left.\frac{dL}{dS}\right|_{S=S_B^*} = B'(W - S_B^* - p)\left(\int_0^{S_B^*} \{G'[B(W - x - p)] - G'[B(W - S_B^* - p)]\}\, B'(W - x - p)h(x)dx\right) \tag{A2}$$

$$[1 - H(S_B^*)]/\{E[B'(W_1)|S_B^*]\}.$$

Now $W - x - p \geqslant W - S_B^* - p$ for $0 \leqslant x \leqslant S_B^*$, so that $G'[B(W - x - p)] \leqslant G'[B(W - S_B^* - p)]$. Hence $dL/dS|_{S=S_B^*} \leqslant 0$. Q.E.D.

Appendix B

The first order condition for B can be written in the form

$$(1 + \mu)[1 - F(S_B^*)]\, E[B'(\tilde{W}_1)] = E\{B'[\tilde{W} - S_B^* - P(S_B^*)]\}\, [1 - H(S_B^*)]. \tag{B1}$$

Let $L = E[A(\tilde{W}_1)]$; it will be shown that $dL/dS|_{S=S_B^*} \leqslant 0$, which implies that $S_A^* \leqslant S_B^*$ assuming L is concave over the required range. Using (B1) to eliminate $(1 + \mu)$, as $A = \lambda B + G$, gives, after simplification,

$$\left.\frac{dL}{dS}\right|_{S=S_B^*} = [1 - H(S_B^*)]\Big\{E[B'(\tilde{W} - S_B^* - P)]\, E\left[\int_0^{S_B^*} G'(\tilde{W} - x - p)\, h(x)\, dx\right] - E[G'(\tilde{W} - S_B^* - P)]\, E\left[\int_0^{S_B^*} B'(\tilde{W} - x - p)\, h(x)\, dx\right]\Big\}\Big/ E[B'(\tilde{W}_1)],$$

9. The issue of attempting to identify the type of individual who would purchase portfolio insurance is addressed in Leland (1980).

where P is evaluated at $S = S_B^*$. For $0 \leq x \leq S_B^*$, $W - x - P \geq W - S_B^* - P$, so that $B'(W - x - P) \leq B'(W - S_B^* - P)$. Therefore, $\int_0^{S_B^*} B'(W - x - P)\,h(x)\,dx \leq B'(W - S_B^* - P)\,H(S_B^*)$. Now $G' < 0$, so that $G'(W - x - P)\,E[B'(\tilde{W} - S_B^* - P)]\,H(S_B^*) \leq G'(W - x - P)\,E[\int_0^{S_B^*} B'(\tilde{W} - x - P)\,h(x)\,dx] \leq G'(W - S_B^* - P)\,E[\int_0^{S_B^*} B'(\tilde{W} - x - P)\,h(x)\,dx]$ for $0 \leq x \leq S_B^*$ and $G'' < 0$. Hence,

$$E\left[\int_0^{S_B^*} G'(\tilde{W} - x - P)\,h(x)\,dx\right] E[B'(\tilde{W} - S_B^* - P)]\,H(S_B^*)$$
$$\leq E[G'(\tilde{W} - S_B^* - P)]\,H(S_B^*)\,E\left[\int_0^{S_B^*} B'(\tilde{W} - x - P)\,h(x)\,dx\right].$$

Therefore, $dL/dS|_{S=S_B^*} < 0$. Q.E.D.

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