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Variances of Security Price Returns Based on High, Low, and Closing Prices*

It has been argued in the literature that improved stock price return volatility measures can be obtained using daily high and low prices in addition to the traditionally used closing prices. In this paper we test the accuracy of the high-low variance estimator and suggest an adjustment procedure to improve upon its practical relevance. We then show how the "adjusted" high-low estimator can be used in a variance prediction rule. This prediction rule could be improved upon even further by including variances implied in option prices.

Introduction

In a recent paper, Garman and Klass (1980) studied the problem of estimating capital asset price volatility parameters. Developing an idea first introduced by Parkinson (1980), they argue that improved volatility measures can be obtained by taking into account the daily high, low, and opening prices in addition to the traditionally used closing prices. This paper is concerned mainly with testing the empirical validity of different volatility estimators based on high, low, and closing prices.

The Garman-Klass Estimator

Garman and Klass develop their estimators on the assumption that the logarithm of stock prices follows a Brownian motion with zero drift. This process is assumed to govern the data, irrespective of whether the market is open or closed. It can then be shown that the minimum variance unbiased volatility estimator on the basis of the daily high (H), low (L), and closing (C) prices can be expressed as[1]

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1. I maintain the notation used by Garman and Klass throughout the paper.

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$$\sigma_4^2 = .511(H_t - L_t)^2 - .019[(C_t - C_{t-1})(H_t + L_t - 2C_t) - 2(H_t - C_t)(L_t - C_t)] - .383(C_t - C_{t-1})^2,$$

where all prices are expressed in logarithmic form.

This variance estimator has an efficiency of 7.4 in comparison to the standard close-to-close variance (σ_0^2) (i.e., its estimation variance is 7.4 times lower). The more practical estimator $\sigma_5^2 = .5(H_t - L_t)^2 - .39(C_t - C_{t-1})^2$ has essentially the same efficiency but eliminates the cross-product terms. Earlier, Parkinson (1980) had shown that simply using the daily high and low prices, the estimator $\sigma_2^2 = (H_t - L_t)^2/4 \log 2$ had an efficiency of 5.2. All these estimators assume that the price path can be continuously monitored. Lacking continuous observation, however, both estimators will be downward biased since the observed high and low prices will respectively understate and overstate the "true" values. Garman and Klass used a simulation to determine how the downward bias is related to the trading volume. They provide a table with adjustment factors to be applied to the empirical estimates for different numbers of daily transactions.

The Data and Some Preliminary Tests

The sample used in the empirical tests consists of the daily high, low, and closing prices for a total of 208 stocks[2] over the 29-quarter period from January 1, 1973, through March 31, 1980. Since the "practical" variance estimator proposed by Garman and Klass is a weighted average of Parkinson's high-low estimator (σ_2^2) and the traditional close-to-close estimator (σ_0^2), we will concentrate on a comparison of the latter two.

As a first test of the relevance of the Parkinson estimator, we addressed the question of whether the high-low prices have the same information content relative to the closing prices across all stocks. An analysis of variance was performed on the ratio $\hat{\sigma}_0^2/\hat{\sigma}_2^2$ for all 208 stocks over the 29 quarters. The F-ratio for a company effect was 13.159, the F-ratio for a time effect was 11.333. Both measures are significant at the 99.9% level. It could therefore be inferred that the relationship between $\hat{\sigma}_0^2$ and $\hat{\sigma}_2^2$ is unstable across stocks and over time.

However, the cross-sectional variation in the variance ratio could be caused by the fact that neither estimator is adjusted for market closures. Garman and Klass do provide us with a table of adjustment factors which should be used when the price path cannot be monitored continuously. Unfortunately, these adjustment factors depend on the number of transactions occurring on a given day—a statistic that is not

2. All stocks were selected out of the list of stocks which had options written on them on March 31, 1980.

readily available. The New York Stock Exchange, however, does publish the monthly average volume of shares per transaction. Using this admittedly crude statistic, daily volume figures can be transformed into the daily number of transactions. This procedure was used to recompute the ratio $\hat{\sigma}_0^2/\hat{\sigma}_2^2$. The analysis of variance yielded an F-ratio of 11.29 for a company effect. Since the crude adjustment procedure did not eliminate the significant cross-sectional differences across companies, it was decided to drop it for the remainder of the study. At this point it is not clear whether the availability of exact transaction data could eliminate the cross-sectional variation (an alternative explanation of the phenomenon could be a systematic difference between nighttime and daytime variance).

The significant cross-sectional differences in information content of σ_2^2 and σ_0^2 suggest that a combination of both estimators using a fixed weighting scheme, as Garman and Klass propose for their "practical" estimator, might not be optimal. To the extent that the high-low estimator does contain relevant information which is not reflected in σ_0^2, an optimal combined estimator would take into account the stock-specific relationships between them. The next section tries to establish whether σ_2^2 does contain additional information. The subsequent section uses the cross-sectional differences in the relationship between σ_0^2 and σ_2^2 to obtain a "better" estimator.

A Direct Comparison between $\hat{\sigma}_0^2$ and $\hat{\sigma}_2^2$

Although it was proven theoretically—under some restrictive assumptions—that $\hat{\sigma}_2^2$ is a more efficient estimator than $\hat{\sigma}_0^2$, it does not follow that the high-low variance estimates are to be preferred over the traditional close-to-close variances. We have already mentioned that the high-low estimator will be biased significantly downward if the stock does not trade sufficiently. In addition, one would expect that bad data points show up more easily in the high and low price quotations. Finally, high and low prices may reflect trades by disadvantaged buyers or sellers and could therefore be less reliable indicators of the true value of the underlying security. While in general the daily range should give us a better indication of the variability of the price, the aforementioned weaknesses could annihilate this advantage.

To decide which of the two estimators is better, we use the following criterion. We compare the predictive ability of both variance estimators. Each one will be tested as a predictor of itself and as a predictor of the other estimator, using a simple regression approach. If one measure would be found to be both the best predictor of itself and of the alternative estimator, it can be proven to be more accurate, in the sense elaborated upon in the Appendix. The first four columns of table 1 summarize the results. It is immediately obvious that $\hat{\sigma}_{0,t-1}$ is

TABLE 1 Cross-sectional Regressions $\hat{\sigma}_{.,n,t} = a + b\, \sigma_{.,n,t-1}$

	$\hat{\sigma}_{0,t} = a + b\, \hat{\sigma}_{0,t-1}$			$\hat{\sigma}_{2,t} = a + b\, \hat{\sigma}_{2,t-1}$			$\hat{\sigma}_{0,t} = a + b\, \hat{\sigma}_{2,t-1}$		
Period	a	b	R^2	a	b	R^2	a	b	R^2
2	.014	1.016	.529	.006	1.099	.625	.029	1.103	.357
	(2.23)	(15.204)		(1.50)	(18.525)		(3.72)	(10.69)	
3	.026	.684	.564	.011	.806	.690	.024	.897	.550
	(5.47)	(16.33)		(3.67)	(21.42)		(4.76)	(15.88)	
4	.035	.970	.590	.020	1.059	.691	.041	1.147	.531
	(5.82)	(17.21)		(5.00)	(21.44)		(6.53)	(15.26)	
5	.033	.604	.512	.022	.662	.667	.047	.626	.359
	(5.82)	(14.69)		(5.50)	(20.29)		(7.43)	(10.75)	
6	.041	.581	.283	.023	.673	.492	.045	.676	.231
	(5.35)	(9.02)		(4.6)	(14.13)		(5.51)	(7.869)	
7	.073	.668	.366	.027	1.064	.615	.058	1.027	.403
	(10.48)	(10.91)		(5.40)	(18.14)		(7.50)	(11.80)	
8	.056	.596	.509	.025	.825	.734	.059	.719	.520
	(9.15)	(14.60)		(6.25)	(23.87)		(10.05)	(14.94)	
9	.015	.769	.565	.002	.843	.792	.023	.832	.617
	(6.14)	(16.36)		(.50)	(28.03)		(4.92)	(18.09)	
10	.029	.596	.549	.017	.654	.781	.032	.677	.566
	(8.27)	(15.68)		(5.67)	(27.11)		(5.52)	(16.51)	
11	.020	.750	.627	.005	.877	.787	.018	.927	.647
	(5.17)	(18.75)		(1.67)	(27.56)		(5.07)	(19.31)	
12	.016	.752	.640	.008	.829	.755	.020	.864	.622
	(4.13)	(19.28)		(2.67)	(25.22)		(3.18)	(18.38)	
13	.024	.770	.504	.018	.866	.679	.023	.939	.569
	(6.12)	(14.53)		(9.00)	(20.88)		(7.15)	(16.47)	
14	.020	.599	.594	.014	.861	.741	.019	.699	.575
	(9.34)	(17.11)		(4.67)	(24.26)		(5.19)	(16.64)	
15	.017	.645	.532	.011	.746	.664	.018	.731	.441
	(4.07)	(15.36)		(3.67)	(20.17)		(6.13)	(12.82)	
16	.022	.770	.576	.010	.917	.684	.018	.922	.572
	(3.88)	(16.74)		(5.00)	(21.10)		(5.29)	(16.46)	
17	.018	.676	.488	.011	.738	.670	.020	.720	.457
	(6.67)	(14.08)		(5.50)	(20.45)		(3.33)	(13.09)	
18	.015	.766	.476	.006	.866	.654	.010	.944	.519
	(3.17)	(13.68)		(2.06)	(19.73)		(4.63)	(14.98)	
19	.031	.496	.389	.021	.618	.427	.031	.573	.347
	(6.85)	(11.53)		(7.04)	(12.39)		(5.07)	(10.42)	
20	.024	.732	.464	.020	.723	.521	.030	.730	.436
	(4.39)	(13.31)		(6.84)	(14.96)		(8.14)	(12.59)	
21	.017	.694	.508	.005	.870	.719	.020	.766	.507
	(5.24)	(14.46)		(2.54)	(22.96)		(6.43)	(14.45)	
22	.025	.784	.473	.020	.805	.550	.034	.750	.394
	(3.09)	(13.52)		(6.76)	(15.87)		(3.23)	(11.54)	
23	.009	.873	.522	.012	.833	.571	.010	.996	.560
	(5.18)	(15.05)		(4.14)	(16.58)		(4.56)	(16.33)	
24	.014	1.033	.675	.005	1.061	.741	.009	1.236	.663
	(4.71)	(20.66)		(1.68)	(24.29)		(3.47)	(20.26)	
25	.019	.559	.629	.015	.591	.737	.018	.688	.629
	(8.16)	(18.63)		(7.45)	(24.02)		(8.29)	(18.59)	
26	.011	.823	.569	.010	.806	.610	.013	.928	.455
	(4.65)	(16.46)		(3.46)	(17.94)		(4.68)	(13.07)	
27	.025	.658	.477	.016	.772	.591	.024	.803	.399
	(5.39)	(13.71)		(5.14)	(17.26)		(8.26)	(11.64)	
28	.026	.818	.637	.017	.880	.723	.024	.970	.559
	(3.73)	(19.02)		(8.26)	(23.17)		(9.41)	(16.17)	
29	.031	.929	.469	.020	1.007	.496	.033	1.072	.397
	(5.24)	(13.46)		(4.34)	(14.245)		(4.27)	(11.66)	

NOTE.—n = 1,208 for the 29-quarter period from January 1, 1973, through March 31, 1980. Numbers in parentheses are t-statistics.

$\hat{\sigma}_{2,t} = a + b\,\hat{\sigma}_{0,t-1}$			$\hat{\sigma}_{6,t} = a + b\,\hat{\sigma}_{6,t-1}$			$\hat{\sigma}_{0,t} = a + b\,\hat{\sigma}_{6,t-1}$			$\hat{\sigma}_{6,t} = a + b\,\hat{\sigma}_{0,t-1}$		
a	b	R^2	a	b	R^2	a	b	R^2	a	b	R^2
.018	.755	.472									
(3.67)	(13.58)										
.025	.503	.474									
(6.10)	(13.63)										
.029	.755	.546									
(5.82)	(15.74)										
.026	.491	.560									
(6.25)	(16.18)										
.030	.477	.410									
(6.11)	(11.96)										
.055	.569	.377									
(9.07)	(11.17)										
.037	.582	.519									
(6.13)	(14.90)										
.004	.706	.595	.006	.819	.753	.015	.737	.590	.009	.830	.679
(.68)	(17.40)		(1.17)	(25.05)		(.96)	(17.14)		(1.52)	(20.85)	
.020	.524	.629	.018	.680	.733	.024	.619	.575	.026	.637	.664
(5.01)	(18.70)		(4.23)	(23.76)		(4.14)	(16.73)		(6.23)	(20.16)	
.015	.631	.603	.003	.907	.787	.013	.809	.690	.015	.806	.657
(3.93)	(17.68)		(.76)	(27.56)		(4.16)	(21.29)		(3.95)	(19.86)	
.010	.655	.640	.011	.807	.761	.016	.743	.688	.012	.807	.690
(3.36)	(19.12)		(3.92)	(25.59)		(3.08)	(21.23)		(3.02)	(21.39)	
.023	.649	.503	.025	.798	.617	.021	.790	.565	.028	.785	.559
(5.62)	(14.43)		(6.12)	(18.2)		(5.69)	(16.46)		(5.67)	(16.16)	
.021	.480	.590	.017	.621	.682	.015	.623	.602	.024	.575	.624
(6.98)	(17.210)		(5.94)	(21.01)		(7.83)	(17.80)		(8.01)	(18.48)	
.018	.556	.571	.010	.764	.652	.015	.655	.480	.016	.711	.645
(5.87)	(16.571)		(3.32)	(19.64)		(3.89)	(13.94)		(5.26)	(19.33)	
.022	.650	.496	.014	.868	.693	.017	.804	.629	.021	.818	.613
(7.21)	(14.244)		(4.76)	(21.56)		(6.31)	(18.70)		(6.95)	(18.07)	
.014	.616	.565	.014	.709	.604	.016	.676	.517	.016	.703	.559
(4.68)	(16.34)		(4.66)	(17.71)		(5.14)	(14.70)		(5.17)	(16.17)	
.016	.611	.455	.009	.848	.595	.011	.824	.519	.014	.779	.533
(5.12)	(13.10)		(2.96)	(17.41)		(3.69)	(14.98)		(3.6)	(15.32)	
.027	.441	.325	.025	.589	.506	.029	.531	.413	.030	.529	.442
(9.12)	(9.97)		(8.12)	(14.53)		(8.06)	(12.07)		(9.98)	(12.77)	
.019	.642	.435	.016	.838	.628	.023	.753	.492	.018	.801	.572
(6.23)	(12.58)		(5.32)	(18.66)		(5.64)	(14.21)		(6.10)	(16.59)	
.012	.641	.476	.011	.799	.691	.015	.744	.566	.014	.736	.604
(3.89)	(13.68)		(3.35)	(21.47)		(3.15)	(16.53)		(4.32)	(17.710)	
.018	.719	.481	.016	.908	.589	.021	.849	.524	.020	.845	.540
(4.26)	(13.84)		(4.06)	(17.18)		(4.40)	(15.16)		(4.98)	(15.54)	
.014	.690	.474	.006	.939	.652	.007	.916	.584	.008	.890	.575
(3.57)	(13.64)		(1.24)	(19.64)		(2.41)	(16.96)		(1.88)	(16.68)	
.015	.804	.620	.006	1.095	.754	.008	1.099	.721	.012	1.018	.693
(3.85)	(18.35)		(1.32)	(25.14)		(3.44)	(22.90)		(3.10)	(21.54)	
.020	.437	.610	.018	.586	.675	.019	.574	.629	.019	.566	.666
(9.82)	(17.96)		(6.06)	(20.68)		(6.36)	(18.52)		(6.12)	(20.26)	
.016	.595	.529	.008	.864	.625	.012	.809	.532	.009	.858	.634
(5.24)	(15.20)		(2.99)	(18.51)		(1.67)	(15.26)		(2.96)	(18.88)	
.028	.481	.408	.020	.751	.602	.022	.701	.525	.024	.703	.543
(9.27)	(11.90)		(6.82)	(17.65)		(7.89)	(15.24)		(8.10)	(15.64)	
.027	.624	.583	.022	.856	.719	.025	.817	.636	.023	.846	.702
(9.10)	(16.96)		(7.16)	(22.96)		(6.67)	(19.0)		(7.86)	(22.05)	
.023	.821	.518	.027	.970	.492	.032	.924	.451	.027	.975	.511
(4.57)	(14.89)		(4.35)	(14.12)		(3.28)	(13.01)		(4.23)	(14.66)	

not a good predictor of $\hat{\sigma}_{2t}$: its use as an independent variable yields an R^2 which is consistently lower than those of the regressions where $\hat{\sigma}_{2,t-1}$ predicts $\hat{\sigma}_{2t}$ (except in the last period). On the other hand, $\hat{\sigma}_{2,t-1}$ is at least as good a predictor of $\hat{\sigma}_{0,t}$ as $\hat{\sigma}_{0,t-1}$ in 11 out of the 28 cases. This indicates that in some periods the high-low estimator more accurately captures the underlying stock price variability than the standard close-to-close estimator. In addition, $\hat{\sigma}_{2,t-1}$ is an extremely good forecaster of $\hat{\sigma}_{2t}$. Avid option traders, who are more interested in the range of the underlying stock price than in its closing value, could therefore use the past $\hat{\sigma}_2^2$ to forecast the future squared range within reasonably accurate limits.

It can therefore be concluded that the high-low estimator may contain information which is not reflected in the close-to-close variance. As mentioned before, an optimal estimator would however incorporate prior knowledge on the cross-sectional variations between $\hat{\sigma}_0^2$ and $\hat{\sigma}_2^2$. The following section further explores this possibility.

An Alternative Variance Estimator

So far we infer: (1) that the information content of the high-low estimator, relative to the close-to-close variance varies from stock to stock; (2) that the estimation error variance of the high-low variance is lower in some periods than for the close-to-close variance. A natural extension would therefore be to adjust $\hat{\sigma}_2^2$ for the cross-sectional differences between both variance measures. More specifically, one could gather information on the prior relationship between $\hat{\sigma}_0$ and $\hat{\sigma}_2$ for a specific stock and incorporate this into the variability estimator for a given quarter. To establish the historical relationship between $\hat{\sigma}_0$ and $\hat{\sigma}_2$ for a given stock, the pairs $(\hat{\sigma}_{0,n,j}, \hat{\sigma}_{2,n,j}), j = t - I, \ldots, t$ are treated as $I + 1$ independent observations on a relation

$$\hat{\sigma}_{0,n,t} = \hat{a}_{n,t} + \hat{b}_{n,t}\, \hat{\sigma}_{2,n,t}, \tag{1}$$

where t = quarter in which variance is estimated and n = stock indicator. One could then use $\hat{\sigma}_{6,n,t} = \hat{a}_{n,t} + \hat{b}_{n,t}\, \hat{\sigma}_{2,n,t}$ as an alternative variance estimator for period t.[3] In this way, the stock-specific relationship between $\hat{\sigma}_{0t}$ and $\hat{\sigma}_{2t}$ is built into $\hat{\sigma}_{6t}$.

The usefulness of $\hat{\sigma}_{6t}$ as an alternative variance estimator was tested using the same procedure as outlined in the previous paragraph. An 8-quarter period was used in running regression (1) (i.e., $I = 7$). This

3. Note that $\hat{\sigma}_{6t}$ is simply a linear transformation of $\hat{\sigma}_{2t}$. To the extent that $\hat{\sigma}_{2t}$ is an inaccurate measure, the transformation cannot force it to become a better predictor of $\hat{\sigma}_{0t+1}$ than $\hat{\sigma}_{0t}$. Note also that the regression procedure which uses $I + 1$ observations to rescale the last observation of the interval assumes serial independence. Given the limited number of observations used in the regression, this assumption is hard to reject on the basis of standard statistical tests.

choice was justified on the ad hoc argument that a sufficiently long period should be used to enable us to accurately estimate the relationship. No alternative lag structure was tried. Columns 5 through 7 of table 1 summarize the results of the regressions which compare the predictive power of $\hat{\sigma}_0$ and $\hat{\sigma}_6$. In 18 out of the 21 quarters, $\hat{\sigma}_{6,t-1}$ was at least as good a predictor of $\hat{\sigma}_{0t}$ as $\hat{\sigma}_{0,t-1}$. In addition, $\hat{\sigma}_{6,t-1}$ could forecast $\hat{\sigma}_{6,t}$ consistently better than $\hat{\sigma}_{0,t-1}$ could predict $\hat{\sigma}_{6,t}$ (except in two periods). We can therefore conclude that $\hat{\sigma}_6$ is a more accurate variance estimator than the traditionally used close-to-close estimator. It was also verified that the (fixed weight) Garman-Klass estimator $\hat{\sigma}_5$ was consistently outperformed by $\hat{\sigma}_6$ in the pairwise comparisons. A correction of the high-low estimator for the cross-sectional variation in its information content can therefore significantly improve its value. The adjustment procedure imbedded in (1) appears to eliminate a large number of the problems which accompany the use of high and low prices.

A Variance Prediction Rule

Although we concluded in the previous paragraph that $\hat{\sigma}_6$ appeared to be a "better" variance estimator than $\hat{\sigma}_0$, it does not follow that the latter will be replaced by it in all circumstances. Indeed, a number of investors who make portfolio decisions on the basis of market closing prices will probably prefer to use a measure which reflects the variability of these prices (rather than the variability of the underlying process).[4] A prediction rule which helps them forecast the variability of these closing prices could therefore be of primary interest to a certain class of investors. It is obvious from the results reported in table 1 that the inclusion of $\hat{\sigma}_6$ in a prediction rule could improve the predictive performance. An optimal prediction rule could include distributed lags on the dependent variable in addition to any other explanatory variables which are considered relevant:

$$\begin{aligned}\hat{\sigma}^*_{0,t} = b\,\hat{\sigma}^*_{6,t-1} + c\,\hat{\sigma}^*_{6,t-2} + d\sum_{i=3}^{8}\frac{\hat{\sigma}^*_{6,t-i}}{6} \\ + e\,\hat{\sigma}^*_{0,t-1} + f\,\hat{\sigma}^*_{0,t-2} + g\left(\sum_{i=3}^{8}\frac{\hat{\sigma}_{0,t-i}}{6}\right)^*,\end{aligned} \tag{2}$$

where * denotes that the variable is defined as a deviation from its cross-sectional mean for a given quarter and therefore has zero observed mean.

4. It is not argued here that it would be rational to do so. It is, however, the case that a number of portfolio managers treat the closing price as a random walk, not as a reflection of a continuous process.

Rather than including $\hat{\sigma}_{.,t-3}, \hat{\sigma}_{.,t-4}, \ldots, \hat{\sigma}_{.,t-8}$, we used the arithmetic average of their values to avoid multicollinearity problems. The decision to use up to eight lagged values for both estimators was again based on ad hoc grounds. No alternative lag structures were tried. Since the coefficients for

$$\hat{\sigma}^*_{6,t-2} \text{ and } \sum_{i=3}^{8} \frac{\hat{\sigma}^*_{6,t-i}}{6}$$

were consistently negative and insignificant for all quarters, both variables were dropped from the regression and the alternative specification was estimated:

$$\hat{\sigma}^*_{0,t} = b\,\hat{\sigma}^*_{6,t-1} + e\,\hat{\sigma}^*_{0,t-1} + f\,\hat{\sigma}^*_{0,t-2} + g\sum_{i=3}^{8} \frac{\hat{\sigma}^*_{0,t-i}}{6}. \quad (3)$$

The results are summarized in column 1 of table 2. All four variables have significantly positive coefficients in most subperiods. In addition, the corrected R^2 is higher in every single quarter than its value in the corresponding regression of table 1. Regression equation (3) could therefore be used as a starting point for a prediction rule: for example, the regression coefficients for quarter 9 can be used as weights to predict the close-to-close variance of quarter 10. Since the (ex post) optimal quarterly regression coefficients fluctuate widely over time, this procedure will obviously not yield accurate results in every single quarter. It was therefore decided to determine the "optimal" weights on the basis of the pooled time series cross-section regression results for all previous quarters. In other words, to obtain the "best" weights to predict the close-to-close variance for quarter 26, we do not use only the regression coefficients for quarter 25, but those for the pooled regression for quarters 9 through 25. The weights for the prediction rule as of time period t are therefore obtained on the basis of the following regression:

$$\sigma^*_{0,n,s} = b_t\,\hat{\sigma}^*_{6,n,s-1} + e_t\,\hat{\sigma}^*_{0,n,s-1} + f_t\,\hat{\sigma}^*_{0,n,s-2} + g_t\sum_{i=3}^{8} \frac{\hat{\sigma}^*_{0,n,s-i}}{6}, \quad (4)$$

where $n = 1, \ldots, N$ is the stock indicator; $s = 9, \ldots, t$ indicates that all previous quarters up to and including period t are included in the pooled regression; and * denotes deviation from the cross-sectional mean for a given quarter.

Column 2 of table 2 shows that the regression coefficients tend to change very little as the time period over which we pool the data grows. A striking result of these pooled regressions is the high weight put on the 3–8-period lag structure, although the other explanatory variables also have statistically significant coefficients.

The regression coefficients of equation (4) can then be used to forecast the close-to-close variance for the next quarter:

$$\sigma_{t+1}^{\text{forecast}} = \hat{b}_t\, \hat{\sigma}_{6,t}^{*} + \hat{e}_t\, \hat{\sigma}_{0,t}^{*} + \hat{f}_t\, \hat{\sigma}_{0,t-1}^{*} + \hat{g}_t \sum_{i=2}^{7} \frac{\hat{\sigma}_{0,t-i}^{*}}{6}. \qquad (5)$$

The accuracy of this forecast is judged on the basis of the R^2 of the following regression: $\sigma_{0,t+1} = \alpha + \beta\, \sigma_{t+1}^{\text{forecast}}$. As can be inferred from column 3 of table 2, the average R^2 of the regression over all 20 quarters is .626 (as compared to an average of .636 for regression [3], which uses the ex post optimal weights in every quarter). Although this still leaves 37% of the cross-sectional variation in the variances unexplained, it indicates that a prediction rule based on historical information can give reasonably accurate results. This prediction rule could be a useful instrument for those investors who want to forecast the future close-to-close standard deviation using historical price information. It has the advantage that the data are readily available and that the implementation does not pose significant computational problems. The striking result is that, over a relatively long history, on average reasonably accurate forecasts of the close-to-close standard deviation can be obtained using historical prices. In the next paragraph we introduce information contained in option prices to improve the prediction rule even further.

The "Best" Variance Prediction Rule

It has been shown in the past (Latane and Rendleman 1976) that standard deviations implied in actual call option prices contain useful information for predicting future stock price variability. An intuitive explanation of this phenomenon derives from the fact that the value of an option depends on the volatility of the underlying stock price over the remaining life of the option. The procedure assumes that investors price options according to the Black-Scholes model (1973) and uses actual call and stock prices to infer the implicit standard deviation (ISD) reflected in the option price. A recent study (Beckers 1981) found that the at-the-money call options contain the most accurate predictive information. In addition, better predictive performance can be obtained if the ISDs are averaged over several trading days. Using these results, we decided to include ISDs in our prediction rule to check whether they added any explanatory power. The ISDs, based on the at-the-money call options, were calculated for the 10 trading days which preceded every quarter. By arithmetically averaging these estimates we obtained a predictor which should reflect the stock price variability over the remaining life of the option. Adding the ISD estimator to equation (3) we obtained the following regression specification:

TABLE 2 Predictive Cross-sectional Regressions for the 21-Quarter Period from January 1, 1975, through March 31, 1980

	$\hat{\sigma}^*_{0,t} = b\,\hat{\sigma}^*_{6,t-1} + e\,\hat{\sigma}^*_{0,t-1} + f\,\hat{\sigma}^*_{0,t-2} + g\sum_{i=3}^{8}\hat{\sigma}^*_{0,t-i}/6$					$\hat{\sigma}^*_{0,n,s} = b_t\,\hat{\sigma}^*_{6,n,s-1} + e_t\,\hat{\sigma}^*_{0,n,s-1} + f_t\,\hat{\sigma}^*_{0,n,s-2} + g_t\sum_{i=3}^{8}\frac{\hat{\sigma}^*_{0,n,s-i}}{6}$					$\hat{\sigma}_{0,t+1} = \alpha + \beta\,\sigma^{\text{forecast}}_{t+1}$		
Period	b	e	f	g	R^2	b	e	f	g	R^2	d	β	R^2
9	.261	.252	.062	.380	.645	.261	.252	.062	.380	.645			
	(1.54)	(1.66)	(.92)	(5.18)		(1.54)	(1.66)	(.92)	(5.18)				
10	−.197	.375	.166	.461	.672	.111	.266	.116	.374	.644	.010	.787	.658
	(−1.37)	(3.32)	(2.80)	(7.13)		(1.00)	(2.84)	(2.56)	(7.68)		(2.07)	(19.90)	
11	.060	.181	.204	.395	.777	.104	.244	.141	.375	.678	−.001	.987	.774
	(.50)	(1.96)	(4.57)	(6.62)		(1.26)	(3.50)	(4.18)	(9.95)		(−.15)	(26.53)	
12	.434	.011	.008	.306	.714	.178	.194	.116	.353	.683	.003	.901	.705
	(3.30)	(.09)	(.14)	(4.32)		(2.53)	(3.22)	(4.02)	(10.9)		(.66)	(22.17)	
13	.631	−.390	.424	.175	.659	.206	.150	.142	.341	.673	.005	.998	.627
	(3.11)	(−2.02)	(4.80)	(2.12)		(3.10)	(2.59)	(5.15)	(11.47)		(1.01)	(18.55)	
14	.184	.183	.110	.215	.666	.223	.139	.139	.314	.670	.004	.818	.667
	(1.49)	(1.59)	(1.71)	(3.64)		(3.78)	(2.67)	(5.57)	(11.89)		(1.14)	(20.30)	
15	−.065	.426	.053	.222	.578	.216	.160	.129	.293	.659	.007	.767	.565
	(−.54)	(4.01)	(.94)	(4.04)		(4.07)	(3.38)	(5.61)	(12.21)		(1.83)	(16.34)	
16	.225	.194	.298	.127	.707	.219	.162	.137	.279	.662	.006	.992	.693
	(2.13)	(2.10)	(4.56)	(2.63)		(4.52)	(3.75)	(6.36)	(12.79)		(1.97)	(21.57)	
17	.299	.058	.254	.142	.566	.225	.156	.145	.266	.655	.009	.898	.561
	(2.16)	(.42)	(3.29)	(2.42)		(4.92)	(3.81)	(7.03)	(13.01)		(2.50)	(16.22)	

TABLE 2 (continued)

Period	$\hat{\sigma}^*_{0,t} = b\,\hat{\sigma}^*_{6,t-1} + e\,\hat{\sigma}^*_{0,t-1} + f\,\hat{\sigma}^*_{0,t-2} + g \sum_{i=3}^{8} \hat{\sigma}^*_{0,t-i}/6$					$\hat{\sigma}^*_{0,n,s} = b_t\,\hat{\sigma}^*_{6,n,s-1} + e_t\,\hat{\sigma}^*_{0,n,s-1} + f_t\,\hat{\sigma}^*_{0,n,s-2} + g_t \sum_{i=3}^{8} \frac{\hat{\sigma}^*_{0,n,s-i}}{6}$					$\hat{\sigma}_{0,t+1} = \alpha + \beta\,\sigma^{\text{forecast}}_{t+1}$		
	b	e	f	g	R^2	b	e	f	g	R^2	d	β	R^2
18	.339	.072	.185	.311	.611	.228	.153	.146	.270	.652	−.002	1.146	.615
	(2.19)	(.51)	(2.19)	(4.42)		(5.22)	(3.89)	(7.34)	(13.87)		(−.49)	(18.14)	
19	.287	−.014	.141	.216	.466	.232	.140	.149	.267	.643	.020	.779	.471
	(1.80)	(−.10)	(1.95)	(3.22)		(5.53)	(3.71)	(7.79)	(14.27)		(5.79)	(13.54)	
20	.174	.220	.096	.376	.612	.229	.143	.147	.271	.641	.012	1.094	.614
	(1.35)	(1.81)	(1.60)	(5.52)		(5.73)	(3.97)	(8.11)	(15.09)		(3.40)	(18.10)	
21	.445	.075	.114	.178	.582	.237	.141	.146	.268	.639	.009	1.040	.582
	(2.89)	(.58)	(1.54)	(2.53)		(6.14)	(4.05)	(8.33)	(15.47)		(2.53)	(16.90)	
22	.077	.213	.226	.510	.636	.234	.141	.149	.274	.637	.009	1.278	.630
	(.40)	(1.32)	(2.76)	(5.89)		(6.19)	(4.18)	(8.73)	(16.19)		(2.32)	(18.75)	
23	.461	−.048	.201	.564	.644	.253	.133	.147	.278	.633	−.006	1.441	.642
	(2.06)	(−.25)	(2.26)	(4.68)		(6.77)	(3.96)	(8.68)	(16.49)		(1.38)	(19.210)	
24	.582	.090	.160	.589	.763	.279	.143	.143	.271	.633	−.011	1.732	.764
	(3.09)	(.54)	(1.78)	(4.48)		(7.49)	(4.26)	(8.48)	(15.91)		(2.64)	(25.84)	
25	−.113	.242	.408	.372	.746	.257	.147	.156	.275	.637	.008	.935	.710
	(−.85)	(1.97)	(6.89)	(4.15)		(7.13)	(4.53)	(9.54)	(16.65)		(2.58)	(22.45)	
26	.035	.686	.094	−.002	.568	.247	.166	.152	.270	.632	.006	1.018	.539
	(.19)	(4.07)	(1.35)	(−.02)		(6.98)	(5.20)	(9.64)	(16.48)		(1.31)	(15.53)	
27	.302	.076	.143	.376	.591	.250	.161	.152	.274	.630	.008	1.050	.593
	(1.68)	(.47)	(1.43)	(3.52)		(7.20)	(5.15)	(9.82)	(16.94)		(2.26)	(17.34)	
28	.204	.452	.052	.207	.659	.252	.170	.147	.271	.630	.016	1.119	.644
	(1.12)	(2.69)	(.83)	(2.68)		(7.41)	(5.55)	(9.72)	(17.13)		(4.33)	(19.31)	
29	−.087	.640	.502	−.033	.503	.236	.196	.159	.257	.616	.025	1.270	.472
	(−.36)	(3.00)	(3.75)	(−.27)		(6.85)	(6.31)	(10.35)	(15.99)		(3.91)	(13.57)	

$$\hat{\sigma}_{0,t} = a + b\,\hat{\sigma}_{6,t-1} + c\,\hat{\sigma}_{0,t-1} + f\,\hat{\sigma}_{0,t-2} + g\sum_{i=3}^{8}\frac{\hat{\sigma}_{0,t-i}}{6} + h\,\mathrm{ISD}_{t-1}. \tag{6}$$

Since the simultaneous inclusion of $\hat{\sigma}_{6,t-1}$, $\hat{\sigma}_{0,t-1}$, $\hat{\sigma}_{0,t-2}$, and ISD_{t-1} created severe multicollinearity problems and as the coefficient of $\hat{\sigma}_{0,t-1}$ was consistently negative and insignificant in most quarters, the alternative specification,

$$\hat{\sigma}_{0,t} = a + b\,\hat{\sigma}_{6,t-1} + c\,\mathrm{ISD}_{t-1} + d\,\hat{\sigma}_{0,t-2} + e\sum_{i=3}^{8}\hat{\sigma}_{0,t-i/6}, \tag{7}$$

is reported in column 2 of table 3. (Since the number of stocks which had options written on them increased over time, the regression results are not directly comparable to those of table 2. For comparison purposes we included the results of regression [3] for the smaller samples in col. 1 of table 3.) It can be inferred that the substitution of ISD_{t-1} for $\hat{\sigma}_{0,t-1}$ resulted in a marginally better predictive performance in most subperiods. A prediction rule for any given quarter t could then be estimated using the following pooled time-series, cross-section regression:

$$\hat{\sigma}^*_{0,n,s} = b_t\,\hat{\sigma}^*_{6,n,s-1} + c_t\,\mathrm{ISD}^*_{n,s-1} + d_t\,\hat{\sigma}^*_{0,n,s-2} + e_t\sum_{i=3}^{8}\frac{\hat{\sigma}^*_{0,n,s-i}}{6} \tag{8}$$

where * denotes deviations from the cross-sectional mean for a given quarter; $n = 1, \ldots, N$ is the stock indicator; and $s = 9, \ldots, t$ indicates that all previous quarters up to and including period t are included in the pooled regression. The results are reported in column 3 of table 3. It is striking that both $\hat{\sigma}_6$ and ISD* obtain an approximately equal weighting (except in the earlier periods) whereas the contribution of $\hat{\sigma}_{0,t-2}$ and of

$$\sum_{i=3}^{8}\frac{\hat{\sigma}_{0,t-i}}{6}$$

are marginal (although significant). When we use these weights to forecast the future close-to-close standard deviation, the accuracy of the forecast can again be assessed on the basis of the R^2 for the regression $\hat{\sigma}_{0,t} = \alpha + \beta\,\sigma_t^{\text{forecast}}$, where

$$\sigma_t^{\text{forecast}} = \hat{b}_{t-1}\,\hat{\sigma}^*_{6,t-1} + \hat{c}_{t-1}\,\mathrm{ISD}^*_{t-1} + \hat{d}_{t-1}\,\hat{\sigma}_{0,t-2} + \hat{e}_{t-1}\sum_{i=3}^{8}\frac{\hat{\sigma}_{0,t-i}}{6}$$

and the coefficients $\hat{b}$, $\hat{c}$, $\hat{d}$, and $\hat{e}$ are those obtained for regression (8). As can be inferred from column 4 in table 3, we obtain an average R^2 of .628 across all 20 periods (whereas the unconstrained regression [7] resulted in an average of .640). The average predictive performance obtained by substituting $\sigma_{0,t-1}$ by ISD_{t-1} increased slightly. (The average predictive R^2 from prediction rule [4] in the smaller sample used in this section is .609.) Given that the computation of ISDs is a costly and time-consuming process—there is no closed form solution for the ISD calculation—some investors may prefer prediction rule (4) over rule (8), although the additional information contained in the ISDs can be significant in any given quarter.

Prediction rule (8) contains all currently available historical price information which relates to stock price variability. There is reasonable evidence that this information is combined in an optimal way such that no alternative combination of the same data will yield a significantly better forecast. It might be that this forecasting rule can be further improved by including fundamental company data or by creatively interpreting patterns in stock price variability.[5] A further exploration of these possibilities is beyond the scope of this paper, however.

Conclusions

This article shows on the basis of empirical tests that the daily price range does contain important new information concerning the stock price variability. It has already been proven theoretically that the high-low range should give a better indication of the stock price variability than the close-to-close change. However, since the information content of the high-low data varies across stocks, its empirical value increases significantly when these cross-sectional differences are incorporated in a new variance estimator. This new estimator appears to be more "accurate" than the traditional close-to-close measure. Conversely, it can be used to forecast the future value of the close-to-close stock price variability. Alternative prediction rules are tested, which incorporate historical information on the different variance measures. Reasonably accurate predictions of the close-to-close variance can be obtained on the basis of the previous history of "adjusted" high-low and close-to-close volatility measures. These predictions can be improved even further by including standard deviations implied in option prices. This prediction rule correctly anticipates on average 63% of the future close-to-close stock price variability.

5. It has been argued before that volatilities tend to regress to the mean and that there is an inverse relationship between the level of the stock price and the variance of return. These factors could possibly be incorporated into a prediction rule.

TABLE 3 **Predictive Cross-sectional Regressions Including ISDs for the 21-Quarter Period from January 1, 1975, through March 31, 1980**

Period	$\hat\sigma_{0,t} = a + b\,\hat\sigma_{6,t-1} + c\,\hat\sigma_{0,t-1} + d\,\hat\sigma_{0,t-2} + e\sum_{i=3}^{8}\hat\sigma_{0,t-i}/6$						$\hat\sigma_{0,t} = a + b\,\hat\sigma_{6,t-1} + c\,\text{ISD}_{t-1} + d\,\hat\sigma_{0,t-2} + e\sum_{i=3}^{8}\hat\sigma_{0,t-i}/6$						$\hat\sigma^*_{0,n,s} = b_t\,\hat\sigma^*_{6,n,s-1} + c_t\,\text{ISD}^*_{n,s-1} + d_t\,\hat\sigma^*_{0,n,s-2} + e_t\sum_{i=3}^{8}\hat\sigma^*_{0,n,s-i}/6$					$\hat\sigma_{0,t} = \alpha + \beta\,\sigma_t^{\text{forecast}}$			Number of Observations
	a	b	c	d	e	R^2	a	b	c	d	e	R^2	b	c	d	e	R^2	α	β	R^2	
9	.001	1.67	−.77	−.32	.23	.718	−.018	.663	.319	−.193	.667	.706	.663	.319	−.193	.167	.706	...	...	...	29
		(2.95)	(−1.55)	(−1.64)	(1.0)			(2.29)	(1.19)	(−1.08)	(.65)		(2.34)	(1.21)	(−1.10)	(.66)					
10	.024	.374	−.065	−.015	.277	.653	.026	.384	−.164	−.030	.359	.663	.388	.151	.041	.132	.643	.030	.548	.602	56
		(1.75)	(−.39)	(−.17)	(2.67)			(3.32)	(−1.29)	(−.35)	(3.24)		(3.13)	(1.18)	(.51)	(1.19)			(9.03)		
11	.011	.116	.074	.101	.426	.698	.005	.143	.166	.073	.348	.714	.315	.170	.056	.206	.665	.004	1.120	.669	62
		(.43)	(.31)	(1.20)	(4.05)			(.98)	(1.79)	(.89)	(3.12)		(3.63)	(2.13)	(.96)	(2.72)			(11.81)		
12	.004	.592	.177	−.050	.109	.717	.003	.694	.160	−.073	.048	.721	.444	.160	−.005	.176	.681	−.002	1.093	.695	106
		(2.68)	(.84)	(−.56)	(1.15)			(6.26)	(1.53)	(−.82)	(.46)		(6.55)	(2.52)	(−.11)	(2.98)			(15.41)		
13	.009	.738	−.417	.145	.307	.669	.008	.273	.114	.131	.270	.662	.392	.146	.022	.221	.674	.010	1.032	.653	116
		(2.91)	(−1.76)	(1.35)	(3.58)			(2.27)	(.86)	(1.18)	(2.94)		(6.70)	(2.56)	(.52)	(4.57)			(14.64)		
14	.005	.271	.132	.118	.177	.711	−.004	.250	.376	.017	.106	.743	.349	.200	.032	.185	.686	.000	.907	.731	146
		(2.37)	(1.17)	(1.90)	(3.21)			(4.12)	(4.37)	(.27)	(1.97)		(7.92)	(4.21)	(.92)	(4.95)			(19.76)		
15	.004	−.021	.362	.135	.207	.566	.000	.212	.163	.123	.203	.559	.330	.197	.047	.186	.663	−.000	.953	.559	156
		(−.13)	(2.29)	(1.81)	(3.09)			(2.08)	(1.64)	(1.62)	(2.85)		(8.20)	(4.63)	(1.46)	(5.69)			(13.98)		
16	.007	.134	.305	.179	.206	.678	.004	.310	.249	.118	.174	.685	.325	.204	.052	.189	.665	.004	1.106	.684	161
		(1.06)	(2.64)	(2.18)	(3.25)			(3.93)	(3.03)	(1.42)	(2.66)		(9.07)	(5.39)	(1.76)	(6.53)			(18.53)		
17	.011	.469	−.080	.221	.115	.553	.008	.313	.233	.192	.026	.577	.326	.208	.068	.165	.651	.007	.942	.566	182
		(3.32)	(−.55)	(2.59)	(1.74)			(3.61)	(3.16)	(2.30)	(.39)		(9.86)	(6.18)	(2.48)	(6.20)			(15.33)		

18	−.000	.561	−.123	.289	.179	.583	−.002	.345	.243	.232	.118	.594	.326	.213	.082	.163	.640	−.002	1.204	.592	188
		(2.78)	(−.66)	(3.07)	(2.29)			(3.78)	(2.29)	(2.40)	(1.47)		(10.45)	(6.59)	(3.08)	(6.41)			(16.43)		
19	.018	.271	.001	.130	.245	.456	.017	.202	.205	.067	.192	.469	.307	.212	.083	.169	.622	.018	.844	.468	192
		(1.59)	(.004)	(1.62)	(3.54)			(2.57)	(2.10)	(.81)	(2.72)		(10.58)	(6.91)	(3.28)	(7.08)			(12.99)		
20	.005	.324	.080	.138	.410	.650	.004	.250	.467	.043	.231	.702	.294	.239	.084	.171	.626	.004	1.296	.691	194
		(2.32)	(.61)	(2.30)	(6.04)			(3.64)	(5.68)	(.75)	(3.28)		(10.95)	(8.29)	(3.56)	(7.57)			(20.72)		
21	.012	.312	.226	.158	.086	.599	.013	.390	.221	.094	.044	.611	.298	.237	.087	.164	.624	.012	.961	.608	195
		(1.95)	(1.67)	(2.11)	(1.19)			(4.11)	(2.93)	(1.30)	(.60)		(11.60)	(8.86)	(3.92)	(7.65)			(17.32)		
22	.008	.181	.136	.216	.476	.623	.009	.259	.104	.190	.451	.624	.298	.227	.097	.176	.620	.014	1.228	.602	190
		(.92)	(.815)	(2.46)	(5.12)			(2.32)	(1.08)	(2.10)	(4.87)		(11.82)	(8.79)	(4.47)	(8.43)			(16.87)		
23	−.004	.503	−.149	.147	.642	.646	−.005	.190	.279	.085	.528	.664	.298	.250	.088	.179	.621	−.003	1.280	.654	191
		(2.29)	(−.78)	(1.61)	(5.37)			(1.79)	(3.12)	(.93)	(4.33)		(12.07)	(10.14)	(4.17)	(8.64)			(18.91)		
24	−.006	.788	−.080	.117	.520	.750	−.004	.460	.351	.072	.329	.766	.319	.300	.078	.153	.634	−.004	1.472	.760	195
		(3.70)	(−.41)	(1.31)	(3.70)			(4.46)	(3.71)	(.83)	(2.26)		(13.11)	(12.65)	(3.73)	(7.41)			(25.31)		
25	.002	−.031	.187	.369	.388	.723	.003	.042	.276	.312	.231	.746	.287	.298	.093	.162	.640	.008	.876	.719	197
		(−.22)	(1.43)	(5.45)	(4.07)			(.69)	(4.34)	(4.70)	(2.34)		(12.70)	(13.35)	(4.65)	(8.17)			(22.34)		
26	.010	−.064	.721	.093	.045	.529	.010	.531	.176	.056	.051	.494	.300	.289	.087	.162	.627	.009	.968	.489	197
		(−.32)	(4.15)	(1.23)	(.31)			(4.14)	(1.63)	(.69)	(.33)		(13.34)	(13.07)	(4.54)	(8.14)			(13.65)		
27	.009	.010	.273	.204	.355	.577	.011	.128	.351	.139	.218	.604	.287	.293	.091	.167	.625	.013	.974	.598	196
		(.06)	(1.70)	(2.13)	(3.49)			(1.59)	(3.97)	(1.47)	(2.20)		(13.24)	(13.67)	(4.84)	(8.54)			(17.00)		
28	.016	.261	.400	.062	.209	.661	.013	.398	.301	.056	.168	.685	.293	.297	.087	.165	.629	.013	1.110	.683	197
		(1.45)	(2.41)	(1.00)	(2.76)			(4.38)	(4.41)	(.93)	(2.27)		(13.93)	(14.61)	(4.88)	(8.79)			(20.60)		
29	.027	.021	.546	.492	.003	.497	.020	.320	.391	.397	−.036	.513	.297	.310	.104	.147	.610	.016	1.315	.502	194
		(.08)	(2.43)	(3.49)	(.02)			(2.05)	(3.43)	(2.79)	(−.28)		(13.67)	(15.04)	(5.61)	(7.59)			(13.91)		

Appendix

Assume that the "true" underlying variance, σ_t, is nonstationary over time. Both $\hat{\sigma}_{0t}$ and $\hat{\sigma}_{2t}$ are estimates of σ_t which contain estimation error:

$$\hat{\sigma}_{0t} = \sigma_t + \epsilon_{1t} \qquad \hat{\sigma}_{0t-1} = \sigma_{t-1} + \epsilon_{1t-1}$$

$$\hat{\sigma}_{2t} = \sigma_t + \epsilon_{2t} \qquad \hat{\sigma}_{2t-1} = \sigma_{t-1} + \epsilon_{2t-1}.$$

A simple (cross-sectional) regression approach is used to determine which estimation is "better." For example,

$$\hat{\sigma}_{2t} = \hat{a} + \hat{b}\,\hat{\sigma}_{2t-1} \qquad \text{coefficient of det } R^2_{22}$$

$$\hat{\sigma}_{0t} = \hat{c} + \hat{d}\,\hat{\sigma}_{2t-1} \qquad \text{coefficient of det } R^2_{02}$$

If $R^2_{22} > R^2_{02}$, then it must be that var $(\epsilon_{1t}) >$ var (ϵ_{2t}) and hence that var $(\hat{\sigma}_{0t}) >$ var $(\hat{\sigma}_{2t})$. Since

$$R^2_{22} = \frac{[\text{cov}\ (\hat{\sigma}_{2t}\ \hat{\sigma}_{2t-1})]^2}{\text{var}\ (\hat{\sigma}_{2t})\ \text{var}\ (\hat{\sigma}_{2t-1})}$$

$$R^2_{02} = \frac{[\text{cov}\ (\hat{\sigma}_{0t}\ \hat{\sigma}_{2t-1})]^2}{\text{var}\ (\hat{\sigma}_{0t})\ \text{var}\ (\hat{\sigma}_{2t-1})}$$

and since

$$\text{cov}\ (\sigma_t\ \epsilon_{2t-1}) = \text{cov}\ (\sigma_{t-1}\ \epsilon_{2t}) = \text{cov}\ (\epsilon_{2t}\ \epsilon_{2t-1}) =$$

$$\text{cov}\ (\sigma_{t-1}\ \epsilon_{1t}) = \text{cov}\ (\epsilon_{1t}\ \epsilon_{2t-1}) \cong 0,$$

$R^2_{22} > R^2_{02}$ implies that

$$\frac{[\text{cov}\ (\sigma_t\ \sigma_{t-1})]^2}{\text{var}\ (\sigma_t) + \text{var}\ (\epsilon_{2t})} > \frac{[\text{cov}\ (\sigma_t\ \sigma_{t-1})]^2}{\text{var}\ (\sigma_t) + \text{var}\ (\epsilon_{1t})},$$

and hence var $(\epsilon_{1t}) >$ var (ϵ_{2t}). Similarly, one can prove that $R^2_{22} > R^2_{20}$ implies that var $(\epsilon_{1t-1}) >$ var (ϵ_{2t-1}).

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