

## Intertemporal Asset Pricing with Heterogeneous Consumers and without Demand Aggregation\*

### I. Introduction

Consumer heterogeneity raises two problems in the derivation of the intertemporal asset-pricing model. First, it is implausible to assume that all assets' returns are multivariate normal (or exhibit separability). Second, the stochastically varying distribution of wealth among consumers is a vector of state variables which may add a large number of parameters to the two-parameter asset-pricing model.

Consider the first problem. The Sharpe-Lintner asset-pricing model (CAPM) is derived under the assumption that all assets' returns are multivariate normal (or, more generally, that they exhibit separability).<sup>1</sup> Whereas it may be plausible to assume that stocks' returns are multivariate normal, it is implausible to assume that all financial assets' returns are multivariate normal. (Financial assets are defined to be the assets in zero net supply.) For example, over a finite time interval a call option's return has a truncated (possibly normal or lognormal) distribution

Consumer heterogeneity raises two problems in the derivation of the intertemporal asset-pricing model. First, it is implausible to assume that all assets' returns are multivariate normal (or exhibit separability). Second, the stochastically varying distribution of wealth among consumers is a vector of state variables which may add a large number of parameters to the two-parameter asset-pricing model. Both problems are resolved in a complete market. Optimality of the competitive equilibrium implies that prices, production, and aggregate consumption are the same as in the equilibrium of a central planner or composite consumer. In the composite consumer's observationally equivalent equilibrium no distributional assumptions are necessary about zero net supply assets. Also the wealth distribution among heterogeneous consumers becomes an irrelevant state variable.

\*I thank Philip H. Dybvig, Jonathan E. Ingersoll, and Edward C. Prescott for helpful criticism on earlier drafts. I remain responsible for errors.

1. Related work includes Mossin (1966), Fama (1970, 1971), Black (1972), and Ross (1976, 1978). The CAPM may be derived under the assumption of quadratic utility without any distributional assumptions. The assumption of quadratic utility resolves the first but not the second of the two problems outlined below.

(*Journal of Business*, 1982, vol. 55, no. 2)

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0021-9398/82/5502-0002\$01.50

and an accident insurance contract has a spiked distribution which is not normal, even over an infinitesimal time interval. If consumers have homogeneous tastes, endowments, and beliefs, this problem is avoided by the simple expedient of closing the markets for financial assets. With the closing of these markets there is no loss of optimality and no change in prices because homogeneous consumers need not trade with each other. The only remaining traded assets are the firms' shares, and it is plausible to assume that their returns are multivariate normal. However, in a heterogeneous consumer economy, as long as there are fewer active firms than states, financial assets are generally indispensable in completing the market and attaining an optimal allocation; the procedure of closing the markets for financial assets is generally invalid.

The second problem arises in an intertemporal economy. The distribution of wealth at date  $t$  among  $m$  heterogeneous consumers (or classes of consumers) is an  $(m - 1)$ -vector of state variables which is stochastic at dates prior to  $t$ . The term structure of interest rates and financial assets' returns generally depends upon these state variables. In the intertemporal extension of the Sharpe-Lintner CAPM, an asset's risk premium is determined not only by its covariance with the market return, but also by its covariance with the  $m - 1$  state variables.<sup>2</sup> The two-parameter CAPM is then replaced by an extended CAPM with  $m + 1$  parameters. Unless the model contains only a small number of parameters, it is empirically difficult to test or apply. Yet in an economy with a large number of heterogeneous consumers we have no reason to limit the parameters to a small number. This problem also is avoided if consumers have homogeneous tastes, endowments, and beliefs. Then wealth distribution is equal across consumers at all future times, and therefore wealth distribution is not a state variable.

Consumer heterogeneity has been identified as the source of the two problems. If we could replace, so to speak, the heterogeneous consumers by a population of homogeneous consumers without changing equilibrium prices, production, and aggregate consumption, the two problems would then be resolved. Indeed, Constantinides (1980) assumed that the heterogeneous consumers have homogeneous beliefs and time-additive utility functions with linear risk tolerance and same exponent. As Rubinstein (1974) has shown, these assumptions imply demand aggregation. Then the heterogeneous consumers may be replaced by one or more homogeneous consumers, referred to as "aggregated" consumers, leaving equilibrium prices, production, and aggre-

2. The intertemporal extension of the Sharpe-Lintner CAPM is developed in Fama (1970), Merton (1973), and Long (1974). Subsequent related work includes Rubinstein (1976), Breeden and Litzenberger (1978), Brock (1978), Cox, Ingersoll, and Ross (1978), Lucas (1978), Stapleton and Subrahmanyam (1978), Breeden (1979), Constantinides (1980), Prescott and Mehra (1980), and Bhattacharya (1981).

gate consumption unchanged. In the observationally equivalent economy of the "aggregated" consumers, the two problems are resolved.

The goal of this paper is to resolve the two problems without resorting to the fairly strong utility assumptions needed for demand aggregation. This is accomplished with the assumption that markets are complete. With complete markets the optimality of the competitive equilibrium states that the heterogeneous consumers may be replaced by a central planner who maximizes a weighted sum of the consumers' utilities. In turn the central planner may be replaced by a "composite" consumer who maximizes a utility function of aggregate consumption. Equilibrium prices, production, and aggregate consumption remain unchanged if the heterogeneous consumers are replaced by the composite consumer. Although existence of a composite consumer is a weaker property than demand aggregation, it is sufficient for the resolution of the two problems.

Prior to our work, in the context of a one-period economy, Dybvig and Ingersoll (this issue) resolved the first of the two problems which arise in a heterogeneous consumer economy by assuming complete markets and state independent utility. Their proof hinges on a result in Dybvig and Ross (1982), that the market portfolio is efficient if markets are complete. In an intertemporal economy Dybvig and Ingersoll's end-of-first-period utility may be interpreted as the derived utility of the future consumption stream. In general this derived utility depends on the distribution of wealth among the heterogeneous consumers and is therefore state dependent. Thus the second of the two problems remains unresolved, unless it is shown that the derived utility is indeed state independent.

The paper is organized as follows. In Section II we define equilibrium in an Arrow-Debreu private ownership economy. The optimality of the competitive equilibrium leads to the formulation of the central planner's problem (P1). Lemma 1 states that the heterogeneous consumers may be replaced by a composite consumer, leaving production, aggregate consumption, and equilibrium prices unchanged. In Section III the general Arrow-Debreu economy is specialized to a one-good, intertemporal economy. Lemma 2 states that, if consumers have time-additive, von Neumann-Morgenstern utilities and homogeneous beliefs, the composite consumer has time-additive, von Neumann-Morgenstern utility also. Lemmata 1 and 2 make possible the derivation of the CAPM in a homogeneous consumer economy, that of the von Neumann-Morgenstern composite consumer. Lemma 3 reduces the composite consumer's intertemporal problem to a one-period problem; under assumptions A1-A5 his derived utility of future consumption has the desirable property of being state independent. This, despite the fact that the heterogeneous consumers' derived utility functions are generally state dependent. Under assumptions A6 and A7

on the technologies, proposition 1 is a statement of the two-parameter CAPM. Proposition 2 replaces the technological assumptions A6 and A7 by either a diffusion process technology (A8) or quadratic utility (A9).

## II. An Arrow-Debreu Economy

Consider a private ownership economy as in Debreu (1959). It consists of  $m$  consumers, indexed  $i = 1, 2, \dots, m$ ;  $n$  firms, indexed  $j = 1, 2, \dots, n$ ; and  $l$  commodities, indexed  $h = 1, 2, \dots, l$ . Consumer  $i$  is endowed with wealth  $(w_{i1} w_{i2} \dots w_{il})$  and shares  $(\theta_{i1} \theta_{i2} \dots \theta_{in})$  satisfying  $\theta_{ij} \geq 0$  and  $\sum^m \theta_{ij} = 1$ . Consumer  $i$  has consumption set  $X_i$ ; consumption  $x_i \equiv (x_{i1} x_{i2} \dots x_{il}) \in X_i$ ; and utility  $U_i(x_i)$ . Firm  $j$  has production set  $Y_j$  and production  $y_j \equiv (y_{j1} y_{j2} \dots y_{jl}) \in Y_j$ . The price vector is denoted  $p \equiv (p_1 p_2 \dots p_l)$ .

An equilibrium is an  $(m + n + 1)$ -tuple  $[(x_i^*), (y_j^*), p^*]$  of points of  $R^l$  such that (1) consumers maximize utility subject to their budget constraint and consumption set, (2) firms maximize profit subject to their production set, and (3) markets clear. Under standard assumptions (closed and convex sets  $X_i$ ,  $Y_j$ , concave utility, etc.) an equilibrium exists and is optimal.

Optimality states that there exist positive numbers  $\lambda_i, i = 1, 2, \dots, m$  such that the solution to (P1) is  $(x_i) = (x_i^*)$  and  $(y_j) = (y_j^*)$ , where

$$\max_{x,y} \sum_{i=1}^m \lambda_i U_i(x_i),$$

subject to

$$y_j \in Y_j, \quad j = 1, 2, \dots, n, \quad (\text{P1})$$

$$x_i \in X_i, \quad i = 1, 2, \dots, m,$$

$$\sum_{i=1}^m x_{ih} = \sum_{j=1}^n y_{jh} + w_h, \quad h = 1, 2, \dots, l.$$

and

$$w_h \equiv \sum_{i=1}^m w_{ih}.$$

Problem (P1) is equivalent to problem (P1)', where

$$\max_y \left[ \max_x \sum_{i=1}^m \lambda_i U_i(x_i) \right],$$

subject to

$$\begin{aligned} y_j &\in Y_j, & j &= 1, 2, \dots, n, \\ x_i &\in X_i, & i &= 1, 2, \dots, m, \\ \sum_{i=1}^m x_{ih} &= \sum_{j=1}^n y_{jh} + w_h, & h &= 1, 2, \dots, l. \end{aligned} \quad (P1)'$$

Define utility  $U(z)$  of aggregate consumption  $z \equiv (z_1, z_2, \dots, z_l)$ ,  $z_h \equiv \sum_{i=1}^m x_{ih}$  by

$$U(z) \equiv \max \sum_{i=1}^m \lambda_i U_i(x_i),$$

subject to

$$\begin{aligned} x_i &\in X_i, & i &= 1, 2, \dots, m, \\ \sum_{i=1}^m x_{ih} &= z_h, & h &= 1, 2, \dots, l. \end{aligned} \quad (P2)$$

Define also problem

$$\max_{y, z} U(z),$$

subject to

$$\begin{aligned} y_j &\in Y_j, & j &= 1, 2, \dots, n, \\ z_h &= \sum_{j=1}^n y_{jh} + w_h, & h &= 1, 2, \dots, l. \end{aligned} \quad (P3)$$

One easily proves the following lemma.<sup>3</sup>

LEMMA 1:

a) The solution to (P3) is  $(y_j) = (y_j^*)$  and  $z_h = \sum_{j=1}^n y_{jh}^* + w_h, h = 1, 2, \dots, l$ .

b) Utility  $U(z)$  is increasing and concave in  $z$ .

c) If  $z_h = \sum_{j=1}^n y_{jh}^* + w_h, h = 1, 2, \dots, l$ , then the solution to (P2) is  $(x_i) = (x_i^*)$ .

d) Given  $\lambda_i, i = 1, 2, \dots, m$ , if the  $m$  consumers are replaced by one "composite" consumer with utility  $U(z)$ , endowment the sum of the  $m$  consumers' endowments, and shares the sum of the  $m$  consumers' shares, the  $(1 + n + 1)$ -tuple  $(\sum_{i=1}^m x_i^*, y_j^*, p^*)$  is an equilibrium.

3. In the context of a recursive equilibrium this result is outlined in Prescott and Mehra (1980).

The derivation of the asset-pricing model is couched in terms of the composite consumer. Lemma 1, together with lemma 2, make possible the derivation of the CAPM in a homogeneous consumer economy, the economy of the composite consumer.

The existence of a composite consumer does not imply demand aggregation for two reasons. First, the composite consumer's demand depends through the  $\lambda_i$ 's on the distribution of endowments and shares among consumers. Second, the composite consumer is defined at the equilibrium prices, and no claim is made in lemma 1 that the composite consumer's demand curve coincides with the aggregate demand curve. Demand aggregation, however, is not necessary in order to derive the capital asset-pricing model: In the next section we derive the CAPM under a set of assumptions which do not necessarily imply demand aggregation.

The existence of a composite consumer holding all shares (i.e., holding the "market portfolio") at equilibrium implies that the market portfolio is "efficient." A portfolio relative to a class of utility functions is defined by Dybvig and Ross (1982) to be efficient, if there exists a utility in the class and an endowment such that a consumer with this utility and endowment holds this portfolio in maximizing his utility. Dybvig and Ross proved that the market portfolio is efficient in a one-good, one-period economy. The existence of a composite consumer implies that the market portfolio is efficient also in a multigood, multiperiod economy, in the sense that there exists an increasing and concave utility function and an endowment, such that a consumer with this utility and endowment holds the market portfolio.

### III. The Capital Asset-pricing Model in an Economy with Heterogeneous Consumers

Assume that the economy extends over  $T$  dates,  $t = 1, 2, \dots, T$ . At date  $t$  the state is a scalar  $s_t, s_t = 1, 2, \dots, S$ ; an event is the vector  $e_t = (s_1 s_2 \dots s_t)$ . There is one good. The location of a commodity is  $h = (e_t, t)$ . Assume

A1. *Beliefs.* Consumers agree on the probability,  $\text{Prob}(e_t)$ , that event  $e_t$  occurs.

A2. *Preferences.* Consumers have time-additive, von Neumann-Morgenstern utility functions

$$U_i(x_i) = \sum_{t=1}^T \sum_{e_t} \text{Prob}(e_t) u_{it}[x_{it}(e_t)], \quad (1)$$

where  $u_{it}$  is increasing and concave.

We now prove the following.

LEMMA 2: Under assumptions A1 and A2 the composite consumer's utility function,  $U(z)$ , defined in problem (P2), is

$$U(z) = \sum_{t=1}^T \sum_{e_t} \text{Prob}(e_t) u_t[z_t(e_t)], \quad (2)$$

where  $u_t$  is increasing and concave.

PROOF: Define

$$u_t[z_t(e_t)] \equiv \max_{\{x_{it}(e_t)\}_i} \sum_{i=1}^m \lambda_i u_{it}[x_{it}(e_t)],$$

subject to

$$\begin{aligned} x_{it}(e_t) &\in X_i, & i &= 1, 2, \dots, m, \\ & & t &= 1, 2, \dots, T, \end{aligned}$$

and

$$\sum_{i=1}^m x_{it}(e_t) = z_t(e_t), \quad t = 1, 2, \dots, T.$$

Since  $\lambda_i \geq 0$ ,  $u_t$  is increasing and concave.

By the definition in problem (P2)

$$U(z) = \max_x \sum_{i=1}^m \lambda_i U_i(x_i),$$

[subject to

$$\begin{aligned} x_{it}(e_t) &\in X_i, & i &= 1, 2, \dots, m, \\ & & t &= 1, 2, \dots, T, \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^m x_{it}(e_t) &= z_t(e_t), & t &= 1, 2, \dots, T] \\ &= \max_x \sum_{i=1}^m \lambda_i \sum_{t=1}^T \sum_{e_t} \text{Prob}(e_t) u_{it}[x_{it}(e_t)] \\ &= \sum_{t=1}^T \sum_{e_t} \text{Prob}(e_t) \max_{\{x_{it}(e_t)\}_i} \sum_{i=1}^m \lambda_i u_{it}[x_{it}(e_t)] \\ &= \sum_{t=1}^T \sum_{e_t} \text{Prob}(e_t) u_t[z_t(e_t)].^4 \end{aligned}$$

4. Phil Dybvig pointed out to me that time additivity of the consumers' utility functions is crucial in this particular proof of the property that the composite consumer's

Q.E.D.

Further assume

A3. *States.* The probability that state  $s_t$  occurs at date  $t$  is independent of  $e_{t-1}$ , that is,

$$\text{Prob}(s_t | e_{t-1}) = \text{Prob}(s_t) \equiv \pi_t(s_t). \quad (3)$$

A4. *Endowments.* Aggregate endowment at date  $t$  is  $w_t$ , independent of  $e_t$ .<sup>5</sup> Define  $y_{jt}^1(e_t)$  as the input and  $y_{jt}^2(e_t)$  as the output of firm  $j$  at date  $t$ . Then

$$y_{jh} = -y_{jt}^1(e_t) + y_{jt}^2(e_t). \quad (4)$$

A5. *Technologies.* Production technology  $j$  has uncertainty resolved by the state, that is,

$$y_{jt}^2(e_t) = f_{jt}(y_{jt}^1(e_{t-1}), s_t) \geq 0, y_{jt}^1(e_{t-1}) \geq 0, \quad (5)$$

and  $f_{jt}$  is concave and increasing in its first argument. The production function  $f_{jt}$  depends on  $e_{t-1}$  through  $y_{jt}^1(e_{t-1})$  but is otherwise independent of  $e_{t-1}$ .

In the next lemma we reformulate the composite consumer's optimization problem as a dynamic program: In every period he chooses aggregate consumption and investment to maximize his expected derived utility of current consumption and of next-period wealth.<sup>6</sup> The derived utility function has two important features: first, it is increasing and concave in next-period wealth; second, at time  $t - 1$  the next-period derived utility,  $V_t$ , depends on wealth,  $W_t$ , but is otherwise independent of the event  $e_t$ .

LEMMA 3: Under assumptions A1 through A5 the composite consumer's problem (P3) becomes

$$\begin{aligned} \max_{\substack{z_{t-1}(e_{t-1}) \\ (y_{jt}^1(e_{t-1}))}} & \left\{ u_{t-1}[z_{t-1}(e_{t-1})] + \sum_{s_t=1}^S \pi_t(s_t) V_t[W_t(e_t)] \right\} \\ & \{ \equiv V_{t-1}[W_{t-1}(e_{t-1})] \} \end{aligned} \quad (6)$$

utility is von Neumann-Morgenstern. He also posed the following problem: Given complete markets, find weaker conditions on the consumers' utility functions that imply that the composite consumer's utility is von Neumann-Morgenstern. This question is closely related to the issues discussed in Dybvig and Ross (1982).

5. Lemma 3 holds under the weaker assumption that  $w_t = w_t(s_t)$ , independent of  $e_{t-1}$ . However, we need to assume A4, i.e., that  $w_t$  is independent of  $s_t$  as well, in order to prove propositions 1, 2.

6. Wealth  $W_t$  is defined in eq. (8) as the value of firms plus the endowment  $w_t$ . Strictly speaking, the definition should include the discounted value of future endowments  $w_{t+1}$ ,  $w_{t+2}$ , . . . .

subject to

$$z_{t-1}(e_{t-1}) + \sum_{j=1}^n y_{j(t-1)}^1(e_{t-1}) = W_{t-1}(e_{t-1}), \quad (7)$$

$$W_t(e_t) = \sum_{j=1}^n f_{jt} [y_{j(t-1)}^1(e_{t-1}), s_t] + \bar{w}_t \quad (8)$$

$$y_{j(t-1)}^1 \geq 0, \quad (9)$$

for all  $t = 2, 3, \dots, T$ , and the boundary condition

$$V_T[W_T(e_T)] = u_T[W_T(e_T)]. \quad (10)$$

The term  $V_t$  is increasing and concave in  $W_t(e_t)$ . Furthermore, given  $W_t(e_t)$ ,  $V_t$  is independent of  $e_t$ , that is,

$$V_t[W_t(e_t), e_t] = V_t[W_t(e_t)]. \quad (11)$$

**Proof outline:** By equation (10),  $V_T$  is increasing and concave in its arguments and is independent of  $e_T$ . First we prove that  $V_t$  is increasing and concave in its arguments and is independent of  $e_t$ , for  $t = T - 1$ . We then claim, by induction, that the same property holds for  $t = T - 2, T - 3, \dots, 2, 1$ . The proof is essentially the one given in Fama (1970). Two deviations from Fama's assumptions are the concavity (instead of linearity) of the production function and the nonnegativity of production inputs. Clearly these assumptions (eqq. [8] and [9]) preserve the convexity of the feasible set of  $z_{t-1}(e_{t-1})$  and  $y_{j(t-1)}^1(e_{t-1})$  and the proof remains valid. A proof of this statement is in Constantinides (1979).

Lemma 3 simplifies the composite consumer's problem in two ways. First, it reduces his multiperiod problem to a sequence of one-period problems. At date  $t - 1$  he chooses aggregate consumption and investment to maximize his expected derived utility of current consumption and next-period wealth. Second, lemma 3 states that derived utility is independent of the distribution of wealth at date  $t$  among the  $m$  heterogeneous consumers.

At date  $t - 1$  the composite consumer invests  $y_{j(t-1)}^1(e_{t-1})$  in firm  $j$ . The rate of return is endogenous but is identified as an exogenous technological variable under the following assumption:

**A6. Constant returns to scale.** Production technology has constant returns to scale, that is,

$$y_{jt}^2(e_t) = y_{j(t-1)}^1(e_{t-1})R_{jt}(s_t). \quad (12)$$

Then the rate of return of the composite consumer's investment in firm  $j$  is  $R_{jt}(s_t)$ . Furthermore, in the competitive economy the price of the  $j$ th firm's stock at date  $t - 1$  and after investment at date  $t - 1$  is  $y_{j(t-1)}^1(e_{t-1})$ . The stock price at date  $t$  and before investment at date  $t$  is

$y_{jt}^2(e_t)$ . Then  $R_{jt}(s_t)$  is also the rate of return on the stock in the competitive economy.

If the rates of return  $R_{jt}(s_t), j = 1, 2, \dots, n$  are multivariate normal, we derive the asset-pricing model as in Sharpe (1964) and Lintner (1965). However, the class of distributions which lead to the CAPM is broader than the multivariate normal class, and a digression into these distributions is in order. Ross's (1978) theory of mutual fund separation motivates the following definition:

DEFINITION: A set of returns,  $\tilde{R}_1, \tilde{R}_2, \dots, \tilde{R}_n$ , is said to exhibit (two-fund) separability if there exist random variables  $\tilde{\eta}$ ,  $\tilde{\epsilon} = (\tilde{\epsilon}_1 \tilde{\epsilon}_2 \dots \tilde{\epsilon}_n)$ , vectors  $(b_1 b_2 \dots b_n)$ ,  $\omega = (\omega_1 \omega_2 \dots \omega_n)$  and scalar  $R_0$ , such that

$$\tilde{R}_j = R_0 + b_j \tilde{\eta} + \tilde{\epsilon}_j, \quad j = 1, 2, \dots, n, \quad (13)$$

$$E[\tilde{\epsilon}_j | \tilde{\eta}] = 0, \quad j = 1, 2, \dots, n, \quad (14)$$

$$\omega' 1 = 1 \text{ and } \omega' \tilde{\epsilon} \equiv 0. \quad (15)$$

Consider an investor allocating his wealth among a riskless asset with return  $R_0$  and  $n$  risky assets with returns  $\tilde{R}_1, \tilde{R}_2, \dots, \tilde{R}_n$  which exhibit separability. The investor maximizes his expectation of a concave utility function of end-of-period wealth. Provided that short sales are allowed, Ross proved that the investor invests in just two funds: the riskless asset with return  $R_0$ ; and a portfolio of the risky assets with weights  $\omega$  and return  $R_0 + \omega' b \tilde{\eta}$ . The investor's portfolio return is independent of  $\tilde{\epsilon}$ , that is, the investor diversifies away the diversifiable risk  $\tilde{\epsilon}$ .

Ross further argued that, since the market portfolio is a linear combination of all investors' portfolios and since each investor's portfolio return is independent of  $\tilde{\epsilon}$ , the market portfolio return,  $\tilde{R}_M$ , is independent of  $\tilde{\epsilon}$ . Thus we may write

$$\tilde{R}_M = R_0 + b_M \tilde{\eta}. \quad (16)$$

We rescale  $\tilde{\eta}$  so that  $b_M = 1$  and substitute in (13) to obtain the asset-pricing model

$$\tilde{R}_i = R_0 + b_i(\tilde{R}_M - R_0) + \epsilon_i, \quad i = 1, 2, \dots, n, \quad (17)$$

or, in expected return form

$$\bar{R}_i - R_0 = b_i(\bar{R}_M - R_0), \quad i = 1, 2, \dots, n. \quad (18)$$

Finally, if covariances are finite,

$$b_i = \text{cov}(R_i, R_M) / \text{var}(R_M). \quad (19)$$

The foregoing discussion may suggest that, if the technological variables  $R_{jt}(s_t), j = 1, 2, \dots, n$ , exhibit separability, we may retrace

Ross's arguments and derive the asset-pricing model. However, separability of the technological variables is insufficient for the following reason: If a set of  $n$  returns exhibit separability, it is not necessarily the case that every subset of the  $n$  returns also exhibit separability, as can be demonstrated by example. Suppose then that the technological variables  $R_{jt}(s_t)$ ,  $j = 1, 2, \dots, n$ , exhibit separability. Given the nonnegativity constraint on investment,  $y_{j(t-1)}^1(e_{t-1}) \geq 0$ , technology  $j$  may or may not be active at date  $t - 1$ , depending on event  $e_{t-1}$ . The subset of the  $n$  technologies which are active at date  $t - 1$  and event  $e_{t-1}$  may not exhibit separability.<sup>7</sup> This observation motivates an assumption on the  $n$  technological variables which is stronger than separability.

**A7. Distribution of technological variables.** The set of technological variables  $R_{jt}(s_t)$ ,  $j = 1, 2, \dots, n$ , and every subset thereof, exhibit separability, as defined by equations (13)–(15).

The most common distribution satisfying assumption A7 is the multivariate normal: If  $n$  variables are multivariate normal, every subset thereof is multivariate normal and exhibits separability. More generally a distribution of returns defined by the property that every linear combination of these returns has a distribution determined by its mean and variance also satisfies assumption A7. These distributions are completely characterized in Chamberlain (1980) and include the multivariate normal as a special case. Finally the multivariate symmetric stable distribution with uniform exponent, discussed in Fama (1971) and Press (1972, chap. 12), satisfies assumption A7.

We define the market portfolio at date  $t - 1$  as the portfolio of all outstanding shares in the competitive economy. It consists of capital  $y_{j(t-1)}^1(e_{t-1})$  invested in technology  $j$ ,  $j = 1, 2, \dots, n$ . The market portfolio return,  $R_{Mt}(e_t)$ , over  $(t - 1, t)$  depends on  $e_{t-1}$  through its composition and depends on  $s_t$  through the returns of its components. Conditional on  $e_{t-1}$ , the expected market return is denoted by  $\bar{R}_{Mt}(e_{t-1})$ .

We denote the return on the riskless asset over  $(t - 1, t)$  by  $r_t(e_{t-1})$ . It is defined in terms of the composite consumer's marginal utility as

$$r_t(e_{t-1}) = \frac{\frac{\partial u_{t-1}}{\partial z_{t-1}}}{\sum_{s_t=1}^s \pi_t(s_t) \frac{\partial V_t}{\partial W_t}}. \quad (20)$$

If a riskless technology exists and is active over  $(t - 1, t)$ , say the technology of firm  $j = 1$ ,  $R_{1t}(s_t)$  is independent of  $s_t$  and  $r_t(e_{t-1}) = R_{1t}$ . We are now in a position to prove our main result.

7. Thus in Constantinides (1980, proposition 1), the phrase "and the distribution of  $\bar{R}_t$  is a separating distribution" must be replaced by "and the distribution of  $\bar{R}_t$ , and every subset thereof, exhibit separability."

PROPOSITION 1: Consider the equilibrium in the Arrow-Debreu economy defined in Section II. Assume that there is only one good and that assumptions A1–A7 hold. Given event  $e_{t-1}$ , let  $j = 1, 2, \dots, n'$  be the subset of active technologies over  $(t-1, t)$ . Then, there exist numbers  $b_1, b_2, \dots, b_{n'}$ , such that

$$R_{jt}(s_t) = r_t(e_{t-1}) + b_j[R_{Mt}(e_t) - r_t(e_{t-1})] + \epsilon_j(e_t), \quad (21)$$

$$j = 1, 2, \dots, n',$$

where

$$E_{s_t} [\epsilon_j(e_t) | R_{Mt}(e_t)] = 0, \quad j = 1, 2, \dots, n'. \quad (22)$$

In expected return form, (21) becomes

$$\bar{R}_{jt} - r_t(e_{t-1}) = b_j[\bar{R}_{Mt}(e_{t-1}) - r_t(e_{t-1})], \quad j = 1, 2, \dots, n'. \quad (23)$$

Furthermore, if covariances are finite

$$b_j = \text{cov}[R_{jt}(s_t), R_{Mt}(e_t)] / \text{var}[R_{Mt}(e_t)] \quad (24)$$

where covariances are conditional on  $e_{t-1}$ .

PROOF: By assumption A7, the subset of variables  $R_{jt}(s_t)$ ,  $j = 1, 2, \dots, n'$ , exhibit separability: there exist random variables  $\tilde{\eta}$ ,  $\tilde{\epsilon} = (\tilde{\epsilon}_1 \tilde{\epsilon}_2 \dots \tilde{\epsilon}_{n'})$ , vectors  $(b_1 b_2 \dots b_{n'})$  and  $\omega = (\omega_1 \omega_2 \dots \omega_{n'})$  and scalar  $R_0$  such that

$$R_{jt}(s_t) = R_0 + b_j \tilde{\eta} + \tilde{\epsilon}_j, \quad j = 1, 2, \dots, n', \quad (25)$$

$$E[\tilde{\epsilon}_j | \tilde{\eta}] = 0, \quad j = 1, 2, \dots, n', \quad (26)$$

$$\omega' I = 1 \text{ and } \omega' \tilde{\epsilon} \equiv 0. \quad (27)$$

We consider separately the case where one of the active technologies is riskless and the case where none of the active technologies is riskless.

Suppose that one of the active technologies is riskless, say technology  $j = 1$ . For this technology,  $b_1 = 0$ ,  $\tilde{\epsilon}_1 \equiv 0$ , and  $R_0 = R_{1t} = r_t(e_{t-1})$ , the riskless rate of return. By lemma 3, the composite consumer maximizes at date  $t-1$  his expectation of a concave utility function of consumption at date  $t-1$  and wealth at date  $t$ . Furthermore, by lemma 3 the utility function is independent of state  $s_t$ . Following Ross's (1978) argument, the composite consumer invests in a portfolio of assets  $j = 1, 2, \dots, n'$  with weights  $\omega_1, \omega_2, \dots, \omega_{n'}$  and in the riskless asset with return  $R_0$ . But there is no exogenous supply of the riskless asset in this economy, other than the riskless technology. Therefore the composite consumer invests in just a portfolio of assets  $j = 1, 2, \dots, n'$  with weights  $\omega_1, \omega_2, \dots, \omega_{n'}$ . The return of this portfolio is the market return, that is,

$$R_{Mt}(e_t) = r_t(e_{t-1}) + b_M \tilde{\eta}, \quad (28)$$

since  $\omega'1 = 1$  and  $\omega'\tilde{\epsilon} \equiv 0$  by (27). We rescale  $\tilde{\eta}$  so that  $b_M = 1$  and substitute in (25) to obtain (21). Equation (22) follows from (26) and (28).

Suppose next that none of the active technologies is riskless. We introduce a fictitious riskless firm with technological variable (in this case, constant) equal to  $r_t(e_{t-1})$ , as defined in equation (20). By construction, the composite consumer is indifferent in the margin between investing in this technology or not. Thus the fictitious technology is active but zero capital is invested in it. The argument proceeds as in the case where one of the active technologies is riskless. As before  $R_0 = r_t(e_{t-1})$ . Since the net investment in the fictitious technology is zero, the composition and return of the market portfolio remains unchanged with the introduction of the fictitious technology. Proceeding as before we derive equations (21) and (22). Q.E.D.

Proposition 1 states that the rates of return of the active firms over  $(t-1, t)$  satisfy the two-parameter Sharpe-Lintner asset-pricing model. No claim is made that the asset-pricing model applies to the rates of return of inactive firms or to financial assets such as options or insurance contracts.

Assumption A6 on constant returns to scale is unnecessary for the derivation of the asset-pricing model. With decreasing returns to scale technologies the price of a firm's stock no longer equals the firm's capital, and stock return no longer equals  $y_{jt}^2(e_t)/y_{jt-1}^1(e_{t-1})$ . Nevertheless the stock price and return are well defined. In order to derive the asset-pricing model we replace assumptions A6 and A7 by either assumption A8 or by assumption A9. Consider first the assumption

A.8. *Decreasing returns to scale and diffusion process.* The production functions take the form

$$y_{j,t+dt}^2(e_{t+dt}) = y_{jt}^1(e_t) + \mu_j dt + \sigma_j dw(t) \quad j = 1, 2, \dots, n, \quad (29)$$

where  $\mu_j \equiv \mu_j[y_{jt}^1(e_t)]$  is increasing and concave;  $\sigma_j \equiv \sigma_j[y_{jt}^1(e_t)]$  is an  $n$ -dimensional vector, and each element of the vector is increasing and concave in  $y_{jt}^1(e_t)$ ; and  $dw(t)$  is the increment of the Wiener process  $w(t)$  in  $R^n$ .

The earlier discussion was couched in a discrete time framework. In a rigorous development the terms need to be redefined and the lemmata reproven in a continuous time framework. Instead we provide a heuristic argument which leads to the asset-pricing model. Under assumption A8, Ito's lemma implies that the returns on the firms' stock are also generated by a diffusion process. Over an interval  $(t, t + dt)$  the returns are multivariate normal and therefore satisfy A7. Repeating the argument which leads to proposition 1 we derive the asset-pricing model without assumption A6. Furthermore, by Ito's lemma the financial assets' returns are generated by a diffusion, and it is easy to show that they also satisfy the asset pricing model.

If neither assumptions A6 and A7 nor assumption A8 hold, the CAPM may be derived under the assumption that consumers have time-additive quadratic utility. Specifically we assume

A9. *Quadratic utility.* Consumers have time-additive and quadratic utility.<sup>8</sup>

It easily follows that the composite consumer's utility is time additive and quadratic. Without any assumptions on the distribution of returns of firms' stock and financial assets, the CAPM holds for all assets. These results are stated as

PROPOSITION 2: Consider the equilibrium in the Arrow-Debreu economy defined in Section II. Assume that there is only one good and that assumptions A1–A5 hold. Also assume that either of assumptions A8 or A9 holds. Then the asset-pricing model of equations (21)–(24) holds for the returns of active firms and all financial assets.

## References

- Bhattacharya, S. 1981. Notes on multiperiod valuation and the pricing of options. *Journal of Finance* 36 (March): 163–80.
- Black, F. 1972. Capital market equilibrium with restricted borrowing. *Journal of Business* 45 (July): 444–54.
- Breeden, D. T. 1979. An intertemporal asset pricing model with stochastic investment opportunities. *Journal of Financial Economics* 7 (September): 265–96.
- Breeden, D. T., and Litzenberger, R. H. 1978. Prices of state-contingent claims implicit in options prices. *Journal of Business* 51 (October): 621–51.
- Brock, W. A. 1978. An integration of stochastic growth theory and the theory of finance. Mimeographed. Chicago: University of Chicago, Department of Economics.
- Chamberlain, G. 1980. A complete characterization of distributions allowing mean-variance portfolio analysis. Mimeographed. Madison: University of Wisconsin—Madison.
- Constantinides, G. M. 1979. Multiperiod consumption and investment behavior with convex transactions costs. *Management Science* 25 (November): 1127–37.
- Constantinides, G. M. 1980. Admissible uncertainty in the intertemporal asset pricing model. *Journal of Financial Economics* 8 (March): 71–86.
- Cox, J.; Ingersoll, J.; and Ross, S. 1978. A rational anticipations intertemporal asset pricing theory. *Econometrica*, in press.
- Debreu, G. 1959. *Theory of Value*. New Haven, Conn.: Yale University Press.
- Dybvig, P. H., and Ingersoll J. E. Mean-variance theory in complete markets. In this issue.
- Dybvig, P. H., and Ross, S. A. 1982. Portfolio efficient sets. *Econometrica* (in press).
- Fama, E. F. 1970. Multiperiod consumption-investment decisions. *American Economic Review* 60 (March): 163–74.
- Fama, E. F. 1971. Risk, return, and equilibrium. *Journal of Political Economy* 79 (January/February): 30–55.
- Lintner, J. 1965. The valuation of risky assets and the selection of risky investments in stock portfolios and capital budgets. *Review of Economics and Statistics* 47 (February): 13–37.
- Long, J. B. 1974. Stock prices, inflation and the term structure of interest rates. *Journal of Financial Economics* 1 (July): 131–70.

8. Quadratic utility  $u_a(x) = x - ax^2/2$ ,  $a > 0$ , implies positive marginal utility only if  $x \leq a^{-1}$ . We assume that constraint  $x \leq a^{-1}$  is nonbinding for all consumers. The need for this assumption is one of the objectionable characteristics of quadratic utility. The derivation of the CAPM under quadratic utility is discussed here solely for completeness.

- Lucas, R. E., Jr. 1978. Asset prices in an exchange economy. *Econometrica* 46 (November): 1426-46.
- Merton, R. C. 1973. An intertemporal capital asset pricing model. *Econometrica* 41 (September): 867-87.
- Mossin, J. 1966. Equilibrium in a capital asset market. *Econometrica* 34 (October): 768-83.
- Prescott, E. C., and Mehra, R. 1980. Recursive competitive equilibrium: the case of homogeneous households. *Econometrica* 48 (September): 1365-79.
- Press, S. J. 1972. *Applied Multivariate Analysis*. New York: Holt, Rinehart & Winston.
- Ross, S. A. 1976. The arbitrage theory of capital asset pricing. *Journal of Economic Theory* 13 (December): 341-60.
- Ross, S. A. 1978. Mutual fund separation in financial theory: the separating distributions. *Journal of Economic Theory* 17 (April): 254-86.
- Rubinstein, M. 1974. An aggregation theorem for securities markets. *Journal of Financial Economics* 1 (Autumn): 225-44.
- Rubinstein, M. 1976. The valuation of uncertain income streams and the pricing of options. *Bell Journal of Economics* 7 (Autumn): 407-25.
- Sharpe, W. F. 1964. Capital asset prices: a theory of market equilibrium under conditions of risk. *Journal of Finance* 19 (September): 425-42.
- Stapleton, R. C., and Subrahmanyam, M. G. 1978. A multiperiod equilibrium asset pricing model. *Econometrica* 46 (September): 1077-96.

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