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On Market Timing and Investment Performance.

I. An Equilibrium Theory of Value for Market Forecasts

I. Introduction

The evaluation of the performance of investment managers is a much-studied problem in finance. The extensive study of this problem could be justified solely on the basis of the manifest function of these evaluations, which is to aid in the efficient allocation of investment funds among managers. However, an equally important latent function of these evaluations is to provide a method of testing the Efficient Markets Hypothesis.¹ If market participants are rational, a necessary condition for superior performance is superior forecasting skills on the part of the performer. Hence, these evaluations can help resolve whether or not the existence of different information among market participants plays an empirically significant role in the formation of equilibrium security prices.

One of the principal applications of modern capital market theory has been to provide a structural specification within which to measure investment performance and thereby to identify

An equilibrium theory for the value of market-timing skills is derived for the case where there are only two possible predictions: either stocks are predicted to outperform bonds or bonds are predicted to outperform stocks. It is shown that the pattern of returns from successful market timing has an isomorphic correspondence to the pattern of returns from following certain option investment strategies. This correspondence is used to derive a unique equilibrium price structure of management fees which are independent of investors' preferences, endowments, or prior probability assessments for stock returns. Sufficient statistics for evaluating market-timing skills are derived, and it is shown that the number of correct forecasts is not a useful statistic for this purpose.

* I thank F. Black and J. Cox for many helpful discussions and R. Henriksson for scientific assistance. Aid from the National Science Foundation is gratefully acknowledged.

1. Because those who believe that they have superior forecasting skills are reluctant to make public the techniques they use, the evaluation of investment performance is essential for "unbiased" testing on the Efficient Market Hypothesis. Fama (1970) provides an excellent discussion of both the Efficient Markets theory and the various attempts to test it.

superior performers if they exist. In this structure, it is usually assumed² that forecasting skills can be partitioned into two distinct components: (1) forecasts of price movements of selected individual stocks (i.e., "microforecasting"); and (2) forecasts of price movements of the general stock market as a whole (i.e., "macroforecasting"). Usually associated with security analysis, microforecasting involves the identification of individual stocks which are under- or overvalued relative to equities generally. In the context of the well-known Capital Asset Pricing Model specification,³ a microforecaster attempts to identify individual stocks whose expected returns lie significantly above or below the Security Market Line. In his strictest form, the microforecaster would only forecast the nonsystematic or "non-market-explained" component of the return on individual stocks. For example, suppose that the random variable return per dollar on security j between time t and $t + 1$, $Z_{jt}(t)$, can be described by

$$Z_{jt}(t) = R(t) + \beta_j[Z_M(t) - R(t)] + \epsilon_j(t), \quad (1)$$

where $Z_M(t)$ is the return per dollar on the market, $R(t)$ is the return per dollar on the riskless asset, and $\epsilon_j(t)$ has the property that its expectation conditional on knowing the outcome for $Z_M(t)$ is equal to its unconditional expectation, that is, $E[\epsilon_j(t)|Z_M(t)] = E[\epsilon_j(t)]$. Then a strict microforecast about security j would be a forecast only about the statistical properties of $\epsilon_j(t)$.

Macroforecasting, or "market timing," attempts to identify when equities in general are under- or overvalued relative to the fixed-income securities. A macroforecaster or "market timer," tries to forecast when stocks will outperform bonds—that is, $Z_M(t) > R(t)$ —and when bonds will outperform stocks—that is, $Z_M(t) < R(t)$. Since a macroforecaster only forecasts the statistical properties of $Z_M(t)$ or $Z_M(t) - R(t)$, his forecasts can only be used to predict differential performance among individual stocks arising from the systematic, or "market-explained," components of their returns, $\{\beta_j[Z_M(t) - R(t)] + R(t)\}$. Of course, both micro- and macroforecasting can be employed to forecast both components of the returns on individual stocks.

Most of the theoretical and empirical research has focused on the performance of portfolio managers who are assumed to be microforecasters (i.e., individual stock pickers).⁴ However, there are notable exceptions. Using both micro- and macroforecasting, Fama

2. See Fama (1972).

3. The "Capital Asset Pricing Model" refers to the equilibrium relationships among security prices which obtain when investors choose their portfolios according to a mean-variance criterion function and have homogeneous beliefs. See Sharpe (1964), Lintner (1965), and Mossin (1966) for the original derivations. For a comprehensive review of the model, see Jensen (1972*b*).

4. See Treynor and Black (1973), Treynor (1965), Sharpe (1966), and Jensen (1968, 1969). Jensen (1972*b*) provides a summary of the empirical studies.

(1972) and Jensen (1972a) examine in detail the theoretical structure of portfolio performance patterns for managers. Jensen also points out the difficulties in identifying empirically the individual contributions of each to overall performance. Grant (1977) derives the effects of market-timing activities on the performance measures employed in the empirical studies which assume microforecasting skills only. Treynor and Mazuy (1966) look for evidence of macroforecasting skills by analyzing return data on mutual funds. Sharpe (1975) simulates the patterns of returns on a portfolio managed by a hypothetically successful market timer. All these studies have in common the fact that they assume a Capital Asset Pricing Model framework.⁵

A fundamental difference between the model presented here and these earlier models is that I assume that the market timer's forecasts take a very simple form, namely, the market timer forecasts either that stocks will earn a higher return than bonds or that bonds will earn a higher return than stocks. By comparison with the Jensen (1972a) formulation, my model may appear to be less sophisticated because the market timer in the model developed here does not forecast how much better the superior investment will perform. However, when this simple forecast information is combined with a prior distribution for returns on the market, a posterior distribution is derived which does permit probability statements about how much better the superior investment will perform. By formulating the problem in this way, I am able to study market timing without assuming a Capital Asset Pricing Model framework, and therefore, for the most part, my conclusions will be robust with respect to assumptions about the distribution of returns.

My study of market timing is broken into two parts. In the first part, which is presented here, I develop the basic model and analyze the theoretical structure of the pattern of returns from market timing. From this analysis, I derive an equilibrium theory of value for market-timing forecasting skills. In the second part, to be presented in a subsequent paper, I use the structure derived here to develop both parametric and nonparametric statistical procedures to test for superior forecasting skills. These tests of investment performance will distinguish market timing from individual stock selection skills.

The principal findings of the first part can be summarized as follows: The pattern of returns from successful market timing will be shown to have an isomorphic correspondence to the pattern of returns from following certain option investment strategies where the ("implicit") prices paid for options are less than their "fair" or market values. That is, the return patterns from successful market timing will be virtually indistinguishable from the return patterns of these option strategies if

5. Specifically, they assume that stock returns are normally distributed and that relative to the "public" information set, securities are priced so as to satisfy the Security Market Line.

the managers of the latter can identify and purchase "undervalued" options.

Using this isomorphic correspondence, we can determine the value of market-timing skills. In making this determination, I begin by analyzing how investors would use the market timer's forecast to modify their probability beliefs about stock returns and, from this analysis, derive both necessary and sufficient conditions for such forecasts to have a positive value. It is shown that the probability of a correct forecast by the market timer is neither a necessary nor a sufficient statistic for determining whether or not his forecasts have positive value. Rather, it is the probability of a correct forecast, conditional upon the return on the market, which serves as such a sufficient statistic.

Under the assumption that the aggregate size of transactions based on the market timer's forecast is small enough to have a negligible effect on market prices, a unique equilibrium price structure of management fees is derived. The derived management fees depend only on the conditional probabilities of a correct forecast and the market prices of options, and therefore, they are independent of investors' preferences, endowments, or prior probability assessments for stock returns. Previous research into the determination of management fees along these lines can be found in Goldman, Sosin, and Gatto (1978) and Goldman, Sosin, and Shepp (1978). However, unlike these earlier analyses, I derive the equilibrium management fees for imperfect market-timing forecasts and obtain my results with no specific assumptions about either the distribution of returns on the market or the way in which option prices are determined.

The paper proceeds as follows: In Section II, I develop the model and derive the equilibrium management fee for the special case where the information set available to all investors (other than the forecaster) is the same and the market timer is a perfect forecaster. Although these assumptions are empirically unrealistic, the analysis of this case provides a useful "benchmark" for interpreting the results derived in the general case. Indeed, the comparative statics propositions proved in this section will apply to the general case. In Section III, the model is developed for the general case where investors may have different information sets, and therefore heterogeneous beliefs about the distribution of returns on the market, and the market timer is an imperfect forecaster. In Section IV, the equilibrium structure of management fees for market timing is derived for this general case.

II. Homogeneous Beliefs and Perfect Forecasting

In this section, I develop the model for the case where there are many investors with homogeneous probability beliefs and a single investor who is a perfect market timer.

At each point in time t , $t = 0, 1, 2, \dots$, let $\phi(t)$ denote the common ("public") information set which is available to all investors. Based upon $\phi(t)$, investors form a probability assessment for $Z_M(t)$, the return per dollar on the market between t and $t + 1$. Because the public information available to all investors is the same, it is reasonable to assume that the probability density function derived by each investor is the same and let $f(Z, t)$ denote this common density function. It is further assumed that this agreed-upon density function satisfies Rational Expectations in the sense that relative to information set $\phi(t)$, the (ex post) path of market returns is consistent with the (ex ante) probability beliefs for that path as expressed by f . There is a default-free bond whose return per dollar between t and $t + 1$ is known with certainty as of date t , and we denote the return per dollar on this security by $R(t)$.

In addition to the public information set $\phi(t)$, one investor has information which he uses to make a market-timing forecast. The nature of this information, when fully exploited, is such that this investor can forecast when stocks will outperform bonds, $Z_M(t) > R(t)$, and when bonds will perform at least as well as stocks, $Z_M(t) \leq R(t)$. In this section, it is assumed that his forecast is always correct. Although this forecast is clearly valuable, its direct sale to other investors is difficult because its dissemination cannot be controlled. Hence, to exploit his information edge, it is assumed that the forecaster forms an investment company or mutual fund which he manages in return for fees. The investment policy of the fund is to invest in the market and bonds according to his forecast. As is the case for "real world" investment companies, investors in this fund do not know in advance either the manager's forecast or the positions he has taken. Of course, ex post, they will know both, and whether or not the forecast was accurate, by simply observing the asset returns. It is further assumed that the total assets invested in the fund are small enough so that purchases and sales by the fund have a negligible effect on market prices.

The usual "frictionless" market assumptions, including no taxes and transactions costs, are made throughout the paper. However, for the analysis in this section only, neither borrowing (to buy securities) nor selling short is permitted. Because of the perfect forecast assumption, in the absence of such restrictions the manager would exhibit infinite demands for securities, which is, of course, inconsistent with the price-taker assumption. Since this derived behavior is caused by the crude assumption of perfect forecasting along with the willingness of others to extend unlimited credit, nothing in terms of insights into the real world is lost by imposing these restrictions here. Indeed, most investment company charters prohibit borrowing or selling short, and all limit to some degree the extent to which either can be carried out. Hence, given these restrictions, the manager will follow the strategy of

investing all the assets in whichever security dominates the other on the basis of his forecast.⁶

I formally represent the forecast at time t by the variable $\gamma(t)$, where $\gamma(t) = 1$ if the forecast is that $Z_M(t) > R(t)$ and $\gamma(t) = 0$ if the forecast is that $Z_M(t) \leq R(t)$. Because $f(Z, t)$ is a rational-expectations density function relative to information set $\phi(t)$, the probability density for the return on the market conditional on the forecast $\gamma(t) = \gamma$, $f^*(\gamma)$, can be written as

$$\begin{aligned} f^*(Z, t | \gamma) &= (1 - \gamma)f(Z, t)/q(t), & 0 \leq Z \leq R(t) \\ &= \gamma f(Z, t)/[1 - q(t)], & R(t) < Z < \infty, \end{aligned} \quad (2)$$

where

$$q(t) \equiv \int_0^{R(t)} f(Z, t) dZ$$

is the probability that the return on the market is less than the return on bonds based upon the public information set $\phi(t)$. Of course, a necessary condition for equilibrium is that $q(t) > 0$. Although the analysis is not Bayesian, I borrow its terminology and refer to $f(Z, t)$ as the prior distribution for the market return and $f^*(Z, t | \gamma)$ as the posterior distribution.

The outside investors in the fund do not know what the forecast will be. Hence, from their perspective, $\gamma(t)$ is a random variable with $\text{prob}\{\gamma(t) = 0\} = q(t)$. The expected value of $\gamma(t)$, $E\{\gamma(t)\}$, is equal to $1 - q(t)$, the unconditional probability that the return on the market will exceed the return on bonds. If $A(t)$ equals the total value of investment in securities by the fund at time t and $F(t)$ equals the total fees paid at the beginning of period t for managing the fund between t and $t + 1$, then the total (gross) dollar amount invested in the fund at time t , $I(t)$, satisfies $I(t) = A(t) + F(t)$.

Although the outside investors do not know what the forecast will be, they do know that the forecast will be correct and that the manager will invest all the fund's assets in the market if $\gamma(t) = 1$ and in bonds if $\gamma(t) = 0$. Therefore, the random variable end-of-period value of the assets, $V(t + 1)$, as viewed by an outside investor can be written as

$$\begin{aligned} V(t + 1) &= \max [A(t)R(t), A(t)Z_M(t)] \\ &= A(t)R(t) + \max [0, A(t)Z_M(t) - A(t)R(t)] \\ &= A(t)R(t) + A(t) \max [0, Z_M(t) - R(t)], \end{aligned} \quad (3)$$

where the distribution function for $Z_M(t)$ is given by $f(Z, t)$.

6. The term "dominates" is used here as it is defined in Merton (1973a, p. 143), namely, security A *dominates* security B if the return on A will exceed the return on B for some possible states of the world and will be at least as large as the return on B in all possible states of the world. Clearly, it is never optimal to invest positive amounts in a dominated security.

For expositional convenience,⁷ suppose that one "unit" (or "share") in the market portfolio is defined such that the current cost of acquiring one share equals \$1. Then, from (3), the end-of-period value of the fund will be identically equal to the end-of-period value of a portfolio which follows the investment strategy of holding $A(t)$ in bonds and one-period call options on $A(t)$ shares of the market portfolio with exercise price per share of $R(t)$.⁸ Such an investment strategy is a specific example of a class called "options-bills" or "options-paper" strategies.⁹

In the absence of management fees, the investment returns from this type of perfect market timing are identical to those that would be earned from the above options-bills strategy (which uses no market-timing information) *if* the call options could be purchased at a *zero* price. To see this, note that in the absence of management fees, the initial investment required in the market-timing strategy to generate the end-of-period dollar return in (3) is $I(t) = A(t)$. To generate the same end-of-period dollar return using the options-bills strategy requires an initial investment of $A(t)$ in bonds *plus* the cost of acquiring the call options on the market portfolio. Since the dollar return (3) can be achieved using the market-timing strategy with an initial investment of only $A(t)$, the economic value of this type of perfect market timing (per dollar of assets held by the fund) is equal to $c(t)$, where $c(t)$ is the market price of a one-period call option on one share of the market portfolio with an exercise price of $R(t)$.

We can rewrite (3) as

$$\begin{aligned} V(t+1) &= A(t)Z_M(t) + \max [0, A(t)R(t) - A(t)Z_M(t)] \\ &= A(t)Z_M(t) + A(t) \max [0, R(t) - Z_M(t)]. \end{aligned} \quad (4)$$

Inspection of (4) shows that the dollar return on the assets of the fund is also identical to the dollar return on a portfolio which follows the investment strategy of holding $A(t)$ shares (at \$1 per share) of the market portfolio and a one-period put option on $A(t)$ shares of the market portfolio with an exercise price per share of $R(t)$.¹⁰ Such an investment

7. Therefore, the "end-of-period" value of one share will be $Z_M(t)$. At that time, the stock is "split" (or "reverse" split) $Z_M(t)$ shares-for-one so that the purchase price of one share is always \$1. Of course, $Z_M(t)$ includes both dividends and capital gains and no distinction is made between the two throughout the paper.

8. An "American" call option gives its owner the right to purchase a specified number of shares at the exercise price on or before the expiration date. If it is not exercised by that time, it becomes worthless. The call option in the text is "European" and therefore can only be exercised on the expiration date. Moreover, it is also "payout protected" (see Merton 1973a, p. 151) and therefore its value is not affected by the composition of the market's return between dividends and capital gains.

9. For a description and analysis of these strategies, see Merton, Scoles, and Gladstein (1978).

10. An "American" put option gives its owner the right to sell a specified number of shares at the exercise price on or before the expiration date. As for the call option discussed in n. 8, the put option in the text is European and payout protected.

strategy is an example of a "protective put" or "insured equity" strategy.¹¹

As with the options-bills strategy, to achieve the end-of-period dollar return (4) using the protective put strategy (with no market-timing information) requires an initial investment of $SA(t)$ in the market portfolio plus $SA(t)g(t)$ in put options where $g(t)$ is the market price of a one-period put option on one share with an exercise price of $R(t)$. Hence, the economic value of this type of perfect market timing (per dollar of assets held by the fund) is also equal to $g(t)$. However, there is no inconsistency in using either the call or put isomorphic relationship to determine the economic value of the information because $c(t) = g(t)$ by the well-known Parity Theorem.¹² (Note that the market prices of the options are the equilibrium prices based upon the public information $\phi[t]$.)¹³

As has been discussed elsewhere,¹⁴ the put option as an instrument is very much analogous to a term insurance policy where the item insured is the value of the underlying stock and the maximum coverage (or "face value") of the policy is the exercise price. Through the derived correspondence of the market-timing strategy with the protective put strategy, we therefore have that the principal benefit of market timing is to provide insurance. The investor who follows the uninformed strategy of always holding the market suffers the capital losses as well as enjoying the gains from market movements. However, the successful market timer earns the gains but is "insured" against the losses.

From (3) or (4), the return per dollar on the fund's assets, $X(t) [= V(t+1)/A(t)]$, can be written as

$$\begin{aligned} X(t) &= \max \{Z_M(t), R(t)\} \\ &= R(t) + \max \{0, Z_M(t) - R(t)\} \\ &= Z_M(t) + \max \{0, R(t) - Z_M(t)\}. \end{aligned} \quad (5)$$

From (3) and (5), the return per dollar to the investors in the fund can be written as

$$V(t+1)/I(t) = X(t)[1 + m(t)], \quad (6)$$

11. For a description and analysis of these strategies, see Merton, Scholes, and Gladstein (in press).

12. If borrowing and lending interest rates are equal, then for payout-protected European options on the same stock with the same exercise price and expiration dates, $g(t) = c(t) - S + E/R(t)$ where S is the current stock price and E is the exercise price. In the case in the text, $S = 1$ and $E = R(t)$, and so, $g(t) = c(t)$. For a proof and discussion of the Parity Theorem, see Merton (1973a, p. 157, theorem 12; 1973b).

13. As was assumed for stocks and bonds in the text, I assume that the market timer's actions have no effect on option prices. Of course, based upon his information set (with no borrowing or selling short), the call option would be worthless if $\gamma(t) = 0$ and the put option would be worthless if $\gamma(t) = 1$.

14. See Merton et al. (1978, pp. 185-88).

where $m(t) \equiv F(t)/A(t)$ is the management fee expressed as a fraction of assets held by the fund. If, as is reasonable to assume, investors behave competitively, then the equilibrium management fee will be such that the market timer extracts all the economic benefit from his differential information. As was shown, the economic value of his forecast per dollar of invested assets is equal to $g(t)$. Therefore, the equilibrium management fee as a fraction of invested assets will satisfy

$$\begin{aligned} m(t) &= g(t) \\ &= c(t). \end{aligned} \quad (7)$$

Expressed as a percentage of gross investment, the fee is given by

$$\begin{aligned} F(t)/I(t) &= m(t)/[1 + m(t)] \\ &= g(t)/[1 + g(t)]. \end{aligned} \quad (8)$$

Another way to see that (7) is the correct equilibrium management fee is as follows: In equilibrium, the cost to the investor for receiving a specified pattern of returns should be the same independent of the manner in which that pattern is created. From the viewpoint of the investor, the fund is a "black box" in the sense that all he can observe is the pattern of returns generated by the fund. He does not observe the forecasts or the resulting transactions made by the manager to create that pattern. So, for example, a manager with no forecasting skills could guarantee the dollar return $V(t+1)$ given in (4) by simply investing $\$A(t)$ in the market and $\$A(t)g(t)$ in put options on the market with exercise price per share of $R(t)$. Viewed as a black box, the dollar returns to this manager's fund would be identical to the dollar returns from the market timer's fund. Of course, the cost to the former of achieving this dollar return would be $A(t) + A(t)g(t) = [1 + g(t)]A(t)$. Equating this cost to the cost of the latter, we have that $[1 + g(t)]A(t) = [1 + m(t)]A(t)$ or $m(t) = g(t)$. At this fee, the investor would be just indifferent between investing in either of the two funds. Since the investor can always choose to follow this protective put strategy, I have also shown that the management fee given in (7) is the *maximum* fee which the market timer can receive.

If the environment were such that investors in the fund had some market power, then they could bargain away from the manager some of the economic value of the forecast, and the management fee would be less than $g(t)$. In this case, the returns earned by investors in the fund would be identical to those earned by following a protective put strategy using no market timing where the investors can identify and purchase "undervalued" puts. Specifically, if λ is defined such that $m(t) \equiv \lambda g(t)$ ($0 < \lambda \leq 1$), then the returns earned by the investors would be "as if" they could purchase puts at $100(1 - \lambda)\%$ less than their "fair" value.

From (7), one could in principle use observed put- and call-option prices to estimate the value of macroforecasting skill. Moreover, this correspondence permits the direct application of the extensive results derived in option pricing theory¹⁵ to a comparative statics analysis of equilibrium management fees. From such an analysis, the following propositions can be established.

PROPOSITION II.1: The management fee is a nondecreasing function of the riskiness (volatility) of the market portfolio.

In Merton (1973a, p. 149, theorem 8), it is shown that a call option's price is a nondecreasing function of the riskiness of its underlying stock where the definition of "more risky" is that of Rothschild and Stiglitz (1970).¹⁶ The proof of the proposition follows directly from (7) because $m(t) = c(t)$. Thus, the more uncertain investors are about the outcomes for the market, the more valuable is the market timer's forecast.

In the analysis, the forecasts are made at discrete points in time ($t = 0, 1, 2, \dots$). Let τ denote the actual length of time between forecasts in some fixed time unit such as "one day." So, for example, if the market timer's forecast is for a month, then $\tau = 30$ (days). In his forecast is for a quarter, then $\tau = 91$ (days). The percentage management fee per forecast period is $m(t)$.

PROPOSITION II.2: The management fee (per forecast period) is a nondecreasing function of the length of the forecast period, that is, $\partial m / \partial \tau \geq 0$.

From (7), $m(t)$ is equal to the value of a call option with length of time until expiration of τ . In Merton (1973a, p. 143, eq. [4]), it is shown that a call option is a nondecreasing function of its length of time until expiration. Proposition II.2 follows directly. Hence, the management fee for 1 month for a manager making 1-month forecasts is at least as large as the management fee for 1 week for a manager making 1-week forecasts. However, this does not imply that the services provided by the manager who makes 1-month forecasts are more valuable than the services provided by the manager who makes 1-week forecasts. Indeed, as the proof of the following proposition demonstrates, quite the opposite is true.

PROPOSITION II.3: If the riskless bond interest rate is nonstochastic over time, then the management fee per fixed unit of time (i.e., the management fee rate) is a nonincreasing function of the length of the forecast period.

15. For a comprehensive survey article on option pricing theory, see Smith (1976).

16. See Rothschild and Stiglitz (1970). Their definition of "more risky" refers to the "total" risk of a security and should not be confused with the "systematic" or "portfolio" risk of a security. See Merton (in press) for further discussion.

Suppose that a market timer will make T forecasts during a fixed time interval τ . Then, the return per dollar of assets invested in his fund for this period, X_1 , can be written as

$$X_1 \equiv \prod_{t=1}^T \max [Z_M(t), R(t)] = \left[\prod_{t=1}^T R(t) \right] \left[\prod_{t=1}^T \max [Q(t), 1] \right],$$

where $Q(t) \equiv Z_M(t)/R(t)$. Consider a second market timer whose forecast interval equals τ and who therefore makes only one forecast for this period. Then, the return per dollar of assets invested in his fund for this period, X_2 , can be written as $X_2 \equiv \max [Z_M, R]$, where Z_M is the return per dollar on the market over the interval τ and R is the return on a riskless security over the interval τ . Clearly,

$$Z_M = \prod_{t=1}^T Z_M(t).$$

By hypothesis, interest rates are nonstochastic, and therefore, to avoid arbitrage,

$$R = \prod_{t=1}^T R(t).$$

Hence, X_2 can be rewritten as

$$X_2 = \max \left[\prod_{t=1}^T Z_M(t), \prod_{t=1}^T R(t) \right] = \left[\prod_{t=1}^T R(t) \right] \max \left[\prod_{t=1}^T Q(t), 1 \right].$$

If either $Z_M(t) \geq R(t)$ for $t = 1, \dots, T$ or $Z_M(t) \leq R(t)$ for $t = 1, \dots, T$, then

$$\max \left[\prod_{t=1}^T Q(t), 1 \right] = \prod_{t=1}^T \max [Q(t), 1].$$

Otherwise,

$$\max \left[\prod_{t=1}^T Q(t), 1 \right] < \prod_{t=1}^T \max [Q(t), 1].$$

Hence, for all possible outcomes, $X_1 \geq X_2$, and for some possible outcomes, $X_1 > X_2$. Therefore, all investors would be willing to pay at least as large a (and generally, larger) management fee (per time interval τ) for the first fund as they would for the second fund.

The most widely used formula for evaluating options was developed in a seminal paper by Black and Scholes (1973). Under the twin assumptions that trading takes place continuously and that the underlying stock returns follow a continuous sample path diffusion process with a

constant variance rate, they use an arbitrage argument to deduce a formula for a call option given by¹⁷

$$f(S, T, E) = S\Phi(x_1) - Ee^{-rT}\Phi(x_2), \quad (9)$$

where $x_1 = [\log(S/E) + (r + \frac{1}{2}\sigma^2)T]/\sigma\sqrt{T}$, $x_2 = x_1 - \sigma\sqrt{T}$, $\Phi(\cdot)$ is the cumulative normal density function, S is the current stock price, T is the length of time until expiration, E is the exercise price per share, r is the instantaneous rate of interest, and σ^2 is the instantaneous variance per unit time of the rate of return on the stock.

If the conditions required for the valid application of the Black-Scholes formula are satisfied, then from (7) we can use formula (9) to evaluate the equilibrium management fee. For the purposes of this paper, $S = 1$, $E = R(t)$, $T = \tau$, and $\exp[-rT] = 1/R(t)$. Hence, under the conditions that trading takes place continuously in time and that the rate of return on the market portfolio follows a diffusion process with a variance rate which is constant over (at least) the forecast interval, the equilibrium management fee can be written as

$$m(t) = 2\Phi[\frac{1}{2}\sigma(t)\sqrt{\tau}] - 1, \quad (10)$$

where $\sigma^2(t)$ is the constant-over-the-forecast-interval variance rate at the time of the t th forecast and τ , as before, is the length of the forecast interval.

PROPOSITION 11.4: If the conditions for the Black-Scholes option formula are satisfied, then the percentage management (per forecast interval) is: (1) an increasing function of the forecast interval; (2) an increasing function of the variance rate on the market; and (3) independent of the level of the interest rate (i.e., the return per dollar on the riskless asset).

The proof follows immediately by differentiating $m(t)$ in (10) with respect to each of these parameters: $\partial m/\partial \tau = \sigma(t) \exp[-\sigma^2(t)\tau/8]/\sqrt{8\pi\tau} > 0$ and $\partial m/\partial \sigma(t) = \sqrt{\tau} \exp[-\sigma^2(t)\tau/8]/\sqrt{2\pi} > 0$, where $\partial m/\partial R(t) = 0$. Part 1 of proposition 11.4 is obviously a special case of proposition 11.2. Indeed, as was shown in Merton (1973a, p. 180, app. 2), part 2 of proposition 11.4 is just a special case of proposition 11.1. Part 3 of proposition 11.4 conflicts with the usual comparative static result that the price of a call option is an increasing function of the interest rate.¹⁸ The reason for the difference is that the exercise price of the call option corresponding to the management fee is an increasing function of the interest rate, whereas in the usual comparative static analysis the exercise price is held fixed. Because the price of a call option is a decreasing function of its exercise price, the positive relationship between the exercise price and the interest rate brings in an

17. This formula applies for a payout-protected European call option which is exactly appropriate for the purposes of the text.

18. See Merton (1973a, pp. 145-46, and theorem 14, p. 167).

offsetting negative effect on value in this case. While I have only shown that the management fee is independent of the interest rate in the special case when Black-Scholes applies, I conjecture that this result will obtain under more general conditions. The basis for this conjecture is that, unlike the call option, the price of a put option is usually believed to be a decreasing function of the interest rate. However, as I have shown, the management fee can also be expressed in terms of a put option's price. Since, whether expressed as a put or a call, the effect of a change in the interest rate on the management fee must be the same, I am led to the belief that it will be independent of the interest rate.

Formula (10) can also be used for the management fee to provide a quantitative example of proposition II.3. Suppose the variance rate for the market returns is a constant σ^2 . Then, from (10), the percentage management fee (per forecast interval) will be the same each period. Let m_n denote the percentage management fee for a fixed time period T where n is the number of forecast periods contained in that fixed time period. That is, the forecast interval τ equals T/n . Because in this case the management fee per forecast period is the same each period, we have from (10) that

$$\begin{aligned} m_n &= [1 + m(t)]^n - 1 \\ &= 2^n \Phi^n(1/2 \sigma \sqrt{T/n}) - 1. \end{aligned} \quad (11)$$

One can verify by differentiation of (11) that $\partial m_n / \partial n > 0$, and, indeed, in the limit of continuous perfect forecasting (i.e., as $n \rightarrow \infty$), $m_n \rightarrow \infty$. That is, from (8), the percentage management for a fixed time period approaches 100% of gross investment.¹⁹ Using (10) and (11), we complete the comparative statics analysis with tables 1 and 2 where the magnitudes of the equilibrium management fees are presented for typical forecast intervals and market volatilities.

To further develop the return characteristics associated with market timing, I shall return to my earlier analysis and compare the pattern of returns from the perfect market-timing strategy with the patterns generated by the "no-information" option strategies used to derive the equilibrium management fee. If the return per dollar from a protective put strategy with no forecasting information is denoted by $Y(t)$, then

$$Y(t) = X(t)/[1 + g(t)], \quad (12)$$

where $X(t)$ is the return per dollar from perfect market timing given in (5). The difference in the expected return because the two strategies can be written as

$$E[X(t)] - E[Y(t)] = g(t)E[Y(t)]. \quad (13)$$

19. This limiting management fee of 100% is caused by the Brownian motion assumption used by Black and Scholes, and it may not obtain for other processes.

TABLE 1 Management Fee per Forecast Interval as a Percentage of Gross Investment

Annual Standard Deviation of the Market Return	Length of the Forecast Interval				
	1 Week	1 Month	3 Months	6 Months	1 Year
10%	.50%	1.13%	1.96%	2.75%	3.62%
15%	.81%	1.70%	2.91%	4.05%	5.64%
20%	1.10%	2.26%	3.83%	5.34%	7.37%
25%	1.37%	2.81%	4.74%	6.59%	8.89%
30%	1.63%	3.34%	5.64%	7.79%	10.65%

TABLE 2 Annual Management Fee Rate as a Percentage of Gross Investment

Annual Standard Deviation of the Market Return	$\frac{R_1}{1 + R_1} \times 100$ Length of the Forecast Interval				
	1 Week	1 Month	3 Months	6 Months	1 Year
10%	22.8%	16.4%	7.6%	5.4%	3.6%
15%	34.6%	18.6%	11.2%	7.9%	5.6%
20%	43.7%	24.0%	14.5%	10.4%	7.4%
25%	51.2%	29.0%	17.7%	12.7%	8.9%
30%	57.5%	33.5%	20.7%	15.0%	10.7%

For even relatively long time intervals, $E[Y(t)] \approx 1$ and $g(t) \ll 1$. For example, if the forecast interval is 1 year and if the expected annual rate of return on the protective put strategy is 15%, then $E[Y(t)] = 1.15$. Using the Black-Scholes formula with a standard deviation for the market of 20%, the 1-year value for $g(t)$ would be .08. If the forecast interval were 1 month, then the corresponding values for $E[Y(t)]$ and $g(t)$ would be 1.012 and .024, respectively. Hence, from (13), the differences in the expected rates of return will be quite significant. In the examples given, the perfect forecasting strategy for an annual forecast will have an expected rate of return of 24%, which is a 60% increase over the 15% for the protective put strategy. For a monthly forecast, the monthly expected rate of return will be 2.4%, which is twice the 1.2% return expected for the protective put.

While the significantly higher expected return from the perfect timing strategy is probably no surprise, the relative characteristics of the higher moments of the two return distributions may be. If $M_n(Z)$ denotes the n th central moment of the random variable Z , then, from (12),

$$M_n[Y(t)] = M_n[X(t)]/[1 + g(t)]^n. \quad (14)$$

Thus, all the higher moments of the perfect timing strategy returns are larger than the corresponding moments of the protective put strategy. Hence, while one might have believed that the perfect market-timing information would result in a significantly smaller standard deviation of return than the "no-information" protective put strategy, it does not, and, in fact, its standard deviation is higher.²⁰ However, to a close approximation, the standard deviations of the two strategies will be the same. For example, if the monthly standard deviation of the protective put strategy were 4%, then the monthly standard deviation of the perfect timing strategy would be 4.1%. Indeed, to a reasonable approximation, $M_n[Y(t)] \approx M_n[X(t)]$ for $n > 2$. To further illustrate this point, I simulated the return experience from the perfect market-timing and protective put strategies with a 1-month forecast interval for the period 1927-78. Summary statistics from these simulations are presented in table 3, where the New York Stock Exchange Index was used for the market and 30-day Treasury bills were used for the riskless asset.²¹

20. Note that this higher standard deviation is fundamental in the sense that there is no other strategy that the market timer could pursue which would lower the standard deviation and not be suboptimal. For example, to reduce his standard deviation by always holding some portion in the riskless security would mean that at those times when $\gamma(t) = 1$ he would be holding positive amounts of a dominated security which is never optimal.

21. The put option premiums used in this simulation were computed using formula (10) with $\sigma(t)$ estimated for each month by the square root of the average of the squared logarithmic returns on the market for the 12 previous months. The return data include dividends. The reader may find it interesting to compare my perfect timing returns with those reported in Sharpe (1975).

TABLE 3 Summary Statistics for Monthly Rate of Return Simulations:
Perfect Market Timing versus Protective Put Strategy,
January 1927–December 1978

Per Month	Market Timer	Protective Put ^a	NYSE Stocks
Average rate of return	2.58%	.55%	.85%
Average excess return	2.37%	.34%	.64%
Standard deviation	3.82%	3.55%	5.89%
Highest return	38.55%	30.14%	38.55%
Lowest return	−.06%	−7.06%	−29.12%
Average compound return	2.51%	.49%	.68%
Coefficient of skewness	4.28	2.84	.42
Average annual compound return	34.65%	6.04%	8.47%
Growth of \$1,000 (1927–78)	\$5,362,212.000	\$21,400	\$67,527

^a These returns also represent the simulated return experience of the market timer's fund after deducting the monthly management fee. This monthly management fee, which is equal to the 1-month put price, averaged 2.02% for the simulation period. However, within this period the fee often varied significantly from this average. The lowest monthly fee was 0.67% and the highest was 7.63%, with an overall standard deviation of 1.20%. The large variations in the put price through time caused the time series estimates of the standard deviation and coefficient of skewness for the protective put strategy to differ from the corresponding estimates for the market-timing strategy by more than the single-period amounts derived in the text.

The preceding analysis was made under the assumption that borrowing to buy securities and selling short are not allowed. Of course, in the real world, such margin transactions are permitted, although there are limits which are set by the Federal Reserve Board under Regulation T. Although the equilibrium management fee will be affected by the margin limits, the basic analysis remains unchanged as long as there are limits.

Let μ ($0 < \mu \leq 1$) denote the margin requirement set by the Federal Reserve. Then, investors may borrow to buy securities up to μ times the value of the securities purchased. So, for example, if $\mu = 0.50$ (a "50%" margin requirement), then for each dollar of securities purchased, the investor can borrow 50 cents. Equivalently, for each dollar the investor provides, he can purchase up to $\$(1/\mu)$, or \$2.00 worth of securities.

The same margin requirement μ also establishes the amount of assurance money that must be posted as collateral for short sales. For each dollar of securities sold short, the investor must put up $\$ \mu$ in addition to the proceeds from the short sale. If $\mu = 0.50$, then for each dollar sold short, the investor must post 50 cents in addition to the dollar received from the sale. So, for each dollar the investor provides, he can sell short $\$(1/\mu)$, or \$2.00 worth of securities.

Both institutions and individual investors are subject to the same margin regulations. However, as noted earlier, investment companies are frequently subject to further restrictions on borrowing and short selling. To take this into account in the analysis, let μ' denote the

margin restriction on the fund where $\mu' \geq \mu$. Otherwise we maintain the usual "frictionless" market assumptions that the (margin) borrowing rate to both individual investors and the fund are the same and equal to the lending rate and that the interest earned on assurance money and proceeds of short sales accrues to the short seller.

PROPOSITION II.5: If the fund is permitted to borrow to buy securities with a margin requirement μ' but is prohibited from making short sales, then the equilibrium percentage management fee (per forecast interval) is given by $m(t) = c(t)/\mu' = g(t)/\mu'$.

The proof of proposition II.5 goes as follows: If the market timer's forecast is that $Z_M(t) > R(t)$, then, of course, the manager will invest as much of the fund's assets in the market as he is permitted to. Hence, for $Z_M(t) > R(t)$, the return per dollar of assets in the fund will be $R(t) + [Z_M(t) - R(t)]/\mu'$. If his forecast is that $Z_M(t) \leq R(t)$, then because he cannot sell short, he will invest all the fund's assets in the riskless bond and the return per dollar of assets in the fund will be $R(t)$. Therefore, from the viewpoint of the outside investors, the random variable return per dollar of assets in the fund can be written as

$$\begin{aligned} X(t) &= \max [R(t), R(t) + [Z_M(t) - R(t)]/\mu'] \\ &= R(t) + \max [0, Z_M(t) - R(t)]/\mu'. \end{aligned} \quad (15)$$

Therefore, if the management fee is $m(t)$, the return per dollar to the outside investor's portfolio, $Z_p(t)$, can be written as

$$\begin{aligned} Z_p(t) &= X(t)/[1 + m(t)] \\ &= \{R(t) + \max [0, Z_M(t) - R(t)]/\mu'\}/[1 + m(t)]. \end{aligned} \quad (16)$$

Consider the option-bills strategy where, for each dollar invested in the riskless asset, $(1/\mu')$ call options on the market with exercise price per share $R(t)$ are purchased. The return per dollar from this strategy will be $\{R(t) + \max [0, Z_M(t) - R(t)]/\mu'\}/[1 + c(t)/\mu']$, where, as before, $c(t)$ is the market price of the call option. By inspection of (16), the return per dollar to the investor in the absence of management fees will be identical to the return on this options-bills strategy where the investor pays nothing for the calls. Hence, by the same arguments used to derive (7), the equilibrium management fee (as a percentage of assets) will equal $c(t)/\mu'$. As was shown earlier, $c(t) = g(t)$. Hence, the management fee can also be expressed as $g(t)/\mu'$. Of course, if the margin requirement is 100%, then $\mu = \mu' = 1$ and the management fee reduces to $c(t)$ as given in (7).

PROPOSITION II.6: If the fund is permitted to both borrow to buy securities and make short sales with a margin requirement μ' , then the

equilibrium percentage management fee (per forecast interval) is given by $m(t) = [c(t) + g(t)]/\mu' = 2c(t)/\mu' = 2g(t)/\mu'$.

The proof of proposition II.6 follows along the lines of the previous proof. As before, if the manager's forecast is that $Z_M(t) > R(t)$, then the return per dollar of assets in the fund will be $R(t) + [Z_M(t) - R(t)]/\mu'$. However, now if his forecast is that $Z_M(t) \leq R(t)$, the manager will short sell the market to the maximum permitted and the return per dollar will therefore be $R(t) + [R(t) - Z_M(t)]/\mu'$. From the viewpoint of an outside investor, the return per dollar of assets in the fund is given by

$$\begin{aligned} X(t) &= \max \{R(t) + [Z_M(t) - R(t)]/\mu', R(t) + [R(t) - Z_M(t)]/\mu'\} \\ &= R(t) + \{\max [0, Z_M(t) - R(t)] + \\ &\quad \max [0, R(t) - Z_M(t)]\}/\mu'. \end{aligned} \quad (17)$$

Hence, if the management fee is $m(t)$, the return per dollar to the outside investor's portfolio can be written as

$$\begin{aligned} Z_p(t) &= \left[R(t) + \{\max [0, Z_M(t) - R(t)] \right. \\ &\quad \left. + \max [0, R(t) - Z_M(t)]\}/\mu' \right] / [1 + m(t)]. \end{aligned} \quad (18)$$

A *straddle* is a put and a call with the same exercise price and expiration date. Consider the "straddle-bills" strategy where, for each dollar invested in the riskless asset, $(1/\mu')$ straddles on the market with exercise price per share $R(t)$ are purchased. Since the market price of a straddle equals $c(t) + g(t)$, the return per dollar from this strategy equals $X(t)/\{1 + [c(t) + g(t)]/\mu'\}$. Hence, by the same arguments used to derive (7), the equilibrium management fee (as a percentage of assets) will equal $[c(t) + g(t)]/\mu'$. Noting that $c(t) = g(t)$, the proof of proposition II.6 is complete.

In summary, we have shown that the returns to a portfolio employing perfect market timing are identical to the returns from following certain option-investment strategies. From this isomorphic relationship, the equilibrium management fees (and, hence, the value of such macroforecasting skills) were determined in terms of the market prices for options. Of course, the assumption that such perfect market-timing forecasts can be made is totally unrealistic. Hence, beginning in the next section, I shall examine the more interesting case where the market timer's forecast can be wrong. However, the analysis of this section will be helpful in providing insights into the more realistic but also more complicated analysis of imperfect macroforecasting.

III. Heterogeneous Beliefs and Imperfect Forecasting

In this section, I shall generalize the model of the previous section by allowing for the possibility that investors have different information

sets and that the market timer's forecast may be wrong. At each point in time t , let $\phi_j(t)$ denote the j th information set, $j = 1, 2, \dots, N$. Without loss of generality, we can associate a particular investor with each information set so that investor j forms his probability assessment for the return per dollar on the market between t and $t + 1$ based upon $\phi_j(t)$. Let $f_j(Z, t)$ denote the probability density function for the return per dollar on the market based upon $\phi_j(t)$. It is assumed that for $j = 1, 2, \dots, N$, f_j satisfies Rational Expectations relative to information set ϕ_j in the sense that the (ex post) path of market returns is consistent with the (ex ante) density f_j . Let $q_j(t)$ denote the probability, based upon $\phi_j(t)$, that the return on the market between t and $t + 1$ will be less than or equal to the return per dollar on the riskless asset, $R(t)$. Then

$$q_j(t) \equiv \int_0^{R(t)} f_j(Z, t) dZ. \quad (19)$$

It is further assumed that, for $j = 1, 2, \dots, N$, $0 < q_j(t) < 1$, and so, for every information set, the return on the market is uncertain and the market neither dominates nor is dominated by the riskless asset.

As in Section II, there is a market timer who forecasts when stocks will outperform bonds, $Z_M(t) > R(t)$, and when bonds will perform at least as well as stocks, $Z_M(t) \leq R(t)$. Unlike in Section II, the market timer's forecast may be wrong. Let $\theta(t)$ be a random variable such that $\theta(t) = 1$ if the forecast is correct and $\theta(t) = 0$ if the forecast is incorrect. Of course, to evaluate the forecast, investors must have some information about the distribution of $\theta(t)$. The information made available to investors will depend upon the information set of the forecaster and his concern with protecting that information as well as the information available to investors from other sources.

For example, one might assume that the forecaster provides the complete joint probability distribution, $p[A(t), \theta(t)]$, where $A(t)$ is a random variable such that $A(t) = 1$ if $Z_M(t) > R(t)$ and $A(t) = 0$ if $Z_M(t) \leq R(t)$. However, this assumption implies that the forecaster has an unconditional or prior distribution for $A(t)$, given by $p[A(t)] = p[A(t), 0] + p[A(t), 1]$, which he may not have. If he does have such a prior, and priors are homogeneous (as in the previous section), then $p[A(t), \theta(t)]$ provides redundant information in the sense that the same information is conveyed by the conditional probabilities $\{p[\theta(t) | A(t)]\}$. To see this, note that $p[A(t), \theta(t)] = p[\theta(t) | A(t)] p[A(t)]$ and, by the homogeneity assumption, all investors know that $p[A(t)] = q(t)$ for $A(t) = 0$ and $p[A(t)] = 1 - q(t)$ for $A(t) = 1$. If, as is assumed in this section, priors are not homogeneous, then by making public $p[A(t), \theta(t)]$, the forecaster reveals his prior distribution $p[A(t)]$. Therefore, by providing the joint distribution, the forecaster gives away valuable information for free to those investors who believe that the forecaster's prior

contains information not incorporated in their own prior. And to those investors who believe that the forecaster's prior does not contain such information, $p[A(t), \theta(t)]$ provides no more information than the conditional probabilities $\{p[\theta(t) | A(t)]\}$. Hence, even when he has such a probability assessment, it is not in the forecaster's economic interest to reveal $p[A(t), \theta(t)]$.

Having ruled out that the forecaster will provide the complete joint probability distribution, we are left to choose between the two conditional probability distributions, $\{p[A(t) | \theta(t)]\}$ and $\{p[\theta(t) | A(t)]\}$. Given that investors have priors about $A(t)$ through the $\{f_j\}$, a natural choice would be the latter, $\{p[\theta(t) | A(t)]\}$. Of course, if investors were assumed to have a prior distribution for $\theta(t)$ instead, then the natural choice would be the former. To provide $\{p[A(t) | \theta(t)]\}$ and $\{p[\theta(t)]\}$ would, of course, be equivalent to providing $p[A(t), \theta(t)]$ and, as will be amply demonstrated, to provide $\{p[\theta(t)]\}$ alone is not sufficient information for investors to evaluate the forecast.

Therefore, it is assumed that investors are given the probability function for $\theta(t)$ conditional on the market return $Z_M(t) = Z$, which I write as

$$\begin{aligned} \text{prob } \{\theta(t) = 1 \mid Z_M(t) = Z\} &= p_1(t) \quad \text{for } 0 \leq Z \leq R(t) \\ &= p_2(t) \quad \text{for } R(t) < Z < \infty, \end{aligned} \quad (20)$$

By not having $p_1(t)$ or $p_2(t)$ in (20) depend upon j , I am assuming that all investors, independent of their information sets, agree on these combinations. In this section, I derive the equilibrium value of such forecasts taking into account the possibility that the accuracy of the forecast may not be independent of the return on the market. Thus, statements such as, "He is a better forecaster in up-markets than he is in down-markets," can be given precise and quantitative meaning in this model. Such a forecaster would be characterized as making forecasts such that $p_1(t) < p_2(t)$. However, in keeping with the hypothesized structure of the information available to the forecaster, $p_1(t)$ and $p_2(t)$ are themselves assumed to be independent of Z . Therefore, while the probability of a correct forecast may depend upon whether or not the return on the market exceeds the return on bonds, it does not otherwise depend upon the magnitude of the market's return.

If, as in the previous analysis, $\gamma(t) = 1$ if the forecast at time t is $Z_M(t) \geq R(t)$ and $\gamma(t) = 0$ if the forecast is that $Z_M(t) \leq R(t)$, then from (20), the probabilities for $\gamma(t)$ conditional upon the realized return on the market can be written as

$$\text{prob } \{\gamma(t) = 0 \mid Z_M(t) \leq R(t)\} = p_1(t) \quad (21a)$$

$$\text{prob } \{\gamma(t) = 1 \mid Z_M(t) \leq R(t)\} = 1 - p_1(t)$$

and

$$\begin{aligned}\text{prob} \{ \gamma(t) = 0 \mid Z_M(t) > R(t) \} &= 1 - p_2(t) \\ \text{prob} \{ \gamma(t) = 1 \mid Z_M(t) > R(t) \} &= p_2(t).\end{aligned}\quad (21b)$$

Therefore, based upon information set ϕ_j , we have from (19) and (20) that, from the viewpoint of investor j , the unconditional probability of a correct forecast by the market timer, $p^j(t)$, can be written as

$$\begin{aligned}p^j(t) &\equiv \text{prob} \{ \theta(t) = 1 \mid \phi_j(t) \} \\ &= q_j(t)p_1(t) + [1 - q_j(t)]p_2(t).\end{aligned}\quad (22)$$

Note from (22) that, in general, the probability of a correct forecast will depend on the investor's information set, and only in the special case when the accuracy of the forecast is independent of the return on the market—that is, $p_1(t) = p_2(t) \equiv p(t)$ —is the probability of a correct forecast, $p^j(t) = p(t)$, independent of j .

From (21a) and (21b), the unconditional probabilities for the forecast variable $\gamma(t)$ can be written from the viewpoint of investor j as

$$\begin{aligned}\text{prob} \{ \gamma(t) = 0 \mid \phi_j(t) \} &\equiv \delta_j(t) \\ &= q_j(t)p_1(t) + [1 - q_j(t)][1 - p_2(t)] \\ \text{prob} \{ \gamma(t) = 1 \mid \phi_j(t) \} &= 1 - \delta_j(t) \\ &= q_j(t)[1 - p_1(t)] + [1 - q_j(t)]p_2(t).\end{aligned}\quad (23)$$

As in Section II, I call $f_j(Z, t)$ the prior distribution of the return on the market for investor j . Based upon this prior, we define the posterior distribution for investor j , $f_j^*(Z, t \mid \gamma)$, to be the probability density for the return on the market conditional upon the forecast $\gamma(t) = \gamma$. Under the hypothesized information structure, we have from (23) that f_j^* can be written as

$$\begin{aligned}f_j^*(Z, t \mid \gamma) &= \left\{ \frac{\gamma[1 - p_1(t)]}{1 - \delta_j(t)} + \frac{(1 - \gamma)p_1(t)}{\delta_j(t)} \right\} f_j(Z, t), \\ &\quad \text{for } 0 \leq Z \leq R(t) \\ &= \left\{ \frac{\gamma p_2(t)}{1 - \delta_j(t)} + \frac{(1 - \gamma)[1 - p_2(t)]}{\delta_j(t)} \right\} f_j(Z, t), \\ &\quad \text{for } R(t) < Z < \infty.\end{aligned}\quad (24)$$

The distribution that investor j would use if, in addition to $\phi_j(t)$, he had the market timer's forecast is f_j^* . Of course, it will be the differences between f_j and f_j^* which will account for investor j 's willingness to pay for the forecast information. The distribution used by the market timer for his *personal* investment decisions if the information available to him, other than his forecast, were information set $\phi_j(t)$ would also be

represented by f_j^* . Note that for perfectly correct forecasts—that is, $p_1(t) = p_2(t) = 1$ —(24) simply reduces to (2) in Section II.

Because the value of the market timer's forecasts will depend upon how those forecasts modify investors' prior distributions, I shall now analyze in detail the differences between the prior and posterior distributions.

While to this point, no restrictions have been imposed upon the conditional probabilities $p_1(t)$ and $p_2(t)$, it seems reasonable to require that they be such that the forecasts are rational. A forecast is said to be *rational* if, given the forecast, no investor would modify his prior in the opposite direction of the forecast. Specifically, investor j 's priors are that bonds will outperform stocks with probability $q_j(t)$ and that stocks will outperform bonds with probability $1 - q_j(t)$. After receiving the market timer's forecast $\gamma(t) = \gamma$, investor j 's revised probabilities for the two events will be $q_j^*(t; \gamma)$ and $1 - q_j^*(t; \gamma)$, respectively, where

$$q_j^*(t; \gamma) = \int_0^{f_j^*(\gamma)} f_j^*(Z, t | \gamma) dZ.$$

Therefore, the forecasts will be rational if, for every investor j , $q_j^*(t; 0) \geq q_j(t)$ and $q_j^*(t; 1) \leq q_j(t)$.

PROPOSITION III.1: A necessary and sufficient condition for a market timer's forecasts to be rational is that $p_1(t) + p_2(t) \geq 1$.

The proof is as follows: From (24), $q_j^*(t; 0) \geq q_j(t)$ if and only if $p_1(t) \geq \delta_j(t)$, and $q_j^*(t; 1) \leq q_j(t)$ if and only if $p_2(t) \geq \delta_j(t)$. So, a necessary and sufficient condition for the forecasts to be rational is that $p_1(t) \geq \delta_j(t)$ for every j . From (23), $p_1(t) \geq \delta_j(t)$ if and only if $[1 - q_j(t)]p_1(t) \geq [1 - q_j(t)][1 - p_2(t)]$. But, $0 < q_j(t) < 1$ for all information sets $\phi_j(t)$, $j = 1, 2, \dots, N$. Hence, $p_1(t) \geq \delta_j(t)$ if and only if $p_1(t) + p_2(t) \geq 1$. However, the condition that $p_1(t) + p_2(t) \geq 1$ is independent of j and is therefore a necessary and sufficient condition for all j 's.

An immediate corollary to proposition III.1 is that if a forecast is *irrational* (i.e., not rational) then, given the forecast, every investor would modify his prior in the opposite direction of the forecast. One should not infer that an irrational forecast has no value. A forecaster who is always wrong is as valuable as one who is always right provided that this "bias" is known by those acting on the forecast. Indeed, the opposite or contrary forecast to an irrational forecast is always rational. To see this, note that the contrary forecast's conditional probabilities of success satisfy $p_1'(t) = 1 - p_1(t)$ and $p_2'(t) = 1 - p_2(t)$. So, if $p_1(t) + p_2(t) < 1$, then $p_1'(t) + p_2'(t) > 1$. Hence, with no loss of generality, it is assumed throughout the rest of the analysis that forecasts are rational and therefore that $p_1(t) + p_2(t) \geq 1$.

As mentioned earlier, for the market timer's forecast to be of value to investor j , it must provide information which would modify his prior

distribution for the return on the market. Therefore, if $f_j^* = f_j$, then the market timer's forecast will have no value to investor j .

PROPOSITION III.2: If the conditional probabilities for a correct forecast given in (20) are such that $p_1(t) + p_2(t) = 1$, then the market timer's forecast will have no value to any investor and hence the value of the forecast information is zero.

The proof of proposition III.2 is as follows: If $p_1(t) + p_2(t) = 1$, then by substitution in (23), we have that $\delta_j(t) = p_1(t) = 1 - p_2(t)$ and that $1 - \delta_j(t) = 1 - p_1(t) = p_2(t)$. Moreover, this value of $\delta_j(t)$ is the same for every investor j ($j = 1, \dots, N$). Substituting for $\delta_j(t)$ in (24), we have that $f_j^*(Z, t | \gamma) = f_j(Z, t)$ for every value of γ and for all j . Thus, independent of the forecast given, each investor's posterior distribution will be equal to his prior distribution. Therefore, for $p_1(t) + p_2(t) = 1$, the market timer's forecast will have no value to any investor and the value of the forecast information is zero.

PROPOSITION III.3: A necessary condition for a rational forecast to have a positive value is that the conditional probabilities given in (20) satisfy $p_1(t) + p_2(t) > 1$.

The proof follows immediately from proposition III.1 and proposition III.2. While it has only been shown that $p_1(t) + p_2(t) > 1$ is a necessary condition for a rational forecast to have positive value, it is also a sufficient condition, as will be shown later in Section IV. Indeed, the larger is $p_1(t) + p_2(t)$, the more valuable is the forecast information.

Proposition III.2 can also be used to confirm the common-sense result that if the market timer's forecasts are random, then the value of such forecasts is zero.

PROPOSITION III.4: If the forecast variable is distributed independent of the return on the market, then the value of the forecast is zero.

By hypothesis, $\gamma(t)$ is distributed independent of $Z_M(t)$. Therefore, $p_1(t) \equiv \text{prob} \{ \gamma(t) = 0 \mid Z_M(t) \leq R(t) \} = \text{prob} \{ \gamma(t) = 0 \mid Z_M(t) > R(t) \} \equiv 1 - p_2(t)$. Hence, $p_1(t) + p_2(t) = 1$, and, by proposition III.2, the forecast has zero value.

So, if forecasts are formed by simply flipping a (not necessarily fair) coin where η is the probability that the forecast will be $\gamma(t) = 0$ and $1 - \eta$ is the probability that the forecast will be $\gamma(t) = 1$, then $p_1(t) = \eta$ and $p_2(t) = 1 - \eta$, and therefore such forecasts will have zero value. Included as a special case of this type of forecast is to *always* forecast that $Z_M(t) \leq R(t)$ —that is, $\gamma(t) = 0$, $p_1(t) = \eta = 1$, and $p_2(t) = 0$ —or to *always* forecast that $Z_M(t) > R(t)$ —that is, $\gamma(t) = 1$, $p_1(t) = \eta = 0$, and $p_2(t) = 1$. Like a stopped clock, such a forecast will sometimes be correct, but it never has any value.

Having established that completely random forecasts have zero value, I shall now go to the other extreme and show that completely predictable forecasts also have zero value.

PROPOSITION III.5: If the forecast variable is itself perfectly forecastable by investor j , then the value of the forecast to investor j is zero.

Suppose investor j uses his information set ϕ_j to predict the market timer's forecast, $\gamma(t)$. Let $\gamma_j(t)$ denote this forecast. By hypothesis, investor j 's prediction is always correct—that is, $\gamma_j(t) = \gamma(t)$ —and therefore, $p_1(t) \equiv \text{prob} \{ \gamma(t) = 0 \mid Z_M(t) \leq R(t) \} = 1 - \gamma_j(t)$ and $p_2(t) \equiv \text{prob} \{ \gamma(t) = 1 \mid Z_M(t) > R(t) \} = \gamma_j(t)$. Hence, $p_1(t) + p_2(t) = 1$ and, by proposition III.2, the value of the forecast is zero.

Like proposition III.4, proposition III.5 is a commonsense result. If investor j can always predict the market timer's forecast, then all the information used by the market timer must be contained in ϕ_j . Since all the information in ϕ_j relevant to the return on the market is captured by f_j , investor j should neither revise his prior in response to the market timer's forecast nor pay for the forecast.

In (20), I assumed that $p_1(t)$ and $p_2(t)$ were the same for all investors. Hence, the hypothesized conditions of proposition III.5 will not be consistent with (20) unless all investors can perfectly forecast $\gamma(t)$. These conditions would be consistent with a more general model where the conditional probabilities $p_1(t)$ and $p_2(t)$ are different for different investors. Since the effect of such a generalization on all the results obtained is discussed at the end of this section, I shall say no more here except to point out one important case where the conditions of proposition III.5 and (20) are consistent. As in Section II, let $\phi(t)$ denote the set of information available to all investors (i.e., the "public information set").

PROPOSITION III.6: If the market timer's forecasts are based solely upon public information, then the value of the forecast is zero.

By its definition,

$$\phi(t) = \bigcap_{i=1}^N \phi_i$$

and therefore, $\phi(t) \in \phi_j$. By hypothesis, the market timer's forecasts are based solely on information contained in $\phi(t)$. Hence, each investor j can use $\phi(t)$, which is contained in ϕ_j , to predict correctly the market timer's forecast $\gamma(t)$. Therefore, by proposition III.5, the value of the forecast to each investor is zero.

The conclusions of propositions III.4, III.5, and III.6 that investors will not pay for random forecasts or for forecasts based upon informa-

tion they already have are hardly counterintuitive. Indeed, because any sensible model should produce these results, proof of these propositions serves as a check on the validity of the model presented here. However, the model does produce other results which may not be so intuitive. For example, the following three propositions point out the "pitfalls" in relying on the probability of a correct forecast as even a qualitative measure of the value of a forecast.

PROPOSITION III.7: The unconditional probability of a correct forecast, $p^j(t)$, being greater than .5 is neither a necessary nor a sufficient condition for the forecast to have a positive value.

To prove that it is not necessary, it is merely necessary to show that it is possible for $p_1(t) + p_2(t) > 1$ and for $p^j(t) \leq 0.5$. Suppose that $q_j(t) = 0.7$, $p_1(t) = 0.4$, and $p_2(t) = 0.7$; $p_1(t) + p_2(t) = 1.1 > 1$ and, from (22), $p^j(t) = 0.49 \leq 0.5$. To prove that it is not sufficient, it can be shown that it is possible for $p_1(t) + p_2(t) = 1$ and for $p^j(t) > 0.5$. Suppose that $q_j(t) = 0.7$ and a "stopped clock" forecast where $p_1(t) = 1$ and $p_2(t) = 0$. Then, $p_1(t) + p_2(t) = 1$ and, from (22), $p^j(t) = q_j(t) = 0.7 > 0.5$.

Let $p_1^j(t)$ denote the unconditional probability of a correct forecast by market timer I from the viewpoint of investor j and let $p_2^j(t)$ be the corresponding probability of a correct forecast by market timer II.

PROPOSITION III.8: $p_1^j(t) > p_2^j(t)$ is neither a necessary nor a sufficient condition for the forecast by market timer I to be more valuable than the forecast by market timer II.

To prove that it is not necessary, I show that it is possible for $p_1^j(t) < p_2^j(t)$; $p_1^j(t) + p_2^j(t) > 1$; and $p_1^{II}(t) + p_2^{II}(t) = 1$. As in the proof of proposition III.7, suppose that $q_j(t) = .7$; $p_1^I(t) = 0.4$; $p_2^I(t) = 0.7$; $p_1^{II}(t) = 1$; $p_2^{II}(t) = 0$. Then, $p_1^j(t) = 0.49 < p_2^j(t) = 0.7$, but $p_1^I(t) + p_2^I(t) = 1.1 > p_1^{II}(t) + p_2^{II}(t) = 1$. To prove that it is not sufficient, one need only interchange I with II in the proof that it is not necessary.

Suppose that investor j makes his own forecast based upon his prior probability assessment for the return on the market f_j . If $\gamma_j(t)$ denotes investor j 's forecast variable, then his forecast will be $\gamma_j(t) = 0$ when $q_j(t) \geq 1 - q_j(t)$ and $\gamma_j(t) = 1$ when $q_j(t) < 1 - q_j(t)$. From the viewpoint of that investor, the unconditional probability of his own forecast being correct is equal to $\max [q_j(t), 1 - q_j(t)]$.

PROPOSITION III.9: $p^j(t) \leq \max [q_j(t), 1 - q_j(t)]$ is a necessary but not a sufficient condition for the market timer's forecast to have zero value.

Necessity can be proven by showing that $p^j(t) > \max [q_j(t), 1 - q_j(t)]$ implies that $p_1(t) + p_2(t) > 1$. From (22), $p_1(t)q_j(t) + p_2(t)[1 - q_j(t)] > \max [q_j(t), 1 - q_j(t)]$. Therefore, $[p_1(t) + p_2(t)] \max [q_j(t), 1 - q_j(t)] > \max [q_j(t), 1 - q_j(t)]$. Hence, $p_1(t) + p_2(t) > 1$. To prove that it is not a

sufficient condition, we simply show that it is possible for $p^j(t) < \max [q_j(t), 1 - q_j(t)]$ and for $p_1(t) + p_2(t) > 1$. As in the previous proofs, suppose that $q_j(t) = 0.7$, $p_1(t) = 0.4$, and $p_2(t) = 0.7$. Then, $p^j(t) = 0.49 < \max [q_j(t), 1 - q_j(t)] = 0.7$ and $p_1(t) + p_2(t) = 1.1 > 1$.

Thus, even if the probability of a correct forecast by the market timer is less than the probability of a correct forecast by investor j using only his prior, one cannot conclude that the value of the market timer's forecast to investor j is zero.

As propositions III.7, III.8, and III.9 amply demonstrate, the frequency with which the market timer's forecasts are correct is in general a very poor statistic for evaluating the forecasts, and this is the case even when investors' beliefs are homogeneous. Using propositions III.2 and III.3, one might summarize by saying, "It is not so much how often the market timer is correct, but when he is correct that determines the value of his forecasts." However, there is one special case when the probability of a correct forecast is a sufficient statistic for value.

PROPOSITION III.10: If the probability of a correct forecast is independent of the return on the market, then $p^j(t) = 0.5$ is a necessary and sufficient condition for the forecast to have zero value.

By hypothesis, we have from (20) that $p_1(t) = p_2(t) \equiv p(t)$. Substituting into (22), we have that $p^j(t) = p(t)$, independent of j . Therefore, $p^j(t) = 0.5$ implies that $p_1(t) + p_2(t) = 2p(t) = 1$. Hence, by proposition III.2, the forecast has zero value. Under the hypothesized condition, $p^j(t) < 0.5$ implies that $2p(t) = p_1(t) + p_2(t) < 1$ and therefore that the forecast is irrational by proposition III.1. However, as discussed earlier, such a forecast has positive value because the contrary forecast which is rational will satisfy $p^j(t) + p^j(t) > 1$. Under the hypothesized condition, $p^j(t) > 0.5$ implies that $2p(t) = p_1(t) + p_2(t) > 1$ and therefore that the forecast has positive value.

Although the probability of a correct forecast is not a reliable indicator of the value of a forecast, it can be used to determine the conditions under which investor j will "agree" with the market timer's forecast. Investor j is said "to agree with the market timer's forecast" if, having received the market timer's forecast, investor j 's own forecast based upon his posterior probability assessment is the same as the market timer's forecast.

From (24), conditional on $\gamma(t) = \gamma$, we have that

$$q_j^*(t; \gamma) = \left[\frac{\gamma[1 - p_1(t)]}{1 - \delta_j(t)} + \frac{(1 - \gamma)p_1(t)}{\delta_j(t)} \right] q_j(t). \quad (25)$$

If $\gamma_j^*(t)$ denotes investor j 's forecast variable based upon his posterior, then, conditional on $\gamma(t) = \gamma$, his forecast will be such that $\gamma_j^*(t) = 0$ when $q_j^*(t; \gamma) > 0.5$ and $\gamma_j^*(t) = 1$ when $q_j^*(t; \gamma) < 0.5$. Thus, investor j

will agree with the market timer's forecast if $\gamma_j^*(t) = \gamma$ where any indeterminacy is resolved by defining $\gamma_j^*(t) = \gamma$ in the singular case $q_j^*(t; \gamma) = 0.5$. Therefore, for investor j to *always* agree with the market timer's forecast, $q^*(t; \gamma)$ must satisfy $q_j^*(t; 0) \geq 0.5$ and $q_j^*(t; 1) \leq 0.5$.

PROPOSITION III.11: A necessary and sufficient condition for investor j to agree with the market timer's forecast, γ , is that $p^j(t) \geq \gamma q^j(t) + (1 - \gamma)[1 - q_j(t)]$.

Investor j will agree with the forecast $\gamma = 0$ if and only if $q_j^*(t; 0) \geq 0.5$. From (23) and (25), $q_j^*(t; 0) \geq 0.5$ if and only if $p_1 q_j(t) \geq [1 - q_j(t)][1 - p_2(t)]$ or, equivalently from (22), if and only if $p^j(t) \geq [1 - q_j(t)]$. This proves the proposition for $\gamma = 0$. Investor j will agree with the forecast $\gamma = 1$ if and only if $q_j^*(t; 1) \leq 0.5$. Again, from (23) and (25), $q_j^*(t; 1) \leq 0.5$ if and only if $q_j(t)[1 - p_1(t)] \leq [1 - q_j(t)]p_2(t)$ or, equivalently from (22), if and only if $p^j(t) \geq q_j(t)$. This proves the proposition for $\gamma = 1$.

It follows immediately from proposition III.11 that investor j will always agree with the market timer's forecast if and only if $p^j(t) \geq \max [q_j(t), 1 - q_j(t)]$. It also follows that investor j will always disagree with the market timer's forecast only if $p^j(t) < \min [q_j(t), 1 - q_j(t)]$. However, this possibility is not consistent with the market timer's forecast being rational.

PROPOSITION III.12: If investor j always disagrees with the market timer's forecast, then the market timer's is irrational and investor j would always agree with the contrary forecast.

From (22), $p^j(t) < \min [q_j(t), 1 - q_j(t)]$ if and only if $p_1(t) + p_2(t) < 1$. Hence, from proposition III.1, the market timer's forecast is irrational. The unconditional probability of the contrary forecast being correct is $1 - p^j(t)$. But $p^j(t) < \min [q_j(t), 1 - q_j(t)]$ implies that $1 - p^j(t) > \max [q_j(t), 1 - q_j(t)]$. Therefore, by proposition III.11, investor j would always agree with the contrary forecast.

In summary, the behavior of investor j toward rational forecasts is to always agree with the market timer's forecast if the unconditional probability of a correct forecast by the market timer is at least as large as the probability of a correct forecast by investor j using his prior distribution. If this condition is not satisfied, then the posterior forecast of investor j , $\gamma_j^*(t)$, will always agree with his own prior forecast, $\gamma_j(t)$.

Because the agreement of investor j 's posterior forecast with the market timer's forecast depends only upon the conditional probability of a correct forecast, it is clear from propositions III.7, III.8, and III.9 that disagreement does not imply that investor j should disregard the market timer's forecast. And such disagreement certainly does not imply that the market timer's forecast is irrational. Indeed, if one were to define a "rational forecast" as one where every investor's posterior forecast agrees with the market timer's forecast, then virtually the only

forecast to satisfy this definition would be one that is always correct, that is, $p_1(t) = p_2(t) = 1$. To see this, note that such a "rational" forecast must satisfy both $p_1(t)/[1 - p_2(t)] \geq \max \{ [1 - q_1(t)]/q_1(t) \}$ and $p_2(t)/[1 - p_1(t)] \geq \max \{ q_2(t)/[1 - q_2(t)] \}$. Hence, with this definition for a rational forecast, virtually all forecasts (including valuable ones) would be "irrational."

To further underscore the difference between "agreement" and "rationality" (as I define it), consider a forecast with $p_1(t) + p_2(t) > 1$ and with an unconditional probability of being correct, $p^*(t)$, which is less than 0.5 for every investor j . From proposition III.7, it is possible for a forecast to have both these properties. Because $p^*(t) < 0.5 \leq \max [q_j(t), 1 - q_j(t)]$ for every investor j , no investor will always agree with the market timer's forecast. Yet, from proposition III.1, this forecast is rational, and indeed, because $p_1(t) + p_2(t) > 1$, it has a positive value. Propositions III.7, III.8, and III.9 made apparent the pitfalls in using the unconditional probability of a correct forecast as an indicator of the forecast's value. This example shows that it is not a reliable indicator of when a forecast is rational. The contrary forecast to the market timer's forecast has an unconditional probability of being correct equal to $1 - p^*(t)$, and in this example all investors would agree that $1 - p^*(t) > 0.5$. However, this contrary forecast is irrational because $p_1(t) + p_2(t) > 1$ implies that $p_1^*(t) + p_2^*(t) = 1 - p_1(t) + 1 - p_2(t) < 1$.

To conclude this section, I shall briefly discuss the effect on the derived results of introducing a more general model where the conditional probabilities given in (20) are different for different investors. As the reader can verify by substituting $p_i^j(t)$ for $p_i(t)$ and $p_2^j(t)$ for $p_2(t)$ in the derivations, there are two basic changes. (1) It is now possible for some investors to view a forecast as having zero value, $p_1^j(t) + p_2^j(t) = 1$, while other investors see it as having positive value, $p_1^j(t) + p_2^j(t) > 1$. (2) My definition of a "rational forecast" loses much of its meaning because it is now possible for some investors to view a forecast as rational, $p_1^j(t) + p_2^j(t) \geq 1$, while others view it as irrational, $p_1^j(t) + p_2^j(t) < 1$. Otherwise the propositions as derived remain intact.

Having established the conditions under which a forecast will have a positive value, I shall now proceed to determine the magnitude of that positive value.

IV. Equilibrium Management Fees and the Value of Market Timing

In the last section, the conditions were established under which a market timer's forecast would or would not have a positive value. It was shown there that the conditional probabilities of a correct forecast, $p_1(t)$ and $p_2(t)$, were sufficient statistics to make that qualitative determination. In this section, I derive the equilibrium value of such forecasts and show that $p_1(t)$ and $p_2(t)$ are also sufficient statistics to make

this quantitative determination. Along the lines of the analysis in Section II, I derive the equilibrium value by showing a correspondence between the returns from a portfolio managed by the market timer and the returns from certain option investment strategies.

As in Section II, the market timer manages an investment company or mutual fund which invests in the market and riskless bonds according to the following strategy: At date t , if his forecast is that bonds will outperform stocks—that is, $\gamma(t) = 0$ —then he will allocate $100 \eta_1(t)\%$ of the fund's assets to the market and $100 [1 - \eta_1(t)]\%$ of the fund's assets to bonds. If his forecast is that stocks will outperform bonds—that is, $\gamma(t) = 1$ —then he will allocate $100 \eta_2(t)\%$ to the market and $100 [1 - \eta_2(t)]\%$ to bonds. It is assumed that the distribution of his forecasts satisfies (20) and that these conditional probabilities along with $\eta_1(t)$ and $\eta_2(t)$ are made known to outside investors. It is further assumed that the market timer's forecasts are rational, that is, $p_1(t) + p_2(t) \geq 1$, and that he acts rationally on his forecasts, that is, $\eta_2(t) > \eta_1(t)$. As in the previous section, investors have different information sets and hence can have heterogeneous probability beliefs about the return on the market.

As before, $\theta(t)$ is a random variable such that $\theta(t) = 1$ if the forecast is correct and $\theta(t) = 0$ if the forecast is wrong. Along the lines of (5) of Section II, the return per dollar on the assets of the fund, $X(t)$, can be written as

$$\begin{aligned} X(t) &= [\eta_2(t) - \theta(t) - \theta(t)\Delta(t)][Z_M(t) - R(t)] + R(t) \\ &\quad \text{for } 0 \leq Z_M(t) \leq R(t) \\ &= [\eta_1(t) + \theta(t)\Delta(t)][Z_M(t) - R(t)] + R(t) \\ &\quad \text{for } R(t) < Z_M(t) < \infty, \end{aligned} \quad (26)$$

where $\Delta(t) \equiv \eta_2(t) - \eta_1(t) > 0$ and the conditional distribution for $\theta(t)$ is given by (20). From (26), the return per dollar to an investor in the fund, $Z_p(t)$, can be written as

$$Z_p(t) = X(t)/[1 + m(t)], \quad (27)$$

where $m(t)$ is the management fee expressed as a fraction of assets held by the fund.

As in Section II, I assume for convenience that one share of the market portfolio has a current price of \$1 and, as before, let $g(t)$ denote the price of a one-(forecast) period put option on one share of the market portfolio with an exercise price of $R(t)$.

I begin my analysis by constructing a specific options-investment portfolio whose returns will be compared with those of the market timer's fund. The specific portfolio strategy is as follows: For each dollar invested in the portfolio, allocate the fractions $\omega_1(t) \equiv \{\eta_1(t) +$

$p_2(t)\Delta(t)/[1+m(t)]$ to the market, $\omega_2(t) \equiv g(t)[p_1(t) + p_2(t) - 1][\Delta(t)/[1+m(t)]$ to put options on the market, and $\omega_3(t) \equiv 1 - \omega_1(t) - \omega_2(t)$ to riskless bonds. The return per dollar on this portfolio, $Z_s(t)$, can be written as

$$Z_s(t) = \omega_1(t)Z_M(t) + \omega_2(t)\{\max[0, R(t) - Z_M(t)]\}/g(t) + \omega_3(t)R(t). \quad (28)$$

Substituting for the portfolio weights, (28) can be rewritten as

$$\begin{aligned} Z_s(t) &= \left\{ \left\{ \eta_1(t) + [1 - p_1(t)]\Delta(t) \right\} [Z_M(t) - R(t)] \right. \\ &\quad \left. + [1 + m(t) - \lambda(t)\Delta(t)]R(t) \right\} / [1 + m(t)] \\ &\quad \text{for } 0 \leq Z_M(t) \leq R(t) \\ &= \left\{ [\eta_1(t) + p_2(t)\Delta(t)][Z_M(t) - R(t)] \right. \\ &\quad \left. + [1 + m(t) - \lambda(t)\Delta(t)]R(t) \right\} / [1 + m(t)] \\ &\quad \text{for } R(t) < Z_M(t) < \infty, \end{aligned} \quad (29)$$

where $\lambda(t) \equiv g(t)[p_1(t) + p_2(t) - 1]$.

If $U(t) \equiv Z_p(t) - Z_s(t)$, then from (26), (27), and (29), we have that

$$\begin{aligned} U(t) &= \left\{ [p_1(t) - \theta(t)]\Delta(t)[Z_M(t) - R(t)] \right. \\ &\quad \left. + [\lambda(t)\Delta(t) - m(t)]R(t) \right\} / [1 + m(t)] \\ &\quad \text{for } 0 \leq Z_M(t) \leq R(t) \\ &= \left\{ [\theta(t) - p_2(t)]\Delta(t)[Z_M(t) - R(t)] \right. \\ &\quad \left. + [\lambda(t)\Delta(t) - m(t)]R(t) \right\} / [1 + m(t)] \\ &\quad \text{for } R(t) < Z_M(t) < \infty. \end{aligned} \quad (30)$$

Define the expectation operator, E_j , such that for any random variable $Y(t)$, $E_j[Y(t)]$ means "the expected value of the random variable $Y(t)$ based upon the information set $\phi_j(t)$." Conditional upon $Z_M(t) = Z$, we have from (20) that investor j 's conditional expectation of $\theta(t)$ can be written as

$$\begin{aligned} E_j[\theta(t) | Z_M(t) = Z] &= p_1(t) \quad \text{for } 0 \leq Z \leq R(t) \\ &= p_2(t) \quad \text{for } R(t) < Z < \infty. \end{aligned} \quad (31)$$

Because $p_1(t)$ and $p_2(t)$ do not depend upon j , we have from (31) that all investors agree on the conditional expectation of $\theta(t)$.

It follows from (30) and (31) that for investor j , the conditional expectation of $U(t)$ will satisfy

$$\begin{aligned} E_j[U(t) | Z_M(t) = Z] &= [\lambda(t)\Delta(t) - m(t)]R(t)/[1 + m(t)] \\ &= \alpha(t) \end{aligned} \quad (32)$$

for all Z . Because $\lambda(t)$, $\Delta(t)$, $m(t)$, and $R(t)$ do not depend upon j , it follows that all investors agree on the conditional expectation of $U(t)$.

From (32), the unconditional expectation of $U(t)$ from the viewpoint of investor j can be written as

$$\begin{aligned} E_j[U(t)] &= \int_0^\infty E_j[U(t) \mid Z_M(t) = Z_M(t) = Z] f_j(Z, t) dZ \\ &= \int_0^\infty \alpha(t) f_j(Z, t) dZ. \end{aligned} \quad (33)$$

Because $\lambda(t)$, $\Delta(t)$, $R(t)$, and $m(t)$ do not depend on Z , $\alpha(t)$ does not depend upon Z , and we have from (33) that

$$\begin{aligned} E_j[U(t)] &= E_j[U(t) \mid Z_M(t) = Z] \\ &= \alpha(t) \end{aligned} \quad (34)$$

independent of the information set ϕ_j . Therefore, all investors agree that the expected value of $U(t)$, conditional on the return on the market, is equal to its unconditional expected value and that this common value is $\alpha(t)$.

Because all investors agree on $\alpha(t)$, we can unambiguously define the random variable $\epsilon(t) \equiv U(t) - \alpha(t)$. From (30) and (34), $\epsilon(t)$ can be expressed as

$$\begin{aligned} \epsilon(t) &= [p_1(t) - \theta(t)] \Delta(t) [Z_M(t) - R(t)] / [1 + m(t)] \\ &\quad \text{for } 0 \leq Z_M(t) \leq R(t) \\ &= [\theta(t) - p_2(t)] \Delta(t) [Z_M(t) - R(t)] / [1 + m(t)] \\ &\quad \text{for } R(t) < Z_M(t) < \infty. \end{aligned} \quad (35)$$

(Note that for $Z_M(t) = R(t)$, $\epsilon(t) = 0$. Otherwise, $\epsilon(t) < 0$ if $\theta(t) = 0$ and $\epsilon(t) > 0$ if $\theta(t) = 1$. Further important properties of $\epsilon(t)$ are summarized in the following proposition.)

PROPOSITION IV.1: If $\epsilon(t)$ is a random variable given by (35), then for each investor j , $j = 1, \dots, N$, (A) $E_j[\epsilon(t)] = 0$; (B) $E_j[\epsilon(t) \mid Z_M(t) = Z] = 0$ for all Z ; (C) $E[\epsilon(t) \mid Z_s(t) = Z] = 0$ for all Z ; (D) $\epsilon(t)$ is uncorrelated with both $Z_M(t)$ and $Z_s(t)$.

From (34), $E_j[U(t)] = \alpha(t)$, $j = 1, \dots, N$. $\epsilon(t) \equiv U(t) - \alpha(t)$, and therefore, $E_j[\epsilon(t)] = 0$, $j = 1, \dots, N$. From (34), $E_j[U(t)] = E_j[U(t) \mid Z_M(t) = Z]$ for all Z and $j = 1, \dots, N$. Therefore, $E_j[\epsilon(t) \mid Z_M(t) = Z] = E_j[U(t) \mid Z_M(t) = Z] - E_j[U(t)] = 0$ for $j = 1, \dots, N$ which proves (B). From (29), $Z_s(t)$ is a function of the random variable $Z_M(t)$ only. Hence, $E_j[\epsilon(t) \mid Z_M(t) = Z] = E_j[\epsilon(t)]$ for all Z implies that $E_j[\epsilon(t) \mid Z_s(t) = Z] = E_j[\epsilon(t)]$ for all Z . Hence, A and B implies C, and D follows immediately from B and C.

In summary, we have that

$$Z_p(t) = Z_s(t) + \alpha(t) + \epsilon(t) \quad (36)$$

where all investors agree on the value of $\alpha(t)$ and on the properties of $\epsilon(t)$ given in proposition IV.1. In the terminology of Rothschild and Stiglitz (1970), $\epsilon(t)$ is "noise" relative to the portfolio return $Z_M(t)$. Hence, the stochastic component of the return on the market timer's fund is equal to the stochastic component of $Z_M(t)$ plus noise where, by inspection of (35), the source of this noise is the error in the market timer's forecast. Indeed if, as in Section II, the market timer were a perfect forecaster, then $p_1(t) = p_2(t) = \theta(t) = 1$, and $\epsilon(t) \equiv 0$.

In the case when the market timer is a perfect forecaster, the equilibrium management fee can be determined immediately from (36) with no further assumptions. Because $\epsilon(t) \equiv 0$, $\alpha(t)$ must equal zero. Otherwise, the market timer's fund will either dominate or be dominated by the portfolio with return $Z_M(t)$, which would violate a necessary condition for equilibrium. Therefore, from (32), the equilibrium management fee must satisfy $m(t) = \lambda(t)\Delta(t)$. Depending upon the leverage position $\Delta(t)$ adopted by the fund, this formula corresponds exactly to the ones given by (7), proposition II.5, and proposition II.6 in Section II.

Unfortunately, for imperfect forecasting—that is, $\epsilon(t) \not\equiv 0$ —(36) alone is not sufficient to determine the equilibrium fee. However, we can derive the equilibrium fee for the case where all the risk associated with $\epsilon(t)$ is diversifiable or nonsystematic risk. This case is particularly relevant because we have assumed throughout the paper that the trades made by the market timer for the fund do not affect market prices. For such information trades to have a negligible effect on prices, the value of securities held by the fund would have to be a very small fraction of the total value of all securities outstanding. However, it will be shown that under this condition, virtually all the risk associated with $\epsilon(t)$ will be diversifiable. Thus, the equilibrium fee derived under the assumption that all the risk of $\epsilon(t)$ is diversifiable will be a close approximation to the "true" equilibrium management fee.

In the well-known Sharpe-Lintner-Mossin Capital Asset Pricing Model, the nondiversifiable or systematic risk of a security is measured by its "beta." Suppose that there are n risky securities in addition to the riskless security and let $X_i(t)$ denote the return per dollar on security i between t and $t + 1$, for $i = 1, 2, \dots, n$. Then, the beta of security i relative to information set ϕ_j is given by

$$\beta_i(t) = \text{cov}_j[X_i(t), Z_M(t)] / \text{cov}_j[Z_M(t), Z_M(t)], \quad (37)$$

where $\text{cov}_j[\cdot, \cdot]$ is the covariance operator based upon information set $\phi_j(t)$.

PROPOSITION IV.2: If, in equilibrium, securities are priced so as to satisfy the Security Market Line, $E_i[X_i(t)] - R(t) = \beta_i(t)[E_j[Z_M(t)] - R(t)]$, for all securities i , $i = 1, \dots, n$ and all information sets ϕ_j , $j = 1, \dots, N$, then the equilibrium management fee is given by $m(t) = \lambda(t)\Delta(t) = [p_1(t) + p_2(t) - 1]g(t)\Delta(t)$.

From (36), $\text{cov}_j [Z_p(t), Z_M(t)] = \text{cov}_j [Z_s(t), Z_M(t)] + \text{cov}_j [\epsilon(t), Z_M(t)]$. From proposition IV.1, all investors agree that $\text{cov}_j [\epsilon(t), Z_M(t)] = 0, j = 1, \dots, N$. Hence, $\beta_p^j(t) = \beta_s^j(t), j = 1, \dots, N$. Therefore, $E_j[Z_p(t)] = E_j[Z_s(t)], j = 1, \dots, N$. But, from (36), this implies that in equilibrium $\alpha(t) = 0$. Hence, from (32), the equilibrium management fee must satisfy $m(t) = \lambda(t)\Delta(t)$.

As is well known, rather specific assumptions about either investors' preferences or the probability distributions of returns are required for the hypothesized conditions of proposition IV.2 to obtain, and for the same reasons, beta is a rather specialized measure of systematic risk. However, using a more general measure of systematic risk developed in Merton (in press), the conclusion of proposition IV.2 can be shown to obtain under much weaker hypothesized conditions.

Let $\psi_j(t)$ denote the set of efficient portfolios relative to information set $\phi_j(t)$ where a portfolio with return per dollar $Z_e^j(t)$ will be a member of $\psi_j(t)$ if there exists an increasing, strictly concave function V whose first derivative satisfies $E_j\{V'[Z_e^j(t)][X_k(t) - R(t)]\} = 0$ for $i = 1, \dots, n$. Thus, $\psi_j(t)$ is the set of all feasible portfolios which might be chosen as an optimal portfolio by a risk-averse, expected-utility-maximizing investor who bases his decisions upon information set $\phi_j(t)$. For any security or portfolio k with return $Z_k(t)$ and any portfolio contained in $\psi_j(t)$ with return $Z_e^j(t)$, we define the systematic risk of security k relative to this efficient portfolio, $b_k^j(t)$, by

$$b_k^j(t) \equiv \text{cov}_j \{Y_j[Z_e^j(t)], Z_k(t)\}, \quad (38)$$

where

$$Y_j[Z_e^j(t)] \equiv \{V'[Z_e^j(t)] - E_j V'[E_e^j(t)]\} / \text{cov}_j \{V'[Z_e^j(t)], Z_e^j(t)\}.$$

As is proved in Merton (in press, theorem III.1), a necessary condition for equilibrium is that

$$E_j[Z_k(t)] - R(t) = b_k^j(t)\{E_j[Z_e^j(t)] - R(t)\} \quad (39)$$

for $j = 1, \dots, N$. Hence, $b_k^j(t)$ and equation (39) play the same role in the general expected utility model that beta and the Security Market Line play in the Capital Asset Pricing model.

PROPOSITION IV.3: If for each information set $\phi_j(t), j = 1, \dots, N$, there exists some portfolio contained in $\psi_j(t)$ whose return $Z_e^j(t)$ satisfies the condition $E_j[\epsilon(t) | Z_e^j(t) = Z] = 0$ for all values Z , then the equilibrium management fee is given by $m(t) = \lambda(t)\Delta(t)$.

From (36), $\text{cov}_j \{Y_j[Z_e^j(t)], Z_p(t)\} = \text{cov}_j \{Y_j[Z_e^j(t)], Z_s(t)\} + \text{cov}_j \{Y_j[Z_e^j(t)], \epsilon(t)\}$. $\text{Cov}_j \{Y_j[Z_e^j(t)], \epsilon(t)\} = E_j\{Y_j[Z_e^j(t)]\epsilon(t)\} = E_j\{Y_j(Z)E_j[\epsilon(t) | Z_e^j(t) = Z]\}$. But, by hypothesis, $E_j[\epsilon(t) | Z_e^j(t) = Z] = 0$. Hence, $\text{cov}_j \{Y_j[Z_e^j(t)], \epsilon(t)\} = 0$ and, from (38), $b_p^j(t) = b_s^j(t)$. Therefore, from (39), a necessary condition for equilibrium is that $E_j[Z_p(t)] =$

$E_j[Z_s(t)]$ for $j = 1, \dots, N$. Thus, a necessary condition for equilibrium is that $\alpha(t) = 0$, from which it follows that $m(t) = \lambda(t)\Delta(t)$.

Because the optimal portfolio of investor j is contained in $\phi_j(t)$, it follows from proposition IV.3 that the equilibrium management fee will be given by $m(t) = \lambda(t)\Delta(t)$ if $E_j[\epsilon(t) | Z^{jk}(t) = Z] = 0$ for all values of Z and $j = 1, \dots, N$ where $Z^{jk}(t)$ is the return per dollar on investor j 's optimal portfolio. Similarly, if $[X^1(t), \dots, X^N(t)]$ denotes the returns per dollar on a set of portfolios which span

$$\bigcup_{j=1}^N \phi_j(t)$$

and if $E_j[\epsilon(t) | X^1(t) = X^1, \dots, X^N(t) = X^N] = 0$ for $j = 1, \dots, N$, then in equilibrium, $m(t) = \lambda(t)\Delta(t)$.

PROPOSITION IV.4: If all investors agree that the market portfolio is an efficient portfolio, then, in equilibrium, $m(t) = \lambda(t)\Delta(t)$.

By hypothesis, the market portfolio is contained in $\phi_j(t)$ for $j = 1, \dots, N$. From proposition IV.1, $E_j[\epsilon(t) | Z_{jt}(t) = Z] = 0$ for $j = 1, \dots, N$ and all values of Z . Hence, the hypothesized conditions of proposition IV.3 are satisfied, and therefore, in equilibrium, $m(t) = \lambda(t)\Delta(t)$.

In summary, if the $\epsilon(t)$ component of the risk of the market timer's fund is completely diversifiable, then the management fee will equal $\lambda(t)\Delta(t)$. The economic interpretation of this fee is very similar to the one presented in Section II for the perfect-forecasting case. If the risk of $\epsilon(t)$ can be "diversified away," then the returns to the fund will be essentially the same as the returns from following the options investment strategy (28) which uses no forecast information. However, in the absence of a management fee, the market timer's fund achieves this pattern of returns without having to "pay" for the options. Therefore, the value of the market timer's forecast skills will equal the value of the options required to follow the options strategy with returns given by (28). To see that this is indeed the case when $m(t) = \lambda(t)\Delta(t)$, note that, as in (8) of Section II, the management fee per dollar of *gross* investment in the fund is equal to $m(t)/[1 + m(t)]$. The value of put options per dollar of investment in the options investment strategy, $\omega_2(t)$, is equal to $[p_1(t) + p_2(t) - 1]z(t)\Delta(t)/[1 + m(t)] = \lambda(t)\Delta(t)/[1 + m(t)]$. Therefore, for $m(t) = \lambda(t)\Delta(t)$, the management fee is just equal to the value of the put options. For the same leverage level, $\Delta(t)$, the number of "free" put options received is smaller for the imperfect forecaster than for the perfect forecaster because $|p_1(t) + p_2(t) - 1| < 1$. In the limiting case of no forecasting skills, $p_1(t) + p_2(t) = 1$, and the number of "free" put options is zero as is the equilibrium management fee.

Having established that $m(t) = \lambda(t)\Delta(t)$ when the risk associated with $\epsilon(t)$ is diversifiable, we now show that this solution will be a good approximation to the exact equilibrium management fee provided that

the value of securities held by the fund is a small fraction of the total value of all securities outstanding.

By inspection of (35), the sources of randomness in $\epsilon(t)$ are the return on the market and forecast error by the market timer. In the proof of proposition IV.1, it was already shown that $E_j[\epsilon(t) | Z_M(t) = Z] = 0$ for all values of Z and $j = 1, \dots, N$. Hence, if $X_i(t)$, $i = 1, \dots, n$, represents the returns on all available securities (other than the fund itself), then there is little loss in generality by assuming that for $j = 1, \dots, N$, $E_j[\epsilon(t) | X_1(t) = X_1, \dots, X_n(t) = X_n] = 0$ for all values of $X_1, \dots, X_n$. This assumption leads to the following important proposition:

PROPOSITION IV.5: Let $d_j^*(t)$ denote the dollar demand for the market timer's fund by investor j in his optimal portfolio at time t . If $E_j[\epsilon(t) | X_1(t) = X_1, \dots, X_n(t) = X_n] = 0$ for all values $X_1, \dots, X_n$, then $d_j^*(t) = 0$ if and only if $m(t) = \lambda(t)\Delta(t)$.

The proof is as follows: If $m(t) = \lambda(t)\Delta(t)$, then $\alpha(t) = 0$ and from (36), $Z_p(t) = Z_s(t) + \epsilon(t)$. By hypothesis and proposition IV.1, we have that $E_j[\epsilon(t) | Z_s(t) = Z, X_1(t) = X_1, \dots, X_n(t) = X_n] = 0$ for all values of $Z, X_1, \dots, X_n$. Hence, from Merton (in press, theorem II.3), we have that $d_j^*(t) = 0$. If $d_j^*(t) = 0$, then the return per dollar on investor j 's optimal portfolio, $Z^{j*}(t)$, will be a linear combination of $X_1(t), \dots, X_n(t)$ and $R(t)$. Therefore, $E_j[\epsilon(t) | Z^{j*}(t) = Z] = 0$ for all values of Z . But, from proposition IV.3, this condition implies that $m(t) = \lambda(t)\Delta(t)$.

Thus, from proposition IV.5, if $m(t) = \lambda(t)\Delta(t)$, then $d_j^*(t) = 0$ for $j = 1, \dots, n$, and the aggregate demand for the fund's shares,

$$D_f(t) = \sum_{j=1}^N d_j^*(t),$$

will equal zero. Therefore, for $m(t) = \lambda(t)\Delta(t)$, all investors will be just indifferent between having the opportunity to invest in the fund or not. From Merton (in press, eq. [III.3]), we have that

$$\partial d_j^*(t)/\partial \alpha(t) > 0 \quad \text{at} \quad d_j^*(t) = 0, \quad j = 1, \dots, N. \quad (40)$$

From (40), for $m(t) > \lambda(t)\Delta(t)$, $d_j^*(t) < 0$ for $j = 1, \dots, N$, and therefore $m(t) = \lambda(t)\Delta(t)$ is the maximum possible equilibrium management fee.

Of course, a profit-maximizing market timer would never choose a management fee exactly equal to $\lambda(t)\Delta(t)$ because at this maximum fee, $D_f(t) = 0$ and he would receive zero revenues. Suppose, however, that in equilibrium, the value of securities held by the fund is very much smaller than aggregate wealth, $W(t)$. Then, $D_f(t) \ll W(t)$. This case is empirically quite relevant because the market value of stocks and bonds in the United States is in excess of \$1 trillion, and therefore, even for a large (e.g., \$1 billion) fund, $D_f(t)/W(t)$ will be quite small. From (40), if $m(t) < \lambda(t)\Delta(t)$, then every investor will choose a positive

investment in the fund, that is, $d_j^+(t) > 0, j = 1, \dots, N$. If the number of investors, N , is large and the distribution of wealth among investors, $[W_j(t)]$, is not perverse, then the fractional allocation of each investor's portfolio to the fund will be positive but small, $0 < d_j^+(t)/W_j(t) \ll 1$. In this case, the limiting management fee associated with incipient demand will provide a close approximation to the equilibrium management fee.

Although prosaic, a good reason for using $\lambda(t)\Delta(t)$ as an approximation to the equilibrium management fee is the observed fact that even the largest of funds represents a small fraction of the total value of securities outstanding. However, further theoretical support for this approximation can be found in the case where there are many independent market timers offering their services to investors. As with other investments, it will be optimal for risk-averse investors to "diversify" and spread their holdings among all the market timers' funds. If the number of such funds is large, then this optimal behavior will lead to a relatively small aggregate demand for any one fund.

Suppose that there are Q market timers and that each provides information and behaves in a fashion similar to the market timer described throughout this paper. Let the subscript q applied to otherwise previously defined variables denote that variable for market timer $q, q = 1, \dots, Q$.

PROPOSITION IV.6: If, for $q = 1, \dots, Q$, the forecast variable for market timer $q, \theta_q(t)$, is independent of the forecast variables of all other market timers, then as the number of market timers becomes large, the equilibrium management fee for market timer q 's fund will almost certainly satisfy $m_q(t) = \lambda_q(t)\Delta_q(t) = [p_{1q}(t) + p_{2q}(t) - 1]g(t)\Delta_q(t)$.

The method of proof is similar to the one used by Ross (1976) to derive his "Arbitrage Theory" of asset pricing. Consider the following type of portfolio strategy: Invest \$1 in the riskless security; and for $q = 1, \dots, Q$, invest $\$ \mu_q/Q$ in the fund of market timer q and go short $\$ \mu_q/Q$ of the portfolio with return per dollar $Z_{sq}(t)$ given by (28). We restrict our attention to only those strategies $\mu = (\mu_1, \dots, \mu_Q)$ which have the property that, for each q , where $q = 1, \dots, Q$, $|\mu_q| \geq 0$ and $|\mu_q|$ has an upper bound independent of Q . This restriction corresponds to the set of "well-diversified" portfolios in the Ross analysis. Let $z(t; \mu, Q)$ denote the return per dollar on the portfolio with strategy μ . From (36),

$$Z(t; \mu, Q) = R(t) + \left| \sum_{j=1}^N \mu_q \alpha_{jq}(t)/Q + \sum_{j=1}^N \mu_q \epsilon_{jq}(t) \right|/Q.$$

By hypothesis, the $[\theta_q(t)]$ are mutually independent. Therefore, it follows from (35) that for $j = 1, \dots, N$ and $q = 1, \dots, Q, E_j[\epsilon_{jq}(t) | \epsilon_1(t) = \epsilon_{11}, \dots, \epsilon_{q-1}(t) = \epsilon_{q-1,1}, \epsilon_{q+1}(t) = \epsilon_{q+1,1}, \dots, \epsilon_Q(t) = \epsilon_Q]$ for all values

$\epsilon_1, \dots, \epsilon_{q-1}, \epsilon_{q+1}, \dots, \epsilon_Q$. Therefore, for each t , every investor agrees that $[S_q(t)]$ form a martingale where

$$S_q(t) \equiv \sum_1^Q \mu_q \epsilon_q(t).$$

Because the variance of $\epsilon_q(t)$ exists and $|\mu_q|$ has an upper bound independent of Q , the variance of $\mu_q \epsilon_q(t)$ has a finite upper bound independent of Q . Therefore, by the Law of Large Numbers for Martingales, every investor agrees that as $Q \rightarrow \infty$, $S_q(t)/Q \rightarrow 0$ with probability one. Hence, as $Q \rightarrow \infty$,

$$Z(t; \mu, Q) \rightarrow R(t) + \left[\sum_1^Q \mu_q \alpha_q(t) \right] / Q$$

with probability one. A necessary condition for equilibrium is that no security dominate any other security. Therefore, as $Q \rightarrow \infty$, $[\sum_1^Q \mu_q \alpha_q(t)]/Q \rightarrow 0$ in equilibrium. But the choice for μ is virtually arbitrary. Hence, satisfaction of this condition will require that as $Q \rightarrow \infty$, for all (but possibly a finite number of) market timers, $\alpha_q(t) \rightarrow 0$. From (32), it follows that almost certainly, $m_q(t) \rightarrow \lambda_q(t) \Delta_q(t)$ as $Q \rightarrow \infty$.

For proposition IV.6 not to be vacuous, it is essential that a significant number of market timers have some forecasting skills. Otherwise, if only a few have such skills, then $p_{1q}(t) + p_{2q}(t) = 1$ for virtually all market timers, and trivially, as $Q \rightarrow \infty$, $m_q(t) = 0 = \lambda_q(t) \Delta_q(t)$ for all but possibly a finite number of market timers. If the Efficient Markets Hypothesis held exactly, then $m_q(t) = 0 = \lambda_q(t) \Delta_q(t)$ for all q , and it would be optimal for investors not to invest in any of the funds. On the other hand, if there is even a very small chance that any given market timer q has some forecasting skill, then $[p_{1q}(t) + p_{2q}(1) - 1]$ will be positive, albeit small, and it will be optimal for every investor to put a small fraction of his wealth in each fund.

The fact that Q is "large" in the sense of proposition IV.6 does not mean that all members of the population or even all investors must engage in market-forecasting activities. That is, $Q \gg 1$ is consistent with $Q/N \ll 1$. For example, $Q = 10,000$ is quite a large number for the purpose of applying the asymptotic conclusions of proposition IV.6 as a good approximation. For the U.S. population 220 million, this would require that only one person in 22,000 be engaged in market forecasting. There are approximately 17 million investors who hold individual stocks in the United States. Hence, for Q to equal 10,000, only one in 1,700 of these active investors need be a market timer. Therefore, market forecasting can be a relatively rare occupation, and yet, the hypothesized conditions of proposition IV.6 could be essentially satisfied.

Independence of the forecast errors among market timers is essential

for proposition IV.6 to obtain. Indeed, if all funds were managed using the same information set, then $\{\epsilon_{qj}(t)\}$ would be perfectly dependent, and as will be shown later in proposition IV.7, the number of funds available will have no effect upon the equilibrium management fees. However, even if there are "families" of market timers where forecasts among members of each family are not independent, proposition IV.6 will still obtain provided there are a large number of independent families.

Proposition IV.6 provides essentially a "demand" reason why the equilibrium size of the market timer's fund would be relatively small. The cost function of the market timer provides a "supply" reason. If beyond some size marginal costs increase, there will be an optimal size for the fund. If this optimal size is small relative to the value of all securities outstanding, then the equilibrium management fee can be approximated by $\lambda(t)\Delta(t)$. The manifest costs of operating the fund, such as salaries, information gathering and processing, bookkeeping, etc., are not likely to produce increasing marginal costs. However, there are other, latent costs associated with size that could cause the marginal cost function to be increasing. As more employees are required to handle an increase in the size of the fund, the risk increases that the proprietary knowledge used by the market timer will be "leaked" to either competitors or the public generally. Such leaks would diminish or even eliminate the rents that the market timer can earn from this knowledge. Even if such direct leaks could be prevented, for a large enough size, the transactions made by the fund will affect market prices. If $A(t)$ is the amount of assets under management, then every change requires a trade of $\$A(t)\Delta(t)$ in the market portfolio and $\$A(t)\Delta(t)$ in the riskless security. Thus, as $A(t)$ gets larger, the market timer must either reduce his response (per unit of assets) to a change in view, $\Delta(t)$, or pay a higher "spread." Either action will diminish the performance of the fund and therefore reduce the management fee that investors will be willing to pay for his services. In a business whose whole value comes from proprietary knowledge which cannot be protected by patents, it is not unreasonable that these latent costs could be significant enough to induce an optimal fund size which is small relative to the value of all securities.

Of course, to determine the optimal fund size and the associated exact equilibrium structure of management fees would require an analysis of the optimal behavior of the market timer. If the objective of the market timer is to maximize the total profits that he receives for his forecasting skills, then to achieve this objective, he can choose both the management fee rate, $m(t)$, and the investment or leverage policy of the fund, $\Delta(t)$. He can also choose to have more than one fund, with a different rate and policy for each fund. In general, the optimal fund size

and management fee will depend upon the cost function and the structure of investors' demand functions for the fund, which in turn depend upon their preferences, endowments, and probability assessments for the returns on all available securities. However, as we now show, neither the leverage chosen for the fund, $\Delta_i(t)$, nor the number of funds made available, K , will affect the total revenues received by the market timer.

PROPOSITION IV.7: Let $m_i^*(t) \equiv M[t, \Delta_i(t)]$ be the optimal management fee rate for the i th fund when the leverage policy chosen for that fund is $\Delta_i(t)$. If investors can borrow or sell short without restriction, then (A) $m_i^*(t) = M[t, 1]\Delta_i(t)$, $i = 1, \dots, K$; and (B) the total management fees received by the market timer are invariant to either the number of funds K or the leverage policy $\Delta_i(t)$ chosen provided that $\Delta_i(t) > 0$.

From (35) and (36), the return to investing $\$I$ in fund i can be written as $Y_i(t) \equiv IZ_{si}(t) + I\alpha_i(t) + I\Delta_i(t)\zeta(t)/[1 + m_i^*(t)]$ where $Z_{si}(t)$ is given by (28) with $m(t) = m_i^*(t)$ and $\Delta(t) = \Delta_i(t)$, $\alpha_i(t) \equiv [\lambda(t)\Delta_i(t) - m_i^*(t)]R(t)/[1 + m_i^*(t)]$, and $\zeta(t) \equiv [p_1(t) - \theta(t)][Z_{si}(t) - R(t)]$ for $0 \leq Z_{si}(t) \leq R(t)$ and $[\theta(t) - p_2(t)][Z_{si}(t) - R(t)]$ for $Z_{si}(t) > R(t)$. Note that the random variable $\zeta(t)$ does not depend upon either $m_i^*(t)$ or $\Delta_i(t)$ and therefore is the same for all the funds, $i = 1, \dots, K$. Consider the following alternative investment of $\$I$: invest $\$I$ in the portfolio with return per dollar $Z_{si}(t)$; invest $\$I\omega_{ik}$ in fund k ; and go short $\$I\omega_{ik}$ of the portfolio with return per dollar $Z_{sk}(t)$ where we pick $\omega_{ik} = \Delta_i(t)[1 + m_k^*(t)]/\{\Delta_k(t)[1 + m_i^*(t)]\}$. The return to this alternative investment, $Y_{ik}(t)$, can be written as $Y_{ik}(t) \equiv IZ_{si}(t) + I\omega_{ik}\{Z_{sk}(t) + \alpha_k(t) + \Delta_k(t)\zeta(t)/[1 + m_k^*(t)]\} - I\omega_{ik}Z_{sk}(t)$. Substituting for ω_{ik} , we can rewrite this expression as $Y_{ik}(t) = Y_i(t) + I[\omega_{ik}\alpha_k(t) - \alpha_i(t)]$. By inspection, if $\omega_{ik}\alpha_k(t) > \alpha_i(t)$, then $Y_{ik}(t) > Y_i(t)$ for every possible outcome, and every investor would be better off to invest in fund k in the alternative strategy than to invest in fund i . Therefore, to avoid fund i being dominated by fund k , $\omega_{ik}\alpha_k(t) \leq \alpha_i(t)$. Of course, to avoid fund k being dominated by fund i , $\omega_{ki}\alpha_i(t) \leq \alpha_k(t)$. Noting that $\omega_{ik} = 1/\omega_{ki}$, we have that for neither fund to be dominated by the other, $\omega_{ik}\alpha_k(t) = \alpha_i(t)$. From the definition of ω_{ik} and $\alpha_i(t)$, we have, therefore, that $m_i^*(t)/\Delta_i(t) = m_k^*(t)/\Delta_k(t)$, for $i, k = 1, \dots, K$. It follows immediately that $m_i^*(t) = M[t, 1]\Delta_i(t)$, which proves A.

Consider a set of optimal management fees $\{m_i^*(t)\}$ that support an equilibrium allocation where an aggregate of $\$I_i(t) (\geq 0)$ is invested in fund i , $i = 1, \dots, K$. For $\omega_{ik}\alpha_k(t) = \alpha_i(t)$, $Y_{ik}(t) = Y_i(t)$, and therefore all investors would be indifferent between investing $\$I$ in fund i or investing $\$I\omega_{ik}$ in fund k and using the alternative strategy. Thus, the same set of optimal management fees that support the equilibrium

allocation $\{I_i(t)\}$ will also support the alternative equilibrium allocation $\{I'_i(t)\}$ where

$$I'_1(t) = \sum_{i=1}^N \omega_{i1} I_i(t)$$

and $I'_i(t) = 0$ for $i = 2, \dots, K$. In the first allocation, the total management fee received by the market timer is

$$\sum_{i=1}^N I_i(t) m_i^*(t) / [1 + m_i^*(t)].$$

In the alternative allocation, the total management fee received by the market timer is

$$\begin{aligned} \sum_{i=1}^N I'_i(t) m_i^*(t) / [1 + m_i^*(t)] &= \sum_{i=1}^N \omega_{i1} I_i(t) m_i^*(t) / [1 + m_i^*(t)] \\ &= \sum_{i=1}^N m_i^*(t) \Delta_i(t) I_i(t) \{ \Delta_i(t) [1 + m_i^*(t)] \} \end{aligned}$$

because $\omega_{i1} = \Delta_i(t) [1 + m_i^*(t)] / \{ \Delta_i(t) [1 + m_i^*(t)] \}$. However, from part A, $m_i^*(t) \Delta_i(t) / \Delta_i(t) = m_i^*(t)$. Hence,

$$\sum_{i=1}^N I'_i(t) m_i^*(t) / [1 + m_i^*(t)] = \sum_{i=1}^N I_i(t) m_i^*(t) / [1 + m_i^*(t)],$$

and the total management fee received is the same under either allocation. Since the choice of fund 1 in the alternative allocation was clearly arbitrary, no conditions on $\Delta_i(t)$, other than it be positive, were imposed, and part (ii) is proved.

In essence, part (ii) of proposition IV.7 states that investors will not pay (the market timer) for financial leverage that they can create for themselves, and it is therefore a special version of the well-known Modigliani-Miller theorem (1958).

Because the positions taken by the K funds described in proposition IV.7 are based on the same (market timer's) information set, their returns will all be perfectly dependent, and there is no "diversification effect" on the management fees as was the case in proposition IV.6. Moreover, if the market timer's forecast is such that a change in the position of one of his funds is warranted, then this same forecast triggers a change in the positions of all his funds. Therefore, if the effect on market prices of a transaction is solely a function of size, then the aggregate cost to all his funds from "spreads" will depend only on the size of the total transactions made by them in response to his forecast. The size of these transactions will be

$$\$ \sum_{i=1}^K A_i(t) \Delta_i(t)$$

in the market portfolio and the same amount in the riskless security where $A_i(t)$ is the value of assets under management in fund i . In the proof of part (ii) of proposition IV.7, it was shown that investors would be indifferent between the allocations $[I_i(t)]$ and $[I'_i(t)]$ among the market timer's K funds. We now show that the size of total transactions in response to the market timer's forecast will be the same under either allocation.

If $I_i(t)$ is the gross investment in fund i , then after taking out management fees, $A_i(t) = I_i(t)/[1 + m_i^*(t)]$. Therefore, the size of the total transactions under the first allocation is given by

$$\sum_1^K I_i(t) \Delta_i(t)/[1 + m_i^*(t)].$$

Similarly, the size of total transactions under the second allocation is given by

$$\sum_1^K I'_i(t) \Delta_i(t)/[1 + m_i^*(t)].$$

Substituting for $I'_i(t)$, we can rewrite this latter expression as

$$\begin{aligned} \sum_1^K I_i(t) \Delta_i(t) \Delta_i(t) [1 + m_i^*(t)] / \{ [\Delta_i(t)] [1 + m_i^*(t)] [1 + m_i^*(t)] \} \\ = \sum_1^K I_i(t) \Delta_i(t) / [1 + m_i^*(t)]. \end{aligned}$$

Therefore, the size of total transactions under either allocation is the same. It follows that neither the leverage chosen for the fund nor the number of funds made available will affect the market timer's transaction costs from spreads. Hence, from this result and proposition IV.7, the total profits earned by the market timer will be invariant to either the number of funds he manages or the leverage policies of the funds.

In contrast to the Jensen model (1972a, sec. 4), we have from proposition IV.7 that, from the investor viewpoint, there is no "optimal" or "preferred" policy $[\eta_1(t), \eta_2(t)]$. All that is required here is that investors know what the policy is and that $\eta_2(t) > \eta_1(t)$. While the maximum management fee, $\lambda(t) \Delta(t)$, will only provide a reasonable approximation to the true equilibrium fee if investment in the fund is a small fraction of total investment, the proof of proposition IV.7 required no such assumption. Therefore, the conclusions of this proposition are valid, independent of the size of the fund. Hence, to analyze the optimal behavior of a market timer and determine the equilibrium value of his information, one can assume without loss of generality that the market timer exploits his information by managing a single fund and that the leverage policy of the fund is $\Delta(t) = 1$.

I close this study with a brief example which suggests that other than in the (approximately) incipient case stressed here, it will not be appropriate to assume that transactions made by the market timer will have a negligible effect on market prices. Therefore, to properly analyze the optimal behavior of a market timer and the associated equilibrium structure of management fees for the nonincipient case, one must develop a significantly more complex model where the market timer and investors in making their optimal decisions take into account the effect of the fund's size on the performance of the fund. Needless to say, no such attempt will be made here.

For purposes of the example, we assume that investors are risk-averse, mean-variance utility maximizers with homogeneous probability assessments. It is also assumed that the distribution of the market timer's forecast error is independent of the return on the market, that is, $p_1(t) = p_2(t) = p(t)$. If it is assumed that the fund's transactions have no effect on market prices, then from a standard mean-variance portfolio analysis, the equilibrium aggregate demand for the fund will satisfy

$$D_f(t)/D_M(t) = \alpha(t)\sigma_M^2(t)/\{\bar{Z}_M(t) - R(t)\} \text{var}[\epsilon(t)] \quad (41)$$

where $D_M(t)$ is the equilibrium aggregate demand for (nonfund) investment in the market, $\bar{Z}_M(t)$ and $\sigma_M^2(t)$ are, respectively, the agreed-upon expected value and variance of the return per dollar on the market portfolio. From (35), we have for $p_1(t) = p_2(t) = p(t)$ and $\Delta(t) = 1$ that

$$\text{var}[\epsilon(t)] = p(t)[1 - p(t)]\sigma_M^2(t)[1 + m(t)]^2. \quad (42)$$

Hence, from (32) and (42), (41) can be rewritten as

$$D_f(t)/D_M(t) = [(2p(t) - 1)g(t) + m(t)][1 + m(t)]R(t) / \{p(t)[1 - p(t)]\bar{Z}_M(t) - R(t)\}. \quad (43)$$

If the market timer has no variable costs,²² then he will choose the management fee which maximizes his total revenues, $m(t)D_f(t)/[1 + m(t)]$. This revenue-maximizing management fee, $m^*(t)$, is given by

$$m^*(t) = \lambda(t)/2 \\ = [2p(t) - 1]g(t)/2. \quad (44)$$

Hence, from (44), (43) can be rewritten as

$$D_f(t)/D_M(t) = \lambda(t)[2 + \lambda(t)]R(t)/\{4p(t)[1 - p(t)]\bar{Z}_M(t) - R(t)\}. \quad (45)$$

By inspection, the equilibrium demand starts at zero when $p(t) = .5$ and the market timer has no forecasting skills and is a strictly increas-

22. Of course, we have neglected throughout our analysis those operating costs common to all investment companies including "passively managed" or "no-information" funds. So, the derived management fee is an "incremental" fee for forecasting skills. The "variable costs" referred to here are the additional trading costs associated with "actively managed" funds.

ing function of $p(t)$. To provide a sense of the magnitude of aggregate demand as a function of $p(t)$, we substitute into (45) the averages for the other parameters based upon a long past history. From Ibbotson and Sinquefeld (1977, p. 19), the annual arithmetic excess return on the market, $Z_M(t) - R(t)$, had an average of 9.2% with a standard deviation of 22.6% for the period 1926-76. The average annual return on U.S. Treasury bills for this same period was 2.4%. Using the historical standard deviation for the market, we derive from (10) an estimate for the 1-year put price, $g(t)$, equal to 0.09. Using these historical values in (45), we have for a 1-year forecaster that

$$D_f(t)/D_M(t) = 3.96 \nu(t)[1 + 0.09 \nu(t)]/[1 - 4 \nu^2(t)], \quad (46)$$

where $\nu(t) \equiv p(t) - 0.5$. Hence, for a market timer with an annual forecasting edge of $p(t) = 0.51$, the equilibrium demand for his fund would be approximately 4% of all other risky asset holdings. And for each percentage point increase in his probability of a correct forecast, the equilibrium demand will increase by fourfold. So, at $p(t) = 0.55$, the fund will represent more than 20% of all other risky assets. Of course, the demand would be even larger if the market timer's forecasting interval were less than 1 year.

Noting that the assumed investment strategy for the fund requires trades of the order of $D_f(t)$, it is simply unrealistic to assume that transactions of the size implied by (46) will not affect market prices. We have, of course, only shown this to be the case in a specific example. However, there is nothing especially pathological about the example. Hence, to properly determine the optimal fund size for a market timer with virtually any forecasting skill at all, one should explicitly take into account the trading or "spread" costs associated with the size of the fund.

In summary, it has been shown that, up to an additive noise term, the pattern of returns from an investment strategy based upon market timing will be the same as the pattern of returns from a "partial protective put" option investment strategy. If this noise term which is caused by forecast error is diversifiable, then, independent of investors' preferences, endowments, or probability beliefs, the equilibrium management fee is given by $m(t) = \lambda(t)\Delta(t) = [p_1(t) + p_2(t) - 1]g(t)[\eta_2(t) - \eta(t)]$. Because the equilibrium management fee is proportional to the price of a put option on the market portfolio, the comparative statics analysis of the equilibrium management fee presented in Section II for the perfect-forecasting case carries over to the imperfect-forecasting case presented here. Finally, in the empirically relevant case where the size of the market timer's fund represents a small fraction of the total value of all securities, $\lambda(t)\Delta(t)$ will provide a close approximation to the exact equilibrium management fee.

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