Uri Dothan and Joseph Williams*
Northwestern University

Education as an Option

To the student education provides options. In practice, premature commitment to a specific occupation or career can be costly. Also, at an early age an individual is often uncertain about his abilities, personal preferences, and even general conditions in the labor market. Education, however, provides the option of postponing commitment to a specific career, plus the further option of choosing among additional careers after graduation. In this paper a general valuation equation for education as an option is derived under uncertainty and solved in several interesting special cases. Throughout, properties of the solutions are described in considerable detail.

I. Introduction

To the student, education provides many benefits. In part, education produces human capital. By attending school and training on the job, the student acquires marketable skills, thereby sacrificing current labor income while augmenting future labor income. Among economists this concept is familiar. Beginning with the seminal work by Mincer (1958) and Becker (1964), models of the accumulation of human capital have been extended to the life cycle and, more recently, to uncertainty.[1] Simultaneously, several of these models have been tested empirically.[2]

To economists, education also serves as a signal. By completing a demanding educational program, the student signals to prospective employers that he possesses greater innate productivity than other potential employees not completing similar programs. This potentially greater productivity the employer then rewards with more favorable conditions of employment,

* We are grateful to S. Bhattacharya and D. Clay for helpful comments on previous drafts.

1. Models of the accumulation of human capital over the life cycle include Ben-Porath (1967), Levhari and Weiss (1974), Blinder and Weiss (1976), and Heckman (1976). Uncertainty appears in Weiss (1972), Levhari and Weiss (1974), and Williams (1978, 1979).

2. For empirical tests of life-cycle models, see Brown (1976) and the references cited therein. A survey of the literature appears in Blaug (1976).

(*Journal of Business*, 1981, vol. 54, no. 1)

0021-9398/81/5401-0001$01.50

in turn giving the student some incentive to complete his schooling. A concept developed by Arrow (1973) and Spence (1974), work on signaling is currently in progress.[3]

In this paper another, related characteristic of education is explored. In part, education acts like an option. For many individuals premature commitment to a specific occupation or career can be costly. Both the time required for training and any direct financial costs are often considerable. Also, at an early age the student is often uncertain about his personal preferences, abilities, and even general conditions in the labor market. In this situation, education appears valuable because it allows the student the option of delaying commitment to a specific occupation or career plus the further option of choosing among additional careers after graduation.

As suggested by the title, this paper focuses on the value of education as an option. In Section II of the paper, a valuation equation for education as an option is derived under plausible conditions. Together with appropriate boundary conditions, this equation uniquely determines for the individual the value of education as a function of several variables. These variables include the student's current evaluation of all subsequent employment opportunities and the current time to graduation. In this problem, the valuation equation is similar to the corresponding equation from the theory of options in finance.[4] However, because both education and employment opportunities represent untraded assets, the derivation differs from the usual hedging argument in finance.

In Section III the general valuation equation for education is solved in one especially interesting special case. Properties of the valuation formula are described and comparative statics are determined. Here, for example, it is shown that the individual's assessment of the personal benefits to education increases with both increasing personal evaluations of alternative employment opportunities and increasing time to graduation. In this solution the individual's assessment of the value of education is independent of both the mean returns to alternative employment opportunities and the riskless rate of interest—except possibly as these parameters affect the personal assessment of potential employment opportunities. In addition, the personal evaluation of education is shown to be increasing with increasing risks to alternative job opportunities, but decreasing with increasing correlation between these risks. Thus, to the individual, education appears to be most valuable as an option when it provides broad training permitting postponed entry into a high-risk occupation.

Section IV adopts a different approach. Because education is an untraded asset, its price or personal value is typically unobservable.

3. See Blaug (1976), Sahota (1978), and the references cited therein.
4. See Black and Scholes (1973), Merton (1973), and the survey by Smith (1976).

Instead, with education only the time the student leaves school, and hence the distribution of schooling across individuals, is identifiable. Accordingly, in this section approximate numerical solutions are computed for the student's optimal time to leave school in the simple educational problem of Section III. Not surprisingly, it is shown that market forces reducing the personal value of education also eventually induce premature departure from school. For example, increasing the value of immediate unskilled employment relative to subsequent skilled employment eventually encourages the student to quit school. Similarly, decreasing the nonpecuniary benefits to education measured net of tuition and foregone wages, decreasing the risks to employment, or, alternatively, increasing the correlation between risks decreases the personal value of education and again encourages early departure from school. In this solution, the optimal time to quit school depends critically on the personal nonpecuniary benefits to education measured net of tuition and foregone wages. Other things equal, the larger this difference, the more likely the student is to leave school later rather than earlier in his educational program. By contrast, for all students the optimal time to quit school is significantly more sensitive to changes in personal preferences, perceived risks, and market conditions earlier rather than later in the educational program.

In Section V the major results from previous sections are briefly summarized. Empirical implications are described and possibilities for future work are discussed. Finally, the Appendix contains technical details of the derivations in Sections II and IV.

II. The Valuation Equation

Consider an individual currently facing a choice between immediate employment and additional education. Past training and experience have qualified this individual for any one of J alternative occupations or careers, each of which requires further training on the job. Suppose that currently, at time $t = 0$, the individual can evaluate all J occupations or careers in terms of the dollar equivalents $x_j(0)$, $j = 1, \ldots, J$. As personal assessments of skill-specific human capital, these dollar equivalents measure the current values of all future wages, adjusted to reflect any differences in nonpecuniary conditions of employment. Because employment opportunities represent untraded assets, the J evaluations depend in general on both the financial wealth and personal preferences of the individual.

If at time 0 the individual selects further schooling, then at any time t prior to graduation at time T, he has the option of leaving school and entering the labor market. Once in the market he chooses among the J available occupations or careers with the current values $x_j(t)$, $j = 1, \ldots, J$. Alternatively, if the student completes his schooling, then upon graduation at time T, he can choose among an additional K occupations or careers, each of which requires additional training on the job. These new jobs, plus the original J jobs, the individual again evaluates in terms of the dollar equivalents $x_j(T), j = 1, \ldots, J + K$.

Clearly, this model is highly simplified. In actual practice, no decision is irrevocable. At some cost the individual can always switch from school to work, back to school, and then on to a new career. Also, in practice, schooling is sequential. When the student completes one educational program, often he has the option of entering subsequent programs. Nevertheless, in this simple model all such complications are ignored. Here commitment to a specific career is extremely costly; by assumption, the individual's decision to leave school is irrevocable. Also, schooling is not sequential; upon graduation the student is forced to seek immediate employment.

Over time economic conditions change. In addition, the individual gradually accumulates progressively more information about his own abilities, personal preferences, and alternative employment opportunities. As a result, the personal evaluations $X = (x_1, \ldots, x_{J+K})'$ of all employment opportunities also evolve through time. When viewed from the initial time 0, these changes are at least partly stochastic. In particular, if all adjustments in X occur continuously in time with independent stochastic increments and plausible restrictions on means and variances, then X follows a diffusion process:[5]

$$I^{-1}(X)dX = (\phi - \omega)dt + \psi dZ \tag{1}$$

Here, $I(X)$ is a diagonal matrix with the nonzero diagonal elements $x_1, \ldots, x_{J+K}$; hence, $I^{-1}(X)dX$ is a vector with the nonzero elements $dx_1/x_1, \ldots, dx_{J+K}/x_{J+K}$ representing, respectively, the instantaneous rates of return on all potential occupations. Also, ϕ and ω are $J + K$–dimensional column vectors determining the expected rates of return on all occupations, while ψ is a $(J + K) \times N$–dimensional matrix specifying the corresponding variances and covariances. Finally, dZ is the infinitesimal increment in an N-dimensional Wiener process.[6]

These equations can be interpreted as follows. Over each infinitesimal interval of time dt, the individual adjusts his evaluation of each future employment opportunity x_j at a rate dx_j/x_j reflecting an expected effect $(\phi_j - \omega_j)dt$ plus a purely random effect $\psi_j dZ$. Here, ϕ_j represents the individual's current expected rate of return per unit of time on skill j when employed in occupation j. In part, this return includes the wage $\omega_j x_j$ paid per unit of time in occupation j. However, while in school the student foregoes this wage, thereby reducing his average rate of return per unit of time on skill j to $\phi_j - \omega_j$. Finally, each

5. The additional restriction is as follows: If Δx_j is the increment in x_j over the interval of time Δt, then the mean and variance of Δx_j are proportional to Δt. For a discussion of diffusion processes, see, for example, Friedman (1975).

6. The elements of Z are independent normal variates with means 0 and variances t.

row ψ_j of ψ determines for the rate of return per unit of time on skill j both the variance $\sigma_j^2 = \psi_j\psi_j'$ and the associated covariances $\sigma_{jk} = \psi_j\psi_k'$ with all other skills $k \neq j$.

Although the stochastic specification (1) may seem quite general, in fact, it imposes several restrictions. For example, if jobs are available only sporadically, then the assumptions in (1) of continuity and independent increments are easily violated. With jobs open only sporadically, the value of immediate employment jumps when jobs open but then falls once positions close. In addition, in most educational programs the student faces a positive probability of failure. If the student fails, the dollar value x_j of each opportunity $j = J + 1, \ldots, J + K$ available only after graduation immediately drops to zero. In this case, the stochastic adjustment in (1) is discontinuous with a negative jump equaling in amplitude the current evaluation $x_j, j = J + 1, \ldots, J + K$.

At the very least, the individual's assessment of the private benefits to education v reflects his evaluation of all employment opportunities X, the current time t, plus some measure of tuition costs and personal preferences. Specifically, let $v(X, t)$ represent for the individual the gross personal value of education. This assessment differs from, say, the net personal value $v(X, t) - \max\{x_j: j = 1, \ldots, J\}$ computed net of the opportunity cost of immediate employment. With this specification, the current time t also identifies the remaining time $T - t$ to completion of the student's educational program. In general, the current value of any educational program depends on both the student's age and the remaining time to graduation.

Suppose for simplicity that the valuation function v for education is at least twice continuously differentiable in X plus at least once continuously differentiable in t. Also, denote by $\gamma(X, t)$ the student's current tuition net of his concurrent nonpecuniary benefits to education, all computed as a fraction of $v(X, t)$. Thus, γv measures in dollars the individual's direct financial costs of education minus his concurrent nonpecuniary benefits from education. In this case, the instantaneous rate of return to education $(dv/v) - \gamma dt$ is calculated from Ito's formula as follows:[7]

$$
\begin{aligned}
\frac{dv}{v} - \gamma dt &= \frac{1}{v}\Bigg[\frac{1}{2}\sum_{j=1}^{J+K}\sum_{k=1}^{J+K}\frac{\partial^2 v}{\partial x_j \partial x_k}\sigma_{jk}x_j x_k \\
&\qquad + \sum_{j=1}^{J+K}\frac{\partial v}{\partial x_j}(\phi_j - \omega_j)x_j + \frac{\partial v}{\partial t} - \gamma v\Bigg]dt \\
&\qquad + \frac{1}{v}\sum_{j=1}^{J+K}\frac{\partial v}{\partial x_j}x_j\psi_j dZ \\
&\equiv (\alpha - \gamma)dt + \lambda dZ,
\end{aligned}
\tag{2}
$$

7. Again see Friedman (1975, p. 90).

where, for notational simplicity, all arguments of all functions are deleted. In (2) the rate of return to education is measured net of all money wages $\omega_j x_j$ foregone as an opportunity cost of continued schooling. Here this rate of return is uncertain because the future values of subsequent employment opportunities are also uncertain. In particular, the rate of return to education exhibits per unit of time the mean $\alpha - \gamma$, the variance $\lambda\lambda'$, and the covariances $\lambda\psi_j'$ with the rates of return to all skills $j = 1, \ldots, J + K$.

The above distinction between pecuniary and nonpecuniary benefits to education is critical. Unlike financial assets, human assets cannot be traded in the market. Consequently, in this simple model individuals cannot realize their pecuniary benefits from education prior to graduation. By contrast, while still in school students face the immediate financial costs of tuition plus foregone wages. If these financial costs exceed the concurrent, personal, nonpecuniary benefits from education, then as shown in Section IV, the student may choose to leave school before graduation. In this case, the educational problem determined by (2) and the conditions specified below has no apparent analytic solution. Instead, the solution must be approximated numerically.[8]

Education is an asset requiring investments of both human and nonhuman wealth. To value this asset, it is sufficient to assume enough market completeness so that the rates of return on the $J + K$ skills can be replicated by $J + K$ portfolios of traded securities. Specifically, suppose first that there exists some set of N securities continuously traded in a capital market completely free of taxes, transaction costs, indivisibilities, and restrictions on short sales. By assumption, these securities, which for simplicity pay no dividends, have prices $P = (p_1, \ldots, p_N)'$ evolving as Ito diffusion processes:

$$I^{-1}(P)dP = \mu dt + \delta dZ. \tag{3}$$

Here, $I^{-1}(P)dP$ is the vector of instantaneous rates of return on the N securities, μ is an N-dimensional column vector denoting the current drift in $I^{-1}(P)dP$, and δ is an $N \times N$–dimensional matrix determining the variance-covariance matrix, $\delta\delta'$ for returns on risky securities.

In our complete capital market, there must exist $J + K$ portfolios of the N risky securities possessing returns instantaneously perfectly correlated with the returns on the $J + K$ skills. Representing these $J + K$ portfolios by the rows of the $(J + K) \times N$–dimensional matrix ξ, the

8. In this model, tuition measured net of all concurrent nonpecuniary benefits to education γ has the same impact as dividends in option pricing models. To value analytically a call option not protected against payouts, dividends must be restricted. See, for example, Merton (1973).

capital market is complete with respect to all uncertainty affecting the returns to education (2) if and only if

$$\xi I^{-1}(P)dP = I^{-1}(X)dX + \omega dt. \tag{4}$$

Together, (3) and (4) imply that

$$\phi = \xi\mu \qquad \text{and} \qquad \psi = \xi\delta. \tag{5}$$

Without specifying the $J + K$ skills satisfying (2), the matrix ψ in (1) can have any value. Consequently, for a complete capital market the matrix δ must have full rank N. That is, after eliminating all redundant securities, the remaining securities with linearly independent returns must equal in number the N independent sources of uncertainty. Finally, each of the $J + K$ portfolios must have percentage investments in the N risky securities summing to one: $\xi 1 = 1$.

Collectively, these assumptions determine the value of education to the individual. The argument, developed in detail in the Appendix, Part I, can be sketched as follows. In our complete capital market, there always exists a portfolio of risky securities possessing returns perfectly locally correlated with the returns to education (2). Thus, a second portfolio π can be constructed from the first portfolio plus the N risky securities which, with appropriate weights, is locally riskless. Given no opportunities in the market for riskless arbitrage, this second portfolio π must earn the riskless rate of return r. Together, (2), (4), and the above restrictions on portfolio π produce the individual's valuation equation for education.

Under the above assumptions, the personal value of continued education v satisfies the partial differential equation:

$$\frac{1}{2}\sum_{j=1}^{J+K}\sum_{k=1}^{J+K}\frac{\partial^2 v}{\partial x_j \partial x_k}\sigma_{jk}x_j x_k + \sum_{j=1}^{J+K}\frac{\partial v}{\partial x_j}(r - \omega_j)x_j - (\gamma + r)v + \frac{\partial v}{\partial t} = 0. \tag{6}$$

This equation, when combined with appropriate boundary conditions, uniquely determines the individual's evaluation of education v as a function of his personal evaluation of all employment opportunities X plus the current time t. As indicated, this valuation equation holds under reasonably plausible conditions.[9]

To determine uniquely the personal value of education to the individual, appropriate boundary conditions must be appended to (6).

9. In fact, the above argument can be generalized as in Dothan and Williams (1979). This would permit the current evaluations of future employment opportunities to be represented as any measurable function of current security prices.

These boundary conditions can be specified in part as follows. At any time t prior to graduation, $0 \leqslant t < T$, the personal value of education must exceed or at least equal the personal value of immediate employment:

$$v(X,t) \geqslant \max \{x_j: j = 1, \ldots, J\}. \quad (7)$$

Were (7) violated, then the student would immediately gain by leaving school and obtaining employment. By contrast, upon graduation at time T, the personal value of education necessarily equals the personal value of immediate employment:

$$v(X, T) = \max \{x_j: j = 1, \ldots, J + K\}. \quad (8)$$

Clearly, (8) applies only if the individual can neither hold multiple jobs nor acquire additional education. In addition to (7) and (8), other boundary conditions may be required to determine v uniquely. Here the precise conditions depend on the specific processes for X. All details are deferred to the next section.

III. The Value of Education

The solution to the boundary value problem (6)–(8) determines the individual's evaluation of the personal benefits to education. Although specification of the boundary value problem is possible under surprisingly general conditions, its solution is not. As is typical with such problems, an exact solution is possible only in special cases. In general, a solution to (6)–(8) must be approximated numerically. Accordingly, in this section an analytic solution is derived for an interesting special case, and its properties are described in detail. In the next section the solution to a more general but more difficult educational problem is approximated numerically for selected values of various parameters.

As shown in this section, the solution to our special problem has some interesting properties. For example, the longer the time to graduation, the more risky the subsequent employment opportunities; and the less correlated these risks are, the more valuable education is to the student as an option. In fact, education appears to be most valuable as an option when it provides broad training permitting postponed entry into a high-risk occupation. By contrast, neither the expected returns to future employment opportunities nor the riskless rate of interest directly affect the personal evaluation of continued education relative to immediate employment. Instead, any effects of these latter parameters are captured by the individual's current evaluation of future employment opportunities.

To simplify the general problem of the previous section, suppose that our individual has access to one of only two occupations or careers.

The first is available at any time ($J = 1$), while the second is obtainable only after graduation ($K = 1$). Also, for simplicity suppose that the variances σ_1^2 and σ_2^2, the covariance σ_{12}, the fractional wages ω_1 and ω_2, and even the percentage tuition net of nonpecuniary benefits from education γ are independent of the personal evaluations x_1 and x_2 of the two employment opportunities. Thus, these parameters depend at most on the current time t. Finally, assume that, for all future periods, the fractional nonpecuniary benefits from education always exceed or at least equal the percentage tuition payments plus foregone wages: $\int_t^T (\gamma + \omega_1)ds \leq 0$, $0 \leq t \leq T$. This last restrictive assumption is critical. Without it, the student may choose to leave school before graduation, thereby rendering inapplicable the analytic solution in this section. Subsequently, in Section IV this restrictive assumption is dropped, and the solution to the more complicated problem with optimal stopping is approximated numerically.[10]

For a unique solution to the above problem, additional boundary conditions are necessary. These conditions stipulate that education is worthless if employment is never available:

$$v(0, 0, t) = 0. \tag{9}$$

Also, relative to each employment opportunity, education has a limited value: $v(x_1, x_2, t)/x_j$ is bounded for all $x_j > 0$, $j = 1, 2$.

Given the above assumptions, the problem of this section can be specified as follows. In (6) transform variables: $y \equiv x_2/x_1$ and $u(y, t) \equiv v(x_1, x_2, t)/x_1$. Also, let $\theta^2 \equiv \sigma_1^2 - 2\sigma_{12} + \sigma_2^2$. This yields

$$\frac{1}{2}\theta^2 y^2 \frac{\partial^2 u}{\partial y^2} + (\omega_1 - \omega_2) y \frac{\partial u}{\partial y} - (\gamma + \omega_1)u + \frac{\partial u}{\partial t} = 0, \tag{10}$$

plus the boundary conditions

$$u(0, t) = 0, \tag{11}$$

$$u(y, T) = \max\{y, 1\}, \tag{12}$$

and

$$u(y, t) \geq 1, \tag{13}$$

with $u(y, t)$ and $u(y, t)/y$ bounded for all $y > 0$. Here it is convenient to regard y as the personal evaluation of skilled relative to unskilled employment and u as the personal value of continued education relative to immediate employment.

To specify the solution to (10)–(13), some additional notation is useful. Accordingly, define

10. These conditions guarantee that the student never leaves school prior to graduation. Similarly, an explicit solution to the option pricing problem of finance is possible only when premature exercise of options is never optimal; see Merton (1973).

$$\Gamma(t) \equiv \int_t^T \gamma ds, \qquad \Omega_j(t) \equiv \int_t^T \omega_j ds, \qquad \Theta^2(t) \equiv \int_t^T \theta^2 ds \tag{14}$$

for $j = 1, 2$. Using this notation, the solution becomes[11]

$$u(y, t) = e^{-\Gamma-\Omega_1}N(d_1) + ye^{-\Gamma-\Omega_2}N(d_2), \tag{15}$$

where $N(\cdot)$ denotes the standardized normal distribution,

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\xi^2/2} d\xi,$$

with

$$d_1 \equiv \frac{\Theta}{2} - \frac{\log y + \Omega_1 - \Omega_2}{\Theta}, \qquad d_2 \equiv \frac{\Theta}{2} + \frac{\log y + \Omega_1 + \Omega_2}{\Theta}.$$

The individual's evaluation of continued education relative to immediate employment u exhibits several interesting properties. In (15) u is strictly increasing and strictly convex in the personal assessment of the relative benefits to skilled versus unskilled employment y:

$$\frac{\partial u}{\partial y} = e^{-\Gamma-\Omega_2}N(d_2) > 0, \qquad \frac{\partial^2 u}{\partial y^2} = \frac{e^{-\Gamma-\Omega_2}N'(d_2)}{\Theta y} > 0. \tag{16}$$

Also, at $y = 0$ the solution (15) satisfies the dropout constraint (13): $u(0, t) = e^{-\Gamma-\Omega_1} \geqslant 1$. As indicated in figure 1, these conditions imply that the value of continued education almost always strictly exceeds the value of immediate employment. In this case, the student's option to quit school early remains more valuable unexercised than exercised. Thus, under our previous assumption that, prospectively, the percentage nonpecuniary benefits from education always exceed the fractional tuition plus foregone wages, $\Gamma + \Omega_1 \leqslant 0$, the student always stays in school until graduation.

The personal value of continued education relative to immediate employment also depends on the remaining time to graduation. For small increments in the time to graduation $\tau \equiv T - t$, (15) implies that

$$\begin{aligned}\frac{\partial u}{\partial \tau} = &-(\gamma + \omega_1)e^{-\Gamma-\Omega_1}N(d_1) - (\gamma + \omega_2)e^{-\Gamma-\Omega_2}N(d_2)\\ &+ \frac{\theta^2}{2\Theta}e^{-\Gamma-\Omega_1}N'(d_1).\end{aligned} \tag{17}$$

The first two terms on the right-hand side of (17) identify the impact on the personal value of extra education relative to immediate em-

11. The solution (15) can be calculated by transforming variables and reducing (6) to the heat equation of classical physics. See, for example, Churchill (1963, chap. 7). Alternatively, the solution (15) can be calculated from (A1) in the Appendix, part 2. The latter procedure is developed in Friedman (1975, chap. 6), and discussed heuristically in Cox and Ross (1976).

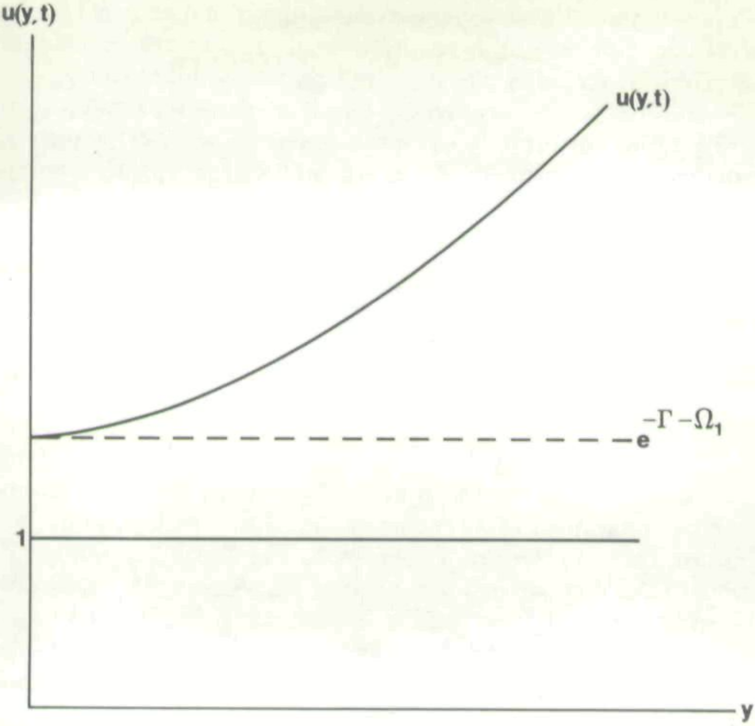

FIG. 1.—Personal value of continued education relative to immediate employment u as a function of the personal value of skilled relative to immediate unskilled employment y.

ployment u of the incremental nonpecuniary benefits net of tuition and foregone wages available over the slightly longer time horizon. Similarly, the last term measures the effect, again on the relative personal value of education, of the slightly longer period over which the student can exercise his option to leave school. If the percentage nonpecuniary benefits to education are never less than the concurrent, fractional tuition plus foregone wages, $\gamma + \omega_1 \leqslant 0$ and $\gamma + \omega_2 \leqslant 0$, then the last term is always positive. Thus, for (17) to be positive, it is sufficient, but not necessary, that $\omega_1, \omega_2 \leqslant -\gamma$. In general, however, the sign of (17) is ambiguous.

Other properties of the student's evaluation of extra education relative to immediate employment are even more striking. Nowhere does the solution (15) depend explicitly on either the expected rates of return to employment, ϕ_1 and ϕ_2, or the riskless rate of interest r. In fact, this irrelevance of both the mean rates of return ϕ and the riskless rate r is not specific to (15). In (6)–(8) all solutions v are obviously independent of ϕ. Moreover, from the easily verified linear homogeneity of v in X, all solutions v/x_j are independent of r for each skill $j = 1, \ldots, J + K$. Of course, as untraded assets, employment opportunities can have personal values X depending potentially on all parameters in the educational problem, including the expected returns to employment ϕ and the riskless rate r. Nevertheless, these properties do suggest that models ignoring uncertainty or, more generally, models emphasizing only the mean returns to education are unlikely to measure accurately the true value of education to the individual.

By contrast, the individual's assessment of the benefits to education depends explicitly on the risks to future employment. In (15) the value of continued education relative to immediate employment u is strictly increasing in the measure of risk Θ^2:

$$\frac{\partial u}{\partial \Theta^2} = \frac{e^{-\Gamma-\Omega_1}N'(d_1)}{2\Theta} > 0. \tag{18}$$

As perceived by the individual, the greater is the risk to employment Θ^2, the more valuable is the option to postpone through additional schooling commitment to a specific career. Thus, with potential occupations or careers of ever greater risk, the flexibility afforded by additional schooling becomes even more valuable to the individual.

The significance of these results is best revealed by separating the measure of risk Θ^2 into its components. Assume temporarily a fixed correlation $\rho \equiv \sigma_{12}/\sigma_1\sigma_2$ between the returns to skill-specific human capital of types 1 and 2. In this case, increasing proportionately the variances σ_1^2 and σ_2^2 over some future period increases the measured risk (14) and hence the value of education (15). In addition, with one variance, say σ_1^2, constant, then increasing the remaining variance during some future period increases both (14) and (15) if $\sigma_2 < \rho\sigma_1$, but decreases both if $\sigma_2 > \rho\sigma_1$. Here the impact of increasing risk on the value of education (15) depends on the balance between risks. Finally, if the correlation ρ between rates of return to the two types of human capital increases again over some future period, then with the variances σ_1^2 and σ_2^2 constant, the total risk (14) and hence the value of education (15) decrease. These and the subsequent comparative statics are summarized in table 1.

The implications of these comparative statics are clear. With increasing risks from early commitment to a specific career, education becomes more valuable. An individual more uncertain about his abilities, personal preferences, and future employment opportunities has a greater incentive to delay commitment to a specific career. Over time as the student learns progressively more about his abilities, preferences, and opportunities, the value of continued schooling diminishes. In addition, the balance between risks is important. For example, generalized training such as liberal arts is more valuable to the student, other things equal, than highly specialized professional or technical training. In short, at least as an option, education appears most valu-

TABLE 1 Comparative Statics for the Values of Education

Increment in	Impact on u	Conditions
y	+	. . .
τ	+	$\omega_1, \omega_2 \leq -\gamma$
ϕ_1, ϕ_2	0	. . .
r	0	. . .
σ_1^2 and σ_2^2	+	$d\sigma_1/d\sigma_2 = \sigma_1/\sigma_2$
σ_1^2, σ_2^2	±	$\sigma_1 \gtrless \rho\sigma_2, \sigma_2 \gtrless \rho\sigma_1$
ρ	−	. . .
γ	−	. . .
ω_1, ω_2	−	. . .

NOTE.—u = personal value of continued education relative to immediate employment; y = personal value of skilled relative to unskilled employment; τ = remaining time to graduation; ϕ_j = mean returns from employment in occupation $j = 1, 2$; σ_j^2 = variance of returns from occupation $j = 1, 2$; ρ = correlation between the returns to occupations 1 and 2; γ = tuition net of nonpecuniary benefits to education as a percentage of the personal value of continued education; ω_j = wage in occupation, $j = 1, 2$ as a percentage of skill-specific human capital.

able when it provides broad training permitting postponed entry into a high-risk occupation.

In this simple model two additional properties are evident. By increasing over some future period the percentage tuition net of nonpecuniary benefits to education γ, the current value of education (15) is decreased. Simultaneously, by increasing over the same future period the fractional wages ω_1 or ω_2 foregone during schooling, the current value of education (15) is decreased. Although neither property is surprising, both have interesting implications. For example, if the perceived nonpecuniary benefits to education net of tuition correlate positively with the educational attainment and financial wealth of the student's parents, then education appears more valuable to students with educated, affluent parents. Similarly, if highly specialized occupations or careers with highly peaked age-earnings profiles yield smaller percentages of total life-cycle earnings in current wages, then students training for such occupations also regard education as more valuable. At least indirectly, both properties have testable implications for observable patterns of schooling across individuals.

IV. The Optimal Time to Quit School

Because education is an untraded asset, its private value to individuals, as computed in the previous section, is generally unobservable. Nevertheless, the previous results still have interesting empirical implications if only because they help to explain observed characteristics of existing educational programs. In the United States these characteristics include a seemingly protracted period of higher education combined with a heavy emphasis on generalized training. Moreover, these properties are most evident among affluent students planning specialized professional careers characterized by highly peaked, highly variable profiles of labor earnings over the life cycle.

By contrast, the distribution of schooling across individuals is directly observable, so that predictions about optimal periods to spend in school are empirically testable. Accordingly, in this section the problem of the previous section is generalized, introducing for the first time the possibility of optimal departure from school before graduation. For this more general, more realistic problem with optimal stopping, solutions are computed characterizing not only the personal value of continued education relative to immediate employment but also the optimal time to quit school. However, because this problem is also more complex, no analytic solution is apparent. Consequently, the solution in this section is approximated numerically.

Our predictions about the student's optimal time to quit school possess some interesting properties. In general, all factors increasing the personal value of continued education relative to immediate employment also induce the student to leave school later. For example, individuals planning professional careers involving either greater specialization, and hence greater risk, or profiles of labor earnings more highly peaked over the life cycle choose to stay in school longer. Similarly, students enrolled in programs which, like liberal arts, provide broader training and thereby permit entry into more diverse occupations also prefer to remain in school longer. Moreover, other things equal, programs charging lower tuition induce students to stay in school longer.

In this section our former problem is modified as follows. Previously, optimal departure from school prior to graduation was precluded by restricting tuition plus foregone wages relative to personal nonpecuniary benefits from education. Specifically, for all future periods, fractional tuition plus wages foregone during school were bounded above by the percentage nonpecuniary benefits from education: $\Omega_1 + \Gamma \leq 0$. Here, this restrictive assumption is dropped, and these parameters are permitted to assume any values.

This single modification produces the following more complicated problem of optimal stopping. Consider the space of points (y, t) representing, respectively, the current personal value of skilled relative to unskilled employment y and the current time t. In this space find the function u, representing as before the current personal value of continued education relative to immediate employment, and the boundary $t = t^*(y)$ with the following properties. For larger values of y lying to the right of the optimal dropout boundary $t = t^*(y)$, the student stays somewhat longer in school. In this case, u again satisfies (10)–(13). By contrast, for smaller values of y on the boundary, the student quits school immediately. In this second case, the personal value of con-

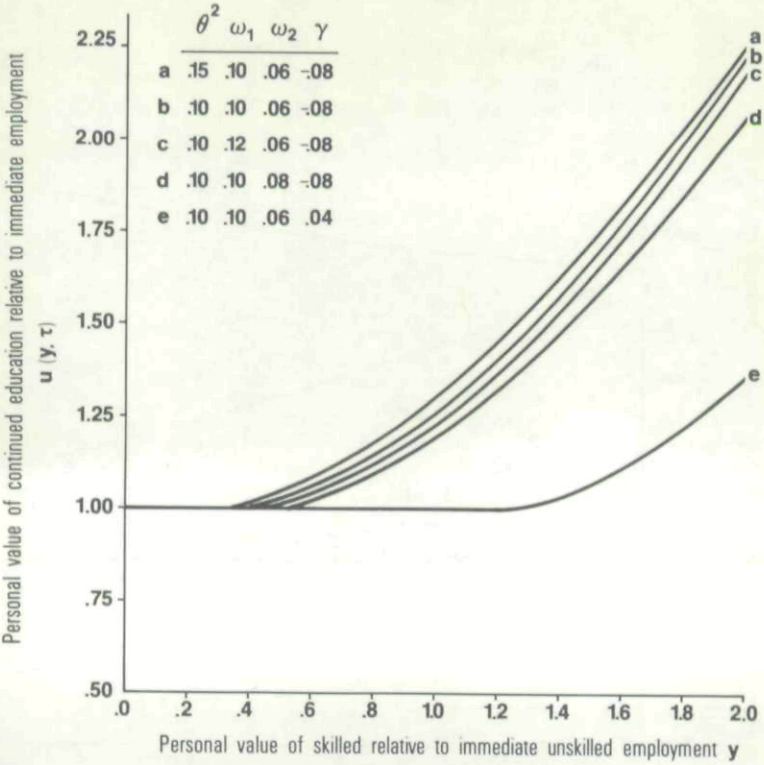

FIG. 2.—Personal value of continued education relative to immediate employment u as a function of the personal value of skilled relative to immediate unskilled employment y ($\tau = 4.0$).

tinued education currently equals the personal value of immediate employment in unskilled labor, so that

$$u(y, t) = 1. \tag{19}$$

Thus, the optimal dropout boundary $t = t^*(y)$ identifies at each time t the maximal current value of skilled relative to unskilled employment y inducing the student to quit school immediately.

The solution to this problem is approximated numerically using the procedure described in the Appendix, part 2. For selected values of various parameters these approximate solutions are sketched in figures 2–4. As shown in figure 2, the personal value of continued education relative to immediate employment u is nondecreasing in the personal value of skilled relative to unskilled employment y. Not surprisingly, only for smaller values of y inducing immediate employment does this properly differ from the previous results in figure 1. As indicated, when

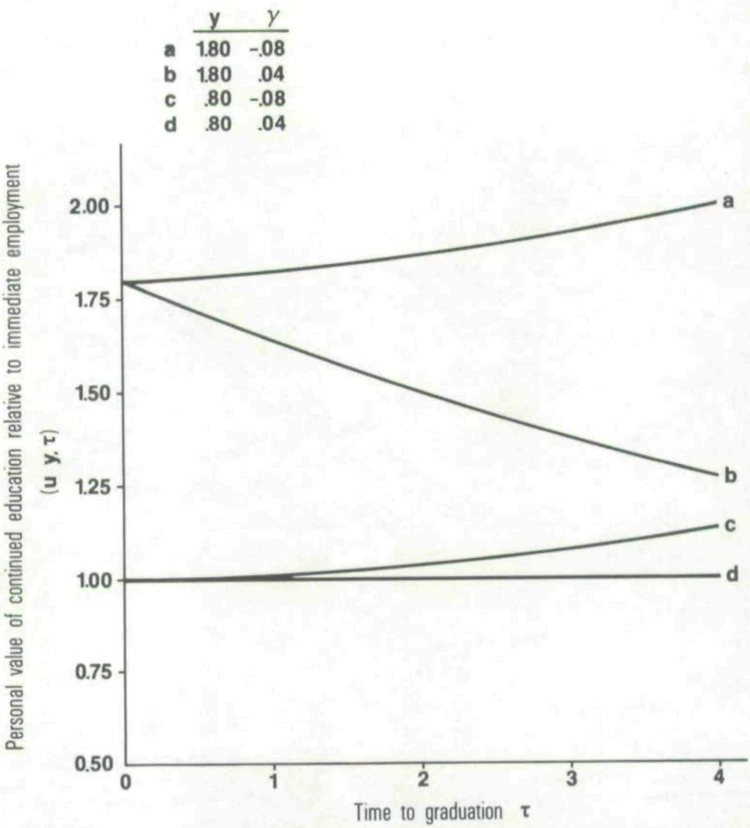

FIG. 3.—Personal value of continued education relative to immediate employment u as a function of the time to graduation τ ($\theta^2 = .10$, $\omega_1 = .10$, $\omega_2 = .06$).

(19) is binding, the relative value of extra education u is independent of the relative value of skilled employment y. Similarly, the comparative statics in figure 2 differ from the previous results only when the student quits school before graduation. For smaller values of y, the personal value of continued schooling relative to immediate employment u is constrained by (19) and therefore invariant to shifts in all parameters.

Figure 3 illustrates the relationship between the personal value of continued education relative to immediate employment u and the remaining time to graduation $\tau \equiv T - t$. As indicated, for smaller values of skilled relative to unskilled employment y inducing immediate departure from school, the relative value of continued education u is independent of the time to graduation τ. In fact, only with the straight line (d) is the constraint (19) binding. By contrast, for larger values of skilled relative to unskilled employment y consistent with continued schooling, the relative value of extra education u is sensitive to the

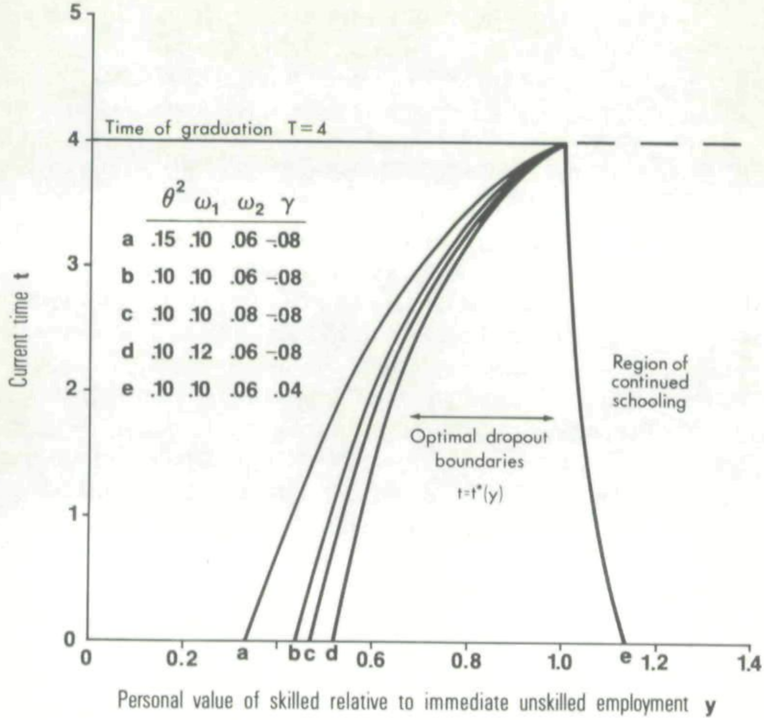

FIG. 4.—Optimal combinations of current times t and current value of skilled relative to unskilled employment y inducing immediate departure from school.

remaining time τ. In particular, with sufficiently small tuition net of nonpecuniary benefits to education γ, the relative value u increases with increasing time to graduation τ, whereas with larger γ the relative value u decreases with increasing τ. The intuition behind this result is suggested by (17). With smaller γ the value of education as an option exceeds its incremental cost of tuition plus foregone wages, whereas with larger γ its incremental cost predominates.

The major results of this section appear in figure 4. Here, for specific values of all relevant parameters, the student's maximal current value of skilled relative to unskilled employment y inducing immediate departure from school at the current time t is graphed as the locus of points $\{(y, t): t = t^*(y)\}$. Because the personal value of continued education relative to immediate employment u increases with increasing personal value of skilled relative to unskilled employment y, the region in which the student currently remains in school lies to the right of the optimal dropout boundary $t = t^*(y)$. Moreover, given the continuous evolution in this model of the relative value y, the region to the left of this boundary cannot be attained without first passing through the boundary. Thus, at any time t when the current value of skilled relative to unskilled employment y first hits the optimal dropout boundary $t = t^*(y)$, the student immediately quits school.

Not surprisingly, the student's optimal time to quit school responds to precisely the same parameters as does the personal value of continued education relative to immediate employment. As indicated in figure 4, reducing the risks to employment θ^2 by reducing the variances of returns to employment, σ_1^2 or σ_2^2, or, alternatively, raising the correlation between these returns ρ induces quicker departure from school. Similarly, increasing either the fractional wages, ω_1 or ω_2, or the percentage tuition net of nonpecuniary benefits from education γ encourages earlier employment. Previously, in figures 2 and 3, the same shifts in parameters were shown to reduce the personal value of continued education relative to immediate employment u. By contrast, in figure 4 neither the expected returns from employment, ϕ_1 or ϕ_2, nor the riskless rate of interest r directly affect the student's optimal time to leave school. Again in figures 2 and 3, these parameters were shown to have no impact on the personal value of extra education relative to immediate employment.

Over time, the student becomes less and less sensitive to differences in perceived risks, personal preferences, and market conditions. In figure 4, as the time to graduation τ diminishes, the maximal value of skilled relative to unskilled employment y inducing immediate employment becomes progressively less sensitive to variations in the perceived risk θ^2, the fractional wages ω_1 and ω_2, and the percentage tuition net of nonpecuniary benefits from education γ. Moreover, this property is most evident within the last year before graduation for variations in the percentage tuition net of nonpecuniary benefits γ.

These results have interesting implications for observable patterns of schooling across individuals, institutions, and occupations. Individuals, for example, with educated, affluent parents may regard more favorably the nonpecuniary benefits to education net of tuition. In this case, other things equal, these individuals value more highly extra education and hence remain in school longer. In addition, some institutions—for example, colleges and universities—prepare students for professional careers involving greater personal commitment and hence greater personal risk. Typically, these institutions also provide broad training, like liberal arts, with diversified opportunities for subsequent employment. By contrast, other institutions attract students by charging lower tuition. In either case, other things equal, individuals attending such institutions regard education as more valuable and thus remain in school longer. Then, too, professions promising over the life cycle more highly peaked profiles of labor earnings often pay out in early stages smaller percentages of total earnings as current wages. Other things equal, individuals again regard training for these occupations as more valuable and hence remain in school longer.

V. Conclusion

To the student education provides options. Very often, specialized training for a specific occupation or career is costly. Also, for many individuals initial uncertainties about personal preferences, abilities, and even general conditions in the labor market are considerable. For such individuals, education—especially formalized schooling—introduces the options of not only postponing commitment to a specific career but also choosing among additional careers after graduation. To the student, this flexibility provided by education is valuable. In part, its value depends on the individual's current assessment of subsequent employment opportunities, the remaining time to graduation, plus various parameters summarizing occupational risks, foregone wages, tuition fees, and personal preferences.

In this paper education is evaluated as an option. Under plausible conditions the argument yields a valuation equation which, subject to appropriate boundary conditions, uniquely determines the individual's current assessment of the personal benefits from continued education. In some interesting special cases, this equation is then solved. In particular, under conditions sufficient to guarantee graduation, our solution is exact; otherwise, the solution is both numerical and approximate. In the numerical approximation both the personal value of education and the optimal strategy for leaving school are specified. Throughout the paper, properties of the solutions and implications for observable behavior are described in considerable detail.

The major conclusions and empirical implications of this paper can be summarized as follows:

1. Given more risky alternatives for subsequent employment, students generally regard continued education as more valuable and therefore prefer to stay in school longer. Thus, younger students more uncertain about their personal preferences, abilities, and opportunities, other things equal, leave school later. In addition, students anticipating careers characterized by either greater specialization and hence greater personal risk or, alternatively, more highly peaked profiles in labor earnings over the life cycle typically choose to stay in school longer.

2. Educational programs providing broader training with more diverse opportunities for subsequent employment are, other things equal, more valuable to individuals. Consequently, such programs induce students to stay in school longer. In particular, during some preparatory period, of, say, undergraduate education, students can often maintain occupational diversification. This then permits individuals to learn more about their personal preferences, abilities, and opportunities before making costly commitments to specific careers.

3. Often, occupations with highly peaked age-earnings profiles initially pay out lower percentages of total life-cycle earnings in current

wages. Other things equal, students training for such occupations, like professional and managerial careers, again regard education as more valuable and thus remain in school longer. Surprisingly, this reverses the usual relationship between cause and effect in models of the accumulation of human capital over the life cycle.

4. Other things equal, educational programs charging lower tuition are more valuable to individuals, thereby delaying departure from school. Also, as graduation nears, students become progressively less likely to leave prematurely programs charging lower tuition relative to programs with higher tuition. In fact, this latter result is related to a more general property. As a student's time to graduation diminishes, his likelihood of early departure from school becomes less and less sensitive to all differences in perceived risks, personal preferences, and economic conditions.

5. Suppose that individuals' nonpecuniary benefits from education net of tuition correlate positively with both the past educational attainment and current financial wealth of their parents. In this case, students with more educated, affluent parents generally leave school later. Moreover, as graduation approaches, students from more educated, affluent families become progressively less likely to leave school before graduation relative to students from all other families.

Because the model in this paper is so simple, many extensions are possible. In the paper the individual's assessment of alternative employment opportunities is presumed to change at most continuously through time with independent stochastic increments. However, if some jobs are available only sporadically or, while in school, failure is possible, then the assumptions of continuity and independent increments are easily violated. For example, with jobs available only sporadically, the value of immediate employment first rises and then falls as positions open and then close. Similarly, if the student faces a positive probability of failure and then fails, the current value of any employment opportunity available only after graduation immediately drops to zero. In turn, this alters the individual's assessment of the personal benefits from continued education and thus the optimal time to quit school.

Other extensions are also possible. For example, in practice schooling is sequential. When a student completes one educational program, often he has the option of either entering a subsequent program or seeking immediate employment. Once employed, he typically then has the further option of training on the job. In addition, no decision about employment is irrevocable. Even if employed full time, the individual can, at a cost, switch careers, or, alternatively, return to school and then switch careers. Consequently, education is often in early stages an option on subsequent options. With such compound options the individual's assessment of both the personal benefits to continued education and the optimal time to leave school are again altered. In turn this has implications for observable patterns of schooling across individuals, institutions, and occupations.

In addition, the above results raise interesting questions about observable patterns of education in general equilibrium. For example, with different schools facing different costs and a competitive educational industry, what bundling or packaging of educational services persists in equilibrium? Again in equilibrium, how easily can students switch between different educational programs? For what period of time can students postpone specialization, how long do degree programs last, and what percentages of students complete different educational programs? If a student gains admission to several schools charging different tuitions but offering different employment opportunities, which should he select? How does this decision differ across individuals and on what characteristics does the decision depend? If, for example, parents' financial wealth is important, what then is the optimal combination of private tuition and public support for higher education? Clearly, many interesting questions remain unanswered, leaving much additional work for the future.

Appendix

I

The valuation equation (6) can be derived as follows. Given that δ has full rank, there exists a portfolio $\upsilon = \lambda\delta^{-1}$ comprised of the N risky securities in (3). By design, this portfolio has a risky rate of return perfectly locally correlated with the rate of return to education (2). Next, construct a second portfolio denoted by the $N + 1$–dimensional row vector π with elements representing, respectively, the N risky securities in (3) plus the portfolio υ. Because this second portfolio includes all components of the individual's human and nonhuman wealth, its percentage investments must sum to one: $\pi 1 = 1$. Moreover, with δ invertible, appropriate allocations among the N risky securities render this portfolio locally riskless:

$$\pi\begin{pmatrix} \delta \\ \lambda \end{pmatrix} = 0.$$

Arbitrage in our frictionless capital market then guarantees that this locally riskless portfolio earns the instantaneously riskless rate of return r:

$$\pi\begin{pmatrix} \mu \\ \alpha + \gamma \end{pmatrix} = r.$$

This characteristic of a capital market without opportunities for riskless arbitrage has been exploited extensively in the financial theory of option prices, as summarized by Smith (1976).

Given these results, the valuation equation for education follows almost immediately. In particular, the above two equations, plus $\pi 1 = 1$, imply that

$$\pi\begin{pmatrix} \mu - r1 & \delta \\ \alpha + \gamma - r & \lambda \end{pmatrix} = 0.$$

Because δ has full rank, there must exist an N-dimensional column vector κ satisfying

$$\begin{pmatrix} \mu - r1 \\ \alpha + \gamma - r \end{pmatrix} = \begin{pmatrix} \delta \\ \lambda \end{pmatrix}\kappa.$$

Deleting κ from these equations gives $\alpha + \gamma - r = \lambda\delta^{-1}(\mu - r1)$. Inserting (2) and (4) into this last equation then yields the valuation equation (6).

II

The solution to the problem of Section IV can be specified as follows:

$$u(y, t) = E\left[\max\left\{1, \eta(T^*)\right\} \exp \int_t^{T^*} (\gamma - \omega_1)ds \middle| y, t\right], \tag{A1}$$

where η is the solution to the stochastic differential equation,

$$d\eta = (\omega_1 - \omega_2)\eta dt + \theta\eta dz, \tag{A2}$$

with the initial condition $\eta(t) = y$. Also, T^* is the minimum of T and the first passage time of η to the boundary $t = t^*(y)$. In (A1) $E[\cdot|y, t]$ denotes the conditional expectation given the current values of y and t. Without the boundary condition (13), this expectation is easily evaluated. In this case, $T = T^*$, so that (A1) simplifies to (15). The stochastic representation of solutions to boundary value problems is developed in Friedman (1975, chap. 6), and described heuristically in Cox and Ross (1976).

In general, however, with condition (13) the expectation (A1) cannot be calculated explicitly. Thus, a numerical approximation is necessary. One simple procedure is as follows. First, approximate (A2) by the discrete process:

$$\Delta\eta = (\omega_1 - \omega_2)\eta\Delta t \pm \theta\eta\sqrt{\Delta t}, \tag{A3}$$

where $\Delta t \equiv (T - t)/N$ and N is a positive integer. In (A3) the Weiner process ξ is replaced by a symmetric random walk. Without the constraint (13), the solution (A1) could easily be approximated by discounting to the current time t, using the discount factor $[1 + (\omega_1 - \gamma)\Delta t]^{-N}$, the average of the terminal values $\max\left\{1, \eta(T)\right\}$ calculated from (A3). However, with (13) the procedure is slightly more complicated. Because both (A1) and the unknown stopping boundary $t = t^*(y)$ must be identified in the solution, it is necessary to proceed step by step with (A3). At each step of the iterative procedure, the two adjacent sample values of η calculated from (A3) are averaged and then divided by $1 + (\omega_1 - \gamma)\Delta t$. If the subsequent discounted average value is below 1, then it is replaced by 1. This process is continued until the discounted terminal sample average $\max\left\{1, \eta(T^*)\right\} \exp\int_t^{T^*}(\gamma - \omega_1)ds$ is identified.

For the values of the parameters indicated in figures 2 and 4, numerical solutions were generated at intervals of .05 in y and .20 in τ. All graphs were drawn by interpolation. The authors will provide details upon request. A similar algorithm appears in Parkinson (1977).

References

Arrow, K. 1973. Higher education as a filter. *Journal of Public Economics* 2 (July): 193–216.

Becker, G. 1964. *Human Capital*. New York: Columbia University Press.

Ben-Porath, Y. 1967. The production of human capital and the life cycle of earnings. *Journal of Political Economy* 75 (August): 352–65.

Black, F., and Scholes, M. 1973. The pricing of options and corporate liabilities. *Journal of Political Economy* 81 (May–June): 637–59.

Blaug, M. 1976. The empirical status of human capital theory: a slightly jaundiced survey. *Journal of Economic Literature* 14 (September): 827–55.

Blinder, A., and Weiss, Y. 1976. Human capital and labor supply: a synthesis. *Journal of Political Economy* 84 (June): 449–72.

Brown, C. 1976. A model of optimal human-capital accumulation and the wages of young high school graduates. *Journal of Political Economy* 84 (April): 299–316.

Churchill, R. 1963. *Fourier Series and Boundary Value Problems*. New York: McGraw-Hill.

Cox, J., and Ross, S. 1976. The valuation of options for alternative stochastic processes. *Journal of Financial Economics* 3 (March): 145–66.

Dothan, U., and Williams, J. 1979. Term-risk structures and the valuation of projects. Manuscript. Evanston, Ill.: Northwestern University.

Friedman, A. 1975. *Stochastic Differential Equations and Applications*. New York: Academic Press.

Heckman, J. 1976. A life cycle model of earnings, learning, and consumption. *Journal of Political Economy* 84 (August): S11–S44.

Levhari, D., and Weiss, Y. 1974. The effect of risk on the investment in human capital. *American Economic Review* 64 (December): 950–63.

Merton, R. 1973. The theory of rational option pricing. *Bell Journal of Economics and Management Science* 4 (Spring): 141–83.

Mincer, J. 1958. Investment in human capital and personal income distribution. *Journal of Political Economy* 66 (August): 281–302.

Parkinson, M. 1977. Option pricing: the American put. *Journal of Business* 50 (January): 21–36.

Sahota, G. 1978. Theories of personal income distribution: a survey. *Journal of Economic Literature* 16 (March): 1–55.

Smith, C. 1976. Option pricing: a review. *Journal of Financial Economics* 3 (January–March): 3–52.

Spence, M. 1974. *Market Signaling*. Cambridge, Mass.: Harvard University Press.

Weiss, Y. 1972. The risk element in occupational and educational choices. *Journal of Political Economy* 80 (November–December): 1203–13.

Williams, J. 1978. Risk, human capital, and the investor's portfolio. *Journal of Business* 51 (January): 65–89.

Williams, J. 1979. Uncertainty and the accumulation of human capital over the life cycle. *Journal of Business* 52 (October): 521–48.