Capital Allocation to Meet Mortality Risks for Life Annuities

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This paper investigates the rules for assessing the capital required by a life annuity portfolio to meet the mortality risks, the longevity risk in particular. Risks other than mortality are disregarded. Rules which could be adopted in the internal models have been discussed. Approaches based on a deferral and matching or an asset and liability logic are further developed in this paper, in which rules that could be adopted in internal models have been discussed. For practical purpose, this approach could represent a good compromise, provided that the relevant assumptions are properly disclosed.

Introduction

Advantages provided by 'large' portfolio sizes in respect of the risk of random fluctuations justify, to some extent, the traditional deterministic approach to mortality, in life insurance calculations. Actually, adopting a deterministic approach to actuarial valuations (i.e., using only expected values in calculating premiums and reserves) is, to some extent, underpinned by the nature of the insurance process, which consists of 'transforming' individual risks through aggregation, thus, lowering the relevant impact.

However, this justification can only be accepted under the assumption that just the risk of random fluctuations in the mortality of insured lives is allowed for. Conversely, other sources of randomness should also be recognized, and hence the existence of risk components other than random fluctuations should be accounted for. When dealing with life annuities (and, in general, with long-term living products), special attention should be devoted to the risk of systematic deviations arising from the uncertainty in representing the future mortality patterns.

Solvency targets and the assessment of the consequent capital requirements, according to a sound risk management approach, clearly witness the need for analyzing all sources of risk and the relevant components. In this regard, it is interesting to note the progressive shift from simple (and, in a modern perspective, rather simplistic) regulatory requirements, based on compact shortcut formulae to more complex calculation structures. The possible use of internal models, which can capture the real risk profile of an insurance company, is a recent, and so far, probably the most important step in this process. An interesting review of the development of solvency requirements (also according to the historical perspective) is provided by Sandström (2006).

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Great attention is currently devoted to the management of life annuity portfolios, both from a theoretical and a practical point of view, in particular because of the growing importance of annuity benefits in private pension plans. Among the risks which affect the pension plans and life annuity portfolios, the longevity risk, which arises from the unknown future trend in mortality at adult and old ages, deserves a deep and detailed investigation, and requires the adoption of proper management solutions. In a risk management perspective, capital allocation constitutes an important tool for financing potential losses, clearly including those originating from unanticipated mortality improvements. For an extensive analysis of mortality dynamics and the impact of longevity risk on pensions and life annuities, the reader can refer to Pitacco *et al.* (2008).

Models for assessing the impact of longevity risk on life annuity portfolios and pension funds, and evaluating the consequent need for capital allocation, have already been proposed in the actuarial literature. In particular, Olivieri and Pitacco (2003) focus on alternative approaches to solvency assessment over a multi-year time horizon. Approaches based, respectively, on a deferral and matching, or an asset and liability logic, proposed therein, are developed further in the present paper, in which rules that could be adopted in internal models are discussed.

In Section 2, tools for the management of risks inherent in a life annuity portfolio (or pension plan) are briefly presented following the approach suggested by the 'Enterprise Risk Management' technique. Section 3 focuses on some aspects of the longevity risk, and the choice of a tractable modeling structure. Solvency targets, and related rules which can be adopted by internal models, are defined and discussed in Section 4. Subsequently, in Section 5 some comparisons among internal rules are presented through numerical examples, and finally, some concluding remarks are presented.

2. The Risk Management of Life Annuities

Several tools can be developed for managing the risks of a life annuity portfolio, which can be analyzed following the approach suggested by the 'Enterprise Risk Management' technique.

The risk management process consists of three basic steps: the identification of risks (mortality risks, market risks, etc.), the assessment (or measurement) of the relevant consequences, and the choice of the most appropriate risk management solution (which, usually, is a combination of various actions). In our discussion, we refer to mortality risks only.

The identification of risks affecting an insurer, can follow the guidelines provided by the supervisory authority, such as those underpinning the European Solvency II project (CEIOPS, 2007 and 2008), or the recommendations suggested by some independent institution, such as the International Actuarial Association (IAA, 2004). Mortality risks can be classified as a sub-set of underwriting risks. The relevant components are: random fluctuations, systematic deviations and catastrophe deviations (see Section 3 for details). The importance of mortality risks, and their components in particular, depends on the specific composition of the portfolio of the insurer, when life annuities are dealt with, and random fluctuations and systematic deviations (the latter also called the longevity risk, for this business), are in particular critical (as we will discuss in Section 3).

A rigorous assessment of risks requires the use of stochastic models. The choice of the most appropriate model depends on the type of risks and the business we are dealing with. For the case of mortality risks, an example is provided in Section 3. The stochastic model should lead to a risk measurement, such as a probability of default (or ruin probability), which is typically referred to in solvency investigations (see Section 4), a risk margin adjusting some expected quantity, which is typically adopted in pricing and reserving (see again Section 4), or some other risk measures (not discussed in this paper).

Risk management solutions for life annuities include a wide set of tools, which can be interpreted, under an insurance perspective, as portfolio strategies aiming at risk mitigation. In this paper, we consider in detail capital allocation only. However, in the remainder of this section, we sketch a comprehensive framework of the several tools available (with reference to mortality risks only).

A portfolio strategy clearly affects the portfolio results. Consider, for example, the annual outflows relating to annuity payments only, which in any event constitute the starting point from which other portfolio results (e.g., profits) may be derived. Reasonably, the insurer sets a threshold amount for the annual outgo, which represents a maintainable level of benefit payment. This amount is financed first by premiums via the portfolio technical provision, and then by shareholders' capital as the result of the allocation policy (consisting of specific capital allocations as well as accumulation of undistributed profits). If the annual outflow is above the threshold level, then a situation of default emerges. In order to keep the relevant probability at an acceptable level, the insurer can adopt various risk management actions. The target can be an increase in the maintainable annual outflow, and thus a higher threshold level, as well as a reduction (and smoothing) of annual outflows in the case of unanticipated improvements in portfolio mortality. Such targets may be reached through loss-control and loss-financing actions (according to the usual risk management terminology).

Loss-control actions are mainly performed via the product design. The insurer should set policy conditions meant to mitigate the loss frequency and lower the severity of the possible loss (i.e., aiming, respectively at loss prevention and loss reduction). Loss reduction requires to take control of the annuity amounts paid out, by adding some flexibility to the life annuity product. One action could be the reduction of the annual amount as a consequence of an unanticipated mortality improvement. However, in this case the product would be a non-guaranteed life annuity, although possibly with a reasonable minimum amount guaranteed. A more practicable tool, consistent with the features of a guaranteed life annuity, consists of reducing the level of investment profit participation when the mortality experience is adverse to the insurer. It is worth stressing that undistributed profits also increase the shareholders' capital within the portfolio, hence raising the maintainable threshold.

Loss-financing actions include a wide range of techniques, in particular risk transfers obtained through reinsurance or alternative arrangements, namely securitization. It is worth mentioning that reinsurance is typically designed for random fluctuations, due to the stronger pooling effect, which can be realized by a reinsurer in respect of what can be reached by an insurer. Conversely, reinsurers are not willing to take risks which are systematic for the overall insurance/reinsurance market, such as the longevity risk. The Alternative Risk Transfer (ART) solutions may help in this regard; in particular, for life annuities, the development of longevity bonds, whose performance is linked to some measure of longevity in a given population, are of interest. The availability of a longevity bond market, which is supposed to be more easily accessible by a reinsurer than an insurer (due to the presumable large size of the deals), could support the development of a reinsurance market for longevity risk. Indeed, reinsurers could hedge the risks taken from insurers with capital market solutions (see Pitacco *et al.* (2008) and the references therein; in particular, Olivieri (2005) for reinsurance arrangements; and Blake *et al.* (2006) and Lin and Cox (2005) for ART solutions).

Among the loss-financing actions, we finally mention natural hedging. This term refers to the possibility of offsetting mortality risks emerging from living benefits with those emerging from death benefits. The diversification effect can be realized inside the life annuity portfolio, allowing for a death benefit combined with the life annuity; such a hedge is usually called 'across time'. Another possibility consists of offsetting risks in different Lines of Business (LOBs), namely life insurance providing death benefits with life annuities. This is the so-called hedging 'across LOBs'. In both the cases, the hedging is based on the fact that the lower is the mortality, the higher is the value of life annuity liabilities, and the lower is the value of life insurance liabilities. Whilst diversification across time can easily be achieved (provided that the policyholder is willing to pay for both living and death benefits), diversification across LOBs can hardly produce a satisfactory effect, due to the different range of ages, and hence the different mortality profiles involved in the life annuity and the life insurance business (Pitacco *et al.*, 2008).

To the extent that the mortality/longevity risks are retained by an insurer, the impact of a poor experience falls on the insurer itself. In order to meet the unexpected amount of obligations, an appropriate level of advance funding may provide a substantial help. For this purpose, the shareholders' capital must be allocated to the life annuity portfolio, and the relevant amount should be determined, so as to achieve the insurer's solvency.

3. Mortality Risks

Mortality risks emerge as deviations in the experienced mortality frequency in respect of the (forecasted) mortality rates. Such deviations may, in particular involve individual or aggregate mortality. In the former case, deviations are simply due to the fact that one individual may outlive or underlive the average lifetime in the population. The relevant risk is the well-known process risk (also named insurance risk or risk of random fluctuations), whose severity reduces as the individual's position becomes negligible in respect of the overall portfolio. The process risk can be hedged by achieving an adequate pooling effect, as well as through traditional risk transfer arrangements, as it reduces risk, as soon as the portfolio consists of similar policies and its size is large enough. We further note that actuaries are familiar with the relevant modeling issues.

Deviations involving aggregate mortality may be permanent or temporary. In the latter case, a sudden jump in mortality rates is experienced, typically due to epidemics, severe climatic conditions, natural disasters, and so on. The catastrophe risk emerges, which is short in time and produces profit for the provider of living benefits. This paper does not address this aspect.

In the case of random fluctuations, the average lifetime experienced by the population, typically turns out to be close to what is expected. Conversely, when mortality frequencies are experienced permanently below or above the expected levels, the average lifetime of the population may be different from what is expected. Such a systematic deviation, which we refer to as a permanent deviation in aggregate mortality, may be the result of either a misspecification of the relevant mortality model or a biased assessment of the relevant parameters. The former aspect is referred to as the model risk, the latter as the parameter risk. The term uncertainty risk is often used to refer to model and parameter risk jointly, meaning uncertainty in the representation of a phenomenon, such as future mortality. When adult or old age is concerned, the uncertainty risk may emerge, in particular because of an unanticipated reduction in mortality rates. In this case, the term longevity risk is used instead of uncertainty risk. The pooling arguments do not apply for its hedging, due to its systematic nature. We further note that modeling issues still constitute an open problem.

In order to represent mortality risks, a stochastic model is required. Since the scope of this paper is solvency for living benefits, we disregard catastrophe risk. The stochastic mortality model should then allow for random fluctuations (process risk), deviations due to the shape of the time-pattern implied by the mortality model, in respect of both age and calendar year (model risk), as well as deviations due to the level of the parameters of the mortality model (parameter risk). Clearly, when dealing with living benefits at adult or old age, appropriate mortality projections should be adopted.

Let $T_{x\tau}$ be the remaining lifetime of an individual with current age, x, born in calendar year τ. Let $F_{x\tau}(t|H(\tau))$ denote the relevant cumulative probability distribution function (cdf), conditional on the assumption $H(\tau)$ about the future behavior of mortality for the cohort τ, i.e., $F_{x\tau}(t|H(\tau)) = \mathbb{P}[T_{x\tau} \leq t|H(\tau)]$. The cdf, $F_{x\tau}(t|H(\tau))$ allows for random fluctuations, given that $H(\tau)$ describes the aggregate mortality trend forecasted for the cohort τ.

A naive, but practical, way to account for the systematic deviations consists of assigning alternative hypotheses regarding the future aggregate mortality trend of cohort τ. An example can be found in projections developed by Continuous Mortality Investigation Bureau (CMIB), addressing the cohort effect and assuming three hypotheses regarding the persistence of such an effect in the future (CMI, 2002 and 2006). Let $\mathcal{H}(\tau)$ denote the set of alternative mortality assumptions for cohort τ. These assumptions may originate from different sets of the relevant parameters of the underlying projection model. In this way, the parameter risk is addressed. Otherwise, alternative assumptions may be given by mortality projections obtained under different procedures. In this case, model risk would also be addressed. However, it is intrinsically difficult to perform an explanatory comparison of different models, so henceforth we refer to the parameter risk only.

Following Olivieri (2001), if we further define a (non-negative and normalized) weighting structure on $\mathcal{H}(\tau)$, then the family $\{F_{x\tau}(t|H(\tau))\,|\,\mathrm{H}(\tau) \in \mathcal{H}(\tau)\}$ allows for both random fluctuations and systematic deviations. Experience providing data for estimating the weighting structure is rarely available, and so personal judgement is often required in this respect. Let $\mathcal{H}(\tau) = \{H_1(\tau), H_2(\tau)\}, \ldots, H_m(\tau)\}$. The weight assigned to the assumption, $H_h(\tau)$ is denoted by ρ_h, with $0 \leq \rho_h < 1$ for $h = 1, 2, \ldots, m$ and $\sum_{h=1}^{m} \rho_h = 1$.

It is worth noting that the structure introduced above can, to some extent, be classified as a static representation of stochastic mortality. Indeed, the notation expresses that the set $\mathcal{H}(\tau)$ and the weights ρ_h are assigned at the valuation time (when the individual is aged x, in the calendar year $x + \tau$). An update based on experience or new information is not embedded in the model. A Bayesian inferential procedure can be set in order to update the weights according to the observed number of deaths (Olivieri and Pitacco, 2002). In order to deal within a dynamic stochastic setting of mortality, either the probability of death or the force of mortality (or some other mortality index) could be modeled as a path of a random process. In the current literature, some proposals have been focusing on this approach, mainly aiming at the pricing of longevity securities. Most investigations deviate from the assumed similarities between the force of mortality and interest rates or simply from the assumption that the market for longevity securities should behave like other aspects of the capital market. The application of some stochastic models developed originally for financial purposes to mortality, is then tested. In particular, interest rate and credit risk models have been considered. The theory of incomplete markets has also been explored (Biffis, 2005; Biffis and Millossovich, 2006; and Cairns *et al.*, 2006; and Pitacco *et al.*, 2008). The basic building blocks of the new theory still require careful discussion and investigation. Indeed, financial models are not necessarily suitable for describing mortality. Given that we are not dealing with a pricing problem, we prefer to retain the simple structure previously introduced, which can anyhow be useful for internal valuations.

Extending the prevailing notation, let ${}_{t}q_{x\tau}^{[H(\tau)]} = F_{x\tau}\left(t\,|\,H(\tau)\right)$. In particular, for $t = 1$ let ${}_{1}q_{x\tau}^{[H(\tau)]} = q_{x\tau}^{[H(\tau)]}$. We then assume,

$$\frac{q_{x\tau}^{[H(\tau)]}}{1-q_{x\tau}^{[H(\tau)]}} = \mathrm{G}^{[H(\tau)]}\left(K^{[H(\tau)]}\right)^{x} \qquad \ldots(1)$$

Hence, the third term of the first Heligman-Pollard law, i.e., the one describing the old age pattern of mortality, is adopted in order to express the mortality profile involved in our calculations. Note that, in particular the relevant parameters are cohort-specific and, further, their value is specific to the adopted mortality assumption for the cohort. Uncertainty then concerns the level of parameters $G^{[H(\tau)]}$ and $K^{[H(\tau)]}$.

We define the seven alternative sets of parameters, which are presented in Table 1. We refer to 1955 as the cohort of the Italian males. Parameters under scenario $H_4(1955)$ have been obtained by fitting Equation 1 to the projected number of males, which indeed refers to Italian male annuitants born in 1955. Assumptions $H_h(1955)$ for $h = 1$, 2 and 3,

Table 1: Parameters for the Heligman-Pollard Law

	$G^{[H_h(1955)]}$	$K^{[H_h(1955)]}$	$\mathbb{E}[T_{65,1955} \mid H_h(1955)]$
$H_1(1955)$	5.561E–07	1.153	19.874
$H_2(1955)$	9.542E–07	1.144	20.442
$H_3(1955)$	1.414E–06	1.137	21.099
$H_4(1955)$	2.005E–06	1.130	21.849
$H_5(1955)$	2.533E–06	1.125	22.711
$H_6(1955)$	2.972E–06	1.121	23.676
$H_7(1955)$	3.294E–06	1.118	24.706

Figure 1: Survival Functions

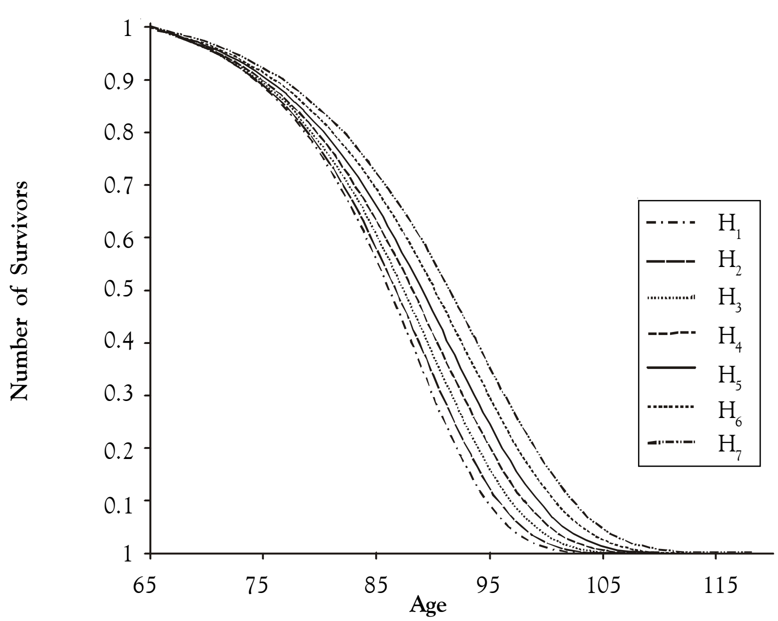
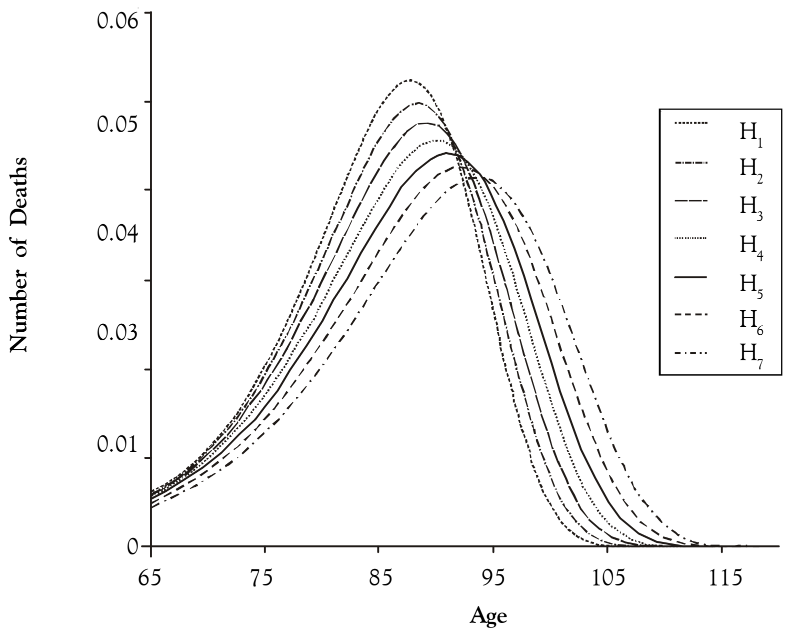


Figure 2: Curves of Deaths



embed some hypotheses of higher mortality (as witnessed by the expected lifetime, $\mathbb{E}[T_{65,1955}|H_h(1955)]$, which are presented in Table 1). Conversely, assumptions $H_h(1955)$ for $h = 5$, 6 and 7, embed some hypotheses of lower mortality ($\mathbb{E}[T_{65,1955}|H_h(1955)]$ in Table 1). Figures 1 and 2 respectively, show the relevant survival functions and curves of deaths. Assumption $H_4(1955)$ is referred to as the best-estimate description of the mortality trend for cohort 1955.

4. Possible Solvency Rules in Internal Models

When dealing with life annuities, capital allocation rules should carefully account for all the risks retained by the insurer. With reference to mortality risks, to which this paper is devoted, longevity risk is in particular critical. However, proper care should also be devoted to other risk components. In this section, we investigate some internal models. Consistently with the aim of the paper, we disregard all but mortality risks. We refer to conventional immediate life annuities, so that there is no allowance for participation in financial or other profits. In order to make results easier to understand, we further assume that no risk transfer has been undertaken. We focus on net outflows, therefore, expenses and related expense loadings are not accounted for.

The portfolio, we refer to, consists of one cohort of immediate life annuities. We assume that all the annuitants are aged x_0 at the time t_0 of issue, and are all entitled to receive the annual amount b at each time t, $t = 1, 2, \ldots$, until death. The lifetimes of annuitants are assumed to be identically distributed and conditional on any given mortality assumption, independent of each other.

Let N_t denote the random number of annuitants at the duration time t, $t = 0, 1, \ldots, \omega - x_0$ (and hence at the beginning of the calendar year $t_0 + t$), with $N_0 = n_0$ a specified number representing the initial size of the portfolio and ω the maximum attainable age. Then we note that $N_{\omega - x_0 + 1} = 0$. The portfolio in effect at time t is defined as $\Pi_t = \{j \mid T^{(j)}_{x_0\tau} > t\}$, where $T^{(j)}_{x_0\tau}$ represents the remaining lifetime of annuitant j at age x_0.

Annual outflows for the portfolio are defined for $t = 1, 2, \ldots, \omega - x_0$, as:

$$B_t^{(\Pi)} = b N_t$$

The present value of future portfolio outflows can be defined as:

$$Y_t^{(\Pi)} = \sum_{h=t+1}^{\omega - x_0} B_h^{(\Pi)} (1+i)^{-(h-t)} = \sum_{h=t+1}^{\omega - x_0} b N_h (1+i)^{-(h-t)}$$

where, for brevity, we have considered a flat term structure of interest rates (hence, i is the annual interest rate).

Let A_t be the amount of portfolio assets at time t and $V_t^{(\Pi)}$ the portfolio reserve (or technical provision) at the same time. These quantities are random at the time of valuation, because of the risks, mortality in particular, borne by the portfolio. Let z be the valuation time ($z = 0, 1, \ldots$). The random path of portfolio assets is recursively described for $t = z + 1, z + 2, \ldots$ as follows:

$$A_t = A_{t-1}\,(1 + i) - b\,N_t \qquad \ldots(2)$$

with A_z given (including both the amount of reserve and capital required according to a chosen solvency rule).

According to current legislation, the portfolio reserve $V_t^{(\Pi)}$ should include a risk margin in order to face the adverse future scenarios. In traditional practice, such a risk margin is implicitly assessed by directly calculating the reserve, $V_t^{(\Pi)}$, as the expected present value of future benefits net of future premiums, according to a conservative description of the future scenario. In explicit terms, the portfolio reserve can be expressed as:

$$V_t^{(\Pi)} = V_t^{(\Pi)[BE]} + RM_t$$

where $V_t^{(\Pi)[BE]}$ is the portfolio reserve calculated according to the best-estimate assumption regarding the future scenario, and RM_t is the risk margin. Reserving formulae leading to an explicit risk margin are currently under discussion.

The quantity M_t, defined as follows,

$$M_t = A_t - V_t^{(\Pi)[BE]},$$

represents the assets available to meet the risks. Thus, M_t should back both the risk margin, to be then included into the portfolio reserve and the required capital. We note that if the risk margin is funded by premiums paid by policyholders (as it should occur on the basis of a going concern and sound management of liabilities), then it also consists of the expected profit of the insurer.

Capital requirements suitable for internal models assume that the insurer can be considered to be solvent in respect of portfolio obligations, if with an assigned (high) probability, assets meet liabilities within a chosen time-horizon, according to a realistic probabilistic structure. Several details need to be specified for the practical implementation of such a definition:

- As far as assets are concerned, reference can be made to assets available to meet risks, M_t, which are required to be non-negative, or to the total amount of assets, A_t, which are required to be not less than the present value of future payments, $Y_t^{[\Pi]}$.
- The time-horizon T may range from a short to medium-term (say, 1 to 5 years), to the residual duration of the portfolio, $\omega + 1 - x_0 - z$ (at time z, $z = 0, 1, \ldots$).
- When a time-horizon longer than one year is adopted, the overall trajectory of assets within the time-horizon can be addressed, or just their final value at time T, is considered.
- The portfolio needs to be defined in a runoff or a going concern perspective. When dealing with life insurance, a runoff approach is more reasonable.

Let z be the time at which solvency is ascertained ($z = 0, 1, \ldots$). According to the possible choices listed above, the following rules can be adopted for assessing the capital required at time z:

$$[R1] \; : \; \mathbb{P}\,[(M_{z+1} \geq 0) \wedge (M_{z+2} \geq 0) \wedge \ldots \wedge (M_{Z+T} \geq 0)] = 1 - \varepsilon_1 \qquad \ldots(3)$$

$$[R2] \; : \; \mathbb{P}\,[M_{Z+T} \geq 0] = 1 - \varepsilon_2 \qquad \ldots(4)$$

$$[R3] \; : \; \mathbb{P}\left[\left(A_{z+1} - Y_{z+1}^{(\Pi)} \geq 0\right) \wedge \left(A_{z+2} - Y_{z+2}^{(\Pi)} \geq 0\right) \wedge \ldots \wedge \left(A_{z+T} - Y_{z+T}^{(\Pi)} \geq 0\right)\right] = 1 - \varepsilon_3 \qquad \ldots(5)$$

where ε_i, $i = 1, 2$ and 3, is the accepted default probability under the chosen rule. Clearly, in all the above-mentioned solvency models, the relevant probability is assessed conditional on the current information at time z.

The difference between rules [R1] and [R2] is apparent. We first note that the quantity M_t originates from the initial capital allocation at time z, whose amount must indeed be assessed through a solvency rule, and from the surplus (possibly negative) emerging year by year. Under rule [R2], only the total surplus in the time interval $[z, z + T]$ is addressed, whilst under rule [R1] its yearly emergence is also checked. When addressing random fluctuations and permanent systematic deviations in the mortality of immediate life annuitants, an annual loss can hardly be recovered just with the flows emerging from the portfolio in future years. Indeed, a negative total surplus is also likely to emerge. Disregarding further the profit sources, as we are doing, outputs from rule [R2] can then be supposed to be similar to those of rule [Rl], so that one of the two can be disregarded. Rule [R2] is more straightforward to implement. Conversely, rule [Rl] is more accurate, as the possible lack of funds is detected sooner. So in the following, we do not consider rule [R2].

With reference to rule [R3], it is worthwhile to rearrange the relevant expression. Recursion of Equation 2 leads to:

$$A_t = A_z (1+i)^{t-z} - \sum_{h=z+1}^{t} b N_h (1+i)^{t-h}$$

The probability in Equation 5 can then be rewritten as:

$$\mathbb{P}\left[\bigwedge_{t=z+1}^{z+T} A_z (1+i)^{t-z} - \sum_{h=z+1}^{\omega-x_0} b N_h (1+i)^{t-h} \geq 0\right]$$

$$= \mathbb{P}\left[\bigwedge_{t=z+1}^{z+T} (1+i)^{t-(\omega+1-x_0)} \left(A_z (1+i)^{\omega+1-x_0-z} - \sum_{h=z+1}^{\omega-x_0} b N_h (1+i)^{\omega+1-x_0-h} \right) \geq 0\right] \quad \text{...(6)}$$

If we note that the quantity,

$$A_z (1+i)^{\omega+1-x_0-z} - \sum_{h=z+1}^{\omega-x_0} b N_h (1+i)^{\omega+1-x_0-h} = A_{\omega+1-x_0}$$

represents the amount of portfolio assets available when the cohort is exhausted, the probability in Equation 6 reduces to:

$$\mathbb{P}\left[\bigwedge_{t=z+1}^{z+T} (1+i)^{t-(\omega+1-x_0)} A_{\omega+1-x_0} \geq 0\right] = \mathbb{P}\left[\bigwedge_{t=z+1}^{z+T} A_{\omega+1-x_0} \geq 0\right] = \mathbb{P}\left[A_{\omega+1-x_0} \geq 0\right]$$

We can further note that:

$$A_{\omega+1-x_0} = M_{\omega+1-x_0}$$

as all obligations have been fulfilled at time $\omega+1-x_0$. So, finally, rule [R3] can be rewritten as:

$$\mathbb{P}\left[M_{\omega+1-x_0} \geq 0\right] = 1-\varepsilon_3 \quad \text{...(7)}$$

It is useful to note that such results hold in particular because: (a) The portfolio is closed to new entrants; (b) The probability in Equation 5 (as well as in Equations 3 and 4) is assessed according to the real world probability distribution of assets and liabilities (so that no risk-adjustment is applied, for example, in a risk-neutral sense); (c) Such probability is implicitly conditional on the information available on the relevant variables at time z (current number of survivors, investment yields, etc.). A similar simplification can however be obtained when more than one cohort is addressed and the term structure of interest rates is not flat.

Comparing rules [*R*l] and [*R*3], the apparent difference consists of the way liabilities are summarized in terms of portfolio reserve in rule [*R*l] and the present value of future payments in rule [*R*3]. A deeper comparison emerges when we consider Equation 7. We assume that in Equation 3 the time-horizon is set at, $T = \omega + 1 - x_0 - z$. In both cases, the total amount of the surplus would be considered. However, under rule [*R*l] its emergence is driven by the reserve (which may smooth annual losses), whilst under rule [*R*3] the initial income and the annual outgoes are simply accounted for. According to a valuation terminology, rule [*R*l] is based on a 'deferral and matching' logic, whilst [*R*3] on an 'asset and liability' approach. Further, rule [*R*l] allows for a preferred time-horizon, whilst under [*R*3] the maximum possible time-horizon is necessarily adopted.

Solving Equation 3 in respect of M_z, through stochastic simulation, one finds the amount of assets required to meet the risks at time z. We denote such an amount as, $M_z^{[R1]}(T)$. Then, $A_z^{[R1]}(T) = V_z^{(\Pi)[BE]} + M_z^{[R1]}(T)$ is the total amount of assets required at time z. Solving Equation 7, again through stochastic simulation, one finds the total amount of assets required at time z, which is denoted as, $A_z^{[R3]}$. The required amount of assets to meet risks at time z is then: $M_z^{[R3]} = A_z^{[R3]} - V_z^{(\Pi)[BE]}$. A numerical investigation is presented and the related results are discussed in Section 5.

5. Comparisons

In this section, some comparisons among the internal rules described in Section 4 are performed through numerical implementation.

We refer to a portfolio with the features described in Section 4. The entry age is, $x_0 = 65$, and the annual interest rate is, $i = 0.03$. For any solvency rule, the accepted default probability has been set to 0.5% level, so that $\varepsilon_1 = \varepsilon_3 = 0.005$. The mortality assumptions are those described in Section 3 (refer Table 1, with weights: $\rho_1 = \rho_7 = 0.05$; $\rho_2 = \rho_3 = \rho_5 = \rho_6 = 0.1$; and $\rho_4 = 0.5$). We note that the best-estimate assumption has been assigned the highest weight, whilst the lowest weight has been assigned to the extreme assumptions among those considered, i.e., $H_1(1955)$ and $H_7(1955)$.

Table 2 presents the individual reserve net of the risk margin, denoted by, $V_z^{(1)[BE]}$. We recall that this quantity is defined as the expected value of future payments, according to the best-estimate assumption of the future scenario. It follows that if n_z is the number of annuitants at time z, then $V_z^{(\Pi)[BE]} = n_z V_z^{(1)[BE]}$.

Table 3 displays the amount of capital (per unit of portfolio reserve net of the risk margin, $V_z^{(\Pi)[BE]}$) that is required according to the solvency rules [Rl] and [R3] for different portfolio sizes. For rule [Rl], the maximum possible time-horizon, i.e., the time to the portfolio exhaustion, $T = \omega + 1 - x_0 - z$, has been chosen. As we would expect from the discussion in Section 4, the two requirements lead to similar outputs, at least when only mortality is addressed. In this case, at least, the outputs suggest that rule [Rl] is to some extent independent of the reserve, when T takes the maximum possible value for the time-horizon. Between random fluctuations and systematic deviations in mortality, the latter are in particular important. Furthermore, due to the fact that the longevity risk arises in the long-term, to some extent the total relevant loss is more important than the timing of its emergence. While interpreting the size of the required capital presented in Table 3, we finally recall that $M_z^{[R1]}(T)$ and $M_z^{[R3]}$ represent the assets required to meet the risks. Such amount must partially back the risk margin, to be included in the technical provision and the required capital.

Table 2: Individual Reserve

Time z	Reserve $V_z^{(1)[BE]}$	Reserve $V_z^{(1)}$
0	15.259	16.342
5	12.956	13.887
10	10.599	11.357
15	8.294	8.874
20	6.167	6.581
25	4.336	4.611
30	2.877	3.050
35	1.807	1.913

Table 3: Assets Required to Meet Risks Based on Rules [Rl] and [R3], Facing Longevity Risk and Random Fluctuations in Mortality

(in %)

Time z	Rule [Rl], with $T = \omega + 1 - x_0 - z$: $\dfrac{M_z^{[R1]}(\omega+1-x_0-z)}{V_z^{(\Pi)[BE]}}$			Rule [R3]: $\dfrac{M_z^{[R3]}}{V_z^{(\Pi)[BE]}}$		
	$n_0 = 100$	$n_0 = 1{,}000$	$n_0 = 10{,}000$	$n_0 = 100$	$n_0 = 1{,}000$	$n_0 = 10{,}000$
0	13.647	10.730	9.819	13.647	10.730	9.819
5	17.588	13.870	12.757	17.573	13.868	12.757
10	22.915	18.240	16.813	22.880	18.240	16.813
15	29.927	23.757	22.125	29.759	23.753	22.125
20	41.014	31.319	29.205	40.742	31.308	29.205
25	56.901	42.052	38.545	56.472	42.025	38.545
30	86.748	57.285	50.622	84.647	57.210	50.605
35	180.337	86.445	66.553	170.443	85.749	66.450

Table 4: Assets Required to Meet Risks Based on Rule [*R*1], Per Unit of Portfolio Reserve (Net of the Risk Margin): $\frac{M_z^{[R1](T)}}{V_z^{(\Pi)[BE]}}$, Facing Longevity Risk and Random Fluctuations in Mortality

(in %)

Time z	Time-Horizon $T = 1$			Time-Horizon $T = 3$		
	$n_0 = 100$	$n_0 = 1{,}000$	$n_0 = 10{,}000$	$n_0 = 100$	$n_0 = 1{,}000$	$n_0 = 10{,}000$
0	0.574	0.473	0.212	1.834	1.063	0.525
5	1.058	0.743	0.376	3.358	1.688	1.006
10	1.951	1.159	0.694	5.162	2.659	1.941
15	3.600	2.034	1.343	8.689	4.679	3.812
20	6.639	3.602	2.559	14.046	8.532	7.301
25	12.246	6.338	4.954	23.787	15.426	13.449
30	22.588	12.168	9.449	46.170	28.301	24.099
35	41.664	26.210	18.265	124.167	54.361	41.379

In Table 4, outputs from rule [*R*l] are investigated for shorter time-horizons. Comparing Table 3 with Table 4, the long-term nature of the longevity risk clearly emerges. We note that, in both Tables 3 and 4, at each valuation time and for each rule, the amount of required assets decreases when a larger portfolio is considered. This is obviously due to the pooling of random fluctuations.

Results displayed in Table 5 address only random fluctuations. In particular, the amount of assets required, has been calculated adopting only the best-estimate mortality assumption, $H_4(1955)$. In contrast, in Table 6, only longevity risk has been accounted for, by assuming that whatever is the realized mortality trend, the actual number of deaths in each year is the same as what we expect under the relevant trend assumption. We note that in the latter case the required amount of assets per unit of portfolio reserve (net of the risk margin) is independent of the size of the portfolio. This is due to the systematic nature of longevity risk. Regarding Table 5, we point out that the random fluctuations accounted for, are not fully comparable to those emerging in Tables 3 and 4. Indeed, in these latter tables a mixture of the random fluctuations, which can be appraised under the several mortality assumptions in $H(1955)$, is addressed. While comparing Table 5 (Lower Panel) with Table 4, it can be seen that, if rule [*R*l] is implemented with a shorter time-horizon, in practice we are mainly accounting for random fluctuations, rather than systematic deviations. Again, this is due to the long-term nature of longevity risk. Tables 5 and 6 do provide us with some useful information. However, it must be pointed out that implementing an internal model allowing for only risk component, represents an improper use of the model itself. Just to sketch the idea, we note that on summing up the results in Tables 5 and 6, for a given rule and portfolio size, we do not get the correspondent results in Table 3 or 4. Thus, some aspects are missed

Table 5: Assets Required to Meet Risks Based on Rules [Rl] and [R3], Facing Mortality Random Fluctuations Only (Mortality Assumption $H_4(1955)$) (in %)						
Time z	Rule [Rl], with $T = \omega + 1 - x_0 - z$ $\frac{M_z^{[R1]}(\omega+1-x_0-z)}{V_z^{(\Pi)[BE]}}$			Rule [R3] $\frac{M_z^{[R3]}}{V_z^{(\Pi)[BE]}}$		
	$n_0 = 100$	$n_0 = 1{,}000$	$n_0 = 10{,}000$	$n_0 = 100$	$n_0 = 1{,}000$	$n_0 = 10{,}000$
0	7.813	2.832	0.879	7.031	2.698	0.800
5	9.983	3.071	1.067	9.436	2.949	1.040
10	12.144	4.040	1.217	11.543	3.759	1.193
15	16.153	5.202	1.544	14.982	4.921	1.462
20	22.343	6.938	2.091	21.292	6.554	1.936
25	29.728	10.388	3.072	28.546	9.642	2.983
30	54.183	16.871	5.547	51.253	16.807	5.152
35	155.859	36.795	11.715	144.058	34.809	11.207
Time z	Rule [Rl], with $T = 1$ $\frac{M_z^{[R1]}(1)}{V_z^{(\Pi)[BE]}}$			Rule [R1], with $T = 3$ $\frac{M_z^{[R1]}(3)}{V_z^{(\Pi)[BE]}}$		
	$n_0 = 100$	$n_0 = 1{,}000$	$n_0 = 10{,}000$	$n_0 = 100$	$n_0 = 1{,}000$	$n_0 = 10{,}000$
0	0.574	0.473	0.171	1.834	0.983	0.378
5	1.058	0.743	0.271	3.358	1.443	0.479
10	1.951	0.932	0.388	5.162	1.957	0.657
15	3.600	1.642	0.583	8.604	2.630	0.932
20	6.639	2.458	0.806	13.304	3.775	1.329
25	12.246	4.633	1.379	19.609	7.129	2.166
30	22.588	7.878	2.804	41.023	13.181	4.168
35	41.664	21.058	7.321	124.167	32.954	10.176

when working with only marginal distributions (as is the case when we address only random fluctuations or systematic deviations).

In Table 6, at any valuation time the required capital is flat in respect to the portfolio size. This suggests a deterministic approach for allocating capital to deal with longevity risk. In particular, the assessment of the required capital could be based on a comparison between the actual reserve (net of the risk margin) and a reserve (again net of the risk margin) calculated under a more severe mortality trend assumption, as it turns out to be the case under some regulatory solvency systems.

Table 6: Assets Required to Meet Risks Based on Rules [Rl] and [R3], Facing Longevity Risk Only (in %)				
Time z	$\frac{M_z^{[R1]}(1)}{V_z^{(\Pi)[BE]}}$	$\frac{M_z^{[R1]}(3)}{V_z^{(\Pi)[BE]}}$	$\frac{M_z^{[R1]}(\omega+1-x_0-z)}{V_z^{(\Pi)[BE]}}$	$\frac{M_z^{[R3]}}{V_z^{(\Pi)[BE]}}$
0	0.110	0.366	9.402	9.402
5	0.247	0.805	12.270	12.270
10	0.531	1.691	16.134	16.134
15	1.105	3.423	21.301	*21.301*
20	2.233	6.681	28.102	28.102
25	4.389	12.511	36.826	36.826
30	8.346	22.240	47.599	47.599
35	15.214	36.932	60.269	60.269

Let $V_z^{(\Pi)[B]}$ be the expected present value of future payments for the portfolio, based on a mortality assumption worse than the best-estimate one, i.e., on a lower mortality hypothesis. Thus,

$$V_z^{(\Pi)[BE]} \leq V_z^{(\Pi)[B]}$$

We note that $V_z^{(\Pi)[B]}$ represents the technical provision, net of the risk margin, based on assumption B. The required amount of assets to meet risks would then be,

$$[R4]: M_z^{[R4]} = V_z^{(\Pi)[B]} - V_z^{(\Pi)[BE]}$$

We note that rule [R4] would account for the longevity risk only. Further, no default probability is explicitly mentioned. However, the mortality assumption adopted in $V_z^{(\Pi)[B]}$ clearly embeds some default probability. The time-horizon implicitly considered is the maximum residual duration of the portfolio, given that this is the time-horizon referred to in the calculation of the reserve. Further, given that $M_z^{[R4]}$ is the difference between the expected values, it turns out to be linear in respect to the portfolio size.

In order to compare rules [Rl], [R3] and [R4], we define the following ratios:

$$QM_z^{[R1]}(T;n_z) = \frac{M_z^{[R1]}(T)}{V_z^{(\Pi)[BE]}}$$

$$QM_z^{[R3]}(n_z) = \frac{M_z^{[R3]}}{V_z^{(\Pi)[BE]}}$$

$$QV_z = \frac{M_z^{[R4]}}{V_z^{(\Pi)[BE]}}$$

The ratios $QM_z^{[R1]}(T;n_z)$ and $QM_z^{[R3]}(n_z)$, depend on the size of the portfolio as they also account for the risk of random fluctuations, whilst the ratio, QV_Z, which addresses only the longevity risk, is independent of the portfolio size. On the other hand, rules [Rl] and [R3] could be implemented considering only the risk of random fluctuations or the longevity risk, as we have illustrated in the calculations presented in Tables 5 and 6, respectively. As it emerges from Table 6, that while addressing only the longevity risk, the ratios $QM_z^{[R1]}(T;n_z)$ and $QM_z^{[R3]}(n_z)$ are independent of the size of the portfolio. This situation suggests a deterministic approach instead of a simulation-based procedure. However, we have already commented that addressing just a component of the mortality risks, represents an improper use of rules [Rl] and [R3]. A further difference between the ratios $QM_z^{[R1]}(T;n_z)$ and QV_Z stands in the possibility to set a preferred time-horizon. Indeed, time-horizons other than the maximum one may be chosen only when rule [Rl] is adopted.

Conclusion

It is not possible to derive general conclusions regarding the comparison between the outcoming levels of ratios, $QM_z^{[R1]}(T;n_z)$ and $QM_z^{[R3]}(n_z)$, on one hand, and QV_Z, on the other. We just consider the results previously illustrated. We can comment on the fact that a simplified rule, such as [R4], which is more direct to implement, may result in an amount of required assets which is too high or too low in relation to the portfolio size and the duration of the portfolio. This is due to having neglected the pooling component, whose impact depends on the portfolio size (as well as age).

Undoubtedly, the advantage of rule [R4] is in its simplicity. Of course, it is possible to find the reserving basis avoiding the unsatisfactory situations (but to be sure, one should first perform the valuation through an internal model, at least for some typical compositions of the portfolio). The possibility to adopt different solvency time-horizons for the different mortality risk components, supports the separate treatment of the components themselves. So, we could choose the maximum possible value for T when accounting for only longevity risk (adopting the simplified rule [R4]) and a short-medium time-horizon, when accounting for random fluctuations only (if rule [Rl] is adopted, with say T = 1 to 5 years). For practical purpose, this approach could represent a good compromise, provided that the relevant assumptions are properly disclosed. If valuation tools other than an internal model are available, or are required for the risk of random fluctuations, then rule [*R*4] is certainly able to capture properly the feature of longevity risk (only). ❖

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