Fair Price Estimation of Basket Default Swap: Evidence from Japanese Market

Fathi Abid* and Nader Naifar**

The aim of this paper is to estimate the fair spread of reconstituted basket credit default swap using Japanese market data. The value of these instruments depends on a number of factors including credit rating of the obligors in the basket, recovery rates, intensity of default, basket size, and the correlation of obligors in the basket. A fundamental part of the pricing framework is the estimation of the instantaneous default probabilities for each obligor. As default probabilities depend on the credit quality of the considered obligor, well-calibrated credit curves are a main ingredient for constructing default times. Similarly, the choice of copula for modeling the correlation of obligors in the basket and the choice of procedures for rare-event simulation govern the pricing of basket credit derivatives. The study has several practical implications that are of value to the financial hedgers and engineers, financial regulators, government regulators, central banks, and financial risk managers.

Introduction

The credit derivative and structured credit markets have grown very rapidly in size and complexity in recent years. The flourishing world of credit derivatives has in turn spurred huge growth in structured products. The demand for more tailored, tradable, and investment-grade instruments have been an important motivation for developments in the structured credit markets. The dominant structured product is basket credit default swaps and Collateralized Debt Obligation (CDO). The most common type of basket default swaps is the First-to-Default Swap (FTDS), where the seller compensates the buyer for any loss of the principal and also, possibly, the accrued interest of the asset in the reference basket which defaults first. The main difference between FTDS and a Credit Default Swap (CDS) is the event causing payout for the contract (in one case, it is the first default of any of a list of names and in the other default of a single name). An n^{th} to default basket default swap gives protection against the n^{th} default in the underlying pool of credits.

In order to understand the significance of developments in the Japanese markets, it is useful to review what is happening in the global market. The global market in credit derivatives is expected to rise to $33 tn by the end of 2008 according to a report to be published by the British Bankers' Association (BBA) at its Credit Derivatives conference.

* Professor of Finance, University of Sfax, Tunisia. E-mail: fathi.abid@fsegs.rnu.tn

** Assistant Professor of Finance, Institute of the Higher Business Studies, University of Sfax, Tunisia. E-mail: doctoratnader@yahoo.fr

The report, based on a survey of market leaders in credit derivatives, predicts that London will remain as one of the world's dominant centers for credit derivative products. According to SIFMA™, Global CDO issuance through the third quarter of 2006, at $322 bn, has exceeded full-year 2005 issuance by 20%. Issuance in the third quarter of 2006, at $117.8 bn, also exceeded issuance in the third quarter of last year by 30%. European Sclerotisation Forum (ESF) forecasts issuance to grow to a new record of €531 bn in 2007, led by Residential Mortgage-Backed Securities (RMBS), Commercial Mortgage-Backed Securities (CMBS), and CDOs. Then, a 16.4% growth rate from €456 bn issued in 2006.

According to *Financial Times*[1], the explosive growth in credit derivatives is bypassing Japan. It accounts for 5% of global activity, estimates the British Bankers' Association, compared with about 40% for London. There are some hopes that the introduction of risk-weighted Basel II capital rules will stimulate the use of CDS for hedging purposes. There is some evidence of this: the country's top bank last year issued a handful of balance sheet collateralized debt obligations referring to Japanese names. But they do not bank on this starting a virtuous cycle of liquidity. Made-in-Japan CDOs are still more likely to involve overseas names to pick up bigger spreads.

Substantial empirical studies are devoted on basket credit default swap. The main problem in the pricing of such instruments is modeling the structure of dependency of the default times. Defaults are rarely observed. Copulas can be introduced to model these correlations by using the correlations of corresponding default time. Li (2000) presents a Gaussian copula method for the pricing of first to default swaps. Other studies of elliptical copulas with higher tail dependence, like the *t*-copula, can be found in Mashal and Naldi (2002). The Marshall-Olkin copula is yet another class of copula functions, which stems from the multivariate compound Poisson process. In this model, individual defaults are constructed from a series of independent common shock. Previous studies on the use of the Marshall-Olkin copula in the context of credit risk modeling include Wong (2000), Duffie and Pan (2001), and Lindskog and McNeil (2003). Hull and White (2001) developed two procedures to pricing trenches of CDO and n^{th} to default swap. The first procedure involves calculating the probability distribution of the number of defaults by a time T and suited to the situation where companies have equal weight in the portfolio and recovery rates are assumed to be constant. The second procedure involves calculating the probability distribution of the total loss from defaults by time T. Tavares *et al.* (2004) present a basket model to deal with the Gaussian copula smile. They combine the copula model (to model the default risk that is driven by the economy) with independent Poisson processes (to model the default risk that is driven by a particular sector and by the company in question). Madan (2004) provides details of the pricing of n^{th} to default contracts using the one factor Gaussian and the Clayton copulas. He modeled the marginal default time densities using Weibull and Frechet families and the joint densities are obtained by using the method of copula. Sicar and Zariphopoulou (2006) study the

[1] *Financial Times*, March 14, 2007.

impact of risk aversion on the valuation of basket credit derivatives. They use the technology of utility-indifference pricing in intensity-based models of default risk. Abid and Naifar (2008) present a copula-based simulation procedures for pricing basket credit default swap. They argue that not only the choice of copula is an important input, but also the choice of procedures for rare-event simulation govern the pricing of basket credit derivatives.

In this study, we present a methodology to estimate the fair spread of basket credit derivatives reconstituted with CDS contracts using Japanese market data. To express dependencies between times of default, Gaussian and Student copulas are considered. We present an application of the Canonical Maximum Likelihood (CML) method for calibrating Student copula.

The Fair Spread of Basket Credit Default Swaps

The valuation of a basket default swap comes down to the calculation of relevant default probabilities. If defaults of the reference entities are independent, a closed-form formula for such probabilities can be derived. If defaults of the reference entities are not independent, then the calculations of first-to-default or n^{th}-to-default probabilities are more difficult. Generally, closed-form formulae are not available and therefore, Monte Carlo simulation is used.

Suppose there is a basket of credit default swap with the following characteristics:

- V : The total value of the basket. $V = \sum_{i=1}^{N} A_i$ with A is the notional amount of each contract;
- N : The number of contracts in the basket;
- δ : The payment frequency. $\delta = 1$ for annual payment frequency;
- T : The maturity date of the basket;
- n : The basket seniority. $n=1$ for first to default basket, $n=2$ for second to default basket; and
- s : The fair price of the basket

According to Galiani (2003), the risk neutral price of the n^{th} to default basket swap is computed by equating the expected value of the discounted premium payment leg (fixed cash flow to be paid till contract expiration T or n^{th} credit event occurs) with the expected value of the discounted default leg (contingent payment in case of default), under the equivalent martingale measure P^*. Under this measure, the price processes of any tradable security, discounted by the money market account, are P^* martingales with respect to some filtration. The premium legs are paid as long as the underlying credit has not defaulted until the maturity of the contract.

The fair spread of the basket credit default swap is given by:

$$s=\frac{E[DL_n]}{E[PL_n]+E[AP_n]}=\frac{\sum_{j=1}^{N}(1-R^n)^T\int_0 \beta(\tau^n)F_{(n)}^{n^{th}=j}(dt)}{\sum_{i=1}^{K}\delta\beta(t)\left[1-F^n(t_i)\right]+\int_{t_{i-1}}^{t_i}\frac{v-t_{i-1}}{t_i-t_{i-1}}\delta\beta(v)F_{(n)}(dv)} \quad \ldots(1)$$

where,

$E[DL_n]$ = The default leg of the n^{th} to default basket default swap[2]

$E[PL_n]$ = The premium leg of the n^{th} to default basket default swap[3]

$E[AP_n]$ = The expected discounted accrued premium

From Equation 1, we notice that the fair price depends on default probabilities, the structure of dependency between default times and the market perception of the loss severity given default through the recovery rate. The estimation of basket credit default swap fair spread using Japanese market data contains the following steps:

- Step 1: Estimate the parameters of the copula functions (Gaussian copula and Student copula) chosen for modeling the structure of dependency among the names in the basket. The estimation of the parameters is described under the subhead "Estimation of Copulas Parameters";

---

[2] The default leg can be expressed as the difference between the expected discounted default payment $E[DP_n]$ and the expected discounted accrued premium $E[AP_n]$. Then, $E[DL_n]=E[DP_n]-E[AP_n]$. With:

$E[DP_n]=E\left[A.\left(1-R^n\right)\beta\left(\tau^n\right)\sum_{j=1}^{N}I_{\{\tau^n\le T\}}\right]=A\sum_{j=1}^{N}\left(1-R^n\right)\int_0^T\beta\left(\tau^n\right)F_{(n)}^{n^{th}=j}(dt)$. We notice that $F_{(n)}^{n^{th}=j}(t)$ is the distribution function of n^{th} basket default relative to the j^{th} defaulter for allowing different recovery rates for the obligors.

$$E[AP_n]=E\left[p\sum_{i=1}^{K}\left(\frac{\tau^n-t_{i-1}}{t_i-t_{i-1}}\delta\right)\beta\left(\tau^n\right)I_{\{t_{i-1}<\tau^n\le t_i\}}\right]=p\sum_{i=1}^{K}\int_{t_{i-1}}^{t_i}\frac{v-t_{i-1}}{t_i-t_{i-1}}\delta\beta(v)F_{(n)}(dv).$$

[3] The present value of the premium leg of the n^{th} to default basket default swap can be computed as follows: $E(PL_n)=E\left[p\delta\beta(t_i)\sum_{i=1}^{K}I_{\{\tau^n>t\}}\right]$. Where $(p=f.A)$ the premium leg as function as the fair spread of the contract (f) as a fraction of is notional amount (A) in basis point, $t_i, i\in\{1,\ldots,k\}$ are the payment dates either until T or until $\tau<T$ in case of default, δ_i is the frequency of payment and $\beta(t)$ is the discount factor. Let $F_{(t)}(t)=P^*\left(\tau^n\le t\right)$ be the distribution function of τ^n, then we can rewrite the premium leg as: $E(PL_n)=p\delta\beta(t)\sum_{i=1}^{K}\left[1-F^n(t_i)\right]$.

- Step 2: Calibrate credit curve for each names in the basket as described under the subhead "Calibration of Credit Curves"; and
- Step 3: Generate correlated default times concisely with the estimated parameters as described under the subhead "Fair Spread Estimation of Reconstituted Basket Credit Default Swap".

Data Description

Tokyo Financial Exchange (TFX) distributes credit default swap reference rates in collaboration with many financial institutions[4]. The indicative rates provided by the designated financial institutions are the rates (premium) of the following standardized credit default swap contracts:

- Five-year maturity;
- An amount of ¥500 mn for notional principle;
- Three credit events such as 'bankruptcy', 'failure to pay', and 'restructuring' (old restructuring); and
- A physical settlement is required in case any credit event occurs.

We refer to Japan Credit Rating Agency (JCR) to obtain rating and industry sector of each reference entity. We also obtain average bond recovery rates a year prior to default for each class of rating. Daily stock prices of each name are obtained from Tokyo Stock Exchange.

We construct an example of basket credit default swap (Table 1) with five names of different ratings and industry sector (Table 2). Our method is applicable to any other reconstituted basket credit default swap. Similarly, we set the notation used throughout this study.

Table 1: Basket Credit Default Swap Description

Basket Default Swap	
Initial Par Value $V = \sum_{i=1}^{N} A_i$ A is the notional amount of the contract	¥2500 mn
Number of obligors: N	5
Start Date: t_o	January 26, 2007
Maturity Date: T	January 26, 2012
Payment Frequency: $\delta = 1$	Annual Payment
Fair Price	s

[4] Fifteen Financial institutions, which support to distribute CDS Reference Rates for the infrastructure development of credit default swap market (Barclays Capital Japan Limited, Bear Stearns (Japan) Ltd., etc.).

Table 2: Basket Credit Default Swap as on January 26, 2007				
Name	**Notional Principle (¥)**	**5-Year CDS Spread**	**Industry Sector**	**Rating**
East Japan Railway Company	¥500 mn	4.86	Transport – Rail	AAA
Bridgestone Corporation	¥500 mn	6.72	Rubber – Tires	AA
Konica Minolta Holdings, Inc.	¥500 mn	18.00	Photo Equipment	A
Sanyo Electric Co., Ltd.	¥500 mn	101.84	Electric Products – Misc	BBB
Japan Airlines Corporation	¥500 mn	233.71	Airlines	BBB

Estimation of Copulas Parameters

The copula function links the univariate margins with their full multivariate distribution. It presents a useful tool when modeling non-Gaussian data since the Pearson's correlation coefficient is adapted for linear dependence and normal distribution. One appealing feature of a copula function is that the margins do not depend on the choice of the dependency structure and so, we can model and estimate the structure of dependency and the margins separately. Copula functions are becoming more and more popular credit correlation modeling due to their simplicity and fast computation.

For n uniform random variables, u_1, u_2, ..., u_n, the joint distribution function C is defined as:

$$C(u_1, u_2, ..., u_n, \theta) = Pr[U_1 \leq u_1, U_2 \leq u_2 ..., U_n \leq u_n] \qquad ...(2)$$

where θ is the dependence parameter.

Copula functions allow us to separate the structure of dependency between default times into two parts: the first part is the specification of the marginal distribution function (the distribution of default time of each obligor). The second part is the choice of the appropriate copula which describes the structure of dependency between default times. Numerous copulas can be found in the literature (Joe, 1997; and Nelsen, 1999). The most commonly applied copula function, especially in finance modeling, is the Gaussian copula[5]. This could be justified by the fact that the multivariate normal distribution has two appealing characteristics: first, their marginal distributions are normal, and second, it can be fully described by their marginal distribution and a variance-covariance matrix. For univariate margins, F_1, …, F_n, which are Gaussians, the dependence structure among the margins is described by a unique normal copula function. Let X_1, …, X_n be random variables which are standard normal distributed with means μ_1, …, μ_n, standard deviations σ_1, …, σ_n and correlation matrix Σ. Then, the distribution function $C_\Sigma(u_1, ..., u_n)$ of the random variables $U_i = \Phi\left(\frac{X_i - \mu_i}{\sigma_i}\right), i \in \{1, ..., n\}$ is a Gaussian

[5] CreditMetrics ™ and KMV models implicitly incorporate copula functions based on the multivariate Gaussian distribution of asset value process.

copula with correlation matrix Σ. Φ(.) denotes the cumulative univariate standard normal distribution function.

The Gaussian copula can be expressed as:

$$C_{\Sigma}(u_1,\ldots,u_n)=\frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{det\Sigma}}\int_{-\infty}^{\Phi^{-1}(u_1)}\cdots\int_{-\infty}^{\Phi^{-1}(u_n)}exp\left(-\frac{1}{2}(\vec{v}-\vec{\mu})^{T}\Sigma^{-1}(\vec{v}-\vec{\mu})\right)dv_1\ldots dv_n \qquad \ldots(3)$$

where Φ^{-1} is the inverse of the standard univariate Gaussian distribution function.

By differentiating Equation 3 with respect to u_1, ..., u_n, we obtain the density of the Gaussian copula[6]:

$$C_{\Sigma}(u_1,\ldots,u_n)=\frac{1}{\sqrt{det\Sigma}}exp\left(-\frac{1}{2}(v_1,\ldots,v_n)(\Sigma^{-1}-I)\begin{pmatrix}v_1\\ .\\ .\\ .\\ v_n\end{pmatrix}\right) \qquad \ldots(4)$$

where $v_i=\Phi^{-1}(u_i), i\in\{1,..,n\}$

If we use a Gaussian copula, we preserve the underlying distribution of the individual random variables but the joint distribution is like a multidimensional Gaussian. This naturally assigns very little weight to the tails. In reality, we find that within the financial markets, tail events occur much more frequently. So we would like a joint distribution which has fatter tails but preserves the same (bell shaped, non-skewed) characteristics of the Gaussian, hence we use the Student copula[7].

$$t_v(x)=\int_{-\infty}^{x}\frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{v\Pi}}\left(1+\frac{s^2}{v}\right)^{-\frac{v+1}{2}}ds \qquad \ldots(5)$$

The Student copula with the correlation matrix Σ and v degrees of freedom is presented in Equation 6.

$$C_{v,\Sigma}(u_1,\ldots,u_n)=\int_{-\infty}^{t_v^{-1}(u_1)}\cdots\int_{-\infty}^{t_v^{-1}(u_n)}k_{v,\Sigma}\left(1+\frac{1}{v}(\vec{v}-\vec{u})^{T}\Sigma^{-1}(\vec{v}-\vec{u})\right)^{-\frac{v+n}{2}}dv_1\ldots dv_2 \qquad \ldots(6)$$

6 Matlab code that generates random numbers from Gaussian copula is presented in Appendix 1.

7 Matlab code that generates random numbers from Student copula is presented in Appendix 1.

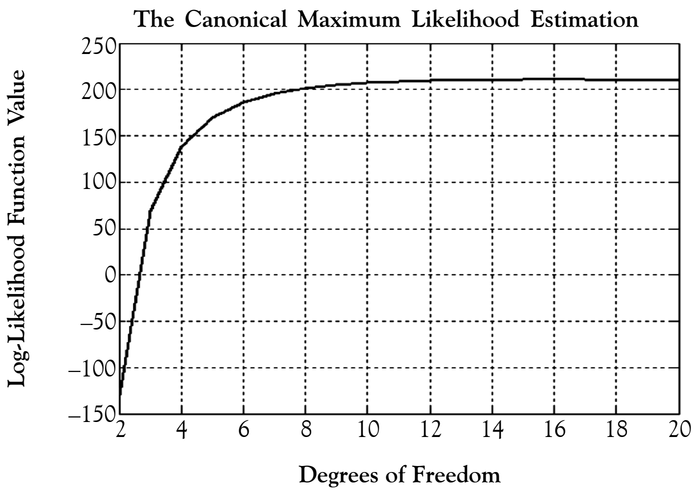
where,

$$k_{v,\Sigma} = \frac{\Gamma\left(\frac{v+n}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{(\Pi v)^n \, det(\Sigma)}}. \quad ...(7)$$

The Gaussian copula is completely determined by the knowledge of the correlation matrix Σ and the parameters involved are simple to estimate. To simulate random variables from Gaussian copula, it is enough to simulate a vector from the standard multivariate normal distribution with correlation matrix Σ and then to transform this vector through a univariate cumulative distributions function so that you can obtain a vector from the chosen copula. The matrix Σ positive definite can be easily determined with the Colicky decomposition in order to calculate a matrix A, such as $AA^T=\Sigma$.

The Student copula is defined by two parameters—the correlation matrix Σ and the number of degrees of freedom v (Figure 1).

Figure 1: Log-Likelihood Function of the Student Copula Density as a Function of Degrees of Freedom



In terms of the appropriate choice for the number of degrees of freedom, it is often necessary to carry out some statistical tests with historical data to ascertain how fat we require the tails to be. Many studies explain how to calibrate Student copula to real market data (Mashal and Zeevi, 2002; Romano, 2002; and Meneguzzo and Vecchiato, 2004). According to Galiani (2003), we find three methods:

1. **Exact Maximum Likelihood (EML) Method:** The idea behind the maximum likelihood parameter estimation is to determine the parameters that maximize the

probability (likelihood) of the sample data. Unlike in the Gaussian copula, the calibration of Student copula via the EML method requires simultaneous estimation of the parameters of the margins and the parameters related to the dependence structure.

2. **The Inference for Margins Method (IFM):** This method has emerged as the preferred fully parametric method because it is close to EML method in approach and is easier to implement. Joe and Xu (1996) present an approach that consists of estimating univariate parameters from separately maximizing univariate likelihoods, and then estimating dependence parameters from separate bivariate likelihoods or from a multivariate likelihood. The Jackknife method is proposed for obtaining standard errors and functions of the parameters. Kim *et al.* (2007) compare the Maximum Likelihood (ML) and IFM methods with a semi-parametric method that treats the univariate marginal distributions as unknown functions. They find that, in terms of statistical computations and data analysis, the semi-parametric method is better than the ML and IFM methods when the marginal distributions are unknown which is almost always the case in practice.

3. **The Canonical Maximum Likelihood (CML) Method:** Both the EML and the IFM require the parametric form of the univariate margins. The CML method does not require any prior assumption on the distribution form of the margins. Genest *et al.* (1995) proposed a semi-parametric procedure for estimating the dependence parameters in a family of multivariate distributions when one does not want to specify any parametric model to describe the marginal distribution. This procedure consists of transforming the marginal observations into uniformly distributed vectors using the empirical distribution function. Then, the copula parameters are estimated by maximization of a pseudo log-likelihood function.

The CML method tend to estimate the unknown parametric marginal $F_n(.)$ for $n=1,...,$ N with the empirical distribution functions $\hat{F}_n(.)$ defined as: $\hat{F}_n(.) = \frac{1}{T}\sum_{t=1}^{T} I_{\{X_{nt} \leq .\}}$ for $n=1,..., N$ and $\{X_{nt} \leq .\}$ represents the indicator function. The CML method is composed with two steps:

- Step 1: Transformation of the data set $X = (X_{1t}, X_{2t},..., X_{Nt})_{t=1}^{T}$ into uniform variates, using the empirical marginal distribution. Then, for $t=1,...,T$;

  $$\hat{u}_t = \left(\hat{u}_1^t, \hat{u}_2^t, ..., \hat{u}_N^t\right) = \left[\hat{F}_1(X_{1t}), \hat{F}_2(X_{2t}), ..., \hat{F}(X_{Nt})\right]$$

- Step 2: Estimation of the vector of the copula parameters α as follows:

  $$\hat{\alpha}_{\text{CML}} = \arg\max_{\alpha} \sum_{t=1}^{T} ln\, C\left(\hat{u}_1^t, ..., \hat{u}_N^t; \alpha\right)$$

This study presents an application of the CML method for calibrating the parameter of the Student copula. Bouyé *et al.* (2000) present an approach based on a recursive optimization procedure for the correlation matrix. This study uses the approach proposed by Mashal and Zeevi (2002) based on the rank correlation estimator given by the Kendall tau. Schweizer and Wolff (1981) show that two standard nonparametric rank correlation (Kendall's tau and Spearman rho) can be expressed solely in terms of the copula function. They are nonparametric measures of dependence since they are independent of the margins.

The Kendall's tau (τ) for two random variables X and Y is the probability of concordance minus the probability of discordance. Suppose (X, Y) and (X^*, Y^*) are two independent realizations of a joint distribution: $\tau = 4\iint_{[0,1]^2} C(u_1,u_2)\,dC(u_1,u_2) - 1$

$$= Prob\left\{\left(X^* - X\right)\left(Y^* - Y\right) \geq 0\right)\right\} - Prob\left\{\left(X^* - X\right)\left(Y^* - Y\right) < 0\right\}$$

For simplicity, it is assumed that the marginal distributions are continuous. Following Genest and MacKay (1986), Kendall tau verifies the following properties:

- $-1 \leq \tau \leq 1$
- τ is invariant under monotone transformations: if f and g are monotone increasing or decreasing functions, then $\tau(f(X), g(Y)) = \tau(X,Y)$
- $\tau = 0$ if X and Y are independent (but not conversely).

According to Lindskog *et al.* (2001), the following theorem is presented:

Theorem 1: Let $X \sim E_N(v, \Sigma, \rho)$. X_i and X_j are continuous for $i, j \in \{1, 2, \ldots N\}$, then:

$$\tau(X_i, X_j) = \frac{2}{\pi}\ \arcsin R_{ij}. \qquad \ldots(8)$$

where $E_N(v, \Sigma, \rho)$ denote the N dimensional elliptical distribution with parameters (v, Σ, ρ), $\tau(X_i, X_j)$ is the Kendall's tau, and R_{ij} denotes the Pearson correlation coefficient for the random variables (X_i, X_j).

As mentioned in Table 2, our basket credit default swap is composed with five names. The calibration procedures contain three steps:

- Step 1: Transform the initial stock prices data X into a set of uniform variates $\hat{U}$ using the empirical marginal transformation;
- Step 2: Estimate the correlation matrix R^{CML} from Equation 8; and
- Step 3: Extract the CML Estimator of the degrees of freedom by maximizing the log-likelihood function of the Student copula:

$$v^{CML} = \underset{v \in (2,\infty)}{\arg\max} \sum_{t=1}^{T} Log C^{Student}(\hat{u}_1^t, \hat{u}_2^t, \ldots, \hat{u}_N^t, R^{CML}, v) \quad \ldots(9)$$

The above method has computational advantages over other methods. This method does not require matrix inversion and therefore has the advantage of being numerically stable in the presence of close-to-singular correlation matrices.

The existing literature on default correlations can be divided into two approaches: the structural approach that models default correlations through companies' asset values; and the reduced-form approach that models default correlations through default intensities. Elizalde (2005) distinguishes three different approaches to model default correlation in the literature of intensity credit risk modeling. The first approach introduces correlation in the firms' default intensities making them dependent on a set of common variables and on a firm-specific factor (conditionally independent default models). The second approach default dependencies arise from direct links among firms (Contagion models). The third model default correlation makes with copula functions. Li (2000) points out that CreditMetrics uses a bivariate normal copula function with asset correlation as the correlation factor. Patel and Pereira (2007) extract common or latent factors that drive companies default correlations using a factor-analytical technique. The results indicate that the common factors, which capture the overall state of the economy, explain default correlations. In KMV methodology, the model and the correlation matrix are computed on asset returns but proxies by equity returns on CreditMetrics. The correlation model in CreditMetrics provides estimates of equity correlations instead of asset correlations. Then, it has become market practice to use equity correlation as a proxy for asset correlation.

The equity prices used in this study is from Tokyo Stock Exchange. It consists of 754 daily observations spanned from January 2, 2004 to January 25, 2007.

$$v^{CML} = \underset{v \in (2,\infty)}{\arg\max} \sum_{t=1}^{T} ln\, C^{Student}\left(\hat{u}_1^t, \hat{u}_2^t, \ldots, \hat{u}_N^t, R^{CML}, v\right).$$

We obtain the following correlation matrix:

1	0.34197	0.17727	0.19895	0.17839
0.34197	1	0.31106	0.33717	0.21573
0.17727	0.31106	1	0.36427	0.25319
0.19895	0.33717	0.36427	1	0.29069
0.17839	0.21573	0.25319	0.29069	1

Calibration of Credit Curves

In the basket default swap pricing, one of the fundamentally important issues is the estimation of default probabilities and correlations. Because default probabilities depend on the credit quality of the considered obligor, well-calibrated credit curves are a main ingredient for constructing default times.

The calibration of credit curves takes into account internal information on credit migrations and default history. There are dozens of rating agencies that provide yearly rating migration and default data for corporations and governments. From this data a rating transition matrix can be constructed. Such a matrix gives transition probabilities for a corporation migrating, in one year, from one rating level to another. Of particular interest is the transition probability to default status.[8]

Let $Q=(q_{ij}, i, j = 1, 2,\ldots, k)$ be a one year rating transition matrix with rating grades $R_1, \ldots, R_{k-1}$ and default state D, where q_{ij} is the probability that an obligor with current rating R_i will migrate to rating R_j in one year.

The aim of this section is to calibrate a credit curve for each rating, where a credit curve for R is a mapping: $t \mapsto P(t,R) = P[R \to D \text{ in time } t]$, with R the rating grades $R_1, \ldots, R_{k-1}$ and default state D. Where $\mapsto$ denotes migration from a rating state to another. Then, $P(t, R)$ is the probability that an obligor with a current rating defaults with the next t years. Suppose a firm's rating follows a time homogeneous Markov process model[9] with the transition probability matrix (as have with Jarrow *et al.* (1997) and other authors). Jarrow *et al.* (1997) model default and transition probabilities by using a discrete time, time-homogenous Markov chain on a finite state space $S=\{1, \ldots, K\}$. The state space S represents the different rating classes. It seems to be a common market practice to model default probability term structures via Markov chain techniques.

Given the one-year $K \times K$ transition matrix M, we are interested in finding a generator matrix such as $M=exp(Q)$. Dealing with the question whether there exists a generator matrix, we use the theorem of Noris (1998) as presented in Appendix 3.

The construction of credit curves, which are compatible with Table 1, is presented on the following steps:

- Step 1: Compute the log expansion $\widetilde{Q} = \widetilde{q}_{ij}$ of the one-year rating transition matrix M;
- Step 2: The calibration of a generator Q-matrix based on the log-expansion of M and a so-called diagonal adjustment (Kreinin and Sidelnikova, 2001);
- Step 3: The approximation of the original matrix M by $exp(Q)$ (Bluhm, 2003; and Bluhm and Overbeck, 2006). The error is:

[8] The method used by Standard and Poor's to estimate credit curves involves two stages. The first stage is the estimation of the probabilities of transitions between different ratings (transition matrix). The second stage is the repeated application of this matrix to determine the credit curves. In both cases, rating transitions are assumed to follow a Markov process, in which transition probabilities are constant over time, and do not depend on the previous rating of the firm.

[9] A process with the Markov property is called a Markov process. A stochastic process has the Markov property if the conditional probability distribution of future states of the process, given all past and the present state, depends only upon the present state.

$$\|M - exp(Q)\|_2 = \sqrt{\sum_{i,j=1}^{K} \left(m_{ij} - (exp(Q))_{ij}\right)^2} \qquad \ldots(10)$$

- Step 4: The credit curves can be read-off from the collection of matrices $exp(tQ)_{t \geq 0}$ by looking at the default columns. Otherwise, we can generate default probability term structures based on the continuous time-homogeneous Markov chain generated by Q via:

$$P(t,R) = (exp\,(tQ))_{row(R),K} \qquad (t \geq 0) \qquad \ldots(11)$$

Where *row*(R) denotes the transition matrix row corresponding to the given rating *R*.

In the present study, we use Japanese data and then we refer to Japan Credit Rating Agency to obtain rating transition matrix and cumulative default rates[10]. We use one year rating transition matrix to compute either marginal or cumulative default probabilities over one or multiple time steps. Marginal default probabilities are computed by applying recursively the standard Markov chain over the different transition states.[11] Credit migration matrices are said to be diagonally dominant, meaning that most of the probability mass resides along the diagonal, most of the time there is no migration.

One-year transition matrix obtained from JCR (for the period January 1995-December 2005)[12] is reduced into another smaller sub-transition matrix (Table 3).

Table 3: Modified Transition Matrix (in %)

RATING	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	95.410	4.590	0.000	0.000	0.000	0.000	0.000	0.000
AA	1.313	93.527	4.423	0.657	0.080	0.000	0.000	0.000
A	0.000	2.014	93.433	3.159	1.316	0.000	0.000	0.079
BBB	0.000	0.000	3.750	92.491	3.513	0.163	0.000	0.083
BB	0.000	0.000	0.000	5.403	77.894	11.353	2.427	2.923
B	0.000	0.000	0.000	4.124	11.681	61.817	5.350	17.028
CCC	0.000	0.000	0.000	0.000	0.000	0.000	0.000	100.000

Table 3 indicates one-year ratings migration probabilities from modified transition matrix. For example, a BBB rated bond has a 3.513% probability of being downgraded to a BB+ rating by the end of one year. To use a ratings transition matrix to calculate the one year default probabilities, we simply take the default probabilities indicated in the last column and ascribe them to bonds of the corresponding credit ratings. For example, with this approach, we would ascribe a BBB rated bond a 0.083% probability of default within one year. Transition matrices can be used to compute credit curves over multiple horizons. This is done by transforming the transition probabilities into marginal default

[10] "JCR Rating Matrices and Cumulative Default Rates", *Rating Perspective*, April 2006.

[11] We use The Risksvr™ calculation engine.

[12] The original one year transition matrix obtained from JCR is presented in Appendix 2.

probabilities, which are then converted into Survival or Hazard rates or expected default probabilities. Table 4 presents cumulative default probabilities for each rating class.

Table 4: Cumulative Default Probabilities Obtained from Markov Process (in%)

Rating	1	2	3	4	5	6	7	8
AAA	0.000	0.000	0.000	0.001	0.004	0.009	0.017	0.030
AA	0.000	0.006	0.024	0.058	0.113	0.193	0.299	0.435
A	0.079	0.194	0.395	0.691	1.078	1.551	2.099	2.715
BBB	0.083	0.294	0.753	1.449	2.353	3.428	4.637	5.950
BB	2.923	9.565	16.591	23.316	29.446	34.891	39.659	43.805
B	17.028	33.249	44.061	51.585	57.050	61.181	64.416	67.022
CCC	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000

We can construct a default probability curve from given default swap rates. Such a method is often called bootstrapping[13]. As we know, the price of single name credit default swap can be affected by marginal probability of default. Hull *et al.*, (2004) mentioned that the credit default swap market is so liquid that we can use the credit default swap spread data to calibrate the default intensity. According to Choudhury (2006), historical evidence suggests that the credit default swap market can anticipate rating downgrades. Take the example of El Paso Corporation[14], the cost of five-year CDS protection more than doubled between September 12 and September 23, 2002 from 575 basis points to 1,250 basis points. Two months later, El Paso Corporation was downgraded from BAA3 to B2. Then, we can construct a default probability curve from default swap rates. Galiani (2003) provides a practical application for extracting the term structure of instantaneous default probabilities. In the corporate CDS market, trading has been concentrated largely in the five-year maturity contract. In our case, we have only five-year credit default swap rates. Therefore, we cannot estimate the instantaneous default probabilities from CDS market data.

Fair Spread Estimation of Reconstituted Basket Credit Default Swap

After the calibration of the credit curves, we can obtain the cumulative default probabilities for any name over any time interval $[0, t]$. Then, we can define the marginal distribution of default times.

Given a credit curve $P(t, R)_{t \geq 0}$, of an asset or obligor with a Rating R up to a given time period t, there exists a unique default times distribution for R-rated obligors.

Effects regarding correlation are particularly strong for some of the most recent innovations in credit markets, namely basket default swap. We use the index i to refer to the i^{th} obligor. The rating assigned to an obligor i will be denoted by $R(i)$ and assume a portfolio of n obligors. Consequently, the joint distribution of default times τ_i:

13 Bootstrapping, iteratively stepping through market quotes to determine the underlying curve, is widely used as the standard method of estimating interest rate curves.

14 El Paso Corporation provides natural gas and related energy products in a safe, efficient, and dependable manner. It is one of North America's largest independent natural gas producers.

$$F(t_1,...,t_n) = P(\tau_{R(1)} \le t_1,...,\tau_{R(n)} \le t_n) \quad ...(12)$$

The joint survival time distribution is given by:

$$S(t_1,...,t_n) = P(\tau_{R(1)} > t_1,...,\tau_{R(n)} > t_n) \quad ...(13)$$

$$= C\left(e^{\left(-\int_0^t h_1(s)ds\right)}, e^{\left(-\int_0^t h_2(s)ds\right)}, ..., e^{\left(-\int_0^t h_n(s)ds\right)} \right) \quad ...(14)$$

For deterministic intensities, this framework converges to Li (2000) model[15]. Number of times the default occurs

$\lambda_i(t) := e^{\left(-\int_0^t h_i(s)ds\right)}$ reach the level of the trigger variables U_i:

$$\tau_i := \inf\{t \ge 0 : \lambda_i(t) \le U_i\} \quad ...(15)$$

The choice of a dependence structure between default times drives the prices of basket default swaps. To simulate the joint default times for the five names, we present the following steps:

- Step 1: Simulate N-dimensional vector of correlated uniform random variates from a copula (C_Σ or $C_{\upsilon,\Sigma}$) as described in Appendix 1. The correlation matrix and the appropriate number of degrees of freedom are estimated as explained under the subhead "The Fair Spread of Basket Credit Default Swaps"; and
- Step 2: Translate the corresponding uniform variates into default time for each obligors. The default dates can now be derived from the uniform random variates. They are given by: $\tau_i = -\dfrac{Ln(u_i)}{\lambda}$.

We have an example of basket credit default swap with five names of different rating and industry sector. We try to price the five-year basket credit default swaps. The default event is triggered off by the n^{th} default in the basket. The seller of the basket credit default swap faces the default payment upon the n^{th} default, and the buyer pays the spread price until n^{th} default or until maturity T. Let (τ_1, τ_2, τ_3, τ_4, τ_5) denote the default order. The fair spread of the basket default swap is given by Equation 1. It is not easy to get the distribution of τ_n so that it is difficult to calculate the expectations in all the legs. Monte Carlo simulation can be used to get the fair spread of the n^{th} basket default swap.

[15] Li (2000) presents a simple and computationally inexpensive algorithm for simulating correlated defaults. His methodology builds on the implicit assumption that the multivariate distribution of default times and the multivariate distribution of asset returns share the same dependence structure, which he assumes to be Gaussian and is therefore fully characterized by a correlation matrix.

Pricing an n^{th} to default basket credit default swap under Gaussian and Student copula using Monte Carlo simulations can be presented in the following steps:

- Step 1: simulate N-dimensional vector of correlated uniform random variates from the corresponding copula with respect to correlation matrix and hazard rates.
- Step 2: Translate the corresponding uniform variates into default time for each obligors.
- Step 3: Sort the credits with respect to their default time τ_i and determine the n^{th} default time τ^n. For first to default swap, we find the first default time $\tau^* = \min_i \tau_i$
- Step 4: Based on specific realization of τ^n, determine the present value of the premium leg $E[PL_n]$.
- Step 5: Determine the present value of the default leg $E[DL_n]$.
- Step 6: Repeat all steps mentioned above until the required number of scenarios has been simulated and the sample average fair spread of the n^{th} to default basket swap as described in Equation 1.

The fair prices of the basket default swap as described under the subhead "The Fair Spread of Basket Credit Default Swaps" under Gaussian an Student Copula and Monte Carlo simulation are presented in Table 5.

Table 5: Fair Spread of First to Default Swap Under Monte Carlo Simulation

Fair Price: First to Default (Gaussian Copula)	Fair Price: First to Default (Student Copula)
327.6322 bp	286.1316 bp

The higher the quality of the obligors, the less likely are the defaults, then, a basket of high quality credit will be cheaper than a basket of low quality credit. We notice that the premiums computed under Gaussian and Student copulas are different. Burtschell *et al.* (2005) find that Gaussian and Student copulas lead to similar premiums for first to default swap when they change the number of names in the basket. Similarly, the differences are minor when they change the rank of default.

Variance reduction has always been a central issue in Monte Carlo experiments. Importance sampling is a variance reduction technique that can be used in the Monte Carlo simulation. The idea behind importance sampling is that certain values of the input random variables in a simulation have more impact on the parameter being estimated than others. If these 'important' values are emphasized by sampling more frequently, then the estimator variance can be reduced. In the Monte Carlo simulation procedures, a path will only result in a default payoff if the n^{th} default occurs before the maturity.

Joshi and Kainth (2004) apply importance sampling to the pricing of n^{th} to default credit swaps within the Li model and obtain stable and sizeable speedups. They show that Monte Carlo simulations in the Li model can be slow to converge and present procedures for accelerating the computation of prices and sensitivities to hazard rates. Glasserman and Jingyi (2005) develop Importance Sampling (IS) procedures for rare-event simulation for credit risk measurement. They focus on the normal copula model originally associated with J P Morgan's CreditMetrics system. Dependence between obligors is captured through a multivariate normal vector of latent variables; a particular obligor defaults if its associated latent variable crosses some threshold. Glasserman and Juneja (2007) have considered the problem of simultaneous estimation of the probabilities of multiple rare events. Successful applications of importance sampling for rare event simulation typically focus on the probability of a single rare event. As a way of demonstrating the effectiveness of an importance sampling technique, the probability of interest is often embedded in a sequence of probabilities decreasing to zero. They show that importance sampling based on exponential twisting produces asymptotically efficient estimates of rare event probabilities in a wide range of problems. Abid and Naifar (2008) argue the choice of copula and the choice of procedures for rare-event simulation govern the pricing of basket credit default swap. They present copulas-based simulations procedures for basket credit default swap under Monte Carlo and Importance sampling simulations.

We notice that the spread of first to default swap change with the structure of dependency and the simulation techniques (Table 6). Then, the choice of copula and the choice of procedures for rare-event simulation also govern the pricing of basket credit derivatives. The advantage of Monte Carlo algorithm is its simplicity and generality. Its main drawback, however, is the quality of the convergence, especially when one computes sensitivities. A good convergence is particularly hard to achieve for credit products since default events are usually rare, and probabilities in the tail of the distribution are difficult to estimate. On the other hand, the direct implementation in closed form is very accurate but is less trivial to implement; it is also very expensive computationally. Indeed, this method is based on enumerating the $2n$ default configurations of the basket and computing the probability of each configuration. This algorithm explodes exponentially as the number of credits increases.

Table 6: Fair Spread of First to Default Swap Under Importance Sampling Simulation

Fair Spread: First to Default (Gaussian Copula)	Fair Spread: First to Default (Student Copula)
316.1415 bp	312.7362 bp

Furthermore, the Gaussian distribution has thin tails compared to other distributions. As we are concerned of default events that are by nature tail events, we use distributions with fat tails such as the *t*-student and Clayton distribution. Similarly, the choice of procedures for rare-event simulation governs the pricing of basket credit derivatives. Joshi and Kainth (2004) show that, for baskets of high quality credits or short-maturity swaps, Monte Carlo simulation produces few paths with n or more defaults; nearly all paths simply return a value of 0. Then, they introduce an importance sampling method that forces all paths to produce at least n defaults. Alternatives to the Monte Carlo simulation are the Clayton copula and the t-student copula under importance sampling procedures for simulation which captures the dependence structure between the underlying variables at extreme values and certain values of the input random variables in a simulation have more impact on the parameters being estimated than others.

Conclusion

The creation, calibration, and pricing of basket credit derivatives raise many technical questions and issues: How to obtain a realistic migration matrix? How can we calibrate credit curve and recovery rate for each obligor? and What simulation procedures are used for rare-event simulation?

In this study, we presented a methodology for fair spread estimation of reconstituted basket default swap using Japanese market data. A fundamental part of the pricing framework is the estimation of default probabilities and the structure of dependency. We presented a copula-based simulation procedure for pricing basket default swap and we estimated parameters from Japanese market data. We referred to the Japan Credit Rating Agency to obtain rating transition matrix and recovery rates. We have considered Gaussian copula and Student copula and studied their impact on basket credit derivative prices. We presented an application of the Canonical Maximum Likelihood Method (CML) for calibrating Student copula. The basic problem of using simulation is that defaults are rare events, and a large number of simulation paths are usually required to achieve a sufficient sampling of the probability space. Subsequently, the choice of copula and the choice of procedures for rare-event simulation govern the pricing of basket credit derivatives. After the calibration of the parameters, we estimated the fair spread of first to default swap under Monte Carlo and importance sampling simulations and Gaussian and student copulas. ❖

References

1. Abid F and Naifar N (2008), "Pricing Basket Credit Default Swap: An Empirically Based Simulation Study", Forthcoming in *The Icfai University Journal of Derivatives Markets*.

2. Bluhm C (2003), "CDO Modelling: Techniques, Examples and Applications", Lecture Notes, Autumn School on Risk Management, HypoVereinsbank, Munich.

3. Bluhm C and Overbeck L (2006), *Structured Credit Portfolio Analysis, Baskets & CDOs*, Chapman & Hall, CRC Financial Mathematics Series, CRC Press.

4. Bouyé E, Durrleman V, Nikeghbali A, Riboulet G and Roncalli T (2000), "Copulas for Finance: A Reading Guide and Some Applications", Working Paper, GRO, Crédit Lyonnais, Paris.

5. Burtschell X, Gregory J and Laurent J P (2005), "A Comparative Analysis of CDO Pricing Models", Working Paper, BNP Paribas, http://laurent.jeanpaul.free.fr/comparative%20analysis%20CDO%20pricing%20models.pdf

6. Choudhury K (2006), "Credit Default Swaps: Development, Pricing and Correlation to Default Risk", Dissertation, http://www.crisil.com/crisil-young-thought-leader-2006/dissertations/Kaushik-Choudhury_Dissertation.pdf

7. Duffie D and Pan J (2001), "Analytical Value at Risk with Jumps and Credit Risk", *Finance & Stochastics*, Vol. 5, No. 2, pp. 155-180.

8. Elizalde A (2005), "Credit Risk Models I: Default Correlation in Intensity Models", Working Paper, CEMFI and UPNA.

9. Galiani S (2003), "Copula Functions and Their Application in Pricing and Risk Managing Multiname Credit Derivative Products", Master's Thesis, King's College London.

10. Genest C and MacKay J (1986), "The Joy of Copulas: Bivariate Distributions With Uniform Marginals", *The American Statisticien*, Vol. 40, No. 4, pp. 280-283.

11. Genest C, Ghoudi K and Rivest L P (1995), "A Semiparametric Estimation Procedure of Dependence Parameters in Multivariate Families of Distributions", *Biometrika*, Vol. 82, No. 3, pp. 543-552.

12. Giese G (2003), "Enhancing CreditRisk+", *Risk*, Vol. 16, No. 4, pp. 73-77.

13. Glasserman P and Jingyi L (2005), "Importance Sampling for Portfolio Credit Risk", *Management Science*, Vol. 51, No. 11, pp. 1643-1656.

14. Glasserman P and Juneja S (2007), "Uniformly Efficient Importance Sampling for the Tail Distribution of Sums of Random Variables", *Mathematics of Operations Research*, Vol. 33, No. 1, pp. 36-50.

15. Hull J and White A (2001) "Valuing Credit Default Swaps II: Modeling Default Correlations", *Journal of Derivatives*, Vol. 8, No. 3, pp. 12-22.

16. Hull J, Predescu M and White A (2004), "The Relationship Between Credit Default Swaps Spreads, Bond Yields, and Credit Rating Announcements", *Journal of Banking and Finance*, Vol. 28, No. 11, pp. 2789-2811.

17. Jarrow R, Lando D and Turnbull S M (1997), "A Markov Model for the Term Structure of Credit Risk Spread", *Review of Financial Studies*, Vol. 10, No. 2, pp. 481-523.

18. Joe H (1997), *Multivariate Models and Dependence Concepts*, Chapman & Hall, London.

19. Joe H and Xu J (1996), "The Estimation Method of Inference Functions for Margins for Multivariate Models", Technical Report 166, Department of Statistics, University of British Columbia.

20. Joshi M S and Kainth D (2004), "Rapid and Accurate Development of Prices and Greeks for n^{th} to Default Credit Swaps in the Li Model", *Quantitative Finance*, Vol. 4, No. 3, pp. 266-275.

21. Kim G, Silvapulle M J and Silvapulle P (2007), "Comparison of Semiparametric and Parametric Methods for Estimating Copulas", *Computational Statistics & Data Analysis*, Vol. 51, No. 6, pp. 2836-2850.

22. Kreinin A and Sidelnikova M (2001), "Regularization Algorithms for Transition Matrices", *Algo Research Quaterly*, Vol. 4, No. 2, pp. 25-40.

23. Li D (2000), "On Default Correlation: A Copula Function Approach", *Journal of Fixed Income*, Vol. 9, No. 4, pp. 43-54.

24. Lindskog F and McNeil A J (2003), "Common Poisson Shock Models: Applications to Insurance and Credit Risk Modelling", *ASTIN Bulletin*, Vol. 33, No. 2, pp. 209-238.

25. Lindskog F, McNeil A and Schmock U (2001), "Kendall's tau for Elliptical Distributions", Working Paper, Risklab, Switzerland, http://www.risklab.ch/Papers.html#KendallsTau

26. Mashal R and Naldi M (2002), "Pricing Multiname Credit Derivatives: Heavy Tailed Hybrid Approach", Working Paper, Columbia Graduate School of Business and Quantitative Credit Reaseach, Lehman Brothers Inc.

27. Mashal R and Zeevi A (2002), "Beyond Correlation: Extreme Co-Movements Between Financial Assets", Working Paper, Columbia Graduate School of Business.

28. Meneguzzo D and Vecchiato W (2004), "Copula Sensitivity in Collateralized Debt Obligations and Basket Default Swaps", *The Journal of Futures Markets*, Vol. 24, No. 1, pp. 37-70.

29. Nelsen R (1999), *An Introduction to Copulas*, Springer Verlag, New York.

30. Noris J (1998), *Markov Chains*, Cambridge University Press, Cambridge.

31. Patel K and Pereira R (2007), "Expected Default Probabilities in Structural Models: Empirical Evidence", *Journal of Real Estate Finance and Economics*, Vol. 34, No. 1, pp. 107-133.

32. Romano C (2002), "Applying Copula Function to Risk Management", Working Paper, http://www.gloriamundi.org/picsresources/cr04.pdf

33. Schweizer B and Wolff E F (1981), "On Nonparametric Measures of Dependence for Random Variables", *Annals of Statististics*, Vol. 9, No. 4, pp. 870-885.

34. Sicar R and Zariphopoulou T (2006), "Utility Valuation of Credit Derivatives and Application to CDOs", Working Paper, Princeton University.

35. Tavares P, Nguyen T, Chapovsky A and Vaysburd I (2004), "Composite Basket Model", Working Paper, Merrill Lynch Credit Derivatives.

36. Wong D (2000), "Copula from the Limit of Multivariate Binary Model", Working Paper, Bank of America Corporation.

Appendix 1

Matlab Codes for Gaussian and Student Copulas

The following algorithm generates random variates $(u_1,\ldots,u_n)$ which are determination of correlated uniform variates on [0,1] from the Gaussian copula with the correlation matrix Σ:

- Find the Cholesky decomposition A of the correlation matrix Σ, such that $\Sigma = A.\, A^T$;
- Simulate n independent standard normal random variates $Z=(z_1,z_2,\ldots,z_n)^T$;
- Set $x = A.Z$;
- Set x back to an n-dimensional vector u of uniform variates on [0, 1] by computing $u=\Phi(x)$. The vector u is a random variate from the n-dimensional Gaussian copula C_Σ.

```
Function y1 = Gaussrnd(rho, N);

This function generates random numbers from Gaussian copula as presented in Figure 2.

rho = The Gaussian copula correlation matrix

N = Number of random numbers to be generated

y1 = Random numbers from Gaussian copula, N by 2 matrix

if N == 0

y1=[];

else

    A = chol(rho);

    z = randn(N, 2)*A;

    u = cdf('Normal', z(:,1),0,1);

    v = cdf('Normal', z(:,2),0,1);

    y1 = [u v];

end
```

(Contd...)

Appendix 1

(...contd)

Matlab Codes for Gaussian and Student Copulas

Figure 2: 1,000 Simulated Standard Uniform Random Variables Under Gaussian Copula

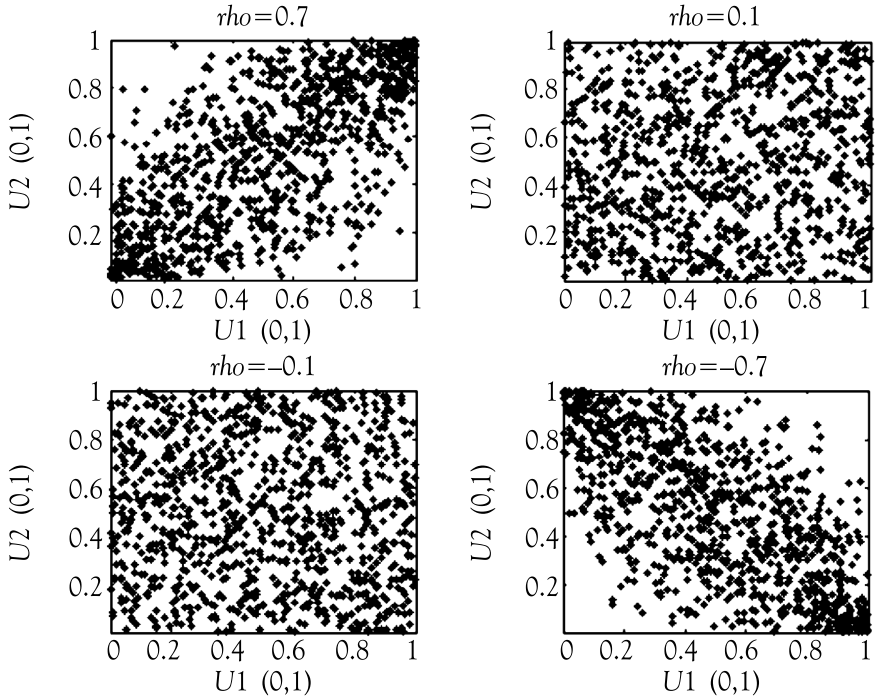
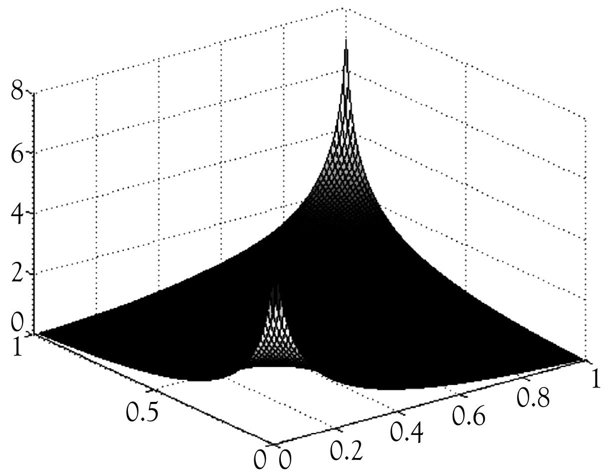


Figure 3: Gaussian Copula Density



(Contd...)

Appendix 1

(...contd)

Matlab codes for Gaussian and Student copulas

The following algorithm generates random variates (u_1, ..., u_n) which are determination of correlated uniform variates on [0, 1] from the Student copula with the correlation matrix and degrees of freedom:

- Find the Cholesky decomposition A of the correlation matrix Σ, such that $\Sigma = A.A^T$;
- Simulate n independent standard normal random variates $Z=(z_1,z_2,\ldots,z_n)^T$;
- Simulate a random variate, s, from χ_v^2 distribution, independent of Z;
- Set $y=A.Z$;
- Set $x = y\sqrt{\frac{v}{s}}$;
- Set x back to an n-dimensional vector u of uniform variates on [0, 1] by computing $u=t_v(x)$. The vector u is a random variate from the n-dimensional Student copula $C_{v,\Sigma}$.

Function y3 = studrnd(*rho*, *dof*, *N*);

This function generates random numbers from Student's *t* copula as presented in Figure 4.

rho = The correlation matrix

dof = Degrees of freedom

N = Number of random numbers to be generated

y3 = Random numbers from Student's t copula, *N* by 2 matrix

```
if N == 0
y3=[];
else
    A = chol(rho);
    z=randn(N, 1);
    s=chi2rnd(dof);
    y=Z'. A
    x=(sqrt(dof)/sqrt(s)).y;
    y3 = tcdf(x, dof);
end
```

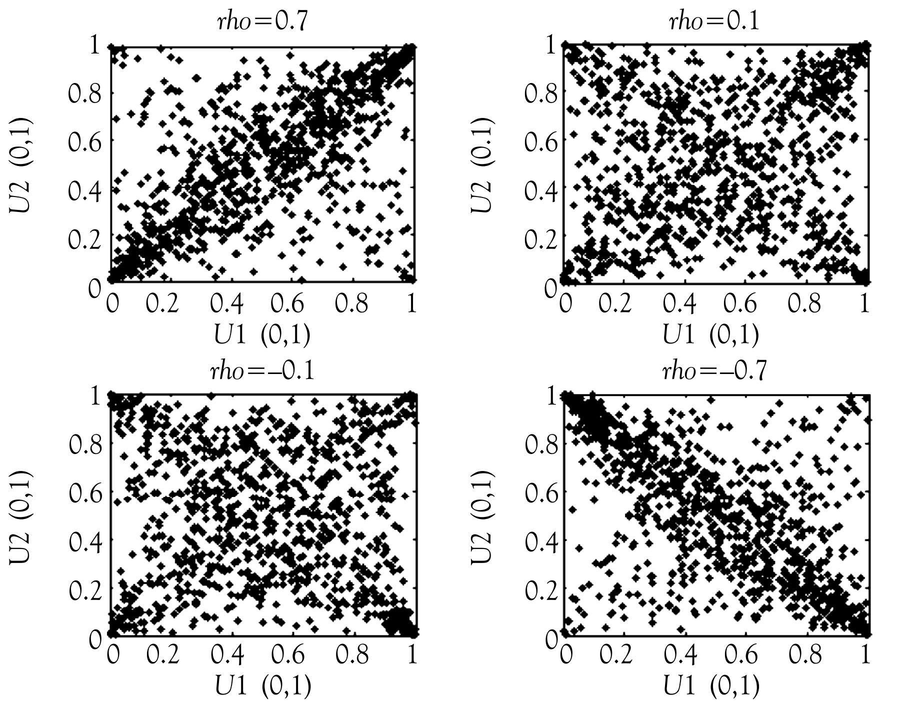
(Contd...)

Appendix 1

(...contd)

Matlab Codes for Gaussian and Student Copulas

Figure 4: 1,000 Samples from *t*-Student Copula with Degree of Freedom Equal to 7



Appendix 2

Table 7: One-Year Rating Transition Matrix (for the Period January 1995-December 2005)

	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-	BB+	BB	BB-	B+	B	B-	CCC or lower	D
AAA	95.41	4.59	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AA+	3.64	91.44	4.92	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AA	0.30	2.73	84.41	8.77	2.85	0.71	0.24	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AA-	0.00	0.36	3.72	84.23	10.42	1.26	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
A+	0.00	0.00	0.00	5.72	83.68	9.15	0.66	0.54	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.24
A	0.00	0.00	0.19	0.08	5.46	84.65	8.22	1.34	0.05	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
A-	0.00	0.00	0.13	0.00	0.25	7.58	84.33	5.99	1.54	0.13	0.00	0.00	0.04	0.00	0.00	0.00	0.00	0.00
BBB+	0.00	0.00	0.00	0.00	0.00	0.62	9.91	78.25	10.38	0.73	0.02	0.00	0.00	0.04	0.06	0.00	0.00	0.00
BBB	0.00	0.00	0.00	0.00	0.00	0.00	0.71	4.99	85.70	7.51	0.71	0.17	0.00	0.00	0.06	0.00	0.00	0.16
BBB-	0.00	0.00	0.00	0.00	0.00	0.00	0.01	1.12	7.85	80.96	7.58	1.88	0.18	0.20	0.05	0.08	0.00	0.09
BB+	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.57	12.74	70.25	10.82	3.56	0.00	0.00	0.21	0.43	0.43
BB	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.90	10.48	67.62	10.00	3.49	2.06	1.59	0.00	2.86
BB-	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	15.07	11.64	34.25	26.03	0.68	0.00	6.85	5.48
B+	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	12.37	4.12	14.43	16.49	47.42	0.00	0.00	5.15	0.00
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	69.84	0.00	6.35	23.81
B-	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	68.18	4.55	27.27
CCC or lower	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

Appendix 3

The Theorem of Noris (1998)

Theorem 1: If the transition matrix M=$(m_{ij}, i, j = 1, 2,\ldots, k)$ is strictly diagonal dominant, i.e.:

$p_{ii} > 0.5$ for every *I*, then the log-expansion

$$\widetilde{Q}_n = \sum_{K=1}^{n} (-1)^{K+1\frac{(M-I)^s}{K}}, n \in N$$

converge to a matrix $\widetilde{Q} = \widetilde{q}_{ij},\ i, j = 1, \ldots, K.$ satisfying:

- $\sum_{j=1}^{K} \widetilde{q}_{ij} = 0$, for every $i=1,\ldots K$.
- $exp(\widetilde{Q}) = M$.

The convergence $\widetilde{Q}_n \to \widetilde{Q}$ is geometrically fast and defines a K×K matrix having row-sums of zero and satisfying $M = exp(\widetilde{Q})$.

The generator of a time-continuous Markov chain is given by a so-called Q-matrix=q_{ij} satisfying the following properties:

- $\sum_{j=1}^{K} q_{ij} = 0$, for every $i=1,\ldots K$. ...(1)
- $0 \leq -q_{ii} < \infty$, for every $i=1,\ldots K$. ...(2)
- $q_{ij} \geq 0$, for all *I*, $j=1,\ldots K$ with $i \neq j$ ...(3)

Theorem 2: The following two properties are equivalent for a matrix $Q \in \Re^{K \times K}$:

- Q satisfies Properties 1, 2 and 3.
- $exp(tQ)$ is a stochastic matrix for every $t \geq 0$, where $exp(.)$ denotes the matrix exponential.

Reference # 37J-2008-12-02-01