Joint Interest Rate Risk Management of Balance Sheet and Hedge Portfolio in a Present Value Perspective

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We present a multi-period mean-variance optimization program, which allows for a joint optimization of the balance and off-balance sheet. Our first finding is the proof of a conjecture of Li and Ng (2000) and Leippold et al., (2003 and 2004) about the equivalence of the original non-separable mean-variance problem and its embedding into a higher dimensional separable problem. We further prove that given a time independent term structure, the one-period and the multi-period problem are equivalent. If the best forecast of the interest rates is the forward rate, we show that it is then optimal to mimic the benchmark strategy. We apply the model first to United Bank of Switzerland (UBS) data and show that the myopic models are not acceptable for key rate delta profile management. Then, we calculate present value of a portfolio for 2005. It follows that the optimal dynamic portfolio strategy leads to a return of 3.12% compared to 2.6% of the myopic model.

Introduction

The management of interest rate risk of the balance sheet is a major task in banks. This task has both an earning and a risk component. Nowadays, some treasurers base the optimal hedge decisions on one-period models, although it is well-known to practitioners that distributing risk and earning over time would improve the performance of the interest rate risk management.

A key obstacle to the diffusion of multi-period models in the finance industry is the absence of economic meaningful and applicable dynamic models. We propose such a model in this paper.

Our model is based on the recent work of Li and Ng (2000), Leippold *et al*., (2003 and 2004). These authors showed how a standard one-period mean-variance extends to a multi-period optimization program. This natural extension possesses a severe mathematical drawback: The dynamic model is no longer separable in the dynamic programming sense which makes a neat analytical solution impossible. The key observation of the authors was to embed the dynamic optimization program into a higher dimensional one, and then to prove that the solutions of the two programs are uniquely related to each other. A first contribution of this paper is to prove that indeed the two programs are equivalent. The authors of the base work only proved one part of the

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equivalence and tacitly assumed that the other part is also true. Given the equivalence, we provide full analytical/semi-analytical solutions for a joint balance sheet and hedge portfolio in the case of linear equalities and inequalities as restrictions. This is to our knowledge, a new contribution, which extends the results in Li and Ng, (2000) and Leippold *et al.*, (2004) in discrete time and Yong and Zhou (1999), and Li *et al.* (2002) in continuous time, where only linear equality restrictions are considered. Our final theoretical contribution is related to the multi-period optimization program if the term structure dynamics is specified. We show first that given a time independent term structure, the multi-period and the one-period model are equivalent. Then we prove that, if the best guess about future interest rates is the forward rate, the optimal balance sheet management has to mimic the benchmark.

Besides these theoretical results, a goal is to show that the approach can be implemented in practice. That for, the abstract notions in the papers Li and Ng (2000) and Leippold *et al.*, (2003 and 2004) need to be adapted. First, to represent the myriad of different balance sheet and hedge instruments generically, we define a linear asset space such that different products are expressed as linear combinations of the base assets. A second task was to formalize the various investment restrictions which are relevant in banks. Besides a pure formalization, restrictions which imply that the set of restrictions is no longer convex and/or state dependent needed to be approximated such that convexity and state independence holds. This specifications allow us finally to test and illustrate the model on a real balance sheet. First, we use public available UBS data to compare the published key rate delta profile with the different model profiles: The dynamic, the static and the benchmark one. In a second application, present values of different models are compared in 2005. The major findings are: The myopic model leads to unrealistic key rate deltas and the dynamic portfolio return dominates all other approaches by more than 1%. These numerical results show that dynamic asset liability management models can add monetary value to the banking firm.

Models

We consider a sequence of time periods $t \in N_0 = \{0, 1, 2,...\}$. Cash flows can only occur at t or at $t + 1$ for some $t \in N_0$ but never in between. This means that 'real world' cash flows occurring during the period $[t, t + 1]$ must be adapted accordingly. All random variables are defined on a probability space. We write P for the objective probability measure and P^* for a risk neutral measure. The information filtration at time A_t is generated by the risky assets realizations from time 0 up to time t. We write $X_t \in A_t$, if the random variable X_t is A_t-measurable, that is, if the discrete stochastic process $(X_t)_{t \in N_0}$ is $(A_t)_{t \in N_0}$ adapted.

We define assets and asset spaces next. To aggregate the myriad of different positions in the balance sheet, each asset is defined by its cash flow profile. The asset space reads:

$$B := \{b = (b^t)_{t \in N_0} \mid b^t \in A_t \text{ for all } t \in N_0\}.$$

Hence, the asset space consists of all possible stochastic A_t-adapted vectors (b^t). If an asset has zero value cash flows for the first t periods, then it belongs to the subspace:

$$B_t := \{b \in B_0 \mid b = (\underbrace{0,\ldots,0}_{t \text{ times}}, b^0, b^1, \ldots)\}$$

Given a cash flow stream at time t, our dynamic approach requires to model the corresponding stream at $t + 1$. Therefore, we define a linear map $\tau := \tau_t : B_t \rightarrow B_{t+1}$ by

$$b = (\underbrace{0,\ldots,0}_{t \text{ times}}, b^0, b^1, b^2, \ldots) \rightarrow \tau(b) := (\underbrace{0,\ldots,0}_{t+1 \text{ times}}, b^1, b^2, \ldots)$$

Some specific assets in the general asset space are of particular interest: Capital, new business assets and hedge instruments. A capital cash flow stream $C_t \in B_t$ is the residuum of all cash flows on the active side of the balance sheet minus the passive side. New business generated in $(t - 1, t]$ is denoted $N_t \in B_t$ and reflects, analogously to the capital C_t, the evolution on both asset and liability side determined by new active products, refinancing, cash management strategy and the clients' switching behavior. An existing hedging instrument at time t reads $H_t \in B_t$ and a new one $S_t \in B_t$. Finally, economic capital $E_t \in B_t$ is the sum of capital and the hedge, i.e.,

$$E_t = C_t + H_t$$

Since performance is typically measured with respect to some benchmark, we write $BM_t \in B_t$ for the benchmark asset. This asset is in applications typically a multiple of some reference asset $bm_t \in B_t$, such as one-month Money Market Instruments, which we normalize to a present value of 1, i.e.,

$$BM_t = PV_t(E_t) bm_t$$

where $PV_t(E)$ denotes the present value at t of E (Appendix 2).

Since the asset space B is a vector space and all assets are elements of this space, we use a set of linearly independent vectors of B, written in matrix form as $B_t = [B_t^1, \ldots, B_t^N]^\dagger$ to represent the different relevant assets at time t. More precisely, for each time t, we consider N linear independent vectors B_t with $B_t^j \in B_t$ defining a basis for S_t, the set of all possible linear combinations at time t of the N base assets. The base asset construction allows us to describe an asset by introducing its nominal values with respect to the base assets. Let A be an arbitrary spanned asset, i.e., $A \in S_t$. Then the asset is uniquely represented by

$$A = \sum_{j=1}^{N} \Phi_t^j(A) B_t^j = \Phi_t(A)^\dagger B_t$$

with a vector of nominal values $\Phi_t(A) \in R^N$ and B_t a vector of base assets. This concludes the formalization of the market instruments.

We consider next the optimization problem of the treasurer or portfolio manager. All mathematical proofs, extensions and details are given in Appendix 1. We assume that the treasurer is a mean-variance optimizer with utility

$$u_{\lambda,w}(x,t) := \lambda_t x - wx^2,$$

where $w \geq 0$ and $\lambda_t \in R$. In an multi-period context, utility is parameterized by the risk aversion index w and a second time-changing parameter λ_t. This additional parameter is needed to embed the multi-period mean-variance problem, which is not separable in the dynamic programming sense, into a larger problem, which is separable in the dynamic programming sense. Hence, the embedded problem is solved using dynamic programming techniques. If one proves that a solution of the embedded problem is always also a solution of the original one and vice versa, we are done. Former works of Li and Ng (2000) and Leippold *et al.*,(2003 and 2004) for example, while stating the complete result, only proved a part of the correspondence between the two problem formulations. Our first theoretical contribution is a complete proof of the correspondence. More precisely, for a set of convex constraints R_t, an affine measurable valuation function valuing the difference between economic capital and benchmark F_t, defined by:

$$F_t(b) := PV_t(b) - PV_t(\tau(BM_{t-1})),$$

discounting by a deterministic function $\beta_t \geq 0$ and $T \in N_1 := \{1, 2, 3,...\}$ the time horizon, the extended optimization problem

$$\max_{\substack{\Phi_t(S_t)\in R_t \\ t=0,...,T-1}} \mathbb{E}_0\left[\sum_{t=1}^{T} \beta_t u_{\lambda,w}(F_t(\mathbf{E}_t))\right] \qquad (P_{\lambda,w}) \qquad ...(1)$$

is for a specific choice of the parameter λ equivalent to a multi-period mean-variance optimization problem:

$$\max_{\substack{\Phi_t(S_t)\in R_t \\ t=0,...,T-1}} \sum_{t=1}^{T} \beta_t \{\mathbb{E}_0[F_t(\mathbf{E}_t)] - wVar_0(F_t(\mathbf{E}_t))\} \qquad (P_w) \qquad ...(2)$$

Denoting by $\mathcal{T}_\beta := \{t \in \{1,..., T\} \,|\, \beta_t > 0\}$ the dates where indeed a valuation occurs, the correspondence theorem reads:

Theorem 1: *For all $t \in \{0,..., T-1\}$, let R_t be a non-empty convex set of control rules, and for all $t \in \{1,..., T\}$, let $a_t := \mathbb{E}_0 [F_t(\mathbf{E}_t)]$ and $\mathbf{E}_t^* := \mathbf{E}_t + S_t^*$ be the optimal economic capital. The programs (P_w) and $(P_{\lambda^*, w})$ have the same solution if and only if*

$$\lambda_t^* = 1 + 2wa_t \,|_{\Phi(\mathbf{E}^*)} \quad \text{for all} \quad t \in \mathcal{T}_\beta$$

Existence of a solution of (2) and its uniqueness are standard results of discrete stochastic optimization, because (P_w) is a convex-quadratic problem (Minoux, 1986).

The last ingredient in the optimization problem (1) is the set of convex constraints R_t. We distinguish between equality and inequality constraints. One equality constraint is the dynamics of the state variable $x_t := \mathrm{PV}_t(\mathbf{E}_t)$. Using the linear capital dynamics:

$$\mathbf{E}_{t+1} = \tau(\mathbf{E}_t) + \tau(S_t) + N_{t+1}$$

and properties of the different cash flow streams in the base assets' representation, we get after some algebra and application of the present value operator

$$x_{t+1} = (v_t + \Phi_t(BM_t)).\, R_{t+1} + Z_{t+1} \qquad \text{...(3)}$$

with the control variable

$$v_t := \Phi_t(S_t) + \Phi_t(\mathbf{E}_t) - \Phi_t(BM_t) \qquad \text{...(4)}$$

and

$$R_{t+1} := PV_{t+1}(\tau(B_t)) \in \mathbb{R}^N, Z_{t+1} := PV_{t+1}(N_{t+1}) \in \mathbb{R}$$

which represent the full valuation at time $t + 1$ of the base assets and of new business written during the period $[t, t + 1]$, respectively.

Although the optimizations with respect to the variables $\Phi_t(S_t)$ in (1) or with respect to v are equivalent, the formulation of the dynamics (3) and (4) in terms of v has a nice interpretation: if v is zero, then capital growth is driven by the growth of the benchmark and the new hedge. In other words, an optimal control v^* represents the correction, which a treasurer has to add to achieve optimality. The proof of the results also shows that using v simplifies the calculation. Besides this dynamic constraint, other constraints matter in practice: Effective Delta Constraints, Transaction Volume or Market Liquidity Constraints, Exposure over Benchmark Constraints, Expected Shortfall Constraints. We consider constraints which are linear or affine in the control variables. Constraints which do not possess these properties are approximated by affine constraints. The expected shortfall risk measure is linearized and the constraints mentioned above are rewritten and recognized to explicitly depend only on the control variable v_t, but not the state variable x_t. This assumption allows us to derive analytic and quasi-analytic closed form for the optimal control rules.

The general technique to tackle the extended optimization problem ($P_{\lambda, W}$) is Bellman's backward induction. That for, we define for time t and state x the dynamics $D.(.)$:

$$D_t(x) := \{x_t = x, x_{s+1} := (v_s + \Phi_s(BM_s)).R_{s+1} + Z_{s+1} (t \le s \le T-1)\}$$

and the value function

$$J_t(x) := \max_{\substack{v_s \in R_s \\ s=t,\dots,T-1 \\ D_t(x)}} \mathbb{E}_t\left[\sum_{s=t}^{T-1} \beta_s u_{\lambda,w}(v_s.R_{s+1} + Z_{s+1})\right]$$

where $x_0 := \mathbb{E}_0^*[PV_0(\mathbf{E}_0)]$

Theorem 2: *Let R_t, $t \in \{0,\ldots,T-1\}$, be a non-empty convex set of control rules. Then, the unique optimal control rule v^* satisfies Bellman's equation*

$$J_t(x) = \max_{v_t \in R_t} \mathbb{E}_t[\beta_{t+1} u_{\lambda,w}(v_t.R_{t+1} + Z_{t+1}) + J_{t+1}(v_t.R_{t+1} + Z_{t+1})]$$

Solution of the Optimization Problem: Equality Constraints Only

The optimal policy for the optimization problem (1) with equality constraints only reads:

Theorem 3: *For all $t = 0, 1,\ldots, T-1$ and a constraint set which consists only of the no-leverage equality constraint $R_t = \{PV_t(S_t) = 0\}$, the optimal policy of (1) reads:*

$$\Phi_t(S_t^*) = v_t^* - \Phi_t(\mathbf{E}_t) + \Phi_t(BM_t) \quad \ldots(5)$$

$$v_t^* = v_t^*\left(\frac{\lambda_{t+1}}{2w}\right) = \frac{\lambda_{t+1}}{2w}\xi_t + \eta_t$$

$$\xi_t := \mathbb{E}_t[R_{t+1}R_{t+1}^\dagger]^{-1}\left\{\mathbb{E}_t[R_{t+1}] - \frac{e_t^\dagger \mathbb{E}_t[R_{t+1}R_{t+1}^\dagger]^{-1}\mathbb{E}_t[R_{t+1}]}{e_t^\dagger \mathbb{E}_t[R_{t+1}R_{t+1}^\dagger]^{-1}e_t} e_t\right\}$$

$$\eta_t := -\mathbb{E}_t[R_{t+1}R_{t+1}^\dagger]^{-1}\left\{\mathbb{E}_t[Z_{t+1}R_{t+1}] - \frac{e_t^\dagger \mathbb{E}_t[R_{t+1}R_{t+1}^\dagger]^{-1}\mathbb{E}_t[Z_{t+1}R_{t+1}]}{e_t^\dagger E_t[R_{t+1}R_{t+1}^\dagger]^{-1}e_t} e_t\right\}$$

with the vector $e_t := PV_t(B_t) \in \mathbb{R}^N$ representing the market value of the base assets at time t.

Applying Theorem 1, we find a value $\lambda^* = (\lambda_1^*,\ldots,\lambda_T^*)$ such that $v_t\left(\frac{\lambda_{t+1}^*}{2w}\right)$ solves the optimization problem (2). In the case of present value valuation, the solution (5) has a similar interpretation as the one in Leippold *et al.*,(2004): The expression

$$\mathbb{E}_t[R_{t+1}] - \frac{e_t^\dagger \mathbb{E}_t[R_{t+1}R_{t+1}^\dagger]^{-1}\mathbb{E}_t[R_{t+1}]}{e_t^\dagger \mathbb{E}_t[R_{t+1}R_{t+1}^\dagger]^{-1}e_t} e_t$$

is the orthogonal projection of base assets' total returns (expected at time t for the period $[t, t+1]$ onto the orthogonal complement to their present values at time t.

Corollary 1: *The dynamic mean-variance optimization (2) of the excess portfolio for a constraint set, which consists only of the no-leverage equality constraint $R_t = \{PV_t(S_t) = 0\}$, has a unique solution which is decomposed into the benchmark $\Phi_t(BM_t)$, the contribution of new business η_t and the contribution of the in force business ξ_t modulated by a scalar factor η_0 which takes into account both risk aversion and new business:*

$$\Phi_t(\mathbf{E}_t^*) = M_0 X_{t+1}\xi_t + \eta_t + \Phi_t(BM_t)$$

where

$$M_0 = M_0(w; R_{t+1}, Z_{t+1}) := \frac{1 + 2w\mathbb{E}_0[\eta_t^\dagger R_{t+1} + Z_{t+1}]}{2w(1 - \mathbb{E}_0[\xi_t^\dagger R_{t+1}])}$$

and the scalar indicator X_t is defined as 1 if $t \in \mathcal{T}_\beta$ and 0 otherwise. Therefore, if the objective function does not have any contributions from the time point $t + 1$ (i.e., $\beta_{t+1} = 0$), then the optimal control rule does not depend on the risk aversion:

$$\Phi_t(\mathbf{E}_t^*) = \eta_t + \Phi_t(BM_t)$$

Some special cases are:

- **No new business case:** If the full valuation of new business vanishes, that is, if $Z_{t+1} = 0$, then

  $$\Phi_t(\mathbf{E}_t^*) = M_0 X_{t+1}\xi_t + \Phi_t(BM_t)$$

  where

  $$M_0 = M_0(w; R_{t+1}) := \frac{1}{2w(1 - \mathbb{E}_0[\xi_t^\dagger R_{t+1}])}$$

- **Infinite risk aversion case:** If we choose an arbitrarily increasing risk aversion parameter, then

  $$\lim_{w \to +\infty} \Phi_t(\mathbf{E}_t^*) = M_0 X_{t+1}\xi_t + \eta_t + \Phi_t(BM_t)$$

  where

  $$M_0 = M_0(+\infty; R_{t+1}, Z_{t+1}) := \frac{\mathbb{E}_0[\eta_t^\dagger R_{t+1} + Z_{t+1}]}{1 - \mathbb{E}_0[\xi_t^\dagger R_{t+1}]}$$

- **Infinite risk aversion and no new business case:** In this case, the optimal allocation is always equal to the benchmark:

  $$\lim_{w \to +\infty} \Phi_t(\mathbf{E}_t^*) = \Phi_t(BM_t)$$

- **Almost risk neutral case:** If we choose an arbitrarily decreasing risk aversion parameter, then the optimal allocation splits into the sum of three contribution, namely, the one coming from the in force business, the one generated by new business, and the one given by the benchmark:

  $$\Phi_t(\mathbf{E}_t^*) = M_0 X_{t+1}\xi_t + \eta_t + \Phi_t(BM_t)$$

where

$$M_0 = M_0(w; R_{t+1}, Z_{t+1} = \frac{1}{2(1-\mathbb{E}_0[\xi_t^{\dagger} R_{t+1}])} O\left(\frac{1}{w}\right) \text{when } w \to 0^+.$$

The above optimization problem considers the possible risk management activities of a treasurer $\Phi_t^*(S_t)$ given the exogenous part Z_{t+1}. This exogenous part reflects different circumstances. The generation of new business is an example. Another exogenous factor is the clients' switching behavior, triggered by the interest rate level, between different balance sheet products. If interest rates are low, one typically observes that clients switch from non-maturing, floating interest rate mortgages into fixed maturity/interest rate ones. Such flows can comprise annually up to 50% or 60% of the assets value in a balance sheet, if interest rates are on an historically very high or low level.

Although new business and a behavioral model can be easily integrated in the problem (1), we prefer to avoid them in the application presented in this paper because the results of the optimization problem become very difficult to interpret. This is due to a non-trivial interference between the exogenous factors and the other primitives of the optimization problem, the chosen preferences, the set of constraints used and properties of the securities.

In summary, we choose $Z_t := 0$ for all t in the application below. This models a dying bank. At first sight, this setup might seem odd. But one often encounters this type of dynamic balance sheet modeling in practice: Treasurers use in month 1 the dynamic solution to make the investment decisions. In month 2, the new banking portfolio and hedge portfolio are reloaded and the optimal dynamic policy starting in month 2 is calculated. The difference between this actual policy and the policy calculated at 1 then captures changes in the balance sheet due to behavior of clients, new business and the realized interest rate level versus the predicted one. Hence, updating of the optimal policy over time is due to purely exogenous factors. This policy is slightly sub-optimal in the sense of dynamic optimization since it neglects the contributions of Z_t for all t to η_0 and to M_0 in the formula for the optimal correction v_0.

Solution of the Optimization Problem: Equality and Inequality Constraints

Several constraints are used in banks to manage interest rate risk of the balance sheet. These constraints define the set R_t and will be extensively treated in Part II of this work. For the time being, we will only assume that they are of the following form:

- **Affine Equalities:** For a given linear operator $E_t: B_t \to \mathbb{R}^n$ and a given inhomogeneity vector $f_t \in \mathbb{R}^n$ the affine equality $E_t(S_t) + f_t = 0$ defines $n \in \mathbb{N}_0$ equality constraints. For example, the no leverage constraint encountered in the previous section:

  $PV_t(S_t) = 0$

is seen to be of this type with the choice $n := 1, E_t := e_t^\dagger = PV_t(B_t)^\dagger$ and $f_t := 0$.

- **Affine Inequalities:** For a given linear operator A_t: $B_t \to \mathbb{R}^m$ and a given inhomogeneity vector $b_t \in \mathbb{R}^m$, the affine inequality $A_t(S_t) + b_t \geq 0$ defines $m \in \mathbb{N}_0$ inequality constraints.

Identifying the linear operators E_t and A_t with their matrix representations with respect to the standard basis in $\mathbb{R}^n$ and respectively $\mathbb{R}^m$ on one hand, and the basis of S_t given by the base assets B_t on the other, the affine constraints define a convex subset of S_t which can be identified with $\{E_t V_t + f_t = 0, A_t v_t + b_t \geq 0\}$. Remark that E_t, f_t, A_t and b_t are in general $\mathcal{A}_t$-measurable random variables. Since the affine restrictions depend explicitly on the optimization variable v_t but not on the state variable x_t, a similar calculation as in the proof of Theorem 3 leads to the following.

Theorem 4: *For all $t = 0,1,\ldots, T-1$ and an affine constraint set $R_t = \{E_t V_t + f_t = 0, A_t v_t + b_t \geq 0\}$, the solution of (1) reads:*

$$\Phi_t(S_t^*) = v_t^* - \Phi_t(\mathbf{E}_t) + \Phi_t(BM_t)$$

$$v_t^* = v_t\left(\frac{\lambda_{t+1}}{2w}, \theta_t^*\right)$$

$$v_t = v_t\left(\frac{\lambda_{t+1}}{2w}, \theta_t\right) = \frac{\lambda_{t+1}}{2w}\xi_t(\theta_t) + \eta_t(\theta_t)$$

$$\xi_t(\theta_t) := -\mathbb{E}_t[R_{t+1}R_{t+1}^\dagger]^{-1}P(\theta_t)\mathbb{E}_t[R_{t+1}]$$

$$\eta_t(\theta_t) := -\mathbb{E}_t[R_{t+1}R_{t+1}^\dagger]^{-1}\{P(\theta_t)\mathbb{E}_t[Z_{t+1}R_{t+1}] + Q(\theta_t)D(\theta_t; f_t, b_t)$$

$$Q(\theta_t) := D(\theta_t; E_t, A_t)^\dagger(D(\theta_t; E_t, A_t)\mathbb{E}_t[R_{t+1}R_{t+1}^\dagger]^{-1}D(\theta_t; E_t, A_t)^\dagger)^{-1}$$

$$P(\theta_t) := \mathbb{I}_{N\times N} - Q(\theta_t)D(\theta_t; E_t, A_t)\mathbb{E}_t[R_{t+1}R_{t+1}^\dagger]^{-1}$$

$$D(\theta_t; X, Y) := \begin{bmatrix} X \\ \operatorname{diag}(\theta_t)Y \end{bmatrix}$$

$$\theta_t^* := \theta_t^*\left(\frac{\lambda_{t+1}}{2w}\right)$$

$$= \arg \min_{\substack{\theta_t \in \{0,1\}^m \\ A_t v_t + b_t \geq 0}} \left\{ \begin{array}{l} \frac{1}{2} v_t\left(\frac{\lambda_{t+1}}{2w}, \theta_t\right)^\dagger \mathbb{E}_t[R_{t+1}R_{t+1}^\dagger] v_t\left(\frac{\lambda_{t+1}}{2w}, \theta_t\right) \\ + \left(-\frac{\lambda_{t+1}}{2w}\mathbb{E}_t[R_{t+1}] + \mathbb{E}_t[Z_{t+1}R_{t+1}]\right)^\dagger v_t\left(\frac{\lambda_{t+1}}{2w}, \theta_t\right) \end{array} \right\}$$

The proof of Theorem 4 closely follows the one of Theorem 3. The minimization problem over a finite set in the closed form solution reflects constraints' bindingness. The expression θ_t^* is a {0, 1}^{m}-valued t-measurable random variable. This variable indicates which inequality restriction binds. The expression $P(\theta_t)$ is an orthogonal projection onto the hyperplane in the policy space v_t. It is defined by the linear equations $E_t V_t = 0$ and diag $(\theta_t) A_t V_t + b_t = 0$. Finally, $v_t^* = v_t^*\left(\frac{\lambda_{t+1}}{2w}, \theta_t^*\left(\frac{\lambda_{t+1}}{2w}\right)\right)$ defines the optimal control policy. The λ-dependence of the optimal solution vanishes for some special cases, where the objective function does not account for a valuation at every partial time step.

Corollary 2: *For all $t \notin \mathcal{T}_\beta$, the solution of (2) under the affine constraint set*

$$R_t = \{E_t v_t + f_t = 0, A_t v_t + b_t \geq 0\}, \text{ reads:}$$

$$\Phi_t(S_t^*) = \eta_t(\theta_t^*) - \Phi(\mathbf{E}_t) + \Phi_t(BM_t)$$

$$\eta_t(\theta_t) := -\mathbb{E}_t\left[R_{t+1}R_{t+1}^\dagger\right]^{-1}\{P(\theta_t)\mathbb{E}_t[Z_{t+1}R_{t+1}] + Q(\theta_t)D(\theta_t; f_t, b_t)\}$$

$$Q(\theta_t) := D(\theta_t; E_t, A_t)^\dagger (D(\theta_t; E_t, A_t)\mathbb{E}_t[R_{t+1}R_{t+1}^\dagger]^{-1} D(\theta_t; E_t, A_t)^\dagger)^{-1}$$

$$P(\theta_t) := \mathbb{I}_{N\times N} - Q(\theta_t)D(\theta_t; E_t, A_t)\mathbb{E}_t[R_{t+1}R_{t+1}^\dagger]^{-1}$$

$$D(\theta_t; X, Y) := \begin{bmatrix} X \\ \text{diag}\,(\theta_t)Y \end{bmatrix}$$

$$\theta_t^* = \arg \min_{\substack{\theta_t \in \{0,1\}^m \\ A_t\eta_t(\theta_t)+b_t \geq 0}} \left\{\frac{1}{2}\eta_t(\theta_t)^\dagger \mathbb{E}_t[R_{t+1}R_{t+1}^\dagger]\eta_t(\theta_t) + \mathbb{E}_t[Z_{t+1}R_{t+1}]^\dagger \eta_t(\theta_t)\right\}$$

For generic times $t \in \mathcal{T}_\beta$, the parameter λ can be chosen by Theorem 1, such that $\Phi_t(S_t)$ solves the dynamic mean-variance problem (P_w). That for, the parameter λ_t has to solve

$$\lambda_t = 1 + 2wa_t^*$$

for all $t \in \mathcal{T}_\beta$. This conditions reads

$$\lambda_t = 1 + 2wa_t^* = 1 + 2w\mathbb{E}_0\left[v_{t-1}^*\left(\frac{\lambda_t}{2w}, \theta_{t-1}^*\left(\frac{\lambda_t}{2w}\right)\right)^\dagger R_t + Z_t\right] \quad \text{...(6)}$$

For the general case, this implicit equation cannot be solved directly in a closed form for λ_t contrary to the equality-constrained optimization problem. But for the special case, when we have only homogeneous constraints and no new business, this can actually be done, at least for a sufficiently small risk aversion.

Lemma 1: *Assume that $f_t = 0$, $b_t = 0$ and $Z_{t+1} = 0$ for all t. Then, it exists a risk aversion $w_0 > 0$ such that for every $w \in [0, w_0]$, the binding constraints Theorem 4 at time t satisfy the formula:*

$$\theta_t^* = \arg \min_{\substack{\theta_t \in \{0,1\}^m \\ A_t \xi_t(\theta_t) \geq 0}} \left\{ \frac{1}{2} \xi_t(\theta_t)^\dagger \mathbb{E}_t[R_{t+1} R_{t+1}^\dagger] \xi_t(\theta_t) - \mathbb{E}_t[R_{t+1}]^\dagger \xi_t(\theta_t) \right\}$$

if $t+1 \in \mathcal{T}_\beta$ and thus depend neither on the risk aversion w nor on the parameter λ_{t+1}.

Therefore, the optimal portfolio satisfies for any risk aversion $w \in [0, w_0]$

$$\Phi_t(\mathbf{E}_t^*) = M_0 \xi_t(\theta_t^*) + \Phi_t(BM_t)$$

where

$$M_0 = M_0(w; R_{t+1}) := \frac{1}{2w(1 - \mathbb{E}_0[\xi_t(\theta_t^*)^\dagger R_{t+1}])}$$

By adding two dummy optimization variables and imposing two additional affine equality constraints, it is possible to provide a solution for the general case in terms of the special case treated in Lemma 1. That for, we extend $v_t = [v_t^1, \ldots, v_1^N]^\dagger$ to $\overline{v}_t := [v_t, v_t^{N+1}, v_t^{N+2}]^\dagger$ for which the linear equality and inequality constraints.

$$\overline{E}_t \overline{v}_t = 0 \quad \text{and} \quad \overline{A}_t \overline{v}_t \geq 0$$

where $\overline{E}_t := [E_t, f_t, 0]$ and $\overline{A}_t := [A_t, b_t, 0]$. While the first dummy variable, v_t^{N+1}, leads to homogeneous constraints for the reformulated optimization problem, the second one, v_t^{N+2}, is introduced to make new business vanish. The valuation at time $t+1$ in the optimization problem 1 can be written as:

$$F_t(\mathbf{E}_t) = \overline{v}_t^\dagger \overline{R}_{t+1}$$

where $\overline{R}_t := \Pi_{N+2,1}^\dagger(R_t, Z_t)$ and $\Pi_{k,l}$ is identified with the matrix representation of the projection

$$\Pi_{k,l} : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k, (a,b) \mapsto \Pi_{k,l}(a,b) := a$$

with respect to the standard basis of $\mathbb{R}^{k+1}$ and $\mathbb{R}^k$ respectively. Hence, we obtain a closed form solution for the general case as well.

Corollary 3: *It exists a risk aversion $w_0 > 0$, such that the unique solution of optimization problem (2) for an affine constraint set $R_t = \{E_t v_t + f_t = 0,\ A_t v_t + b_t \geq 0\}$ is given for any risk aversion $w \in [0, w_0]$ and any time t for which $t+1 \in \mathcal{T}_\beta$ by:*

$$\Phi_t(\mathbf{E}_t^*) = \overline{M}_0 \Pi_{N,2} \overline{\xi}_t(\overline{\theta}_t^*) + \Phi_t(BM_t)$$

where

$$\overline{\mathrm{M}}_0 = \overline{\mathrm{M}}_0(w;\overline{\mathrm{R}}_{t+1}) := \frac{1}{2w(1-\mathbb{E}_0[\overline{\xi}_t(\overline{\theta}_t^*)^\dagger\overline{\mathrm{R}}_{t+1}]}$$

$$\overline{\theta}_t^* = \arg \min_{\substack{\overline{\theta}_t\in\{0,1\}^{m+2} \\ \overline{\mathrm{A}}_t\overline{\xi}_t(\overline{\theta}_t)\geq 0}} \left\{\frac{1}{2}\overline{\xi}_t(\overline{\theta}_t)^\dagger\,\mathbb{E}_t[\overline{\mathrm{R}}_{t+1}\overline{\mathrm{R}}_{t+1}^\dagger]\overline{\xi}_t(\overline{\theta}_t) - \mathbb{E}_t[\overline{\mathrm{R}}_{t+1}]^\dagger\,\overline{\xi}_t(\overline{\theta}_t)\right\}$$

$$\overline{\xi}_t(\theta_t) := \mathbb{E}_t[\overline{\mathrm{R}}_{t+1}\overline{\mathrm{R}}_{t+1}^\dagger]^{-1}\overline{\mathrm{P}}(\overline{\theta}_t)\mathbb{E}_t[\overline{\mathrm{R}}_{t+1}]$$

$$\overline{\mathrm{Q}}(\overline{\theta}_t) := D(\overline{\theta}_t;\overline{\mathrm{E}}_t,\overline{\mathrm{A}}_t)^\dagger(D(\overline{\theta}_t;\overline{\mathrm{E}}_t,\overline{\mathrm{A}}_t)\mathbb{E}_t[\overline{\mathrm{R}}_{t+1}\overline{\mathrm{R}}_{t+1}^\dagger]^{-1}D(\overline{\theta}_t;\overline{\mathrm{E}}_t,\overline{\mathrm{A}}_t)^\dagger)^{-1}$$

$$\overline{\mathrm{P}}(\overline{\theta}_t) := \mathbb{I}_{(N+2)\times(N+2)} - \overline{\mathrm{Q}}(\overline{\theta}_t)\mathrm{D}(\overline{\theta}_t;\overline{\mathrm{E}}_t,\overline{\mathrm{A}}_t)\mathbb{E}_t[\overline{\mathrm{R}}_{t+1}\overline{\mathrm{R}}_{t+1}^\dagger]^{-1}$$

$$D(\overline{\theta}_t;X,Y) := \begin{bmatrix} X \\ \mathrm{diag}\,(\overline{\theta}_t)Y \end{bmatrix}$$

$$\overline{\mathrm{E}}_t := [E_t, f_t, 0]$$

$$\overline{\mathrm{A}}_t := [A_t, b_t, 0]$$

The Optimization Problem for Specific Term Structures

The previous results hold for arbitrary term structures. We prove in this section two results for specific interest rate models. The results are both interesting from a theoretical and practitioner's point of view.

Theorem 5: *Let R_t, $t \in \{0,\ldots, T-1\}$ be a non-empty convex set of control rules. If the term structures of interest rates are stochastic independent over time, then for any vector of weights $\beta \in \mathbf{R}^T$, the solution of the dynamic mean-variance optimization problem (2) is equal to the sequence of solutions of the static one-period mean-variance optimizations for the same points in time.*

It is a well-known observation that the time dependence of interest rates becomes very weak as soon as the rates under consideration are more distant than one month. Such a wide time spacing of rates is hardly encountered in interest rate risk management of the balance sheet. Therefore, the theorem shows that a treasurer typically is acting suboptimal if a one-period model is used. On the contrary, for strategic planning purposes of the bank, wider time periods are considered than the treasurer's ones. The theorem shows that for this kind of activities a static approach suffices.

Theorem 6: *Let R_t, $t \in \{0,\ldots, T-1\}$ be a non-empty convex set of control rules. If at every point in time the conditional expectation of the term structure of interest rates is equal to the forward structure and no restrictions but the no leverage one are imposed, and if the present value of new business vanishes, then the solution of the dynamic multiperiod mean-variance problem, that is the optimal economic fixed income capital, is, for any vector of weights $\beta \in \mathbf{R}^T$ and any risk aversion $w>0$, always equal to the benchmark. If other convex restrictions are imposed, then the optimal solution is the unique feasible minimizer for the conditional variance of the future excess portfolio present value.*

Theorem 6 illustrates the importance of the benchmark. If the treasurer believes that the best interest rate guess is the forward rate and if he neglects new business, then the optimal earnings and risk management of the balance reduces to the identification of the most appropriate benchmark.

Applications

In all applications, we consider a balance sheet where new business is set equal to zero.

Key Rate Profile

We compare the UBS key rate delta profile with the optimal dynamic and myopic profile. To achieve this goal, we measure the treasury profit and loss at the end of every sub-period, that is $\beta_t := 1$ for all $t=1,\ldots, T := 4$ months. We further only consider equality constraints. We analyze the balance sheet and the hedging portfolio of UBS Group Treasury focussing on the CHF positions, whose aggregated first order interest rate sensitivities on December 31, 2004 are available from the UBS home page in internet (UBS, 2005). This data are shown in Table 1. Using this data we construct in Table 2 an arbitrary but realistic example by distributing the aggregated sensitivities over the different key rates. The benchmark is a 3Y-annuity. The sensitivities are determined by the standard procedure described in the following.

Table 1: First Order Interest Rate Sensitivities in kCHF

Key Rates	3M	6Y	1Y	2Y	3Y	4Y	5Y	6Y
Sensitivities	–165	–87	27	–62	–25	30	85	113
Key Rates	7Y	8Y	9Y	10Y	15Y	20Y	30Y–	Total
Sensitivities	70	–22	–33	–47	1	0	0	–116

Table 2: Aggregated First Order Interest Rate Sensitivities in kCHF

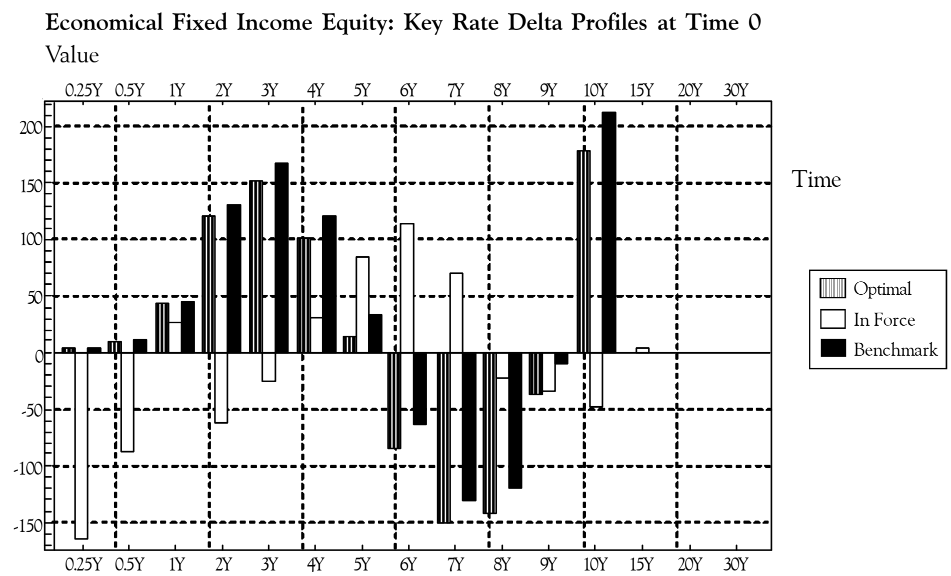
Time Buckets	OM – 1M	1M – 3M	3M – 12M	1Y – 5Y	5Y –	Total
Sensitivities	–166	1	–87	55	81	–116

We choose an Arrow-Pratt absolute risk aversion coefficient $w_{AP}=50$ and set our risk aversion coefficient to

$$w := \frac{w_{AP}}{2PV_0(\mathbf{E}_0)}$$

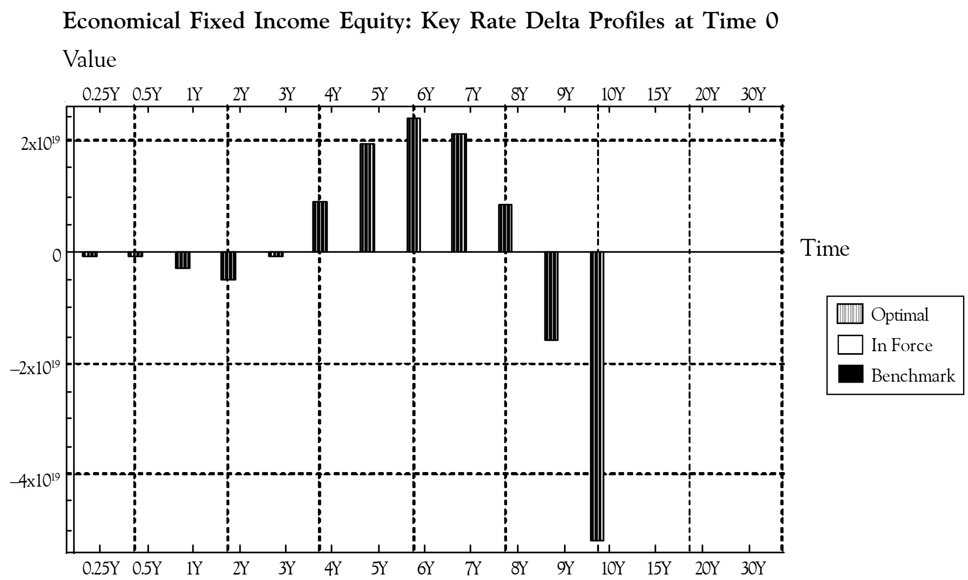
The data needed to calculate the key rate deltas for the net bank exposure of Figure 1 are the cash flows of the instruments and the swap term structure. Cash flows for aggregated balance and off balance sheet positions are projected on a monthly bases until run-off. Figure 1 shows a balance sheet model where all prepayments are collected in a single position, i.e., a one month money market account for example. How is the stochastic cash flow profile generated? For fixed maturity and fixed rate products,

Figure 1: Dynamic Optimal Key Rate Deltas at Time 0 (in thousands CHF). The 'In Force' Profile is the Public Available UBS Profile



the profile are deterministic and known. For all other maturing products, an interest rate model is used to generate the cash flows for each product and each payment date. For non-maturing positions a prepayment model is used. Finally, all cash flows are aggregated to a cash flow stream representing the economic fixed income equity. This stream and the present Swap term structure are inserted into the definition of key rate delta to give the profile shown in Figure 1. The term structure in this example is modeled according to the (M2) procedure explained in application 'Profit and Loss'. The results in Figures 1 and 2 for the four portfolios—the 'In Force' (i.e., the UBS-proxy), the benchmark and both the optimal dynamic and myopic portfolios—are: First, the magnitudes of the key rates using the myopic optimal portfolio are too much high, i.e., the key rate exposure will not be acceptable to the treasurer (Figure 2). Therefore, the myopic model is rejected in this application based on UBS data for first order risk management. Second, the dynamic optimal key rates are comparable with the 'In Force' values of UBS. Therefore, the dynamic model output is a reasonable benchmark for the treasurer in interest rate risk decision-making. The model output is indeed a challenge for the treasury, since the 'In Force' profile is by and large the opposite compared to the model outputs, i.e., if a positive key rate for given maturity holds in reality, the optimal model key rates possess a negative sign.

Figure 2: Myopic Optimal Key Rate Deltas at Time 0 (in thousands CHF). The 'In Force' Profile is the Public Available UBS Profile



Sharpe Ratios

In this example, we compare Sharpe ratios for different allocations. We consider a time horizon $T = 4$ weeks. We set $\beta_4 := 1$ and $\beta_1 = \beta_2 = \beta_3 := 0$ and choose a 3Y annuity as a benchmark. The term structure in this example is modeled according to the (M2) procedure explained in application 'Profit and Loss'. As in the example 'Key Rate Profile' the Arrow-Pratt risk aversion is $w_{AP} = 50$. Time $t = 0$ corresponds to January 1, 2005 and time $t = 4$ to February 1, 2005. Assuming vanishing risk-free rate, we can define the Sharpe Ratio of an asset $b \in B$ between time 0 and t as

$$SR_t(b) := \frac{E_0[b]}{Vol_0(b)}$$

Table 3 shows Sharpe ratios of different portfolios for different periods. The Sharpe ratio for period [0, 4] is maximized by the portfolio consisting in long optimal dynamic

Table 3: Sharpe Ratios for Different Portfolios for Different Periods in Sharpe Ratios

Portfolio	Time 1	Time 2	Time 3	Time 4
Dynamic BM	–108.458	–113.058	–127.848	–113.826
Static BM	–105.838	–1.371	–12.595	–14.123
Optimal Dynamic	–0.174	–0.119	–0.144	–0.128
Optimal Static	–0.028	0.228	0.280	0.329
Dynamic (Optimal–BM)	108.317	75.404	61.646	53.472
Static (Optimal–BM)	105.810	1.599	1.813	1.854

allocation and short benchmark: This result is in line with the optimization problem P_w and the choice of betas.

Profit and Loss

In this sophisticated example, we compare and backtest dynamic versus myopic decision-making for a dying bank, i.e., cash flows from maturing positions or prepayment are reinvested in the base assets to beat the benchmark. We consider a time horizon $T = 4$ weeks. The term structure model is given by a not recombining tree, which is generated according to the following procedure, which guarantees that the term structures at different times are stochastic independent for the model (Ml) and dependent model (M2). For every $t = 1, 2,..., T$:

1. Consider a sufficient long Swap term structure history with frequency t and a term structure forecast for time t, which we interpret as the at time 0 expected term structure at time t. This forecast can be obtained, for instance, by means of econometric techniques involving macro-economical leading indicators. After that:
2. **Model (Ml):**
   a. Determine the historical total returns of synthetic zero bonds whose maturities are the treasury key rates. Calculate the empirical covariance matrix $\sum_t^{ret}$.
   b. Perform the principal component analysis of $\sum_t^{ret}$. The first three factors, that is the first three components of the vector valued random variable F_t^{ret} corresponding to the largest eigenvalues of $\sum_t^{ret}$ are functionalss of the well-known shift, twist and butterfly factors explaining 90% of the term structure movement. Let U_t denote the $N \times 3$ matrix containing the (normalized) first three eigenvectors of $\sum_t^{ret}$.
   c. Multiply the matrix U_t^{ret} with all the results of a Monte Carlo simulation of $F_t^{ret;1,2,3}$ under the standard multivariate normal distribution assumption to obtain simulations of the total returns of the zero bonds.
   d. Using the expected term structure for t at time 0 determine the implied expected total returns of the zero bonds. Subtract from the realizations of the total returns, their arithmetical mean and add the expected value. The new realizations of the total returns are consistent at expected value level with the term structure forecast.
   e. Derive from the realizations of the zero bond returns the implied discount factors describing the term structure at time t.
3. **Model (M2):**
   a. Determine the historical first differences of continuous spot rates for treasury key rates. Calculate the empirical covariance matrix $\sum_t^{int}$.

b. Perform the principal component analysis of $\sum_t^{int}$. The first three components of the vector-valued random variable F_t^{int} corresponding to the bigger eigenvalues of $\sum_t^{int}$ are the shift, twist and butterfly factors. Let U_t^{int} denote the $N \times 3$ matrix containing the (normalized) first three eigenvectors of $\sum_t^{int}$.

c. Multiply the matrix U_t^{int} with all the results of a Monte Carlo simulation of $F_t^{int;1,2,3}$ under the standard multivariate normal distribution assumption to obtain simulations of the first differences for continuous spot rates for treasury key rates.

d. Using the expected term structure for t at time 0 and the term structure at time zero, rescale the realizations of the first differences of rates in order to make them consistent with the term structure forecast.

By construction, the simulated term structures are stochastic independent over time for Ml and dependent for M2. As expected in view of Theorem 5 the solution dynamic problem in Ml case is equal to the sequence of static one period optimization problems (myopic case). We set $\beta_4 := 1$ and $\beta_1 = \beta_2 = \beta_3 := 0$ and choose a 3Y annuity as a benchmark. As in 'Key Rate Profile', the Arrow-Pratt risk aversion is $w_{AP}: = 50$. At the beginning of every month of 2005 we allocate now 10 Mia CHF according to the following strategies:

- **Static, i.e., A Single Annual Period:** At time $t = 0$, an investment decision is taken according to mean-variance with an investment horizon of one month with no possibility of reallocation in between.
- **Myopic:** On the basis of term structure model (Ml), a sequence on one period static allocations is chosen to reallocate every week.
- **Dynamic:** On the basis of term structure model (M2), a dynamic strategy is chosen to reallocate every week.

The realized present values of the three different strategies as well as of the benchmark are shown in Figure 3 (on a monthly basis) and in Figure 4 (year cumulation). The findings are: First, the static portfolio approach is by far the worst performing one: Its annual performance is around 1%, whereas the most performing model reaches more than 3%. Second, the inability of the static model to distribute risk over time is apparent in the monthly P&L 3 in October, 2005. In this month, the interest rate model forecast leads to a monthly loss for all models, but the extreme loss of more than 100 Mio CHF resulted for the static model. Third, the dynamic model outperforms the myopic model. The annual return of the dynamic model is 3.1%, whereas the myopic model has a return of 2.6%. The myopic return is also dominated by the benchmark return, where the serial interest rate correlation is considered. But if independence is assumed in the benchmark model, the myopic model leads to a higher return.

Figure 3: Monthly P&L (Mio CHF)

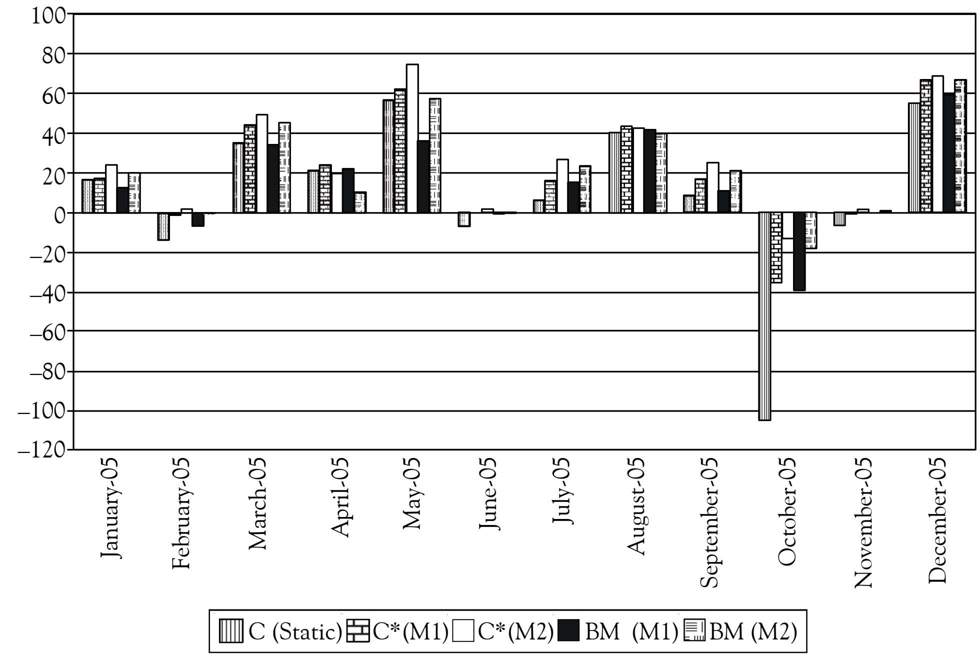
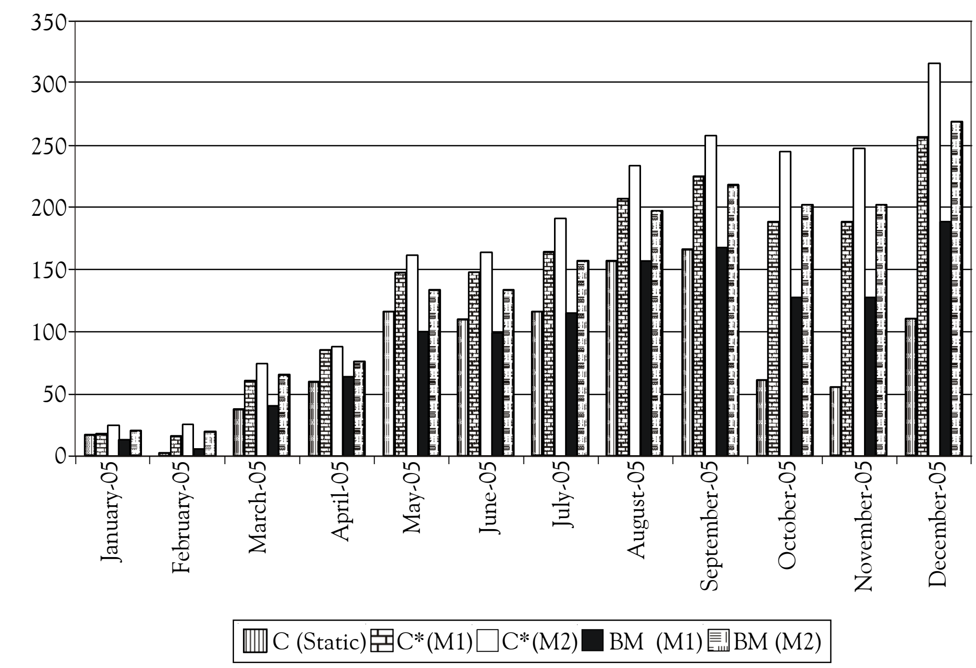


Figure 4: Cumulated P&L 2005 (Mio CHF)



Conclusion

We presented a dynamic ALM model, which is implementable in the financial industry. We first had to fill a major gap in the representation theorem of Li and Ng (2000) and Leippold *et al*., (2003 and 2004), which *de facto* conjectures the equivalence of the original multi-period non-separable mean-variance problem with its embedding into a higher dimensional separable problem. This conjecture is now proved.

Next, to allow for an implementation of real life balance sheets, we had to derive a generic representation for the cash flows of the different products. Using this representation, we derived closed form and semi-analytical solutions of the dynamic mean-variance problem with affine equality and inequality constraints. That for, some non-convex constraints were substituted by convex ones and all non-affine constraints needed to be linearized in order to restore affinity and state independence.

As a corollary, we then proved that for a stochastic independent term structure, the optimal allocation of the dynamic model and its one-period static one coincide. We compare the different models—dynamic, myopic and benchmark—in two applications. In the first one, the key rate delta profiles of the portfolios are compared with the public available UBS profile. It follows that the myopic model leads to non-acceptable key rates. The magnitudes are by far too large compared to the real rates. The dynamic rates are by contrast of the same order of magnitudes but they are 'opposite' to the published rates: To a positive published key rate delta, a negative optimal rate is opposed. In the second application, annual present values for the different models are calculated. It follows that the dynamic portfolio model leads to a more than 1% higher return than all other approaches. This demonstrates the superiority of dynamic models over myopic ones using real financial instruments in real financial market.❖

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Appendix 1 : Proofs

Proof of Theorem 1

Proof: With the state and control variables

$$x_t := PV_t(\mathbf{E}_t), v_t := \Phi_t(S_t) + \Phi_t(\mathbf{E}_t) - \Phi_t(BM_t)$$

the constraint set reads

$$R = \{v := (v_0, \ldots, v_{T-1}) \,|\, x_{s+1} := (v_s + \Phi_s(BM_s)).R_{s+1} + Z_{s+1} \ \ (t \le s \le T-1),$$
$$eq_s(x_s, v_s) = 0, ineq_s(x_s, v_s) \le 0, (0 \le s \le T-1)\}$$

The functionals *eq* and *ineq* define convex equalities or inequalities. Since the valuation $F_t(\mathbf{E}t) = x_t - PV_t(\tau(BM_{t-1}) = v_{t-1}.R_{t+1} + Z_{t+1}$ is a well-defined function of the state and control variables, the two optimization problems can be written as:

$$\max_{v \in R} \sum_{t=1}^{T} \beta_t \{\mathbb{E}_0[F_t(\mathbf{E}_t)] - wVar_0(F_t(\mathbf{E}_t)\} \quad (P_w)$$

$$\max_{v \in R} \sum_{t=1}^{T} \beta_t \{\lambda_t \mathbb{E}_0[F_t(\mathbf{E}_t)] - w\mathbb{E}_0(F_t^2(\mathbf{E}_t)]\} \quad (P_{\lambda,w})$$

We note that the restriction set R and the dynamics are the same for both problems. With the abbreviations:

$$a_t := a_t(v) := \mathbb{E}_0[F_t(\mathbf{E}_t)] \qquad b_t : b_t(v) := \mathbb{E}_0[F_t^2(\mathbf{E}_t)]$$
$$a := (a_1, \ldots, a_T) \qquad b := (b_1, \ldots, b_T),$$

(Contd...)

and the functions

$$\pi = \pi(a,b) := \sum_{t=1}^{T} \beta_t \{a_t - wb_t + wa_t^2\}$$

$$\pi_\lambda = \pi_\lambda(a,b) := \sum_{t=1}^{T} \beta_t \{\lambda_t a_t - wb_t\},$$

the optimization problems (P_w) and $(P_{\lambda,w})$ are written in a compact form as

$$\max_{v \in R} \pi(a(v), b(v)) \qquad (P_w)$$

$$\max_{v \in R} \pi_\lambda(a(v), b(v)) \qquad (P_{\lambda,w})$$

The Jacobian and Hessian matrices of π and π_λ are

$$\left[\frac{\partial \pi}{\partial(a,b)}\right] = [\beta_1(1+2wa_1),\ldots,\beta_T(1+2wa_T),-w\beta_1,\ldots,-w\beta_T]$$

$$\left[\frac{\partial \pi_\lambda}{\partial(a,b)}\right] = [\beta_1\lambda_1,\ldots,\beta_T\lambda_T,-w\beta_1,\ldots,-w\beta_T]$$

$$\left[\frac{\partial^2 \pi_\lambda}{\partial^2(a,b)}\right] = \text{diag}\,(2w\beta_1,\ldots,2w\beta_T,0,\ldots,0)$$

To prove the theorem, we follow the logic: Theorem 1 is proved if and only if the following statements A, B and C are proved:

$$(A): \lambda^* = 1 + 2wa^* \Rightarrow (v^* \text{ solves } (P_w) \Rightarrow v^* \text{ solves } (P_{\lambda^*,w}))$$

$$(B): \lambda^* = 1 + 2wa^* \Leftarrow (v^* \text{ solves } (P_{\lambda^*,w}) \Rightarrow v^* \text{ solves } (P_w))$$

$$(C): \lambda^* = 1 + 2wa^* \Rightarrow (v^* \text{ solves } (P_{\lambda^*,w}) \Rightarrow v^* \text{ solves } (P_w))$$

The steps (A) and (B) were already proved in Li and Ng (2000) and Leippold *et al.*, (2003), but there part (C) was lacking. Without loss of generality, we assume that valuations are done at the end of every period, i.e., $\mathcal{T}_\beta = \{1,\ldots, T\}$.

(Contd...)

We prove (A).

The Hessian of π is positive semi-definite and π is convex. Therefore, for each $(a^*, b^*) \in \mathbf{R}^{2T}$ we have:

$$\pi(a,b) \geq \pi(a^*,b^*) + \frac{\partial \pi}{\partial(a,b)}(a^*,b^*).(a-a^*,b-b^*)$$

which for any λ^* satisfying $\lambda^* = 1 + 2wa^*$ implies

$$\pi(a,b) - \pi(a^*,b^*) \geq \sum_{t=1}^{T} \beta_t \{(1+2wa_t^*)(a_t - a_t^*) - w(b_t - b_t^*)\}$$
$$= \pi_{\lambda^*}(a,b) - \pi_{\lambda^*}(a^*,b^*)$$

Assume that $v^* \in R$ is a solution of (P_w) and define $\lambda^* := 1 + 2wa(v^*)$. If v^* does not solve $(P_{\lambda^*,w})$, then a portfolio $v \in R$ exists, such that

$$\pi_{\lambda^*}(a,b) - \pi_{\lambda^*}(a^*,b^*) > 0$$

Hence,

$$\pi(a,b) > \pi(a^*,b^*)$$

follows which contradicts the assumption that v^* is a solution of (P_w). Therefore, v^* solves $(P_{\lambda^*,w})$. This proves (A).

We prove (B).

We proved in (A) that every solution $v^*(w)$ of (P_w) also defines a solution $v^*(\lambda; w)$ of $(P_{\lambda,w})$ for a specific choice $\lambda = \lambda^* \in \mathbf{R}^T$. Therefore, the problem (P_w)

$$\max_{v \in R} \pi(a(v), b(v))$$

reduces to an unrestricted optimization problem

$$\max_{\lambda \in \mathbf{R}^T} \pi(a(v^*(\lambda;w)), b(v^*(\lambda;w))),$$

in the variable $\lambda \in \mathbf{R}^T$. Assuming $v^*(\lambda^*;w) = v^*(w)$, for the value $\lambda = \lambda^*$

$$\left(\frac{\partial}{\partial \lambda}\pi(a(v^*(\lambda;w)), b(v^*(\lambda;w)))\right)\Bigg|_{\lambda=\lambda^*} = 0$$

(Contd...)

holds, or more explicitly

$$\left(\frac{\partial\pi}{\partial a}\frac{\partial a(v^*(\lambda;w))}{\partial\lambda}+\frac{\partial\pi}{\partial b}\frac{\partial b(v^*(\lambda;w))}{\partial\lambda}\right)\Big|_{\lambda=\lambda^*}=0 \qquad \text{...(7)}$$

In the same way, the restricted problem

$$\max_{v\in R}\pi_{\lambda^*}(a(v),b(v)),$$

with the solution $v^*(\lambda^*;\ w)$, reduces to the unrestricted problem:

$$\max_{\lambda\in\mathbf{R}^T}\pi_{\lambda^*}(a(v^*(\lambda;w)),b(v^*(\lambda;w))).$$

By definition of $v^*(\lambda;\ w)$, for $\lambda = \lambda^*$ the equality

$$\left(\frac{\partial}{\partial\lambda}\pi_{\lambda^*}(a(v^*(\lambda;w)),b(v^*(\lambda;w)))\right)\Big|_{\lambda=\lambda^*}=0,$$

follows, or more explicitly

$$\left(\frac{\partial\pi_{\lambda^*}}{\partial a}\frac{\partial a(v^*(\lambda;w))}{\partial\lambda}+\frac{\partial\pi_{\lambda}}{\partial b}\frac{\partial b(v^*(\lambda;w)}{\partial\lambda}\right)\Big|_{\lambda=\lambda^*}=0 \qquad \text{...(8)}$$

Equations 7 and 8 can be written in the form

$$\begin{cases} Ly_1=0 \\ Ly_2=0, \end{cases}$$

with

$$y_1:=\left[\frac{\partial\pi}{\partial a},\frac{\partial\pi}{\partial b}\right]^{\dagger}\Big|_{\lambda=\lambda^*}\in\mathbf{R}^{2T}$$

$$y_2:=\left[\frac{\partial\pi_{\lambda^*}}{\partial a},\frac{\partial\pi_{\lambda^*}}{\partial b}\right]^{\dagger}\Big|_{\lambda=\lambda^*}\in\mathbf{R}^{2T}$$

$$L:=\left[\left(\frac{\partial a}{\partial\lambda}\right)^{\dagger},\left(\frac{\partial b}{\partial\lambda}\right)^{\dagger}\right]\Big|_{\lambda=\lambda^*}:\mathbf{R}^{2T}\to\mathbf{R}^T$$

(Contd...)

The vectors y_1 and y_2 are elements of the kernel of the linear map L. Since $dim\ ker(L) + dim\ Im(L) = dim\ \mathbf{R}^{2T} = 2T$ and $dim\ Im(L) = T$, y_1 and y_2 are elements of a T–dimensional subspace of $\boldsymbol{R}^{2T}$. Since the last T components of the vectors coincide, the two vectors must be identical. We have in particular:

$$\left.\frac{\partial \pi}{\partial a}\right|_{\lambda=\lambda^*} = \left.\frac{\partial \pi_{\lambda^*}}{\partial a}\right|_{\lambda=\lambda^*}$$

But this is equivalent to

$$1 + 2wa(v^*(\lambda^*; w)) = \lambda^*$$

We proved (B).

We prove (C).

If the solution $v^*(\lambda^*;\ w)$ of

$$\max_{v \in R} \pi_{\lambda^*}(a(v), b(v))$$

solves

$$1 + 2wa(v^*(\lambda^*; w)) = \lambda^*$$

then we have

$$\frac{\partial \pi}{\partial a}\Big|_{\lambda=\lambda^*} = \frac{\partial \pi_{\lambda^*}}{\partial a}\Big|_{\lambda=\lambda^*}.$$

This implies

$$\begin{aligned}
&\left(\frac{\partial}{\partial \lambda}\pi(a(v^*(\lambda;w)), b(v^*(\lambda;w)))\right)\Big|_{\lambda=\lambda^*} \\
&= \left(\frac{\partial \pi}{\partial a}\frac{\partial a(v^*(\lambda;w))}{\partial \lambda} + \frac{\partial \pi}{\partial b}\frac{\partial b(v^*(\lambda,w))}{\partial \lambda}\right)\Big|_{\lambda=\lambda^*} \\
&= \left(\frac{\partial \pi_{\lambda^*}}{\partial a}\frac{\partial a(v^*(\lambda;w))}{\partial \lambda} + \frac{\partial \pi_{\lambda^*}}{\partial b}\frac{\partial b(v^*(\lambda,w))}{\partial \lambda}\right)\Big|_{\lambda=\lambda^*} \\
&= \left(\frac{\partial}{\partial \lambda}\pi_{\lambda^*}(a(v^*(\lambda;w)), b(v^*(\lambda;w)))\right)\Big|_{\lambda=\lambda^*} \\
&= 0
\end{aligned}$$

(Contd...)

since $\lambda = \lambda^*$ is a solution of

$$\max_{\lambda \in \mathbf{R}^T} \pi_{\lambda^*}(a(v^*(\lambda; w)), b(v^*(\lambda; w)))$$

This shows that $\lambda = \lambda^*$ is a stationary point of the function

$$\pi(a(v^*(\lambda; w)), b(v^*(\lambda; w))),$$

in $\lambda \in \mathbf{R}^T$. Therefore, $\lambda = \lambda^*$ solves

$$\max_{\lambda \in \mathbf{R}^T} \pi(a(v^*(\lambda; w)), b(v^*(\lambda; w)))$$

which is equivalent to

$$\max_{v \in R} \pi(a(v), b(v))$$

which is a concave problem. Thus, the stationary point $\lambda = \lambda^*$ is the unique maximizer. In other words, $v^*(\lambda^*; w) = v^*(w)$. This proves (C).

Proof of Theorem 2

Proof: Existence and uniqueness of an optimal control rule v^* are standard results of stochastic control theory (Minoux, 1986). The valuation at time $t+1$ reads:

$$F_{t+1}(\boldsymbol{E}_{t+1}) = v_t^{\dagger} R_{t+1} + Z_{t+1}$$

with $R_{t+1} = V_{t+1}(\tau(B_t))$, $Z_{t+1} = V_{t+1}(N_{t+1})$ the valuation of the time shifted base asset and new business. The following calculation shows that the value process satisfies the Bellman equation:

$$J_t(x) = \max_{\substack{v_s \in R_s \\ s=t,\ldots,T-1 \\ D_t(x)}} \mathbb{E}_t\left[\sum_{s=t}^{T-1} \beta_s u_{\lambda,w}(v_s.R_{s+1} + Z_{s+1})\right]$$

$$= \max_{\substack{v_t \in R_t \\ x_{t+1} = v_t.R_{t+1} + Z_{t+1}}} \max_{\substack{v_s \in R_s \\ s=t+1,\ldots,T \\ D_{t+1}(x_{t+1})}} \mathbb{E}_t[\mathbb{E}_{t+1}[\beta_{t+1} u_{\lambda,w}(v_t.R_{t+1} + Z_{t+1})) + \sum_{s=t+1}^{T-1} \beta_s u_{\lambda,w}(v_s.R_{s+1} + Z_{s+1})]]$$

$$= \max_{\substack{v_t \in R_t \\ x_{t+1} = v_t.R_{t+1} + Z_{t+1}}} \mathbb{E}_t\left[\beta_{t+1} u_{\lambda,w}(v_t.R_{t+1} + Z_{t+1}) + \max_{\substack{v_s \in R_s \\ s=t+1,\ldots,T \\ D_{t+1}(x_{t+1})}} \mathbb{E}_{t+1}\left[\sum_{s=t+1}^{T-1} \beta_s u_{\lambda,w}(v_s.R_{s+1} + Z_{s+1})\right]\right]$$

(Contd...)

$$= \max_{\substack{v_t \in R_t \\ x_{t+1}=v_t.R_{t+1}+Z_{t+1}}} \mathbb{E}_t[\beta_{t+1} u_{\lambda,w}(v_t.R_{t+1} + Z_{t+1})) + J_{t+1}(v_t.R_{t+1} + Z_{t+1})]$$

$$= \max_{v_t \in R_t} \mathbb{E}_t[\beta_{t+1} u_{\lambda,w}(v_t.R_{t+1} + Z_{t+1}) + J_{t+1}(v_t.R_{t+1} + Z_{t+1})]$$

The commutation of maximization with conditional expectation is a non-trivial step justified in Chapter 9.2. of (Minoux, 1986).

Proof of Theorem 3

Proof: The key to the solution of $(P_{\lambda,w})$ is Bellman's equation of Theorem 2, which can be simplified due to the particular choice of the constraints:

$$R_t = \{PV_t(S_t) = 0\}$$

The definition of the benchmark implies that the present value of the economic capital and of the benchmark are the same, i.e., $PV_t(\mathbf{E}_t) = PV_t(BM_t)$. Therefore, the constraints read

$$R_t = \{e_t^{\dagger} v_t = 0\}$$

with $e_t := PV_t(B_t)$ the present value of the base assets. The Bellman equation for time $t = T-1$ leads to the one-period problem

$$J_{T-1}(x) = \max_{v_{T-1} \in R_{T-1}} \mathbb{E}_{T-1}[\beta_T u_{\lambda,w}(v_{T-1}^{\dagger} R_T + Z_T)]$$

whose solution readily follows:

$$v_{T-1}^* = \frac{1}{2w} \mathbb{E}_{T-1}[R_T R_T^{\dagger}]^{-1} \Bigg\{ (\lambda_{t+1} \mathbb{E}_{T-1}[R_T] - 2w \mathbb{E}_{T-1}[Z_T R_T])$$

$$- \frac{e_{T-1}^{\dagger} \mathbb{E}_{T-1}[R_T R_T^{\dagger}]^{-1} (\lambda \mathbb{E}_{T-1}[R_T] - 2w \mathbb{E}_{T-1}[Z_T R_T])}{e_{T-1}^{\dagger} \mathbb{E}_{T-1}[R_T R_T^{\dagger}]^{-1} e_{T-1}} e_{T-1} \Bigg\}$$

Using an induction argument, the formula for the optimal policy (5) follows. The parameters $\lambda = (\lambda_1, \ldots, \lambda_T)$ are finally fixed in such a way that $\Phi_t^*(S_t)$ solves problem (P_w). That for, the parameters have to satisfy

$$\lambda_t = 1 + 2wa_t^*$$

for all $t \in \mathcal{T}_\beta$ where

(Contd...)

$$a_t^* = \mathbb{E}_0[F_t(\mathbf{E}_t^*)] = \mathbb{E}_0[v_{t-1}^*.R_t + Z_t] = \frac{\lambda_t}{2w}\mathbb{E}_0[\xi_{t-1}^\dagger R_t] + \mathbb{E}_0[\eta_{t-1}^\dagger R_t + Z_t]$$

which implies for all $t \in \mathcal{T}_\beta$

$$\lambda_t = \frac{1 + 2w\mathbb{E}_0[\eta_{t-1}^\dagger R_t + Z_t]}{1 - \mathbb{E}_0[\xi_{t-1}^\dagger R_t]}$$

Proof of Theorem 5

Proof: The objective function and static mean-variance optimization problem for the time interval $[t, t + 1]$ read

$$I_t(x) := \mathbb{E}_t[F_t(\mathbf{E}_t)] - w\ \mathrm{Var}_t(F_t(\mathbf{E}_t))) \text{ and respectively } \max_{v_t \in R_t} I_t$$

The weighted objective functions for the sequence of static problems can be simplified due to the assumed stochastic independence:

$$\begin{aligned}\sum_{t=1}^{T}\beta_t I_t &= \sum_{t=1}^{T}\beta_t\{\mathbb{E}_{t-1}[F_t(\mathbf{E}_t)] - w\ \mathrm{Var}_{t-1}(F_t(\mathbf{E}_t))\} \\ &= \sum_{t=1}^{T}\beta_t\{\mathbb{E}_{t-1}[F_t(\mathbf{E}_t)] - w\mathbb{E}_{t-1}[F_t^2(\mathbf{E}_t)] + w\mathbb{E}_{t-1}[F_t(\mathbf{E}_t)]^2\} \\ &= \sum_{t=1}^{T}\beta_t\{\mathbb{E}_0[F_t(\mathbf{E}_t)] - w\mathbb{E}_0[F_t^2(\mathbf{E}_t)] + w\mathbb{E}_0[F_t(\mathbf{E}_t)]^2\}\end{aligned}$$

The final expression is identical to $J_0(x_0)$, if the sequence of optimal static solutions is inserted. But $J_0(x_0)$ is the optimal value of the objective function for the dynamic mean-variance optimization problem. By Bellmann's Theorem 2, the two solutions must be identical.

Proof of Theorem 6

Proof: If the zero spot rates at time $t+1$ are expected at time t to be equal to the corresponding forward rates, then the expected total return at time t for the period $[t, t+1]$ of any fixed income portfolio is equal to the discrete zero spot rate at time t for term to maturity $(t+1) - t = 1$. Therefore, the total returns of all portfolios are equal and the conditional expectation of the excess total return of any portfolio over any benchmark vanishes. In particular,

(Contd...)

Appendix 1: Proofs

(...contd)

$$\mathbb{E}_{t-1}[PV_t(\tau(\mathbf{E}_{t-1} + S_{t_1} - BM_{t-1}))] = 0$$

If we assume that the valuation of new business gives no contribution, that is, if

$$\mathbb{E}_{t-1}[PV_t(N_t)] = 0$$

then, the dynamic mean-variance objective function reads as

$$-w\sum_{t=1}^{T}\beta_t \; Var_0(PV_t(\mathbf{E}_t - BM_{t-1}))$$

Bellman's equation leads to T recursive optimization problems (Theorem 2):

$$\min_{v_t \in R_t} Var_t(v_{t-1}.R_t) \qquad (t = 0,\ldots,T-1)$$

This is the minimum of a positive definite quadratic form, which is attained for a unique feasible $v_t^* \in R_t \subset \{PV_t(S_t) = 0\}$, if the restriction set R_t is convex. In particular, if $R_t = \{PV_t(S_t) = 0\}$, then $v_t^* = 0$, i.e., $\Phi_t(\mathbf{E}_t^*) = \Phi_t(BM_t)$ for all t.

Appendix 2: Sensitivities

Definition 1: *Let at time t be given an interest rate term structure R = R(t, τ) (where τ ≥ 0 denotes the term to maturity) be given. The stochastic present value of an asset b ∈ B is defined by:*

$$PV_t(b) := PV_t(b,R) := \sum_{s \geq t} \mathbb{E}_t^{*,s}[b^s]e^{-R(t,s-t)(s-t)}$$

Thereby, the symbol $\mathbb{E}_t^{*,s}[.]$ denotes the s-forward risk neutral conditional expectation at time t (Björk, 2004; Chapter 24.4). We next describe possible perturbations of the term structure. $Let\{\delta R^{\tau_1}(t,\tau),\ldots,\delta R^{\tau_n}(t,\tau)\}$ be a partition of unity which describes the perturbation of the term structure $R(t, \tau)$ at the key rates $\{\tau_1,\ldots,\tau_n\}$ $=: \mathcal{T}$. (This time grid has typically not an uniform spacing, since relative more weight is given to key rates closer to the present than to those which are further distant in the future). In other words, $R = R(t, T)$ is mapped into

$$R_\varepsilon := R_\varepsilon(t,\tau) := R(t,\tau) + \varepsilon.\delta R(t,\tau)$$

(Contd...)

where $\varepsilon = (\varepsilon_1,...,\varepsilon_n)$. and $\delta R(t,\tau) := (\delta R^{\tau_1}(t,\tau),....,\delta R^{\tau_1}(t,\tau))$. Then, the present value is approximated as a function of ε by a second order Taylor polynomial for $\varepsilon \to 0$. This gives us

$$PV_t(b,R_\varepsilon) = PV_t(b,R) - \sum_{j=1}^{n} \varepsilon_j KR\delta_{t,\tau_j}(b,R)$$
$$+\frac{1}{2}\sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j KR\gamma_{t,\tau_i,\tau_j}(b,R) + O(|\varepsilon|^3)$$

with the terminology:

Definition 2: *Let $b \in B$ be an asset, $R = R(t, \tau)$ a term structure and $\{\delta R^{\tau_1}(t,\tau),...,\delta R^{\tau_n}(t,\tau)\}$, the above partition of unity. The key rate deltas and gammas are then defined by:*

$$KR\delta_{t,\tau}(b,R) := \sum_{s \geq t}(s-t)\mathbb{E}_t^{*,s}[b^s]e^{-R(t,s-t)(s-t)}\delta R^{\tau}(t,s-t)$$

$$KR\gamma_{t,\tau,\sigma}(b,R) := \sum_{s \geq t}(s-t)^2\,\mathbb{E}_t^{*,s}[b^s]e^{-R(t,s-t)(s-t)}\delta R^{\tau}(t,s-t)\delta R^{\sigma}(t,s-t)$$

as well as the effective delta and gamma:

$$\delta_t(b,R) := \sum_{\tau \in T} KR\delta_{t,\tau}(b,R)$$
$$\gamma_t(b,R) := \sum_{\tau,\sigma \in \mathcal{T}} KR_{\gamma_{t,\tau,\sigma}}(b,R)$$

To map an instantaneous parallel shift of the term structure one sets $\varepsilon = (\varepsilon_1,...,\varepsilon_n) = (\in,..., \in)$ and gets for $\in \to 0$:

$$PV_t(b,R_\varepsilon = PV_t(b,R) - \in \delta_t(b,R) + \frac{1}{2}\in^2 \gamma_t(b,R) + O(|\in|^3)$$

It is customary for a treasurer to express the interest rate sensitivities as monetary quantities, that is, as instantaneous present value variations generated by an instantaneous term structure shock of 1 basis point *(bp* := 0.0001) for the key rates considered. Therefore, first order interest rate sensitivities are obtained by multiplying key rate deltas by *bp* and, analogously, second order interest rate sensitivities are the product of key rate gammas by bp^2. Aggregation of sensitivities for a time bucket means summation over the key rates in the bucket considered.

Reference # 37J-2008-09-04-01