Empirical Analysis of Credit Risk Regime Switching and Temporal Conditional Default Correlation in Credit Default Swap Valuation: The Market Liquidity Effect

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This paper extends the debate concerning Credit Default Swap (CDS) valuation to include time-varying correlation and covariances. Traditional multivariate techniques treat the correlations between covariates as constant over time; however, this view is not supported by the data. Since financial data does not follow a normal distribution because of its heavy tails, modeling the data using a Generalized Linear Model (GLM) incorporating copulas emerge as a more robust technique over traditional approaches. This paper also includes an empirical analysis of the regime switching dynamics of credit risk in the presence of liquidity, following the general practice of assuming that credit and market risk follow a Markov process. The study is based on CDS data obtained from Bloomberg that spans over the period from January 1, 2004 to August 8, 2006. The empirical examination of the regime switching tendencies provides quantitative support to the anecdotal view that liquidity decreases as credit quality deteriorates. The analysis also examines the joint probability distribution of the credit risk determinants across credit quality, through the use of a copula function which disaggregates the behavior embedded in the marginal gamma distributions, so as to isolate the level of dependence captured in the copula function. The results suggest that the time-varying joint correlation matrix performs far superior, compared to the constant correlation matrix; the centerpiece of linear regression models.

Introduction

This paper emphasizes on the examination of multivariate outcomes of various marginal distributions interacting simultaneously with each other in the risk-neutral correlated hazard process of the study's referenced entities. Initial observation of the study's credit and market risk data tend to suggest that dependencies vary over time, and are not constant. According to Sun *et al.* (2006) a copula is a multivariate distribution with uniform marginal distributions in the interval [0, 1]. Copulas allow for the construction of time-varying joint conditional distributions (Patton, 2006a and 2006b), which permits the evolution and subsequent evaluation of conditional correlation between financial asset's credit quality and their explanatory variables. Incorporation of time variation in the joint conditional default correlation may be achieved through time-varying

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conditional marginal distributions[1] (transitional matrices) and through variations in the evolution of the copula parameters over time.

This paper models the copula dynamics of the transition between credit risk, market risk and market liquidity conditions, which will enable us to derive the implied future (n-step ahead) Credit Default Swap (CDS) premia distribution from the historical CDS prices. The study's main innovations are: (a) exploring and introducing a copula based valuation approach for pricing credit risk which includes a liquidity measure for the CDS premium, and (b) examining and using the joint credit risk and liquidity regime switching dynamics of credit risk premia to explain investors' behavior in investing in credit risky products. The paper develops and implements a multifactor Markov chain model, using time-varying historical transition matrices and a set of latent credit risk explanatory variables. Unlike earlier studies that focused solely on using copulas to evaluate correlated default processes, this paper shows how to calibrate future CDS prices using copula theory that includes a financial liquidity proxy. Further, the paper looks at credit risk and liquidity migration to determine whether higher credit and liquidity risk exhibits higher levels of volatility.

Next, the paper reviews the literature concerning the use of copulas in financial applications. Then, it introduces the theoretical foundations of the models and discusses the Markov switching model and the transition parameters. It also discusses the application of copula theory to credit risk analysis, parameterization of the study's t-copula and the n-step ahead calibration procedure. Next, it gives a brief description of the CDS data and the various explanatory variables. Then, it presents the main empirical findings regarding the copula model and the regime switching analysis, finally it summarizes the findings and proposes areas for future research.

Literature Review

Quite recently there has been increased interest in the use of copula analytics in credit risk analyses (Li, 2000; Das and Geng, 2004; Burtschell *et al.,* 2005; Luciano, 2005; and Bandreddi *et al.,* 2006). Copulas are tools for understanding relationships among multivariate outcomes. It is a function that links univariate marginals to their full multivariate distribution. Copula functions, which were introduced to probability theory in 1959 (Sklar, 1959), measure correlation dependencies, or association, embedded in the underlying marginal distributions.

The normal distribution assumption has long dominated the study of multivariate distributions. Frees and Valdez (1998) suggest that the normal distribution assumption was ideal in probability analysis because: (a) their underlying marginal distributions are also assumed to be normal; and (b) dependencies between random variables can be fully described by the model's implied correlation coefficient and underlying marginal distributions. The assumption of normality is widespread in the finance literature,

[1] The study uses two matrix structures in the fitted t-copula. Both a uniform correlation (EX) and a time-varying (AR–1) structure is used to test the changes in the covariates over the sample period.

as evidenced in a number of asset pricing theories that assume normally distributed asset returns. However, Sun *et al.* (2006) states that the distributions of financial asset prices exhibit heavy tails, which calls into question the assumption of a normal distribution assumed in most pricing theories.

Copulas have been used extensively in survival analysis and actuarial analytics. Since the assumption of marginal normality is not consistent with observed financial data, given the data's asymmetric properties, a number of recent studies have attempted to use copulas to model dependent risk in *n*-variate credit risk distributions (Embrechts *et al.*, 1999; Das *et al.*, 2006; and Sun *et al.*, 2006). The copula method for understanding multivariate distributions has a relatively short history in credit risk analysis, with most of the credit risk applications arising during the last five years. Further, Das *et al.* (2006) suggests that an important feature of copulas is its ability to permit varying degrees of tail dependence (i.e., the extent to which the correlation between random variables arises from extreme observations).

One of the earlier researchers to introduce credibility theory to financial data was Li (2000), who in using copula functions to determine joint probabilities, introduced the use of copula functions to Collateralized Debt Obligations (CDO) credit risk valuation analysis. Sun *et al.* (2006) used a multivariate model, based on a copula function to address the drawback of the normality assumption in financial asset pricing. Hull and White (2006) used a one-factor copula model to derive implied CDO quotes. Das *et al.* (2006) used copula analysis to evaluate default risk at the portfolio level (multivariate distribution) and more importantly evaluated default dependencies among issuers using a copula function to separate the marginal behavior from the correlated dependencies.

The choice of copula function to be utilized in any study is critical, given the number of copulas available to researchers. Kole *et al.* (2006) found some issues with the Gaussian copula, in that it did not capture the dependence among extreme events, which was also shown in Embrechts *et al.* (1999). In general, the choice of copula function depends on the statistical fit to the data. Estimation can be accomplished using either a parametric or semi-parametric approach.

The application of copula functions to credit markets has increased in recent years. Abid and Naifar (2005) applied copulas in their analysis of the impact of equity market volatility on CDS rates. They found that the dependence structure between CDS rates and stock return volatility was asymmetric with positive skew, and displayed right tail dependence, best modeled using a Gumbel copula. Further, they found that high rated entities (AAA) have a weaker dependence on equity volatility compared to low rated entities. Frey *et al.* (2001) analyzed the use of copulas in modeling credit portfolio losses. They show that the copula of the latent variables determine higher order joint default probabilities for the group of obligors, illustrating the extreme risk of multiple defaults present in the referenced portfolio. This illustrated that traditional, linear correlation is not adequate when seeking to describe the dependence between defaults in a portfolio.

Cherubini and Luciano (2002) used copulas to evaluate default puts and credit switch contracts. They found that by using copulas to represent the dependence structure, one can accurately devise the so-called super-hedging strategies for counterparty risk. They also found that the choice of copula function can have a significant impact on the resulting evaluation of counterparty risk; overvaluation in some cases, undervaluation in others. Cherubini (2004) later applied copulas in evaluating counterparty risk in swap transactions. He found that dependence affects both the level and slope of credit spreads, particularly for institutions paying fixed premium. Das and Geng (2004) used copula functions to model, simulate and assess the joint default process of their referenced set of issuers. They found that it is imperative to capture the interdependence of marginal distributions and copula to achieve the best joint distribution depicting default.

Mashal and Naldi (2002) examined the estimation, simulation and pricing of multi-name contingent instruments. Multi-name instruments have payoffs that are contingent upon the default realization in a portfolio of names. They found that some copulas allow for an accurate estimation of the tail-dependencies for joint extreme events. In particular, their findings suggested that the *t*-copula has non-trivial tail dependence and therefore allows for more extreme joint events. Mashal and Zeevi (2002) found that the fit of the *t*-copula is generally superior to that of the Gaussian copula due to the ability of the *t*-copula to better capture extreme values often observed in financial data. They found that the *t*-copula captures extreme comovements regardless of the marginal behavior of the individual assets.

The choice of the *t*-copula for the analysis was fairly straightforward for two main reasons: (a) Das and Geng (2004) found by fitting a number of copulas (Frank, Gumbel, Clayton and the *t*-copula) to CDS data that the t-copula had the best fit; and (b) the analysis of the study showed that the data exhibits symmetric tail dependencies, which is best suited for a *t*-copula. The fitting of the *t*-copula with the symmetric tail behavior makes it possible to test whether times of increased dependency can also be characterized by changes in both tails of the distribution. However according to Rodriguez (2007), in order to capture these dependence structures, the copula that describes it must be time varying. Following the pioneering work by Patton (2006a and 2006b) in time-varying copula structures, we introduce a AR–1 matrix structure to the copula model and a uniform dependence matrix structure to test if a time-varying approach best predicts future CDS premia.

Schönbucher and Schubert (2001) provided insights into the connection between default dependencies and the joint dynamics of default intensities which are implicitly specified by specifying the default dependency. They illustrate the use of copula functions to specify the dependency structure between the individual obligors and the portfolio of obligors, without affecting the calibration of the other obligors or the dependency structure.

Model Specification

This section lays out the general model structure and the components of the model used in the study. The methodology used abstracts from actuarial credibility theory, to estimate credit risk transition matrices in a multivariate Markov framework for valuing credit risk. Model 1 evaluates the regime switching transitions of credit risk, market liquidity conditions as represented by the bid-ask asymmetries and market risk. A Markov chain model is fitted to observed CDS prices, market risk (spot rate) and the market liquidity proxy to determine the transitional matrices needed for the regime switching analysis.

Model 2 presents the multivariate *t*-copula function that is implemented as a part of the Generalized Linear Model (GLM)[2] framework to evaluate the joint conditional default probabilities. As discussed later, using a *t*-copula function, we separate the estimation of the marginal distribution from the estimation of the joint distribution. A gamma marginal distribution is considered for each issuer's hazard rates, combined with a *t*-copula. The *t*-copula is fitted with two different correlation structures to capture the differences in the dependency structure of the CDS premia. In addition, the model is extended from the traditional literature to include a liquidity measure.

We begin by establishing the complete probability space which is represented by a filtered probability space (Ω, F, P), where Ω is the state space of all possible credit states, F is the σ-algebra representing measurable events, and P is the empirical probability measure. Information evolves over the trading interval according to the augmented right-continuous complete filtration $\{F_t : t \in [0, \tau]\}$. Let $E(\bullet)$ denote expectation with respect to the probability measure P. Further, let τ represent the time period $(0, 1, 2, 3, \ldots, \infty)$ of all economic activities, for a set of n correlated corporate credit risk of varying credit qualities.[3] Table 2 shows that credit risk has a strong positive correlation across credit qualities.

Model 1 – Markov Switching Model

A number of studies suggest that market and credit risk are positively correlated, implying that as market risk increases, the likelihood of default of the firm also increases (Jarrow and Yildirim 2001; and Dunbar, 2007). Similarly, as illustrated by Dunbar (2007), market liquidity plays a unique role in enhancing credit markets. High credit risk products[4] tend to be less liquid than low credit risk assets because of investors risk appetite or levels of risk aversion. Both these phenomena should generate differing regimes (states) of credit risk, thereby differing extremes in the hazard functions of the underlying referenced entities. This external bifurcation of credit risk into two regimes assigns

[2] The general linear model can be seen as an extension or generalization of linear multiple regression (OLS). Generalized Linear Models (GLMs) are used for regression modeling of non-normal data such as financial and actuarial data. GLM assumes a parametric distribution family for the dependent variable but then allows the mean parameter to be a function of the covariates.

[3] The credit default swap qualities vary across the spectrum of referenced entities, these ranges from AAA to C on the S&P rating scale.

[4] Assets of poor credit quality such as the ones with a S&P credit rating of B or C.

state 1 to investment grade securities (S&P AAA, AA, A, BBB) and state 2 to high yield securities with a (Standard and Poor's) S&P rating of BB and B.

To determine credit risk regime switching, a Markov regime switching framework with time-varying transition probabilities is specified, like that of Das *et al.* (2006), where a regime switching model was used to capture the regime switching effects of high and low probability of default in credit risk products. Markov switching models exhibit a number of useful statistical properties such as their treatment of non-linearities, non-stationary distributions, serial correlation and the fat-tailed distributions of volatile equity prices.

The Markov switching model of the study was used to capture the probability effects of liquidity changes along the lines of credit quality. The methodology assumes that the distribution of market CDS prices is governed by a two-state first-order Markov switching process. Each state is characterized by high or low variances and means that correspond to either regimes. As stated above, state 2 is characterized by high market risk and low market liquidity, this obligor class includes referenced entities with credit quality BB and B on the S&P credit scale. Economic theory suggests that credit risk and market risk are positively correlated; hence high market risk implies high credit risk. Similarly, credit risk is inversely related to market liquidity (Dunbar, 2007), so as credit risk increases (deterioration of credit quality), market liquidity disappears. State 1 is the opposite of state 2, with low market risk and high levels of market liquidity.

The probability that CDS prices and volatility is either in state 1 or 2 at time t is a function of the earlier state in t–1. Regime (state) 1 of the Markov model indicates a situation of high variance and high mean, whereas regime (state) 2 implies low variance and low mean. This analysis should supplement the financial literature on the behavior of rational investors in financial markets. Rational investors are presumed to invest in highly volatile assets only if the risk premium of the volatile asset is high enough to compensate for the investors' investment risk. Given the preceding discussion, the historical distribution of observed CDS market prices is assumed to be a collection of both states 1 and 2, where the collection is determined by a probabilistic transition between the two states.

The hazard rate process across all referenced entities are assumed to follow a diffusion volatility model which can be represented as follows:

$$\lambda_{r_t^i, l_t^i} = \kappa^i_{r,l}\left[\theta^i_{r,l} - \lambda^i_{r,l}\right]dt + \sigma^i_{r,l}\sqrt{\lambda^i_{r,l}(t)}\,d(t)\in(t), \qquad r, l = \{Hi, Lo\} \qquad \text{...(1)}$$

where, $\theta^i_{r,l}, \kappa^i_{r,l} \neq 0$, and $\sigma^i_{r,l} > 0$ are positive constants. $\kappa^i_{r,l}$ is the rate of mean reversion, $\theta^i_{r,l}$ represents the mean premia returns to investors, and $\sigma^i_{r,l}$ is the volatility parameter. In determining regimes from the sample data, the average hazard rates across all issuers are first computed.

As discussed earlier, the regimes or states are indexed by r_t (the spot interest rate process) and l_t (the market liquidity process), which is either a high or low representing the two levels of economic conditions. The process r_t gives knowledge of the macro economy, implying information of future macroeconomic conditions. Whereas l_t on the other hand, describes the evolution of the level of liquidity available to the various credit qualities in the market at given points in time. A logistic model is then used to generate a transition matrix which is then used to determine the probability of switching between regimes.

The transition matrix is as follows:

$$\begin{pmatrix} p_{kij}(r,l)_{Lo} & 1-p_{kij}(r,l)_{Lo} \\ 1-p_{kij}(r,l)_{Hi} & p_{kij}(r,l)_{Hi} \end{pmatrix} \qquad ...(2)$$

where $p_{kij}(r,l)=\dfrac{e^{(\alpha_{r,l})}}{1+e^{(\alpha_{r,l})}}$, r, $l \in \{Lo, Hi\}$

p_{ij} denotes the probability of state i transitioning to state j after k steps, and where $p_{kij} \geq 0 \, \forall i$ and j.[5] Alternatively, the n-step transition shown in Equation 3 satisfies the Chapman-Kolmogorov equations for any $0 < k < n$.

$$P_{ij}^{n}(r,l)=P\sum_{r\in S} p_{ir}^{k} p_{rj}^{(n-k)} \qquad ...(3)$$

All parameters are estimated using maximum-likelihood, and the transition probabilities defined in Equation 2 are the observed conditional risk-neutral transition probabilities for the process. This regime switching model is then fitted across all τ-risk classes using the stochastic process outlined in the diffusion process of Equation 1. Equation 4 presents the regime shifting model for the risk classes:

$$\lambda_{n,r_t^i,l_t^i} = \kappa_{n,r,l}^{i}\left[\theta_{n,r,l}^{i} - \lambda_{n,r,l}^{i}(t)\right]dt + \sigma_{n,r,l}^{i}\sqrt{\lambda_{n,r,l}^{i}(t)}d(t) \in (n,t), \; r,l = \{Hi, Lo\} \qquad ...(4)$$

where $n=\{1,...,6\}$

Model 2 – Copula Dynamics Under the Generalized Linear Model[6]

Here, we show how copulas can be applied to incorporate the dependence structure of credit risk. Before fitting the copula model, a Hierarchal Linear Model (HLM) is first fitted to examine the inter-credit class and intra-credit class dependencies.

[5] $\sum_{j=1}^{\tau} p_{ij} = 1 \forall i$

[6] For a discussion on GLM *see* Sun *et al.* (2006).

The HLM framework makes use of a nested structure that allows effects to vary from one context to another. In hierarchical data, entities in the same group (credit class) are likely to be more similar than entities in different groups. Due to this, variations in outcome may be due to differences between groups, rather than individual differences within a group. Thus, variance component models, where disturbance may have both a group and an individual component, can be of help in analyzing data of this nature. Within these models, individual components are independent, but they are perfectly correlated within the groups.

Model 2 looks at the normality assumption that is widely used in the analysis of financial data. As illustrated in Table 4, credit risk data exhibit heavy tails, suggesting that extreme values in the data are more likely to occur than in normally distributed data. Further, the preceding discussion suggests that the use of copulas in finance allows researchers to model the effects of the underlying marginal distribution separately from their dependency structures, i.e., the joint probability distributions.

The dependency structure of the CDS premia across the sample period for several distinct entities can be determined. So, from our proposed dataset of τ-risk classes of referenced entities credit risk, we can determine each issuer's marginal historical distribution, and using a t-copula[7], we can construct the joint distribution with a desired correlation structure. Given the implied marginal distribution of the issuers, the time-varying joint conditional probabilities can be computed by choosing copulas with time-varying parameter values.

In finance, pricing of a risky asset involves the determination of the asset's joint risk-neutral default probabilities and the historical marginal distribution of the referenced entities market price of credit risk. Under the study's risk neutral measure, the historical price of credit risk, η_t linking both the risk-neutral probability distribution and the historical distribution Y of x_i, depends only on the explanatory variables r_t and l_t. In setting up the obligor's joint risk-neutral probability distribution functions we begin by specifying τ potential outcomes, x_1, x_2,…, x_τ for the i^{th} obligor class[8] of credit risk. This joint probability distribution function can be described as follows:

$$C_i\left(x_1, x_2, \ldots, x_\rho\right) = \Pr\left(X_1 \le x_1,\ X_2 \le x_2, \ldots, X_\rho \le x_\rho\right) \quad \ldots(5)$$

where:

(a) The function C is a copula.

(b) X is a uniform random variable, $X \in R^n = \{X_1, \ldots, X_n\}$

(c) x is the observed corresponding historical distribution.

7 Sklar (1959) suggests that every continuous joint distribution can be represented as a unique copula and the marginal distributions, which makes this technique very useful in finance, in general and multivariate credit risk analysis, in particular.

8 Obligor class refers to the S&P's credit risk ratings AAA, AA, A, BBB, BB and B, where AAA is most secure and B is least secure.

From Equation 5 the copula probability distribution function[9] can be derived in a straightforward manner from the probability density function through integration and can be represented as follows:

$$C\left[F_1\left(x_1\right), F_2\left(x_2\right),..., F_\tau(x_\tau)\right] = \Pr\left(X_1 \le x_1, X_2 \le x_2,..., X_\tau \le x_\tau\right) = F\left(x_1, x_2,..., x_\tau, \lambda_\tau^\theta\right) \quad ...(6)$$

where,

(a) Risk class $n = 1,..., \tau$

(b) $\lambda_t^Q = y_i \beta$

(c) y_i is a $K \times 1$ vector of known explanatory variables (short rate and market liquidity proxy).

(d) β is a $K \times 1$ vector of unknown parameters.

(e) Each member of the n-variate distribution has its own marginal distribution $F_i(x_i)$, $i = 1,2,..., n$

Equation 6 defines a copula enhanced multivariate distribution function evaluated at the observed historical distribution $x_1, x_2,..., x_\tau$ with marginal distributions $F_1, F_2, ..., F_\rho$. The risk-neutral default process in the reduced-form model framework, used by a number of researchers such as, Jarrow and Yildirim (2001) can be specified as λ_t^Q[10], which according to Das *et al.* (2006) is a joint probability stochastic process. For this study, the cumulative hazard function used to generate the hazard rates is derived as an integral of a hazard function, represented as:

$$\lambda_{kij}^Q = -\ln\left(1 - P_{kij}\right) \ge 0 \quad ...(7)$$

where P_{kij} is the probability of default for the various credit ratings of the study. P_{kij} is derived from the transition matrix represented by Equation 2.

Equation 6 is established under the assumption that the marginal distribution function $F(\bullet)$, for the observations of market price of credit risk is common up to a systematic component, λ_t^Q, that is known up to n parameters. Credit risk default is assumed to be random following a Poisson process, hence it is further assumed that $F(\bullet)$ is from the natural exponential family of distributions, which encompasses the normal, Poisson and gamma distributions, such that the probability density function for the i^{th} obligor class at time t can be expressed as:

$$f\left(y_i, \lambda_i^Q\right) = e^{\left(\frac{(y_i\lambda_i) - \alpha(\lambda_i)}{\phi} + \psi(y_i, \phi)\right)} \quad ...(8)$$

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[9] The corresponding probability density function is $f_i\left(x_i,..., x_\tau\right) = c\left[F_1\left(x_1\right)..., F_\tau\left(x_\tau\right), \lambda_\tau^Q\right]\prod_{t=1}^{\tau} f_t\left(x_t\right)$.

[10] $\lambda_t^Q = \lambda_0 + \lambda_1 r_t + \lambda_2 l_t$

where the functions α and ψ are chosen to represent a particular distribution and ϕ is a known dispersion factor. Following Frees and Wang (2005), we use a canonical link function to link λ_t^Q to the systematic component so that, $\alpha(\lambda_i^Q) = E(y_i)$ and $\lambda_i^Q = x_i'\beta = g(Ey_i)$. Now that Equation 8 has been specified, the GLM uses these along with the copula covariates as input. Assuming independence among credit risk class, the copula dependence can be estimated parametrically using maximum likelihood. Given the copula marginal distribution function, $F_{it}(y_{it}, x_{it}, \beta, \gamma)$, with an accompanying density function, $f_{it} = f(y_{it}, x_{it}, \beta, \gamma)$, with parameter vector θ, then the log-likelihood function of the i^{th} credit risk class can be written as follows:

$$l_i(y_i, x_{i1}, \ldots x_{i\tau}, \beta, \gamma, \theta) = \sum_{t=1}^{T_i} \ln f_{it}(y_i \lambda_i^Q) + \ln c(F_1, F_2, \ldots, F_i, \theta) \qquad \ldots(9)$$

where γ is the scale and shape parameter; β is the estimated marginal parameter, and x_{it} is the explanatory credit risk data.

The log likelihood function using the GLM framework can thus be written as,

$$l_i = \alpha_0 + \sum_{t=1}^{T_i} \frac{y_i x_i'\beta - \alpha(x_i'\beta)}{\phi} + \ln c(F_1, F_2, \ldots, F_{iT}) \qquad \ldots(10)$$

Substituting the copula density function into the GLM framework illustrated by Equation 9, yields an expression for the log-likelihood of the i^{th} credit risk class. Following Sun *et al.* (2006) the parameters β, γ, and θ were estimated by estimating the sum of the likelihood function.

Implied CDS Premia from the Copula Predictive Distribution

Following Frees and Wang (2005), the t-copula used in this paper is parameterized by r degrees of freedom, and the correlation matrix $-\Sigma_\tau$. Frees and Wang (2005) showed that the number of parameters to be evaluated is dependent on the matrix structure adopted. Suppose the correlation matrix associated with the multivariate distribution $\{Y_1, Y_2, Y_3, \ldots, Y_\tau, Y_{\tau+1}\}$ is given by:

$$\begin{pmatrix} \sum_\tau & P_{\tau+1g\tau} \\ P_{\tau+1} & 1 \end{pmatrix} \qquad \ldots(11)$$

where,

(a) Σ_t describes the correlation between $\{Y_1, Y_2, Y_3, \ldots, Y_\tau\}$

(b) $P_{\tau+1g\tau}$ describes the correlation between the implied $Y_{\tau+1}$ and the observed credit premia $\{Y_1, Y_2, Y_3, \ldots, Y_\tau\}$.

(c) Y_t is a multivariate t-distribution with r degrees of freedom; the accompanying marginal distribution is a t-distribution with r degrees of freedom denoted by G_r.

So to formulate the joint conditional density function, we first derive the conditional variance which is expressed as:

$$\sigma^2_{\tau+1g\tau} = 1 - P'_{\tau+1g\tau} \sum\nolimits_{\tau}^{-1} P_{\tau+1g\tau} \qquad \text{...(12)}$$

Using Equation 12, the joint conditional density function for the implied CDS premia distribution is therefore given as:

$$f\left(Y_{\tau+1} \middle| Y_1, Y_2, \ldots, Y_\tau\right) = g_r\left(\frac{V_{\tau+1} - P_{\tau+1g\tau} \sum\nolimits_{\tau}^{-1} \nu}{\sigma_{\tau+1g\tau}}\right) \frac{f\left(Y_{\tau+1}, \theta_{\tau+1}\right)}{g_r\left(V_{\tau+1}\right) \sigma_{\tau+1g\tau}} \qquad \text{...(13)}$$

where,

(a) $V_t = G_r^{-1}\left(F_t\left(Y_t\right)\right)$, $t = 1, 2, 3, \ldots, \tau + 1$; G_r is a distribution function of credit premia.

(b) $F_t(Y_t)$ and $f(Y_t)$ are both cumulative mass functions.

(c) Y_t is a multivariate t-distribution with r degrees of freedom; the accompanying marginal distribution is a t-distribution with r degrees of freedom denoted by G_r.

Given the preceding discussion the marginal premia distribution is modeled as a two parameter gamma distribution with density,

$$f(Y;\alpha,\gamma) = \frac{\gamma^{\alpha-1}}{\gamma^{\alpha}\,\Gamma(\alpha)} e^{\left(-\frac{Y}{\gamma}\right)} \qquad \text{...(14)}$$

The dependence structure is modeled by a t-copula with r degrees of freedom, and the log-likelihood function for firm i over τ years is expressed as:

$$l_i(\alpha,\gamma) = \alpha_0 + \sum_{t=1}^{T_i} \frac{y_i x_i' \beta - \alpha\left(x_i' \beta\right)}{\phi} + \ln c\left(F_1, F_2, \ldots, F_{iT}\right) \qquad \text{...(15)}$$

The study assumes that the gamma distribution parameters are constant across the obligors in our sample, so there are four parameters to be estimated for the fitted t-copula. These are ρ_1, α, γ and r.

Description of the Data

This section provides a brief description of the data used in this paper. CDS bid-ask[11] and mid-price data for the pricing estimates were obtained from Bloomberg. In addition, Bloomberg was also used to obtain the weekly US Treasury note and bill prices that are required for parameter estimation of the spot rate process. The Bloomberg CDS dataset comprises of quotes for contracts of maturities, 1 to 15 years. The sample period covered in the study comprises of 135 weeks of default swap quotes per referenced entity.

The data analyzed is based on weekly observations from January 2, 2004 to August 8, 2006, where $t = \frac{1}{365},\ldots,\bar{t}$. One observation made during this stage of exercise, is the fact that prior to 2002 the CDS market was not as liquid and active as it currently is. Hence, reasonable data were not in abundance prior to 2003. The data comprises of a mixture of \$29 denominated AAA, AA, A, BBB, BB and B CDSs issued by 29 Fortune 500 companies, across several industries chosen to stratify the various industry groupings such as, cable/media, financial, insurance, US banks, telecom, energy, retail, technology and manufacturing.

Quotations are available only on days when there is some level of liquidity in the market as evidenced either through trades or by active market making by a dealer. Bloomberg was also used to obtain the CDS characteristics such as maturity dates, coupon percentages and seniorities. In reality, since most of the CDS trading activity is within the five-year time to maturity group, the price quotes on the five year CDS premia is used in the study's pricing analysis. For an issuer to be included in the sample, it must have at least 130 weekly observations of its five-year CDS data points. As a result of this selection technique, the CDS dataset used in this study covers 29 issuers with an average of 133 weekly observations per issuer, for maturities of 1, 3, 5, 7 and 15 years respectively. Additionally, the sample period was subdivided into 11 sub-periods of 13 weeks of daily observations, so as to allow the copulas to determine changes in the covariates over the sub-periods of the larger sample period.

As stated earlier, the market liquidity proxy is derived from the bid-ask spreads. In any market that is in equilibrium, there will generally be a difference between the best quoted ask price and the best quoted bid price. That difference is called the bid-ask spread (or bid-offer spread). For the market liquidity proxy, this study follows the approach used in Dunbar (2007) and uses the percentage bid-ask spread, which is the bid-ask premia divided by the mid-price. Tang and Yan (2006) suggest that bid-ask spreads measure trading costs that compensate market makers for the risk of adverse selection and hedging costs. Depending upon the market, the bid-ask quotes may be expressed as actual prices, yields, implied volatilities, etc. The average of the bid and ask prices is called the mid-offer price.

Discussion of Empirical Results

Descriptive statistics of the 29 firms included in this study are presented in Table 1. Daily market observations from January 2, 2004 to April 10, 2006 were broken into

[11] The bid-ask prices are consensus quotes among market participants regarding the value of the CDS.

11 subgroups, each having 13 weeks of daily observations, resulting in 174 observations of market premia. 13 weeks of observations from April 11, 2006 to August 8, 2006 were held back for the out-of-sample portion for the copula prediction model. From Table 1, we see that the average CDS premia varies from year to year. The table also show that variability moves inversely with credit quality.

Table 2 summarizes the Hierarchical Linear Model (HLM) marginal analysis correlations among CDS prices across credit qualities and cross-sectional residuals over time. As discussed in the subsection: Model 2 – Copula Dynamics Under the Generalized Linear Model, the HLM has a nested structure that allows effects to vary from one context to another. In hierarchical data, entities in the same group (credit class) are also likely to be more similar than entities in different groups. Due to this, the variations in the outcome may be due to differences between groups, rather than individual differences within a group. As a result of this tendency, Table 2 looks at variations between groups rather than within groups.

In Table 2, the results in the upper right corner above the diagonal represent the correlations of the residuals of the predicted marginal model. The results in the lower left of the diagonal represent the correlation of actual CDS premia observations over the sample sub-periods. The observed premia show strong correlations, while correlations of the predicted residuals vary from period to period. The results show that any model that ignores temporal dependencies does not provide an appropriate fit to the relationships being explored. Further, though temporal dependencies from period to period are high, they are not constant across the sample period. Hence, as stated earlier the dependencies among variables over time cannot be ignored, neither can we underscore the importance of time-varying covariates.

Table 3 presents the results of the regime switching model that incorporates the liquidity proxy variable in the hazard functions of the firms considered in the present study. Following Das *et al.* (2006), average hazard rates across all issuers are first computed using the square root drift diffusion process, represented by Equation 4 in the subsection: Model 1 – Markov Switching Model. On one hand, the results are as expected and are consistent with prior findings that did not include a measure of market liquidity. The data from the high regime model indicates that a higher average return is required in order to compensate investors for the higher levels of risk. Conversely, investors are prepared to take a lower average return on investments bearing lower levels of volatility.

While on the other hand, the inclusion of liquidity as an explanatory variable in the CDS valuation model adds an entirely new dimension to the regime switching model. In Table 3, we see that while investors are being compensated for taking higher levels of credit risk with higher average returns, this average return declines with credit quality. This is because as credit risk increases the credit risk premium paid by protection seekers increases, so the cost of this added credit risk protection reduces potential returns to the investor.

Table 1: Summary Statistics of the 5-Year CDS Premia Showing Credit Ratings Average Premia and Variability Among Credit Classes

Issuer	Ticker	Industry	S&P Rating	Moodys Ratings	Mean CDS Premia	Maximum CDS Premia	Minimum CDS Premia	Standard Deviation
General Electric	GE	Industrial	AAA	Aaa	24.47	34.66	15.99	5.28
Altria	MO	Consumer	BBB+	Baa2	118.11	177.19	54.46	45.18
Aetna	AET	HealthCare	A–	A3	33.29	44.26	21.81	8.49
Ace Insurance	ACE	Insurance	A–	A3	44.69	67.87	27.09	11.54
Alcan	AL	Mining	BBB+	Baa1	33.77	51.07	24.20	8.78
Alcoa	AA	Mining	A–	A2	26.94	41.20	18.67	6.93
Altell	AT	Telecom	A–	A2	38.54	54.80	24.70	9.91
American Express	AXP	Financial Services	A+	A1	23.63	29.76	18.22	4.13
American International Group	AIG	Insurance	AA	Aa2	23.05	37.74	17.75	5.60
Arrow Electronics	ARW	Electronics/Wholesale	BBB–	Baa3	87.41	128.93	60.06	27.02
Bristol-Myers Squibb	BMY	Pharmaceutical	A+	A1	24.73	39.33	13.94	8.51
Cendant	CD	Rental and Leasing	BBB+	Baa1	59.54	97.40	43.70	18.45
Caterpillar	CAT	Industrial	A	A2	21.99	28.88	16.31	4.12
Cingular	AT&T	Telecom	A	Baa1	99.52	295.99	22.83	86.17
Capital One	COF	Financial Services	BBB	A3	46.11	68.19	26.24	15.43
IBM	IBM	Computer	A+	A1	20.47	26.93	14.60	4.04
WalMart	WMT	Consumer	AA	Aa2	13.84	17.86	9.20	2.79
Target	TGT	Consumer	A+	A2	18.98	28.83	11.52	6.97
Dow Chemical	DOW	Manufacturing	A–	A3	33.57	51.16	22.57	10.09
Washington Mutual Bank	WAMU	Financial Services	A	A2	39.06	51.06	29.84	6.23
Viacom	VIA	Cable	A	Baa3	49.02	75.49	24.00	15.00
Carnival Corporation	CCL	Entertainment	A–	A3	32.62	46.51	24.64	8.41
Lucent	LU	Manufacturing	B	B1	245.29	408.75	29.58	105.04

(Contd...)

Table 1: Summary Statistics of the 5-Year CDS Premia Showing Credit Ratings Average Premia and Variability Among Credit Classes *(...contd)*

Issuer	Ticker	Industry	S&P Rating	Moodys Ratings	Mean CDS Premia	Maximum CDS Premia	Minimum CDS Premia	Standard Deviation
Starwood Resorts	HOT	Hotel	BB+	Ba2	122.70	202.50	63.35	35.39
Unum Provident Group	UNM	Insurance	BB+	Ba1	148.47	265.73	76.12	62.78
Nordstrom	JWN	Consumer	A	Baa1	34.17	50.90	23.78	7.16
Haliburton	HAL	Industrial	BBB+	Baa1	42.57	75.23	24.31	17.82
Marriott	MAR	Hotel	BBB+	Baa2	40.40	54.10	27.57	8.58
XL Capital	XL	Insurance	AAA	A3	44.53	52.35	35.75	5.42

Note: The Mean, Minimum, Maximum and Standard Deviation cover the period of the study.

Table 2: Credit Default Swap Correlations

	CDS Period 1	CDS Period 2	CDS Period 3	CDS Period 4	CDS Period 5	CDS Period 6	CDS Period 7	CDS Period 8	CDS Period 9	CDS Period 10	CDS Period 11
CDS Period 1	**1.0000**	–0.0512	–0.3678	–0.5442	–0.5520	–0.3837	0.1833	–0.5132	–0.0814	0.6858	–0.4028
CDS Period 2	0.9528	**1.0000**	–0.3957	0.0139	–0.2621	–0.2824	–0.2825	–0.0107	–0.6088	–0.1529	–0.3259
CDS Period 3	0.9188	0.9947	**1.0000**	0.4118	0.4145	0.6810	–0.1119	0.6181	0.4550	–0.3024	0.5071
CDS Period 4	0.8883	0.9835	0.9968	**1.0000**	0.8609	0.7058	–0.2283	0.4883	0.0820	–0.8563	0.4631
CDS Period 5	0.9747	0.9895	0.9773	0.9627	**1.0000**	0.8178	–0.0114	0.5047	0.2133	–0.8014	0.6731
CDS Period 6	0.9882	0.9696	0.9471	0.9258	0.9931	**1.0000**	0.1430	0.6336	0.4450	–0.5329	0.8624
CDS Period 7	0.9878	0.9403	0.9100	0.8839	0.9765	0.9949	**1.0000**	–0.1272	0.2667	0.2232	0.2286
CDS Period 8	0.9856	0.9284	0.8955	0.8675	0.9691	0.9909	0.9992	**1.0000**	0.5162	–0.6216	0.7340
CDS Period 9	0.9829	0.9140	0.8780	0.8481	0.9591	0.9850	0.9973	0.9992	**1.0000**	0.0728	0.6069
CDS Period 10	0.9845	0.9202	0.8854	0.8564	0.9634	0.9876	0.9983	0.9997	0.9999	**1.0000**	–0.4355
CDS Period 11	0.9867	0.9372	0.9067	0.8805	0.9752	0.9940	0.9998	0.9996	0.9979	0.9988	**1.0000**

Note: The table displays correlations of CDS prices for the 11 subperiods of the study. From the table it is apparent that the multivariate average CDS prices are not independent.

Table 3: Results of Regime Switching Model Indexed by Market Risk and Liquidity

Credit Quality	Parameters					
	Regime 1			Regime 2		
	κ_{HI}	θ_{HI}	σ_{HI}	κ_{LO}	θ_{LO}	σ_{LO}
AAA	0.740	2.623	0.020	0.824	0.075	0.010
t-stat.	2.39	178.39	4.48	2.53	68.09	4.47
AA	0.718	2.614	0.089	0.766	0.077	0.043
t-stat.	1.93 4	0.03	4.47	2.01	15.63	4.47
A	0.699	2.056	0.091	0.674	0.139	0.052
t-stat.	2.57	34.66	4.47	2.44	14.95	4.47
BBB	0.065	1.915	0.112	0.116	0.216	0.090
t-stat.	0.27	0.58	4.47	0.53	0.51	4.47
BB	0.680	1.283	0.029	0.733	0.325	0.020
t-stat.	7.77	78.52	4.47	8.62	60.54	4.48
B	0.142	1.941	0.059	0.595	0.126	0.013
t-stat.	0.58	8.55	4.47	4.21	48.30	4.47

Note: Regime 1 represents high risk/high returns and Regime 2 represents low risk/low returns.

Due to the high level of dependencies between the variables (as shown by Table 2), a further attempt is made to look at the upper and lower tail dependencies. Table 4 shows that there is a high level of tail dependencies in the upper and lower quartiles of the computed default probabilities. This characteristic best suits a *t*-copula which is then fitted to the sample data.

Table 5 presents the results of the covariates fitting the *t*-copula model which uses two different correlation matrix structures to show the effects of a constant, versus a time-varying joint covariance structure, in credit risk forecasting. The results show that the correlation coefficients (ρ) for both the specifications of the predictive model were statistically strong with *p*-values of less than 1% level. The fact that ρ is statistically significant, provides strong statistical evidence that the correlation structure of the financial data is not independent. Further, these findings compliment the findings of earlier analysis presented in Table 2, which suggest that the CDS data display high level of dependencies across the sample period. With the exception of R_t—the spot rate on interest which proxies the macro economy—all other coefficients are statistically significant at less than 1% level of significance.

These results strengthen the view that the spot rate and liquidity are useful predictors of credit risk. Economic theory suggests that credit risk and market risk are positively

Table 4: Tail Dependencies in the CDS Premia Data						
	Lower Percentile Tail Dependency					
	AAA	**AA**	**A**	**BBB**	**BB**	**B**
AAA	1.000	0.971	0.998	0.801	0.970	0.918
AA	0.971	1.000	0.983	0.634	1.000	0.795
A	0.998	0.983	1.000	0.766	0.982	0.894
BBB	0.801	0.634	0.766	1.000	0.632	0.973
BB	0.970	1.000	0.982	0.632	1.000	0.794
B	0.918	0.795	0.894	0.973	0.794	1.000
	Upper Percentile Tail Dependency					
	AAA	**AA**	**A**	**BBB**	**BB**	**B**
AAA	1.000	0.925	0.787	0.965	0.822	0.995
AA	0.925	1.000	0.962	0.992	0.977	0.958
A	0.787	0.962	1.000	0.922	0.998	0.844
BBB	0.965	0.992	0.922	1.000	0.943	0.986
BB	0.822	0.977	0.998	0.943	1.000	0.875
B	0.995	0.958	0.844	0.986	0.875	1.000

Note: Both upper and lower percentile shows high levels of dependencies among credit ratings.

Table 5: Summary of Maximum Likelihood Estimates of *t*-Copula Parameters						
Parameter	**AR–1 Correlation Matrix**			**EX Correlation Matrix**		
	Coefficient	**Std. Error**	***p*-value**	**Coefficient**	**Std. Error**	***p*-value**
ρ	0.9839	0.0061	0	0.9719	0.0108	0
Intercept	6.8919	2.1232	0.0006	6.6754	1.9886	0.0004
R_t	0.0020	0.0536	0.4851	0.0101	0.0374	0.3934
Liq	–1.5678	0.5076	0.0010	–1.5085	0.4790	0.0008
α	23.1196	8.0865	0.0021	23.7059	8.0765	0.0017
r	2.1486	0.7722	0.0027	2.0129	0.6916	0.0018
AIC	974.7000	–	–	1008.8700	–	–

correlated, so an increase in credit risk will lead to an increase in market risk. Similarly, the negative coefficient for liquidity indicates that as credit risk increases, market liquidity falls off. We see this phenomenon in the market for securities, where high yield securities is less liquid than their investment grade counterparts. To compare the goodness of fit of both the models the AIC[12] technique is used. The result shows that the AR–1[13] time-varying correlation matrix gives a much better fit to the data than the constant correlation (EX)[14] matrix model. These findings support the hypothesis that a model that best captures the time-varying joint conditional correlations across time, is a better fit to financial data than the one that assumes constant covariates.

Traders, speculators, portfolio managers are interested in predictive methodologies that produce accurate and reliable forecasts of CDS premium. Current multivariate techniques provide point estimates and predictive intervals that are most appropriate for normal distributions. Since CDS data is not normal, copulas are used to obtain the predictive distribution (see discussion in the subsection: Implied CDS Premia from the Copula Predictive Distribution).

Table 6 presents the simulation results of the predictive *t*-copula model used in the study. The table presents the results of the t-copula fitted with a constant joint correlation matrix, consistent with the assumption of traditional multivariate analysis. The table also

Table 6: Empirical Results of CDS Predictive Estimates from the *t*-Copula Model

Issuers	Actual CDS Premia	Predicted CDS Premia	
		Model 1: Constant Correlations	Model 2: Time-Varying Correlations
Altria	54.4610	68.4822	63.7125
GE	15.9900	30.8940	20.5989
Aetna	29.6240	30.7100	29.3459
ACE	36.4660	40.6982	36.2411
Alcan	30.9690	31.1159	33.1364
Alcoa	21.0450	23.3363	24.8489
Altell	53.5330	41.7641	45.1203
American Express	18.2170	22.8554	19.5336
AIG	18.2250	20.4799	17.4940
Arrow Electronics	62.8980	67.1101	64.5184
Bristol-Myers-Squibb	13.9390	12.3316	13.5419
Cendant	54.9590	62.0015	78.1635
Caterpillar	16.7910	16.7296	17.3426
Cingular	22.8330	35.8996	23.7369

(Contd...)

12 Defined as AIC = –2*log (maximum likelihood) + 2*(number of estimated parameters). A smaller AIC value indicates a better statistical fit to the data, or a better model.

13 This is a time series representation of temporal relationships, which shows that credit risk at t_0 has diminishing influence on credit risk in later periods, t_n.

14 The exchange structure or uniform correlation matrix (discussed at length in Frees (2004)) indicates that covariates within a credit class does not change over time, or exhibits constant correlation.

Table 6: Empirical Results of CDS Predictive Estimates from the *t*-Copula Model *(...contd)*			
Issuers	**Actual CDS Premia**	**Predicted CDS Premia**	
		Model 1: Constant Correlations	**Model 2: Time-Varying Correlations**
CapitalOne	26.2750	25.7546	24.2826
IBM	14.6020	12.2117	14.2246
WalMart	9.1980	5.9703	8.5244
Target	11.5190	10.9676	11.8339
Dow Chemical	22.5740	26.9606	26.9012
Viacom	49.1890	65.0839	65.5109
Carnival Corporation	29.5780	33.0708	32.0097
Lucent	138.9580	213.1035	131.4094
Starwood Resorts	81.6660	124.4306	108.9638
Nordstrom	23.7830	27.5370	26.1198
Haliburton	30.6820	29.0880	31.7721
Marriott	30.9250	35.1615	33.9705
XL Capital	45.1770	44.5439	41.9858
Washington Mutual Bank	29.8390	37.0511	33.8988
Unum Provident Group	81.6090	104.9778	89.5346
Mean Absolute Difference	–	**7.7520**	**3.1980**
Std. Deviation	**27.6000**	**42.0000**	**30.1000**

Note: Model 1 was estimated using a constant correlation (EX) framework, while Model 2 used time-varing correlations (AR–1) to derive its outsample predictions.

presents the results of a time-varying joint correlation matrix, which shows that the predictive capability of the model is by far superior to the constant correlation matrix. For the model utilizing a constant correlation matrix, the mean absolute difference is approximately 7.8, whilst the model fitted with time-varying correlations has a mean absolute difference of approximately 3.2. Model 2 of Table 6 also has a standard deviation of 30.1, which is very close to the standard deviation (27.6) of the actual CDS data. So, from the results presented in Tables 5 and 6, we can infer that the joint time-varying conditional correlation model performs better than the constant correlation model, which is often assumed in multivariate analyses.

The findings regarding the use of copulas to model dependencies over time are consistent with the results obtained by Sun *et al.* (2006), who found that long tailed longitudinal data are best fitted with a copula that is capable of separating the multivariate joint distribution into interdependent probabilities and marginal distributions.

Conclusion

Dependency structures vary over time and are not constant as in traditional multivariate analyses. In this paper, we incorporated a copula function to model the 'heavy-tail' dynamics of credit risk data. The copula was used to model the dependencies over time. To test the multivariate normality assumption, we explored two different specifications of the covariance matrices across the referenced entities of the study. In addition, to test the time-varying joint conditional and constant correlation hypotheses, we developed alternative constant and time-varying correlation matrices that were fitted to the copula methodology. Recent empirical work by Duffie *et al.* (2007) on term structures of conditional probabilities of corporate default used covariate estimates that assumed normality in the underlying data, however our findings clearly illustrate that allowing dependency structures to vary over time is superior to fixed correlation parameters.

Secondly, several studies have looked at the regime switching characteristics of credit risk, but none looked at the effects of liquidity on the Markov switching dynamics of the CDSs. The study shows that as credit risk increases, investors demand a higher average return for assuming greater levels of risk. However, this return moves inversely with credit quality due to the fact that CDS premium that investors pay to protection sellers, increases at a magnitude greater than the corresponding increase in expected returns (because of increasing illiquidity as credit quality decreases).

Prior work by Das *et al.* (2006) and others indicate that investors require a higher average return to take on higher levels of risk; however these results ignored the level of returns across credit classes in the presence of a liquidity measure. So an important extension to both the Markov switching and copula dynamics discussion is the inclusion of liquidity as a determinant of financial asset pricing and investor psyche. From prior research we know that investors take on higher levels of risks because of the higher levels of average returns, but our findings now suggest that these average levels of returns tend to change with the levels of liquidity of financial assets.

Further, credit risk and liquidity migration across credit classes suggests that higher credit and liquidity risk generally exhibits higher levels of volatility. The regime switching analysis indicates that the increase in credit risk across credit classes is met with an increase in illiquidity and a simultaneous increase in volatility. The increased volatility or risk level as the credit quality falls off, partly explains the need for credit risk insurance or CDS protection, across credit quality, to make these investments attractive to potential investors. ❖

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Reference # 37J-2008-06-03-01