Integrating Market and Credit Risk in Stochastic Portfolio Optimization

José Luiz Barros Fernandes*, José Renato Haas Ornelas**
and Marcelo Yoshio Takami***

This paper has two main objectives. The first is to propose a measure to integrate the market and credit risk. We define a way to convert credit risk into market risk, and then define an integrated risk measure. Based on this integrated measure, an Adjusted Sharpe Index is defined as a metric to compare various portfolios in the surface frontier in terms of financial efficiency. Second, several methodologies of estimating efficient portfolios were evaluated, under the perspective of a global long-term investor, incorporating estimation risk and evaluating the effect of credit risk in the model selection. The results support the use of the Michaud (1998) resampling methodology as it offers better results in terms of financial efficiency, allocation stability and diversification. Using the Michaud approach, the efficient surface frontier is defined and generated.

Introduction

Two well-known issues in asset allocation problems, using Markowitz optimization are: the instability of the output asset allocation, and the absence of other sources of risk (despite market risk) in the analysis. These drawbacks make the results of the model misleading and is a reason why Markowitz-based models have not played the important role that they should in the global portfolio management.

The first issue derives from the fact that the output portfolios are driven by the risk/return estimation, which usually generates an unfeasible dynamic of the asset allocation. The most famous problem with this technique is the substitution problem, where two assets with the same risk but slightly different expected returns generate strongly different asset allocations. The optimizer would give all the weight to the asset with the higher expected return, leading to very unstable asset allocation. This is particularly problematic when we have an investor with a long-term investment horizon (e.g., pension funds, and central banks). In this case, the asset allocation stability is much more relevant as the investor often wants to follow a passive strategy.

Recent literature has tried to overcome the previous problem of leading to unfeasible portfolios where some of the proposed models are discussed. The main focus of those models is to find out how realistic portfolios can be created considering that the values used for risk and return are not deterministic but just estimates. It should be noted that the misspecification of returns is much more critical than that of the variances (Zimmer and Niederhauser, 2003).

* Advisor, *Banco Central do Brasilia*, Brazil. E-mail: jluiz.fernandes@bcb.gov.br
** Advisor, *Banco Central do Brasilia*, Brazil. E-mail: jrenato.ornelas@bcb.gov.br
*** Advisor, *Banco Central do Brasilia*, Brazil. E-mail: marcelo.takami@bcb.gov.br

Despite the vast literature that addresses the asset allocation problem, the consideration of a long investment horizon in the global markets perspective has been neglected. This is probably due to the main interest in the short run, and the lack of reliable international financial data. However, it is considered that several market participants, characterized by a more conservative long-term profile, would be willing to adjust their portfolios or benchmarks in a more efficient way, and in this sense, the available techniques are less suitable.

Another neglected aspect in portfolio selection models is the effect of credit risk. The traditional asset allocation approach considers the volatility of past returns as the only risk factor. However, some asset classes have other types of risks, and so there are risk premia embedded in their returns to compensate for these additional sources of risk. If those risk premia are not taken for the analysis, the results of the model tend to be distorted with credit portfolios, for instance, dominating the default-free portfolios. Business Information Systems (BIS) (2003) has been concerned about the risk integration and has observed two important trends on the basis of a survey of 31 financial institutions in 12 jurisdictions: (1) greater emphasis on the management of risk on an integrated firm-wide basis; and (2) related efforts to "aggregate" risks through mathematical risk models.

This paper has two main objectives. First is to propose a measure to integrate the market and credit risk, which could be extended to any source of risk that has an identifiable premium. A method to convert credit risk into market risk resulting in an integrated risk measure is defined. Based on this integrated measure, an Adjusted Sharpe Index is defined as a metric to compare various portfolios in the efficient surface in terms of financial efficiency.

Second, several methodologies of estimating efficient portfolios were evaluated, under the perspective of a global long-term investor, incorporating estimation risk into mean-variance portfolio selection and evaluating the effect of credit risk in the model selection. The results support the use of the Michaud (1998) resampling methodology as it provides better results in terms of financial efficiency, allocation stability and diversification. Using the Michaud Approach, the efficient surface frontier in a three-dimensional space where market risk, credit risk and expected return are the axis were defined and generated.

This paper has several contributions: first, a generic measure is proposed that integrates the market and credit risk into a single measure; second, a metric is proposed to compare portfolios with different market and credit risks thereby, providing an objective way to price the trade-off between the previous risks; third, from a range of eight usual models of portfolio optimization, an empirical out-of-sample test is provided to define which of them suits better for a global long-term investor; forth, we include in the model selection evaluation, a restriction for credit risk and so taking into account the most relevant two sources of risk (market and credit risk); and finally, we define and generate the efficient surface frontier and so providing several practical insights in terms of asset allocations. It is important to stress that the empirical evaluations are based on a comprehensive database of global assets which provide robust results.

The paper offers a literature review: describing the other methodologies of asset allocation and discussing the effects of credit risk, defines the efficient surface, and proposes an integrated risk measure that incorporates market and credit risk. Based on this measure, an Adjusted Sharpe Index is also proposed that incorporates credit and market risks. Later it presents the empirical study, describing the data and the implementation; and provides the results followed by a conclusion which reviews the main achievements.

Literature Review

Portfolio Optimization Models

The most relevant existing approaches to portfolio optimization are reviewed under the same framework. The following notational conventions are used, as in Feldman and Reisman (2002): constants and variables will be typed in italic font, operators and functions in regular straight font, and vectors and matrices in boldface straight font. The models to be addressed are given in the Table 1.

Table 1: List of Portfolio Optimization Models

Mnemonic	Model
Mark	Markowitz (1952) Portfolio Selection Model
Jorion	Jorion (1986) Bayesian Informative Prior Model
Bayes	Bayesian Diffuse Prior Model (1986)
Kempf	Kempf *et al.* (2002) Bayesian Informative Prior Model
Risk Adj	Black and Litterman (1992) Risk Adjusted Model
Equil	Black and Litterman (1992) Equilibrium Model
Horst	Horst *et al.* (2002) Adjusted Risk Aversion Model
Mich	Michaud (1998) Portfolio Resampling Model

The Markowitz (1952) Model

The Markowitz's mean-variance model is the classical paradigm for portfolio allocation. By using variance as a measure of risk, Markowitz formalized the intuition that investors optimize the trade-off between returns and risks. A market with N risky securities and one risk-free security is assumed. Let R_i, $i = 1, ..., N$, be the random rates of return of the risky securities 1, ..., N, and let R_F, be the constant return of a risk-free security. In this model, the probability distributions of the rates of return are not specified but means and variances that are real finite numbers are assumed. The variance-covariance matrix of the risky securities is denoted as $\mathbf{V}$, a positive definite matrix, which implies that there are no redundant securities.

Let a portfolio $\mathbf{a}$, of the $N + 1$ market securities, be an $N \times 1$ vector of real numbers, a_i, $i = 1, ..., N$, where a_i is the weight of the risky security i in the portfolio. In this case, the rate of return of the portfolio is given by

$$R_a = (1 - \mathbf{a}^T\mathbf{1})R_F + \mathbf{a}^T\,\mathbf{R}$$

where $\mathbf{1}$ is an $N \times 1$ vector of ones, $\mathbf{R}$ is an $N \times 1$ vector of the risky securities' rates of return, and superscript 'T' denotes the transpose operator. Observe that weights of all $N + 1$ securities in portfolio 'a' add up to one, and short positions are permitted. The expected (mean) rate of return and the variance of the rate of return of portfolio 'a' are

$$E(R_a) = R_F + \mathbf{a}^T \mathbf{X}$$

$$VAR(R_a) = \mathbf{a}^T \mathbf{Va}$$

where $\mathbf{X}$ is the $N \times 1$ vector of excess risky security expected rates of return over R_F.

Following Markowitz's (1952), Portfolio Theory, the term *Portfolio Frontier* is used for the set of portfolios that have the lowest standard deviation for any given expected return, which is the same as to solve the following minimization problem:

$$\underset{a}{\mathrm{Min}} \frac{1}{2} \mathbf{a}^T \mathbf{Va} \approx$$

subject to

$$E(R_a) = \mathbf{a}^T \mathbf{X} = x$$

where for simplicity x, $x = g - R_F$ are defined to be the excess return of the exogenous desired return, g, over the risk-free rate 0. In the presence of a risk-free asset, the efficient frontier (given by the efficient portfolios) becomes a straight line—the Capital Market Line (CML).

Throughout this paper, the defining of the tangency portfolio—the portfolio that is both on the CML and on the portfolio frontier of risky assets hyperbola—is of interest. As shown in Feldman and Reisman (2002) the weights for the tangency portfolio are given by:

$$\mathbf{a} = ((\mathbf{V}^{-1}\mathbf{X})^T \mathbf{1})^{-1} \mathbf{V}^{-1} \mathbf{X}$$

Moreover, the weights of the hyperbola portfolio with the lowest standard deviation are given by:

$$\mathbf{a}_0 = ((\mathbf{V}^{-1}\mathbf{1})^T \mathbf{1})^{-1} \mathbf{V}^{-1} \mathbf{1}$$

The mean-variance model has been criticized on several grounds as it rests on assumptions of quadratic preferences and/or elliptical symmetric return distributions (Manganelli, 2006a, 2006b). Moreover, it assumes that the risk and return are known with certainty and not estimates which imply estimation risk. The following models try to overcome some of the previous drawbacks of the Markowitz model.

The Jorion (1986) and Diffuse Prior's Model

This model offers a simple empirical Bayes estimator that should outperform the sample mean in the context of a portfolio. The main idea is to select an estimator with average minimizing properties relative to the loss function (the loss due to estimation risk). Instead of the sample mean, an estimator obtained by 'shrinking' the means toward a common value is proposed (the average return for the minimum variance portfolio), which should lead to a decreased estimation error.

The specification of Jorion is characterized by two assumptions. First the prior mean is identical across all N risky assets. Second the estimation risk is proportional to the innovation risk. Following this model, the optimal portfolio choice should be based on the predictive function of the vector of future rates of return, and the estimated expected rates of return and expected covariance matrix are given by

$$E(R_a) = (1-w)\mathbf{X}+w\mathbf{1a}_0\mathbf{X}$$

and

$$\mathrm{VAR}(R_a) = \mathbf{V}\left(1+\frac{1}{Y+\lambda}\right)+\frac{\lambda}{Y(Y+1+\lambda)}\frac{\mathbf{11}^{\mathrm{T}}}{\mathbf{1}^{\mathrm{T}}\mathbf{V1}}$$

where Y is the number of return observation used to estimate the mean and

$$w=\frac{\lambda}{Y+\lambda}$$

Observe that λ is the precision of the prior belief that the prior mean is identical across all N risky assets (the average return for the minimum variance portfolio). Therefore, this approach will outperform the classical sample mean because it relies on a richer model and includes the sample mean as the special case $\lambda = 0$ (diffuse prior).

In the diffuse prior's case, the shrinkage estimator $w = 0$ and the moments reduce to

$$\mathrm{E}(R_a) = \mathbf{X}$$

and

$$\mathrm{VAR}(R_a) = \mathbf{V}\left(1+\frac{1}{Y}\right)$$

However in the general case ($\lambda \neq 0$) the shrinkage estimator is given by

$$w=\frac{N+2}{(N+2)+(\mathbf{X}-\mathbf{a}_0\mathbf{X1})^T Y\mathbf{V}^{-1}(\mathbf{X}-\mathbf{a}_0\mathbf{X1})}$$

From the previous formulation it is observed that when there are a short number of return's observations (Y is small), the shrinkage estimator seems to offer better results (w is higher), intuitively due to the lower confidence on the sample mean. As Y is increased (w gets lower), the relevance of the prior belief is diminished and the model tends to follow the classical two step approach of Markowitz. Jorion (1986) shows that the model's improvement is more relevant for $Y < 100$.

The Kempf et al. (2002) Model

Similar to Jorion, this model also assumes that the prior mean is identical across all N risky assets, however Kempf's model considers estimation risk as a second source of risk, determined by the heterogeneity of the market, τ^2, given by the standard deviation of the expected returns across risky assets. The estimated expected rates of return and expected covariance matrix are given by

$$E(R_a) = [\mathbf{V}+\mathbf{Y}\tau^2\mathbf{I}]^{-1}Y\tau^2\mathbf{X}+[\mathbf{V}+\mathbf{Y}\tau^2\mathbf{I}]^{-1}\mathbf{V1a}_0\mathbf{X}$$

and

$$\mathrm{VAR}(R_a) = \mathbf{V}[\mathbf{1}+\tau^2[\mathbf{V}+\mathbf{Y}\tau^2\mathbf{I}]^{-1}]$$

where **I** denotes the identity matrix. In this case, as in Jorion (1986), the conditional expected return is a weighted sum of the prior and the historical average, however the weighting factors are no longer scalars but matrices. Kempf *et al.* (2002) show that this model generates more efficient portfolios when compared to Jorion or the classical two steps; and this result is robust to different estimation periods.

The Black Litterman (1992) Model

Black and Litterman (1992) posit that the consideration of the global Capital Asset Pricing Model (CAPM) equilibrium can significantly improve the usefulness of asset allocation models, as it can provide a neutral starting point for estimating the set of expected excess returns needed to drive the portfolio optimization process. This paper considers two of their approaches: risk-adjusted equal means and equilibrium approach.

Risk-Adjusted Equal Means: Considered a naïve approach to defining a neutral reference point, assumes that different risky assets have the same expected return per unit of risk, where the risk measure is the volatility. So, expected excess returns can be estimated by a linear regression where the independent variable is the standard deviation of each asset.

$$\mathbf{X} = \alpha\sigma + \varepsilon, \qquad \varepsilon \approx = \mathrm{N}(0,1)$$

where σ is a vector of the risky assets' standard deviations. The main problem with this approach is that it does not consider the correlation between risky assets when defining the expected excess returns.

Equilibrium Approach: In this case, the set of expected returns are the ones which "clear the market", if all investors have identical views. Therefore, expected returns are given by the CAPM equilibrium model. An asset is just rewarded by the additional systematic risk it adds to the market portfolio. The formulation is given in the following way

$$\mathbf{X} = \beta\mathbf{M} + \varepsilon$$

and

$$\beta_i = \frac{\mathrm{Cov}(\mathbf{X}_i, \mathbf{M})}{\mathrm{Var}(\mathbf{M})}, \; i = 1, \ldots, N$$

where **M** is a vector of excess return of the market portfolio. This model overcomes the problem of not considering risky assets' correlation of the previous model. The main inconvenience of this approach is related to the selection of a reasonable market portfolio.

The Horst et al. (2002) Model

Horst *et al.* (2002) propose a new adjustment in mean-variance portfolio weights to take into account the estimation risk. The adjustment amounts to using a pseudo risk-aversion, rather than the actual risk aversion which depends on the sample size, the number of assets in the portfolio and the curvature of the mean-variance frontier. The pseudo risk aversion is always higher than the actual one and this difference increases with the uncertainty about the expected return estimations. Horst *et al.* (2002) propose the pseudo risk aversion to be given by

$$\alpha = \left(\mathbf{V}^{-1}\mathbf{X}\right)^{\mathrm{T}}\mathbf{1}\left(1 + \frac{\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1}}{\left(\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1}\right)\left(\mathbf{X}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{X}\right) - \left(\left(\mathbf{V}^{-1}\mathbf{X}\right)^{\mathrm{T}}\mathbf{1}\right)^2}\frac{N-1}{Y}\right)$$

and so the weights for the tangency portfolio are given by

$$\mathbf{a} = \alpha^{-1}\mathbf{V}^{-1}\mathbf{X}$$

It is observed that this approach is similar to the diffuse prior Bayesian, the one presented earlier however, in that case the adjustment factor is

$$\alpha = (\mathbf{V}^{-1}\mathbf{X})^{\mathrm{T}}\,\mathbf{1}\left(1 + \frac{1}{Y}\right)$$

With a diffuse prior, the adjustment also depends on the number of observations, but the Horst *et al.* (2002) adjustment also takes into account the curvature of the mean-variance frontier capturing the intuition that estimation risk should be more important when errors in the expected returns are very costly in volatility terms.

The Resampling Technique (Midland, 1998)

Portfolio sampling allows an analyst to visualize the estimation error in traditional portfolio optimization methods. Suppose, both the variance-covariance matrix **V** and the excess return vector **X** are estimated by using Y observations, it is important to remember that the point estimates are random variables and so another sample from the same distribution would result in different estimates.

Sherer (2002) suggests that sampling from a multivariate normal distribution (a parametric method termed Monte Carlo simulation) is a way to capture the estimation error. In this sense, **X** and **V** would be just the expected values for a multivariate normal distribution. If we just consider two assets, the probability density function for a multivariate normal distribution would be given by

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2\left(1-\rho^2\right)}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2\rho xy}{\left(\sigma_x\sigma_y\right)}\right)\right)$$

By repeating the sampling procedure n times, we get n new sets of optimization inputs, and then a different efficient frontier. The resampled weight for a portfolio would then be given by

$$\mathbf{a}^{\mathrm{Resamp}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{a}_i$$

The resampled portfolios should reflect a greater diversification (more assets enter in the solution) than the classical mean-variance efficient portfolio and should also exhibit less sudden shifts (smooth transitions) in allocations as return requirements change. Both characteristics are desirable for practitioners.

Recent literature has shown unambiguous results in favor of resampled portfolios in out of sample analysis (Markowitz and Usmen, 2003; Pawley, 2005; Wolf, 2006). However Harvey *et al.* (2006), evaluating Bayes vs. resampling methods, posit that the choice of risk aversion drives the results. Kohli (2005) concludes that despite the fact that there are no conclusive advantages or disadvantages of using resampling as a technique to obtain better returns, resampled portfolios do seem to offer higher stability and lower transaction costs, the two crucial features for long-term investors' choices.

Integration of Market and Credit Risk: As seen earlier, the only risk factor considered was the volatility of returns. However, some asset classes have other types of risk, and so there are risk premia embedded in their returns to compensate for the additional risk taking. This is the case of credit risk, liquidity risk, and legal risk. Therefore, in a Risk-Return framework with the volatility being the only risk considered, assets with premium for other type of risks will dominate the others. In the case of credit risk, corporate bonds tend to dominate treasury bonds, since market risk will be similar, but the risk premium for the credit risk and the corresponding returns of corporate bonds are expected to be higher.

Generally, market and credit risks are the most important sources of risk in terms of the impact on profit and loss at the portfolio level. Credit risk is the uncertainty of changes in value related to a default or a change in the credit quality of the corresponding counterparty, while uncertainty in market risk originates essentially from the volatility of market prices. Credit risk is also considered one of the sources of market risk (Duffie and Singleton, 2003). Moreover, credit risk is generally assessed over longer horizons than market risk, because the number of events of default or enhancement and deterioration of credit quality is not sufficient enough along short horizons to allow in making the credit assessments at statistical significance.

Besides, information required for calculation of each type of risk is different (for credit risk, we need probability of default and, for market risk, expected change in market value). Therefore. practitioners treat credit and market risks separately, and their integration is not common. However, as higher shares of a portfolio have been allocated into high-yield assets, incorporating credit risk within an integrated view becomes mandatory.

As the current reality of the financial world (BIS, 2003) and the recent academic trend are recognized, this paper tackles the issue of integration between market and credit risks. Jarrow and Turnbull (2000), Duffie and Singleton (2003), Hou (2003), and Kuritzkes *et al.* (2003) provide a framework for an integration approach. In the proposal for portfolio optimization, Markowitz (1952) framework is expanded by incorporating a credit risk dimension. In this sense, expected default was used as the credit risk measure for its intuitive appeal and for its ease of computation. The riskier a portfolio is, the higher the expected default. Besides, a more practical approach was adopted because the computation of any other default measure would be too time-consuming.

The Efficient Surface Frontier

A simultaneous assessment of market risk and credit risk is provided by building an efficient surface frontier, which is a three-dimensional figure representing the set of efficient portfolios. The three dimensions are: expected return, expected volatility and expected default. The inclusion of a third dimension may represent an increase of economic welfare as long as there is a credit risk premium. It is worth noting that the whole premium for the credit yield curves is assumed to be due to credit risk. However, as can be seen in Huang and Huang (2003), credit risk is not the only factor that affects the premium of the credit curves. In this case, the additional return from credit should more than compensate the underlying credit risk. Besides, the correlation between market and credit risks should be less than one, i.e., there must be diversification by risk type and, consequently, an expansion of the set of feasible allocations opportunities for the investor.

Assuming zero correlation between market and credit risk, the weights of all the portfolios are calculated in the efficient surface frontier just by including the restriction $E\ (d) \leq y$ to the minimization problem:

$$\underset{a}{\mathrm{Min}} \frac{1}{2} \mathbf{a}^{T} \mathbf{V} \mathbf{a}$$

subject to

$$E\ (R_a) = \mathbf{a}^T \mathbf{X} = x$$

$$E\ (d) = \mathbf{a}^T \mathbf{D} \leq y$$

where d is the default rate of the portfolio and $\mathbf{D}$ is the vector of default probabilities for each asset. The previous optimization problem is repeated for different levels of expected default rates in order to generate the surface. In the presence of a risk-free asset, the efficient surface frontier, for each level of credit restriction, also consists of a straight line—the Capital Market Line (CML).

An Integrated Measure of Market and Credit Risk

So far the credit and market risks were treated separately. However, some studies have tried to integrate market and credit risks into a single number (Jarrow and Turnbull, 2000; and Duffle and Singleton, 2003). Here, a way is proposed to convert credit risk into market risk, creating an integrated measure of risk. Also, this new measure of risk is used to build an Adjusted Sharpe Index to consider market and credit risks. In fact, the methodology is general, and can be applied to other types of risk that create a risk premium, and not only to credit risk.

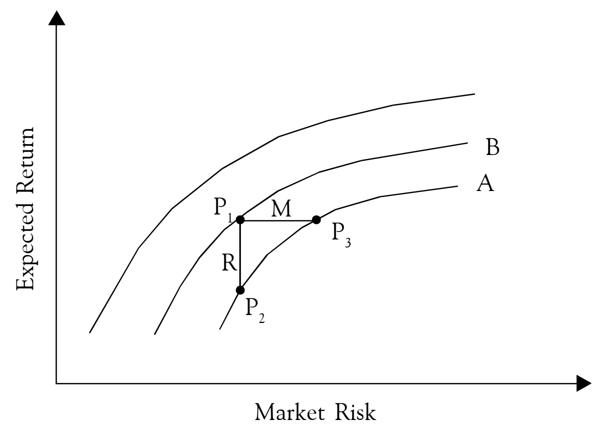
The idea is to convert the credit risk into market risk using the return's dimension as a reference. The integrated risk measure is the sum of the market and the credit risks converted to market risk in the way described below.

Consider, in the Market Risk x Return plane, a yield curve A with a degree of credit risk C_A, and a yield curve B with credit risk C_B, higher than C_A (Figure 1).

A portfolio P_1 is also considered in the yield curve B, and a portfolio P_2 on the yield curve A, both with the same market risk. The credit risk difference (C_B–C_A) of portfolios P_1 and P_2 can be associated[1] with a premium R that is the vertical distance P_1–P_2. This premium

[1] In fact, this premium can be associated to other types of risk, like liquidity risk.

Figure 1: Trade-off Market and Credit Risk



R can also be obtained by moving from portfolio P_2 to the right, over the curve A, until reaching a portfolio P_3 with the same return of portfolio P_1. This portfolio P_3 will have a higher market risk than portfolios P_2 (and P_1), with the same credit risk of P_2, so that the premium R can also be obtained with the difference of the market risk of portfolios P_1 (or P_2) and P_3, which is called M. Therefore, as the premium R can be obtained either by increasing the market risk in M or the credit risk in (C_B–C_A), it can be argued that both risks are equivalent. This will be the definition of risk equivalence.

The Integrated Risk Measure (*IRM*) will use this equivalence. First, it is defined as the Conversion Value (*CV*) of a credit risk ε in terms of market risk of a portfolio P_i in the following way:

$$CV(\varepsilon, P_i) = MR(P_j) - MR(P_i)$$

Given that:

$$\varepsilon = (CR[P_i] - CR[P_j]) > 0 \text{ and } R(P_j) = R(P_i)$$

where

$MR(P)$ is the market risk of a portfolio P

$CR(P)$ is the credit risk of a portfolio P

$R(P)$ is the expected return of a Portfolio P

The idea is that the market risk differential of portfolios i and j generate the same excess return of the credit risk differential (ε), so that both are equivalent. It is noted that this is a local trade-off of market and credit risks, and is valid in the neighborhood of P_i with radius ε.

Then an Integrated Risk Measure (IRM) is defined which will sum the converted credit risk from the default-free curve up to the credit curve of the portfolio to estimate the integrated risk. The definition is recursive:

$IRM\ (P_i, \varepsilon) = IRM(P_j, \varepsilon) + CV(P_i, \varepsilon)$

where

$IRM(P) = MR(P) \leftrightarrow P$ in the default-free curve

$\varepsilon = (CR[P_i]–CR[P_j]) > 0$

The calculation depends on the ε factor that is the credit risk difference between the curves. In fact, this factor can be understood as the precision of the conversion value summation.

In terms of financial efficiency, the traditional Sharpe Index is a measure of the mean excess return per unit of market risk. It does not capture all the effects from other types of risk. In order to overcome this pitfall, we propose a ratio of mean excess return per unit of integrated risk

Our ratio has on the numerator the excess return, as in the traditional Sharpe Index. The denominator has the integrated measure of risk, i.e., the market risk plus the credit risk converted into market risk in a recursive way, as described in the previous section. The Adjusted Sharpe Index (*ASI*) is defined as follows:

$ASI(P_i, \varepsilon) = (R[P_i, \varepsilon)]–R[SR])/IRM(P_i, \varepsilon)$

where

SR is the short-term default-free interest rate.

It is noted that the *ASI* of any portfolio in the default-free curve is calculated in the same way as the traditional Sharpe Index, since there is no credit risk to be converted. As the credit exposure of a portfolio increases, one should expect an increase on the traditional Sharpe Index, since returns tend to increase and the volatilities are little affected by higher credit risk. The Adjusted Sharpe Index, however, captures this higher credit risk in the integrated risk measure, located in the denominator of the ratio.

It is observed that the metric takes into account market and credit risks and can be used to price the trade-off between these two sources of risk. Moreover, it clarifies the portfolio choice problem in an integrative framework and so it has multiple practical usages.

Empirical Study

Data and Implementation

Tests are based on two samples: a Global Assets Sample; and a US Credit Sample. The Global Assets Sample consists of monthly data of 15 Lehman Brothers, JPMorgan international total return indices and also 3 stock indices, from June 1998 to July 2006. All total return time series are calculated on US-Dollar basis and a 3-month US T-Bill rate is used when calculating excess returns.[2] Table 2 presents the description of each asset considered. This sample is used for the optimization model selection as it is a diversified group of assets in terms of currency, market and instrument type.[3]

[2] Our evaluation is under a US based investor perspective.

[3] The results for the US Credit Sample were slightly different due to the high correlation assets. However under the perspective of a Global investor the Global sample is more representative.

Table 2: Descriptive Information						
Name	**Mnemonic**	**Market**	**Currency**	**Instrument Type**	**E[R] (%)**	**Std. (%)**
Asset Backed Securities	ABS	USA	USD	Bonds	0.470	0.778
DAX30	DAX	Germany	EUR	Stocks	0.391	5.721
EMBI Plus	EMBI	Emerging	USD	Bonds	0.908	3.384
Treasuries EUR 1-5years	EUR1-5	Euro Zone	EUR	Bonds	0.531	2.865
Corporates	Corp	USA	USD	Bonds	0.490	1.090
FTSE100	FTSE	UK	GBP	Stocks	0.442	3.492
Global Index	Global	Global	USD	Global	0.462	1.630
Gold	Gold	USA	USD	Currency	1.098	7.908
High Yield Bonds	HY	USA	USD	Bonds	0.463	2.272
Bank Deposits EUR 1M	DepEUR	Euro Zone	EUR	Cash	0.427	2.402
Bank Deposits JPY 1M	DepJPY	Japan	JPY	Cash	0.188	2.663
Bank Deposits USD 1M	DepUSD	USA	USD	Cash	0.308	0.163
Mortgage Backed Securities	MBS	USA	USD	Bonds	0.467	0.765
S&P500	S&P	USA	USD	Stocks	0.572	3.707
Treasury Bill 1-3 month	Tbill	USA	USD	Bonds	0.278	0.151
Treasuries US 1-5years	US1-5	USA	USD	Bonds	0.387	0.671
Guilts	GLT	UK	GBP	Bonds	0.656	2.619

Note: This Table presents descriptive information of each asset considered in the analysis. UK stands for United Kingdom. EMBI stands for Emerging Markets Bond Index. Expected return is the mean monthly return calculated over the sample period. Std is the monthly standard deviation also calculated over the sample period.

The usual positive relation between market risk (standard deviation) and return is to be observed here. Also, this sample is diversified in terms of country, instrument type and risk, thereby providing an adequate database for evaluating portfolio optimization models.

The US Credit Sample consists of monthly data of 10 Lehman Brothers US total return indices plus the S&P500, for the period from January, 1993 to March, 2007. All total return time series are calculated on the basis of US dollor and a three-month US T-Bill rate is used when calculating excess returns. Table 3 presents the description of each asset considered. The main purpose of this sample is to isolate any home bias effect and verify if it has any influence in our integrated financial efficiency measure behavior.

Table 3: Descriptive Information						
Name	Mnemonic	Market	Currency	Instrument Type	E[R] (%)	Std. (%)
US Treasuries	UST	USA	USD	Bonds	0.510	1.295
Corporates Aaa	CorpAaa	USA	USD	Bonds	0.559	1.382
Corporates Aa	CorpAa	USA	USD	Bonds	0.558	1.325
Corporates A	CorpA	USA	USD	Bonds	0.560	1.372
Corporates Baa	CorpBaa	USA	USD	Bonds	0.580	1.507
High Yield Ba	HYBa	USA	USD	Bonds	0.699	1.535
High Yield B	HYB	USA	USD	Bonds	0.631	2.039
High Yield Caa	HYCaa	USA	USD	Bonds	0.614	3.402
High Yield CaD	HYCaD	USA	USD	Bonds	0.974	5.022
US Universal	USUni	USA	USD	Bonds	0.543	1.056
S&P500	SP500	USA	USD	Equities	0.772	3.956

Note: This Table presents descriptive information of each asset considered in the analysis. Expected return is the mean monthly return calculated over the sample period. Std is the monthly standard deviation also calculated over the sample period.

First, the performance of the eight optimization strategies presented in literature review are analyzed, over the Global sample. The estimation period *(p)* in an out of sample analysis is varied. The parameters are estimated using monthly return observations of the past *p* months. Then the efficient risky portfolio is defined and held for the next *(e)* months. Then the parameters are re-estimated and the portfolio weights are adjusted. The models' performance is evaluated considering a short selling constraint.

To judge the financial performance of the strategies their empirical Sharpe-Ratios are computed, and the stability of the allocation weights generated by the model are evaluated. The mean of the standard deviations of each asset's allocation weights are used for the period as a proxy for dispersion. To infer the diversification, the mean Herfindahl index is used, given by the sum of the squared asset allocation weights. Finally, the portfolio optimization is considered with different credit risk constraints, using the expected default as the credit risk measure. In this way, we can assess if the choice of optimization method is sensitive to the credit risk exposure.

Once the best model is reached, an efficient surface is generated for both samples and the Adjusted Sharpe Index is calculated, discussing the results. Next part of the paper presents the results.

Results

Models Evaluation: The Sharpe Index of the different strategies calculated over the realized return time series of each model, are presented in Table 4 for the total period from 1998 to 2006, considering $e = 1$ or 6; and p varying from 60 to 72 months. Three levels of credit limit: A, Baa and Ba were considered.

Table 4: Sharpe Index								
Credit Risk	Mark	Jorion	Kempf	Horst	Bayes	Risk Adj	Equil	Mich
Credit Risk: A								
60-1	0.187	0.181	0.246	0.212	0.182	0.185	0.112	0.230
72-1	0.197	0.199	0.185	0.160	0.190	0.162	0.071	0.296
60-6	0.152	0.154	0.176	0.180	0.143	0.185	0.101	0.217
72-6	0.178	0.207	0.207	0.192	0.178	0.162	0.065	0.254
Credit Risk: Baa								
60-1	0.258	0.217	0.227	0.291	0.262	0.185	0.112	0.261
72-1	0.238	0.300	0.270	0.275	0.232	0.162	0.071	0.365
60-6	0.191	0.180	0.198	0.206	0.192	0.185	0.101	0.234
72-6	0.267	0.309	0.296	0.300	0.272	0.162	0.065	0.316
Credit Risk: Ba								
60-1	0.261	0.204	0.230	0.294	0.265	0.185	0.112	0.246
72-1	0.249	0.297	0.241	0.270	0.249	0.162	0.071	0.349
60-6	0.188	0.179	0.208	0.212	0.200	0.185	0.101	0.285
72-6	0.275	0.295	0.307	0.306	0.275	0.162	0.065	0.328

Note: This Table presents the Sharpe Index of the efficient portfolio generated by each estimation model. The Sharpe Index is calculated by dividing the excess return observed divided by the standard deviation. Credit risk refers to the constraint considered in the optimization problem.

In terms of financial efficiency, the resampling method (Mich) seems to improve the other models' results, especially if we consider a larger evaluation period. This result is robust for different levels of credit constraints.[4] It is important to notice that among the remaining models, none of them seems to clearly outperform. When the dispersion in the optimal allocations are taken into account, the Global sample is reached and the results are presented in Table 5.

It is clearly seen that Mich is among the models which generate the lower dispersion. Mich outperformed the traditional Markowitz as expected, in all the cases. A low dispersion is a relevant characteristic for an asset allocation methodology as it implies lower transaction costs with lower needs for rebalancing the portfolio. Table 6 evaluates the diversification of the optimum portfolios generated by each model and Mich offered the most diversified portfolio with a herfindahl index of about 0.34, while its value for Markowitz, for instance was over 0.44.

In general it can be stated that Mich offered better results for a long-term passive short selling constrained investor when considering financial efficiency, stability and diversification. The previous result is robust for different estimation and evaluation periods; and credit level constraints.

[4] Compared to the standard Markowitz, from the 12 cases, the Mich just offered worse results for the situation of credit risk Ba and e–p = 60–1.

Table 5: Dispersion								
Credit Risk	Mark	Jorion	Kempf	Horst	Bayes	Risk Adj	Equil	Mich
Credit Risk: A								
60-1	0.043	0.027	0.031	0.029	0.042	0.013	0.020	0.029
72-1	0.028	0.018	0.022	0.020	0.028	0.019	0.016	0.017
60-6	0.050	0.032	0.035	0.032	0.050	0.017	0.021	0.031
72-6	0.029	0.018	0.023	0.022	0.029	0.012	0.017	0.016
Credit Risk: Baa								
60-1	0.038	0.023	0.028	0.025	0.038	0.010	0.020	0.026
72-1	0.027	0.016	0.019	0.018	0.026	0.014	0.016	0.015
60-6	0.044	0.028	0.033	0.030	0.045	0.015	0.021	0.030
72-6	0.021	0.014	0.018	0.014	0.020	0.018	0.017	0.017
Credit Risk: Ba								
60-1	0.037	0.023	0.028	0.024	0.037	0.022	0.020	0.027
72-1	0.026	0.017	0.020	0.018	0.026	0.013	0.016	0.014
60-6	0.045	0.029	0.033	0.028	0.040	0.029	0.021	0.029
72-6	0.021	0.013	0.018	0.015	0.021	0.019	0.017	0.014

Note: This Table presents the dispersion in the asset allocation weights given by its mean standard deviation calculated over the whole out of sample period.

Table 6: Herfindahl Index								
Credit Risk	Mark	Jorion	Kempf	Horst	Bayes	Risk Adj	Equil	Mich
Credit Risk: A								
60-1	0.452	0.542	0.552	0.570	0.458	0.905	0.783	0.353
72-1	0.431	0.589	0.546	0.542	0.434	0.906	0.922	0.339
60-6	0.486	0.589	0.566	0.611	0.494	0.908	0.821	0.339
72-6	0.443	0.617	0.578	0.580	0.443	0.910	0.917	0.347
Credit Risk: Baa								
60-1	0.469	0.572	0.573	0.586	0.471	0.816	0.783	0.388
72-1	0.455	0.608	0.574	0.586	0.458	0.818	0.922	0.366
60-6	0.482	0.587	0.586	0.596	0.488	0.811	0.821	0.395
72-6	0.452	0.644	0.600	0.621	0.467	0.817	0.917	0.371
Credit Risk: Ba								
60-1	0.469	0.564	0.569	0.588	0.472	0.544	0.783	0.372
72-1	0.454	0.609	0.572	0.587	0.454	0.541	0.922	0.358
60-6	0.474	0.583	0.582	0.585	0.467	0.540	0.821	0.370
72-6	0.438	0.681	0.587	0.607	0.438	0.545	0.917	0.373

Note: This Table presents the mean herfindahl index calculated over the optimum portfolio allocation weights considering the whole out of sample period.

The Efficient Surface Frontier: From the evaluation of models, a clear advantage of the Michaud methodology is observed for defining the efficient frontier. With this in mind, the efficient surface frontier was generated, adding the credit risk constraint in the analysis(all available data for the estimation period were considered). The results for the Global sample are presented in the following Figures:

Figure 2 presents the three-dimensional efficient surface plotted along the expected return, expected volatility and expected default dimensions. Figure 3 shows two-dimensional efficient frontiers along the expected return and the expected volatility dimensions, each efficient frontier corresponding to a different value of expected default. In other words, it represents the projection of the three-dimensional efficient surface of Figure 2 on the expected return-expected volatility plan.

From these Figures, we observe: i) a trade-off effect between return and market risk; ii) a trade-off effect between return and credit risk; and iii) a substitution effect between market risk and credit risk.

The Figures show that the inclusion of the credit risk restriction results in one Markowitz-like efficient frontier for each credit risk value. Then, these curves are concatenated along the credit risk axe to form the overall efficient surface. The main outcome is that the set of opportunities for the investor is enlarged by a significant amount. This evidence can be realized by observing Figure 3 and verifying how the efficient curves compounding the efficient surface shift to the left side as the credit risk increases.

Therefore, the highest-credit risk-efficient curve is represented by the leftmost curve, while the lowest-credit risk-efficient curve, by the rightmost one. The incorporation of credit risk make efficient some market risk-total return combinations, which otherwise would have been deemed inefficient, had the credit risk restriction been not included in the optimization model. Furthermore, this effect is more remarkable in the middle part

Figure 2: Efficient Surface Frontier (Global Sample)

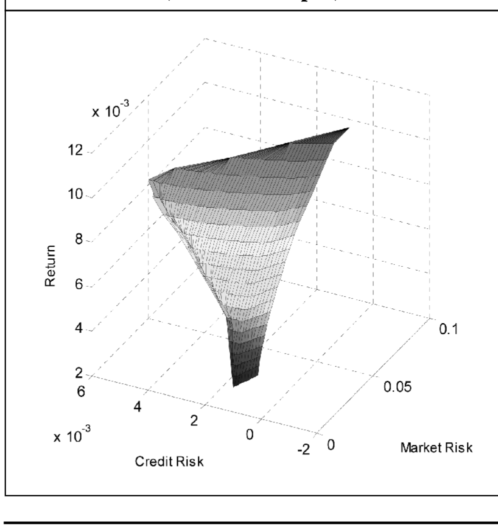
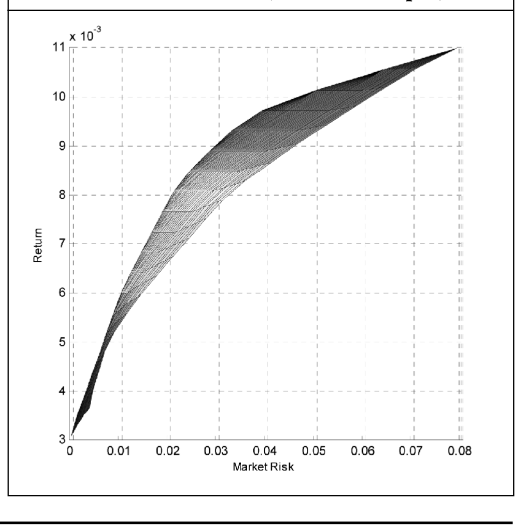


Figure 3: Cross-Sections of the Efficient Surface Frontier (Global Sample)



of the market risk domain, where diversification factor plays its major role. On the other hand, in the lowermost and uppermost parts of market risk range, credit bets result in almost no increase in the efficient returns, i.e., in these parts of market risk range only the trade-off effect can be observed between return and market risk. This can be explained, on the lowermost part, by the fact that high-credit risk portfolios cannot achieve a market risk close to the minimum variance portfolio and thus do not enter in the frontier. On the uppermost part, this is explained by the existence of assets with high expected return and market risk and no credit risk, specifically equities and gold. These assets are efficient substitutes for the high credit risk assets.

Besides, the domain of valid market risks is slightly augmented, i.e, there is a reduction of the minimum feasible market risk.These lower market risk points become attainable by increasing credit risk, which is another evidence of risk diversification. Figures 4 and 5 are analogous to Figures and 2 and 3, respectively, i.e., and present the results for the US sample: the shape of the surface and curves in Figures 4 and 5 is similar to the ones in Figures 2 and 3, respectively, i.e., there are two trade-off effects between risk and return and the substitution effect, as well. Nonetheless, a narrower range of feasible points were obtained in the market risk and return dimensions (about half the values in comparison to Figures 2 and 3) for the US sample in comparison to the Global sample. Figure 5 shows that the maximum return difference between the highest-credit risk-efficient curve and the lowest-credit risk-efficient curve is approximately 0.75 bps, while this difference in the Global sample (Figure 3) is around 1 bps. Figure 5 also indicates that the interval within the market risk range with the most striking diversification effects is 8 bps, while this interval in the Global sample (Figure 3) is around 3 bps. The Global sample consists of a more comprehensive range of asset than the US sample and this fact may explain why a larger set of opportunities for the Global sample exist.

Figure 4: Efficient Surface Frontier (US Credit Sample)

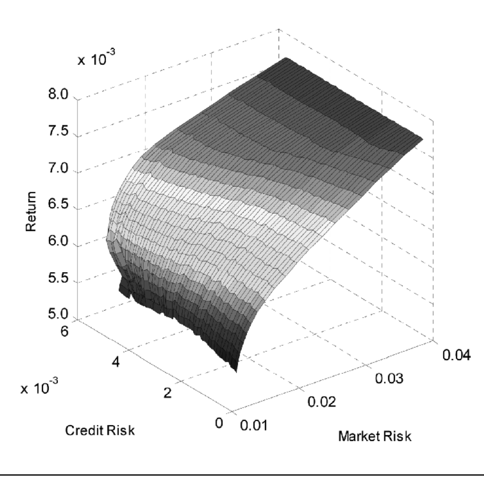
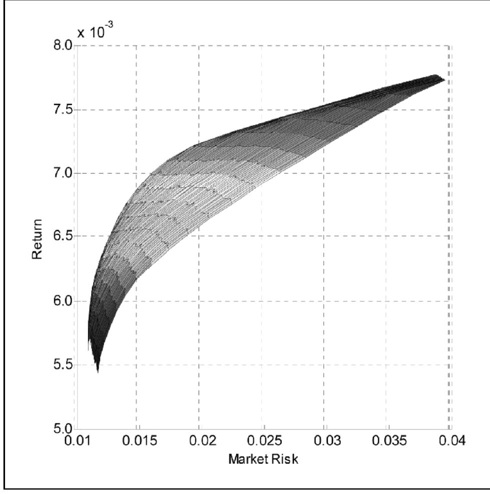


Figure 5: Cross-Sections of the Efficient Surface Frontier (US Credit Sample)



Adjusted Sharpe Index: In the empirical study, a method was proposed to convert credit risk into market risk, creating an integrated measure of risk. The ASI as a way to price the trade-off between market and credit risk. To show the empirical behavior of both ratios, calculation was done for the two sets of data used before, the Global and the US data.

Figure 6 shows the behavior of the Traditional Sharpe Index for the Global data in a three-dimensional graph, while Figure 7 also shows the Sharpe Index with only the credit risk axis. In Figure 7, each curve represents a level of market risk, with the highest risk at the bottom and the lowest at the top. As expected, the Sharpe Index increases with the credit risk. However, as can be seen from Figure 7, this inclination decreases as credit risk grows. This is due to the lack of diversification when high levels of credit risk are reached. It is also seen that the ratio decreases with market risk. The market risk negative relationship between market risk and Sharpe Index is derived from the aspect of the yield curve.

Figures 8 and 9 are equivalents to 6 and 7, but show results for the US Credit sample. It is seen that the general behavior is the same, although there is some noise when high levels of credit risk are reached. This may be due to the lower number of assets and their high correlation, which makes the curves' shapes to be less smooth. Regarding market risk, we see a behavior similar to the Global data sample except for the lower market risk region, when the slope is positive instead of negative.

It is concluded that if credit risk is not considered in the analysis, portfolios with credit risk will dominate others, except for very low or very high level of market risk, when the lack of diversification will diminish the effects of credit risks.

Figure 10 shows the Adjusted Sharpe Index against market and credit risk in a three-dimensional Graph for the Global sample. The modified ratio is not increasing with credit risk, as in the traditional Sharpe Index. Rather, it has a slight decreasing function of the credit risk, except for portfolios with low market risk. Figure 11 also shows

Figure 6: Sharpe Index Over the Efficient Surface Portfolios (Global Sample)

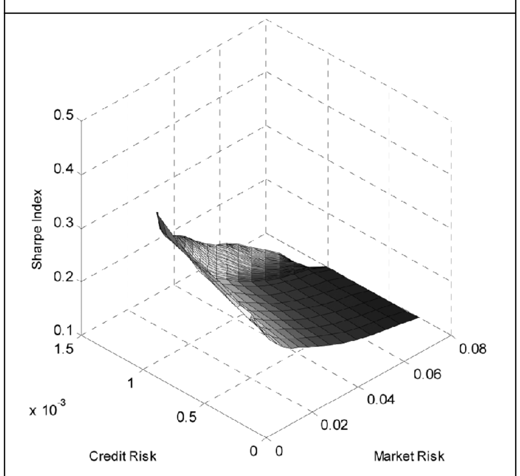
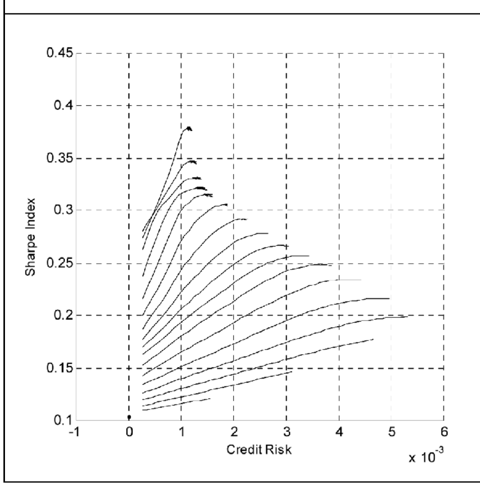


Figure 7: Cross-Sections of the Sharpe Index Over the Efficient Surface Portfolios (Global Sample)



these features, with the curves on the top (those with lower market risk) increasing, and with the curves in the bottom (higher market risk) slightly decreasing.The maximum ratio belongs to the region with low market risk, around 0.027.

Figures 12 and 13 show the Adjusted Sharpe Index for the US sample, and are analogous to 10 and 11. It is seen that the general behavior is the same. The maximum ratio belongs to the region with low market risk and high credit risk.

Comparing the Traditional Sharpe Index with Adjusted Sharpe Index, that a change of pattern can be identified. While the Traditional one suffers exogenous effects of credit risk that makes it an increasing function of credit risk, the Modified one incorporates the

Figure 8: Sharpe Index Over the Efficient Surface Portfolios (US Credit Sample)

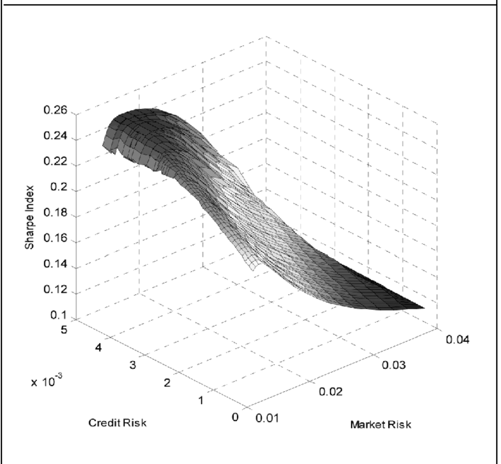


Figure 9: Cross-Sections of the Sharpe Index Over the Efficient Surface Portfolios (US Credit Sample)

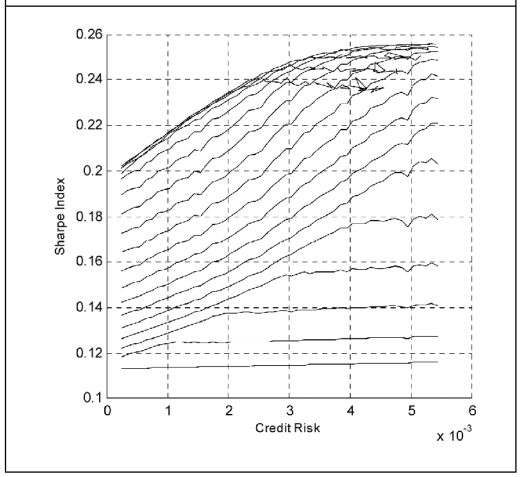


Figure 10: Adjusted Sharpe Index Over the Efficient Surface Portfolios (Global Sample)

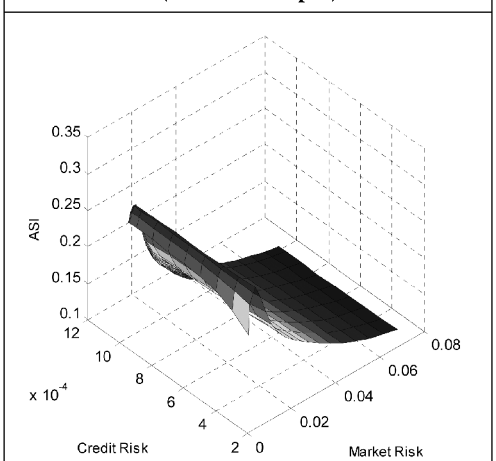
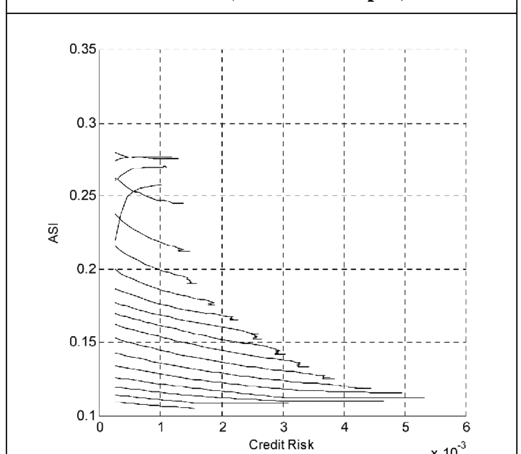


Figure 11: Cross-Sections of the Sharpe Index Over the Efficient Surface Portfolios (Global Sample)



credit risk. The result of this incorporation is a ratio that is an increasing function of credit risk when there is an area of low market risks and decreasing otherwise. Thus, the optimal level of credit risk can be identified when choosing a portfolio.

Since the Modified Sharpe Index considers the credit risk, it is a better tool to evaluate the trade-off risk-return of portfolios, being adequate, for example, to rank fixed income mutual funds with credit risk exposure.

Finally, in order to assess the empirical behavior of the risk-return trade-off of the new integrated risk measure with expected returns, they can be calculated for the two sets, before the Global and the US data. Results are in Figures 14 and 15.

Each Figure has several lines, and each line represents a level of market risk. The lower-left end of each line contains the portfolios with lower credit risk, while the upper-right end contains the higher credit risk portfolios. As expected, there is a positive relationship between returns and integrated risk. It can also be identified, on the lower-right part of the figures, that portfolios with high credit risk and low market risk tends to dominate the others, while on the upper-left area portfolios with higher credit risk and high market risk are dominated by others. In the middle-range of the figures, the break-even of these effects is seen, where portfolios seem to be equalized, with nobody dominating nobody.

Figure 14: Risk-Return Trade-off (Global Sample)

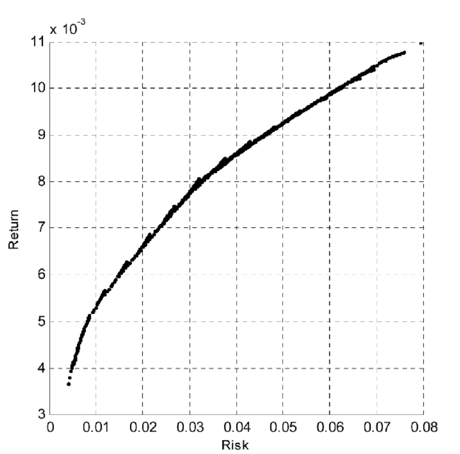
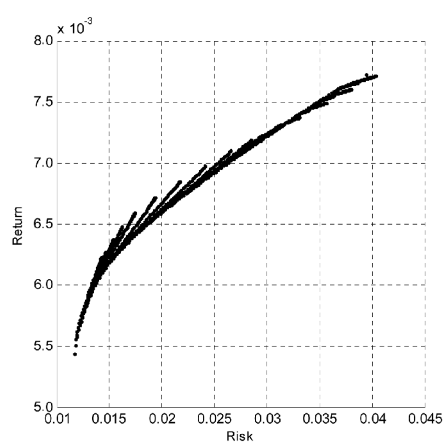


Figure 15: Risk-Return Trade-off (US Credit Sample)



Conclusion

This paper had two main goals: first to propose an integrative measure of financial efficiency which takes into account the market and credit risks, and so pricing the trade-off among the previous risks. The results indicate that decisions based on the traditional Sharpe Index are misleading when assets with different levels of credit risk are in the analysis. On the other hand, the Adjusted Sharpe Index seems to correct the previous distortion.

And the second goal is to evaluate eight portfolio optimization models in terms of financial efficiency, stability of the allocation and diversification of the portfolio. In an out-of-sample analysis, the results indicated that for a US based global investor, Michaud's methodology outperformed the others when a global diversified set of assets is considered. This result is robust for different levels of credit risk constraints (Michaud, 1998).

The empirical approach can be generalized in two ways. First, any other risk factor that generates a premium in terms of expected return can be incorporated in the analysis as an additional dimension. Thus, not only the credit risk but other risks would be converted into market risk in the way we described in an integrated measure of market and credit risk. Second, other measures of market and credit risk other than volatility and expected default can be used. For instance, VaR-based measures of risk may be used, although the numerical methods to generate the efficient surface frontier would be quite more complex. ❖

Acknowledgment: *The authors would like to thank Antonio Francisco Silva Junior and Paulo Cacella for very helpful comments. The views expressed in this work are those of the authors and do not reflect those of the Banco Central do Brasilia or its members.*

References

1. BIS (2003), Trends in Risk Integration and Aggregation, Basel Committee on Banking Supervision.
2. Black Fisher and Robert Litterman (1992), "Global Portfolio Optimization", *Financial Analysts Journal*, Vol. 48, No. 5, pp. 28-43.
3. Duffie Darrell and Kenneth Singleton (2003), *Credit Risk,* Princeton University Press.
4. Feldman David and Haim Reisman (2002), "Simple Construction of the Efficient Frontier", *Forthcoming, European Financial Management.*
5. Horst Jenke, Roon Frans and Bas Werker (2002), "*Incorporating Estimation Risk in Portfolio Choice*", CentER Working Paper No. 65. Available at SSRN: http://ssrn.com/abstract=244695
6. Hou Yuanfeng (2003), "Integrating Market Risk and Credit Risk: A Dynamic Asset Allocation Perspective", Yale University, Working Paper.
7. Huang Jing-zhi and Huang M (2003), "How Much of the Corporate-Treasury Yield Spread is Due to Credit Risk?", 14th Annual Conference on Financial Economics and Accounting (FEA), Texas Finance Festival. Available at SSRN: http://ssrn.com/abstract=307360
8. Jarrow Robert and Stuart Turnbull (2000), "The Intersection of Market and Credit Risk", *Journal of Banking and Finance,* Vol. 24, pp. 271-299.
9. Jorion Philippe (1986), "Bayes-Stein Estimation for Portfolio Analysis", *The Journal of Financial and Quantitative Analysis,* Vol. 21, No. 3, pp. 279-292.

10. Kempf Alexander, Kreuzberg Klaus and Christoph Memmel (2002), "How to Incorporate Estimation Risk into Markowitz Optimization", Operations Research Proceedings 2001, Berlin *et al*. 2002.

11. Kohli Jasraj (2005), "An Empirical Analysis of Resampled Efficiency", Thesis submitted to the Faculty of the Worcester Polytechnic Institute.

12. Kuritzkes Andrew, Shuermann Til and Scott Weiner (2003), "Risk Measurement, Risk Management and Capital Adequacy in Financial Conglomerates", in Herring R and Litan R (Eds.), Brookings-Wharton Papers in Financial Services, pp. 141-194.

13. Manganelli Simone (2006a), "Asset Allocation by Penalized Least Squares", Working Paper.

14. Manganelli Simone (2006b), "A New Theory of Forecasting", Working Paper Series, ECB, No. 584.

15. Markowitz Harry (1952), "Portfolio Selection", *The Journal of Finance,* Vol. 7, pp. 77-91.

16. Markowitz Harry and Nilufer Usmen (2003), "Resampled Frontiers versus Diffuse Bayes: An Experiment", *Journal of Investment Management*, (Fourth Quarter), Vol. 1, No. 4, pp. 9-25.

17. Michaud Richard (1998), *Efficient Asset Management*, Harvard Business School Press, Boston, MA.

18. Sherer Bernd (2002), "Portfolio Resampling: Review and Critique", *Financial Analysts Journal*, Vol. 58, No. 6, pp. 98-109.

19. Wolf Michael (2006), "Resampling vs. Shrinkage for Benchmarked Managers", Institute for Empirical Research in Economies, University of Zurich, Working Paper Series, No. 263.

20. Zimmer Christian and Beat Niederhauser (2003), "Determining an Efficient Frontier in a Stochastic Moments Setting", *Universidade de Sao Paulo, Departamento de Administraeao*, Working Paper Series N° 03/010.

Reference # 37J-2008-03-01-01